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Signal analysis of gravitational wave tails

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Abstract. This paper investigates, by means of analytical and numerical methods, the level at which gravitational wave “tails”, generated by inspiralling compact binary systems, could be detected in future laser interferometric detectors. Gravitational wave tails result from the non-linear interaction of the usual quadrupole radiation, generated by an isolated system, and of the static monopole field associated with the total mass of the system. Their contribution in the field at large distances from the system arises at the three halves post-Newtonian approximation. Their inclusion in the matched filters of the future data analysis of inspiralling binary signals has been recently proposed [1]. We consider here a family of matched filters comprising the effects of tails, and depending on a set of “test” parameters including, as a particular independent parameter, the total mass that is in factor of the wave tail. By filtering the output of the detector, composed of the useful signal superposed with some Gaussian noise, through this family of filters, we obtain a set of measured values of the parameters. Detecting the effects of tails is then equivalent to attribute, with some confidence level, a *non-zero* value to the total mass parameter. The $1 - \sigma$ error bar in the measurement of the total mass is computed as a function of the optimal signal-to-noise ratio (SNR). We find, in the case of an inspiralling neutron-star binary (and of a simplified model of noise in a laser interferometric detector), a minimal required SNR of about 40 – 50 for detection of the effects of tails at the $1 - \sigma$ level. We conclude that the effects of tails will probably be detected in future laser interferometric detectors (say, in a second generation of detectors next to the first generation of VIRGO and LIGO detectors).

1. Introduction

The late inspiral and final coalescence stages of neutron-star or black-hole binary systems should be routinely observed by future broadband gravitational-wave detectors (see [2],[3] for reviews). The estimated number of final coalescence events of neutron stars is a few per year out to a distance of 100 Mpc [4]-[6]. At this distance, the first generation of VIRGO and LIGO detectors [7],[8] might observe the waves with a signal-to-noise ratio (SNR) of order 10 (see e.g. [9]); and future “advanced detectors” could achieve very large SNRs indeed (see [10] for discussion).

Optimal signal processing will be used in order to extract the useful gravitational signal out of the detector noise. This means that the raw output of the detector will be correlated with a family of matched (or Wiener) filters. A matched filter is a filter whose Fourier transform is equal to that of the signal divided by the power spectral density of the noise. A family of filters is required to deal with the unknown parameters of the signal, for instance, the two masses of the binary. Each filter in the family corresponds to a particular set of “test” parameters, and by maximization of the correlation over these parameters, one identifies some “measured” parameters, whose expectation values are equal to the parameters of the signal (for high enough SNR). This technique requires both that the spectral density of the noise in the detector is measured, and that the form of the expected signal is known. In particular, the accuracy with which the signal is known decides the accuracy in the measurement of the parameters [11]. Thus, taking higher-order (post-Newtonian) theoretical corrections into account in the construction of matched filters allows a more accurate measurement of the parameters, and allows also the determination of new parameters. This is interesting both because one will obtain more information concerning the source, and because one will be able to *detect* higher-order effects predicted by general relativity.

In a recent paper [1], it was proposed to take the particular effects associated with the *tails* of gravitational waves into account in the construction of the matched filters of inspiralling binary signals in laser interferometric detectors like VIRGO and LIGO. The present paper is devoted to the investigation, by means of analytical and numerical methods, of the level at which the effects of tails could be detected. As we shall see, the prospects are excellent for detecting these effects in, say, the advanced generation of detectors next to the first generation of VIRGO and LIGO detectors.

The existence of gravitational wave tails is a well-established prediction of general relativity. The tails are created by non-linear interaction between the time-varying multipole moments of the source (for instance, the quadrupole moment) and its static

total mass M [12]–[19]. Physically, the waves undergo a continuous scattering, as they propagate outward from the source, off the Schwarzschild space-time associated with M . This phenomenon is in fact common to the propagation of any integer spin field, like the electromagnetic field, on the Schwarzschild background [20]–[28]. In the gravitational case, the existence of wave tails is but one aspect of a fundamental property of the gravitational interaction, namely that its propagation proceeds both on and *within* the local light cone, i.e., at any velocity smaller than or equal (on average) to the velocity of light [29]–[35].

At large distances from the source (where the detector is located), the contribution of the wave tail is dominantly of post-Newtonian order ε^3 with respect to the usual quadrupole radiation, where ε is the ratio of a typical velocity in the source and of the velocity of light. The wave tail, at this order, has been computed in [18],[36] and has been shown to be consistent, in the sense of energy conservation, with a related contribution [37],[38] in the radiation reaction forces acting within the source. The wave tail involves, besides the total mass M , the components of the quadrupole moment of the source integrated over the whole interval of time extending from the retarded time $t - r/c$ up to arbitrary remote times in the past (this reflects the fact that gravity propagates on average at any velocity smaller than or equal to c). Thus, the wave tail appears to be a *non-local* in time, or “hereditary”, functional of the source’s parameters. By contrast, the lowest-order post-Newtonian radiation (including the usual quadrupole radiation) is purely local, depending on the source’s parameters at time $t - r/c$ only. We shall sometimes refer to the latter local component of the wave as the wave *front*.

The Fourier transform of the wave tail at large distances from the source has a particularly simple form derived in [1] (see also [51],[58]). This form shows that both the *amplitude* and the *phase* of the wave front are modified, in the Fourier domain, by small tail-induced corrections depending on the considered frequency. (Note that these corrections are valid only as long as the post-Newtonian assumptions are valid, i.e. only for *low* enough frequencies corresponding to much larger wave lengths than the Schwarzschild radius of the source.) In particular, the modification of the phase of the wave front introduces a new effect when the radiation is not monochromatic, as is the case of the radiation generated by an inspiralling binary. This effect basically results in a small tail-induced shift of the relative positions of the different frequency components of the wave along the line of sight from the source to the detector, or, equivalently, in a small tail-induced relative delay of the instants of arrival of these frequency components at a given distance of the source.

By including the effects of tails in the matched filters of inspiralling binaries, we introduce the total mass M of the source as a new parameter in addition to the parameters of the wave front. Note that the mass M is not, in the case of inspiralling binaries, independent of the parameters of the wave front. Indeed, the latter parameters include separately the usual chirp mass $\mathcal{M} = \mu^{3/5} M^{2/5}$ of the quadrupole radiation (where μ is the reduced mass of the binary), and also for instance the parameter $\nu = \mu/M$ entering in the first post-Newtonian radiation [39]–[44]. Thus, from these parameters, we deduce $M = \mathcal{M}\nu^{-3/5}$. In this paper, we propose to use a family of matched filters depending on a family of test parameters including the total mass M in front of the wave tail as a parameter which is *independent* of the parameters of the wave front, and then to maximize the correlation by independently varying all these parameters, including M . The reason for proceeding in this way is to ensure that, if the effects of tails do *not* exist, the optimal filter of the family will be able to find a value of M compatible with zero together with some definite values of the other parameters (e.g., the parameters $\mathcal{M}$ and ν). If the effects of tails do exist, the optimal filter will measure a non-zero value of M , and it will be checked that this value is eventually consistent with the measured values of the parameters of the wave front (see [45] for discussion).

Considering the mass M in factor of the wave tail as an independent parameter in the matched filters has a further advantage. Indeed, the anticipated one-sigma precision of the measurement of M , and thus of the measurement of the tail effect itself, can be analytically computed, in the limit of high SNR, from the component of the covariance matrix corresponding to M . Let us denote by $\sigma_M^2 = C_{00}$ the variance of M , chosen to be the 00 component of the covariance matrix $C_{\alpha\beta}$, where α, β range over all the parameters. The level at which the tail effect will be detected (at the $1 - \sigma$ confidence level) is simply the level at which σ_M will get smaller than the value of M itself. Using this criterion, we compute in this paper that this will happen when the optimal SNR of the signal will get larger than a minimal value equal to 23 in the case of white noise, and to 47 in the case of “coloured” noise appropriate for shot noise in the standard recycling configuration of a laser interferometric detector with Fabry-Perot cavities. These figures are for a neutron-star binary with $M = 2.8M_\odot$. For a black-hole binary with $M = 10M_\odot$, we obtain 6.4 in the case of white noise, and 13 in the case of coloured noise.

The detection of tails will necessitate a thorough data analysis of the outputs of (preferably) several detectors operating at a time. Here we simulate this data analysis, assuming that a preliminary (non-optimal) analysis has been performed on

line, yielding some preliminary values of the parameters. We construct a lattice of filters by the method of [46],[47] (one dimension in the lattice corresponds to the total mass M), and we filter through it the output of the detector, composed of a known signal added to simulated Gaussian noise. By repeating the filtering process for a large number of realizations of noise, we obtain the distribution of the measured values of M , and in particular the standard deviation of the measured M around its mean. When the optimal SNR increases, the mean value tends to M , and the standard deviation tends to zero. Using the same criterion as above, we obtain that the tail effect should be seen, when $M = 2.8M_{\odot}$, at a SNR of about 25–30 in the case of white noise, and of about 40–50 in the case of coloured noise. When $M = 10M_{\odot}$, the SNRs are less than about 15. All these values are in good agreement with the analytical computations.

We think that the detection of tails is challenging, but is quite within reach in future detectors. Let us assume for discussion that three neutron-star coalescence events occur per year out to a distance of 200 Mpc. This is the standard estimate of Phinney [5], who gives also upper and lower limits of three events per year out to 1 Gpc and 23 Mpc respectively. (This standard estimate is in agreement with other classic estimates [4],[6], but a recently published estimate [48] is 100 events per year out to 200 Mpc — a value 30 times more optimistic than the standard estimate.) Let us also assume that the currently planned VIRGO and LIGO detectors will observe these events with a SNR of order 10. (We are perhaps slightly optimistic in assuming such a SNR, but the values quoted in the literature range from about 10 [9] up to about 40 [3].) Then, we can roughly estimate that VIRGO and LIGO will observe $3(10/40)^3 = 1/20$ events per year with a SNR larger than or equal to 40, minimal value required for detecting the tails. This rate of events is too small, but it drastically depends on the assumed standard estimate (with the estimate of [48] one gets about one event per year). The number of coalescence of black holes, or of a neutron star and a black hole, is not known, but could be comparable with the number of coalescences of neutron stars [5],[6]. The minimal value $\text{SNR} \sim 10 - 15$ obtained in this paper for detecting the tails in the case of a black-hole coalescence with $M = 10M_{\odot}$ could thus be reached even in the first generation of VIRGO and LIGO detectors. Finally, advanced generations of detectors should detect the tails at a reasonably high rate. For instance, the advanced LIGO detector envisaged by Finn and Chernoff [10] should detect one event each two years with a SNR equal to 40 (value computed by using the standard estimate for neutron-star coalescences in equation (5.5) of [10]).

The plan of this paper is as follows. In the next two sections we summarize some formulas concerning the generation of gravitational wave tails by isolated systems

(section 2), and the theory of optimal filtering (section 3). Section 4 is devoted to the analytical computation of the covariance matrix of the signal (including the tail), and section 5 to the numerical simulation of the detection of tails. A non-optimal filtering matched on the wave front of the signal is examined in an appendix.

2. Generation of gravitational wave tails

This section is a review of the relevant formulas (taken from [36] and [1]) concerning the generation of gravitational wave tails by an isolated system, at large distances from the system where the detector is located. We shall denote by $h(t)$ that particular combination of the components of the wave which is directly felt by the detector (e.g., $h(t) = \delta L(t)/L$ is the relative variation of the arms' length of a laser interferometric detector), and we shall refer to it as the gravitational wave, or signal.

The wave $h(t)$ is given, at the post-Newtonian order $\varepsilon^3 \sim c^{-3}$ beyond the “Newtonian” quadrupole radiation (where ε is the ratio of a typical velocity in the source and of the velocity of light), and at first order in the inverse of the distance r from the source, by the expression [36]

$$h(t) = h_0(t) + \frac{2GM}{c^3} \int_{-\infty}^t dt' \left[\ln \left(\frac{t-t'}{2b} \right) + \frac{11}{12} \right] \frac{d^2 h_0}{dt^2}(t'). \quad (2.1)$$

This expression is derived in a “radiative” coordinate system, covering the regions at large distances from the source, and defined in (2.4) below. Throughout this paper, we shall work consistently with the precision at which the expression (2.1) is valid. That is, we shall consistently neglect all terms of post-Newtonian order $O(\varepsilon^4) = O(c^{-4})$ beyond the quadrupole radiation, and/or of order $O(r^{-2})$ in the inverse of the distance from the source. Thus, for simplicity's sake, we shall not mention in (2.1), and in other formulas below, the neglected terms $O(\varepsilon^4)$ and/or $O(r^{-2})$.

The wave $h(t)$ is composed of two distinctive parts. The first part $h_0(t)$ can be referred to as the “wave front”. It is defined by the equation

$$h_0(t) = \frac{G}{rc^4} \{ F_+(p_i p_j - q_i q_j) + F_\times(p_i q_j + q_i p_j) \} \frac{d^2 K_{ij}}{dt^2}(t - r/c, \mathbf{n}) , \quad (2.2)$$

where we denote by $\mathbf{n}$, $\mathbf{p}$, $\mathbf{q}$ an oriented orthonormal triad, with $\mathbf{n}$ the unit vector directed from the source towards the detector and $\mathbf{p}$, $\mathbf{q}$ two unit polarization directions in the plane orthogonal to $\mathbf{n}$. The two coefficients F_+ and $F_\times$ denote the standard

detector beam-pattern functions which are for instance given, in the case of a laser interferometric detector, by equations (104) of [2]. (Note that the functions F_+ and $F_\times$ depend on the choice $\mathbf{p}, \mathbf{q}$ of polarization directions, but of course $h_0(t)$ itself is independent of this choice.) The tensor $K_{ij}(u, \mathbf{n})$, where $u = t - r/c$ is the retarded time, is related to the (trace-free) multipole moments of the source (mass type moments $I_{ij}(u), I_{ijk}(u), \dots$ and current type moments by $J_{ij}(u), J_{ijk}(u), \dots$) by

$$\begin{aligned} K_{ij}(u, \mathbf{n}) = & I_{ij}(u) + \frac{1}{c} \left[\frac{1}{3} n_k I_{ijk}^{(1)}(u) + \frac{4}{3} \varepsilon_{kl(i} J_{j)k}(u) n_l \right] \\ & + \frac{1}{c^2} \left[\frac{1}{12} n_k n_l I_{ijkl}^{(2)}(u) + \frac{1}{2} \varepsilon_{kl(i} J_{j)km}^{(1)}(u) n_l n_m \right] \\ & + \frac{1}{c^3} \left[\frac{1}{60} n_k n_l n_m I_{ijklm}^{(3)}(u) + \frac{2}{15} \varepsilon_{kl(i} J_{j)kmn}^{(2)}(u) n_l n_m n_n \right]. \end{aligned} \quad (2.3)$$

(We denote $I^{(n)}(u) = d^n I(u)/du^n$ and $T_{(ij)} = \frac{1}{2}(T_{ij} + T_{ji})$.) The multipole moments of the source are given at retarded time u by explicit integrals extending over the compact-supported stress-energy distribution of matter in the source at the *same* retarded time u . The expressions of the moments of the source are given in [49],[50] (see also [36],[1]). In the limit $c \rightarrow \infty$, the tensor (2.3) reduces to the usual Newtonian quadrupole moment of the source.

The second part in the wave $h(t)$, which is purely of post-Newtonian order ε^3 , can be referred to as the “wave tail”. As is clear from its expression, the wave tail is, in contrast with the wave front, a *non-local* in time functional of the stress-energy tensor of the source, depending on its values not only at the simply retarded time $u = t - r/c$, but also at all times anterior or equal to u . (Note that the precision given by the first term in (2.3) is sufficient to compute the wave tail, which is already of small order ε^3 .) Subject to weak conditions concerning the emission of radiation at very early times $t' \rightarrow -\infty$, one can easily show that the wave tail in (2.1) is given as a convergent integral (see [37],[36],[1]). We have chosen, following [1], to include the term corresponding to the constant 11/12 in the wave tail although this term is a local functional of the source. The constant b in the logarithm of the integrand of the wave tail is an arbitrary positive constant having the dimension of time. It enters the relation between the radiative coordinate system t, r used by the experimenters at large distances from the source and the harmonic coordinate system t_H, r_H covering the source:

$$t = t_H - \frac{2GM}{c^3} \ln \left(\frac{r_H}{cb} \right). \quad (2.4a)$$

$$r = r_H. \quad (2.4b)$$

By substituting (2.4) into (2.1), we eliminate the dependence on the constant b [36]. (Note that the value of the constant 11/12 depends on the coordinate system chosen in (2.4) to cover the source. For instance, the value is 17/12 instead of 11/12 if we use in (2.4) Schwarzschild coordinates instead of harmonic coordinates; compare [1] and the work of Poisson [51].)

The Fourier transform of the gravitational wave $h(t)$ has been derived in [1]. Let us denote by $\tilde{h}(\omega)$ and $\tilde{h}_0(\omega)$ the Fourier transforms of $h(t)$ and of its wave front $h_0(t)$ (where $\omega = 2\pi f$ denotes the angular frequency):

$$\tilde{h}(\omega) = \int_{-\infty}^{+\infty} dt h(t) e^{i\omega t} \quad ; \quad \tilde{h}_0(\omega) = \int_{-\infty}^{+\infty} dt h_0(t) e^{i\omega t} . \quad (2.5)$$

Since $h(t)$ and $h_0(t)$ are real, we have $\tilde{h}(-\omega) = \tilde{h}^*(\omega)$ and $\tilde{h}_0(-\omega) = \tilde{h}_0^*(\omega)$ where $*$ denotes the complex conjugation. Then, by taking the Fourier transform of (2.1), one obtains

$$\tilde{h}(\omega) = \tilde{h}_0(\omega) \left\{ 1 + \frac{2GM}{c^3} \left[\frac{\pi}{2} |\omega| + i\omega \ln(2|\omega|b') \right] \right\} , \quad (2.6)$$

where $|\omega|$ denotes the absolute value of ω . (Note that this expression is only valid for low frequencies ω such that $GM\omega/c^3$ is of small post-Newtonian order $O(\varepsilon^3)$.) In (2.6), the constant b' is related to the constant b by

$$b' = b \exp \left[C - \frac{11}{12} \right] \quad (2.7)$$

where C denotes the Euler constant $C = 0.577\dots$

It is sometimes more useful, and clearer, to write the tail-induced corrections in (2.6) in the form of a real amplitude times a complex exponential. Neglecting higher-order post-Newtonian terms, we can write

$$\tilde{h}(\omega) = \tilde{h}_0(\omega) \left[1 + \frac{\pi GM}{c^3} |\omega| \right] e^{i\theta(\omega)} \quad (2.8)$$

where the square brackets represent a tail-induced modification of the amplitude of the wave, and where

$$\theta(\omega) = \frac{2GM\omega}{c^3} \ln(2|\omega|b') \quad (2.9)$$

represents a tail-induced modification of its phase. The latter modification of the phase introduces a differential effect in the case where the radiation contains more than one frequency in the Fourier domain (see [1]). This effect can be rigorously analyzed when the radiation spectrum is composed of several well-separated wave packets. Two wave

packets, centred around two different frequencies ω_1 and ω_2 (and belonging to the “same” wave), propagate with the same group velocity, but their relative positions are slightly shifted along the line of sight. (The position of the wave packet is the position of its maximum of amplitude in space.) They differ by a small quantity equal to $2GM/c^2$ times the difference of the *logarithms* of their central frequencies ω_1 and ω_2 . Put another way, the two wave packets arrive at a given distance of the source with a small relative delay equal to $(2GM/c^3) \ln(\omega_1/\omega_2)$, with the wave packet corresponding to the lower frequency arriving first. In the case of the radiation of an inspiralling binary (which contains a whole continuum of frequencies because the orbit is decaying), the tail-induced phase (2.9) implies in the time series a phase $\theta[\Omega(t)]$ where $\Omega(t)$ is the instantaneous frequency of the radiation (see the appendix B of [1] for the computation in the case of a decaying orbit). The signal $h(t)$ thus acquires a phase difference with respect to its wave front $h_0(t)$ growing, as the frequency of the signal increases, like $\Omega(t) \ln \Omega(t)$. It appears therefore to be important to include the tail effects, and notably the phase (2.9), in the matched filters of the data analysis of inspiralling binary signals.

Finally, it will be convenient in this paper to make a definite choice of the constant b' appearing in (2.6) (and thus a choice of a particular radiative coordinate system). We shall relate b' to the “seismic cutoff” of a laser interferometric detector, defined to be the frequency ω_s , below which the terrestrial seismic noise prevents any detection, by posing

$$b' = \frac{1}{2\omega_s} . \quad (2.10)$$

(In a more general situation, we can simply think of ω_s as being a typical frequency at which the detector is operating.)

3. Summary of optimal filtering techniques

For more details about the theory of optimal (or matched) filtering, we refer to the book of Wainstein and Zubakov [52] and, in the context of gravitational radiation, to e.g. [2],[53]–[55],[46],[47]. The raw output of the detector $o(t)$ is the superposition of the useful signal $h(t)$ given by (2.1) and of noise $n(t)$:

$$o(t) = h(t) + n(t) . \quad (3.1)$$

The noise is assumed to be a stationary Gaussian random variable, with zero expectation value,

$$\overline{n(t)} = 0 , \quad (3.2)$$

and with (supposedly known) frequency-dependent power spectral density $S_h(\omega)$, using customary notation, satisfying

$$\overline{\tilde{n}(\omega)\tilde{n}^*(\omega')} = 2\pi\delta(\omega - \omega')S_h(\omega) , \quad (3.3)$$

where $\tilde{n}(\omega)$ is the Fourier transform of $n(t)$ in convention (2.5). In (3.2) and (3.3), we denote by an upper bar the average over many realizations of noise in a large ensemble of detectors. From (3.3), we have $S_h(\omega) = S_h^*(\omega) = S_h(-\omega) > 0$.

Looking for the useful signal $h(t)$ in the output of the detector $o(t)$, the experimenters construct the correlation $c(t)$ between $o(t)$ and a filter $q(t)$, i.e.

$$c(t) = \int_{-\infty}^{+\infty} dt' o(t') q(t + t') , \quad (3.4)$$

and divide $c(t)$ by the square root of its variance (or correlation noise). Thus, the experimenters consider the ratio

$$\sigma[q](t) = \frac{c(t)}{(\overline{c^2(t)} - \overline{c(t)}^2)^{1/2}} = \frac{\int_{-\infty}^{+\infty} \frac{d\omega}{2\pi} \tilde{o}(\omega) \tilde{q}^*(\omega) e^{i\omega t}}{\left(\int_{-\infty}^{+\infty} \frac{d\omega}{2\pi} S_h(\omega) |\tilde{q}(\omega)|^2 \right)^{1/2}} , \quad (3.5)$$

where $\tilde{o}(\omega)$ and $\tilde{q}(\omega)$ are the Fourier transforms of $o(t)$ and $q(t)$. The expectation value (or ensemble average) of this ratio defines the filtered signal-to-noise ratio (SNR)

$$\rho[q](t) = \overline{\sigma[q](t)} = \frac{\int_{-\infty}^{+\infty} \frac{d\omega}{2\pi} \tilde{h}(\omega) \tilde{q}^*(\omega) e^{i\omega t}}{\left(\int_{-\infty}^{+\infty} \frac{d\omega}{2\pi} S_h(\omega) |\tilde{q}(\omega)|^2 \right)^{1/2}} . \quad (3.6)$$

The optimal filter (or Wiener filter) which maximizes the SNR (3.6) at a particular instant $t = 0$ (say), is given by the matched filtering theorem as

$$\tilde{q}(\omega) = \gamma \frac{\tilde{h}(\omega)}{S_h(\omega)} , \quad (3.7)$$

where γ is an arbitrary (real) constant. The optimal filter (3.7) is matched on the expected signal $\tilde{h}(\omega)$ itself, and weighted by the inverse of the power spectral density of the noise. The maximum SNR, corresponding to the optimal filter (3.7), is given by

$$\rho = \left(\int_{-\infty}^{+\infty} \frac{d\omega}{2\pi} \frac{|\tilde{h}(\omega)|^2}{S_h(\omega)} \right)^{1/2} = \langle h, h \rangle^{1/2} . \quad (3.8)$$

This is the best achievable SNR with a linear filter. In (3.8), we have used, for any two real functions $f(t)$ and $g(t)$, the notation

$$\langle f, g \rangle = \int_{-\infty}^{+\infty} \frac{d\omega}{2\pi} \frac{\tilde{f}(\omega) \tilde{g}^*(\omega)}{S_h(\omega)} \quad (3.9)$$

for an inner scalar product satisfying $\langle f, g \rangle = \langle f, g \rangle^* = \langle g, f \rangle$.

In practice, the signal $h(t)$ or $\tilde{h}(\omega)$ is of known form (given by (2.1) or (2.6)) but depends on an unknown set of parameters which describe the source of radiation, and are to be measured. The experimenters must therefore use a whole family of filters analogous to (3.7) but in which the signal is parametrized by a whole family of “test” parameters which are a priori different from the actual source parameters. Thus, one will have to define and use a *lattice* of filters in the parameter space (see section 5). The set of parameters maximizing the SNR (3.6) is equal, by the matched filtering theorem, to the set of source parameters. However, in a single detector, the experimenters maximize the ratio (3.5) rather than the SNR (3.6), and therefore make errors on the determination of the parameters, depending on a particular realization of noise in the detector. If the SNR is high enough, the measured values of the parameters are Gaussianly distributed around the source parameters, with variances and correlation coefficients given by the standard covariance matrix, the computation of which we now recall. Since the optimal filter (3.7) is defined up to an arbitrary multiplicative constant, it is convenient to treat separately a constant amplitude parameter in front of the signal (involving, for instance, the distance of the source). We shall thus write the signal in the form

$$\tilde{h}(\omega; \mathcal{A}, \lambda_\alpha) = \mathcal{A} \tilde{k}(\omega; \lambda_\alpha) \quad (3.10)$$

where $\mathcal{A}$ is an amplitude parameter. In the case of an inspiralling binary, $\mathcal{A}$ is for instance given by (4.15) below. The function $\tilde{k}$ depends only on the other parameters, collectively denoted by λ_α where the greek label α ranges for convenience on the values $0, 1, \dots, N$. The family of matched filters (or “templates”) we consider is defined by

$$\tilde{q}(\omega; {}_t\lambda_\alpha) = \gamma' \frac{\tilde{k}(\omega; {}_t\lambda_\alpha)}{S_h(\omega)} \quad (3.11)$$

where ${}_t\lambda_\alpha$ is a set of test parameters and γ' is arbitrary. (In section 5 we shall make a suitable choice of γ' so as to simplify the numerical simulation.) Note that these parameters are assumed to be all independent, even though the source parameters λ_α may not be so. By substituting (3.11) into (3.5) and choosing $t = 0$, we get, with the notation (3.9),

$$\sigma({}_t\lambda) = \frac{\langle o, k({}_t\lambda) \rangle}{\langle k({}_t\lambda), k({}_t\lambda) \rangle^{1/2}}. \quad (3.12)$$

(Note that σ is in fact a function of both the parameters λ_α and ${}_t\lambda_\alpha$.) Now the experimenters choose as their best estimate of the source parameters λ_α the *measured* parameters ${}_m\lambda_\alpha$ which among all the test parameters ${}_t\lambda_\alpha$ (independently) maximize (3.12), i.e. which satisfy

$$\frac{\partial \sigma}{\partial {}_t\lambda_\alpha}({}_m\lambda) = 0, \quad \alpha = 0, 1, \dots, N. \quad (3.13)$$

Assuming that the SNR is high enough, we can work out (3.13) at first order in the difference between the actual source parameters and the measured ones,

$$\delta\lambda_\alpha = \lambda_\alpha - {}_m\lambda_\alpha. \quad (3.14)$$

As a result, we obtain

$$\delta\lambda_\alpha = \mathcal{C}_{\alpha\beta} \left\{ -\langle n, \frac{\partial h}{\partial \lambda_\beta} \rangle + \frac{\langle n, h \rangle}{\langle h, h \rangle} \langle h, \frac{\partial h}{\partial \lambda_\beta} \rangle \right\} \quad (3.15)$$

where a summation is understood on the dummy label β , and where the matrix $\mathcal{C}_{\alpha\beta}$ (with $\alpha, \beta = 0, 1, \dots, N$) is the inverse matrix of the so-called Fisher information matrix $\mathcal{D}_{\alpha\beta}$ given by

$$\mathcal{D}_{\alpha\beta} = \langle \frac{\partial h}{\partial \lambda_\alpha}, \frac{\partial h}{\partial \lambda_\beta} \rangle - \frac{1}{\langle h, h \rangle} \langle h, \frac{\partial h}{\partial \lambda_\alpha} \rangle \langle h, \frac{\partial h}{\partial \lambda_\beta} \rangle \quad (3.16)$$

(we have $\mathcal{C}_{\alpha\beta}\mathcal{D}_{\beta\gamma} = \delta_{\alpha\gamma}$). In the right-hand-sides of (3.15)-(3.16), the signal is equally (with this approximation) parametrized by the measured or actual parameters. Since the noise is Gaussian, so are, by (3.15), the variables $\delta\lambda_\alpha$'s (indeed, the $\delta\lambda_\alpha$'s result from a linear operation on the noise variable). The expectation value and quadratic moments of the distribution of these variables are readily obtained from the facts that $\overline{\langle n, f \rangle} = 0$ and $\overline{\langle n, f \rangle \langle n, g \rangle} = \langle f, g \rangle$ for any deterministic functions f and g (see (3.2) and (3.3)). We then obtain

$$\overline{\delta\lambda_\alpha} = 0, \quad (3.17)$$

$$\overline{\delta\lambda_\alpha \delta\lambda_\beta} = \mathcal{C}_{\alpha\beta}. \quad (3.18)$$

Thus, the matrix $\mathcal{C}_{\alpha\beta}$ (inverse of (3.16)) is the matrix of variances and correlation coefficients, or covariance matrix, of the variables $\delta\lambda_\alpha$. The probability distribution of the $\delta\lambda_\alpha$'s reads as

$$P(\delta\lambda_\alpha) = \frac{1}{\sqrt{(2\pi)^{N+1} \det \mathcal{C}}} \exp \left\{ -\frac{1}{2} \mathcal{D}_{\alpha\beta} \delta\lambda_\alpha \delta\lambda_\beta \right\}, \quad (3.19)$$

where $\det \mathcal{C}$ is the determinant of $\mathcal{C}_{\alpha\beta}$.

Finally, we deal with the amplitude parameter $\mathcal{A}$ of the signal. The experimenters attribute to $\mathcal{A}$ the measured value

$${}_m\mathcal{A} = \frac{\sigma({}_m\lambda)}{\langle k({}_m\lambda), k({}_m\lambda) \rangle^{1/2}} . \quad (3.20)$$

It can be shown from this expression that the difference $\delta\mathcal{A} = \mathcal{A} - {}_m\mathcal{A}$ between the actual and measured values of $\mathcal{A}$ is a Gaussian variable with moments

$$\overline{\delta\mathcal{A}} = 0 , \quad (3.21)$$

$$\overline{\delta\mathcal{A}^2} = \frac{\mathcal{A}^2}{\langle h, h \rangle} \left\{ 1 + \frac{\mathcal{C}_{\alpha\beta}}{\langle h, h \rangle} \langle h, \frac{\partial h}{\partial \lambda_\alpha} \rangle \langle h, \frac{\partial h}{\partial \lambda_\beta} \rangle \right\} , \quad (3.22)$$

and with correlations with the variables $\delta\lambda_\alpha$ given by

$$\overline{\delta\mathcal{A} \delta\lambda_\alpha} = -\mathcal{A} \frac{\mathcal{C}_{\alpha\beta}}{\langle h, h \rangle} \langle h, \frac{\partial h}{\partial \lambda_\beta} \rangle . \quad (3.23)$$

We can check from (3.18) and (3.22)-(3.23) that the “complete” covariance matrix associated with the “complete” set of parameters $\lambda_A = \{\lambda_\alpha, \mathcal{A}\}$ including the amplitude parameter is simply the inverse of the “complete” information matrix $\langle \partial h / \partial \lambda_A, \partial h / \partial \lambda_B \rangle$ (where A, B range on all the parameters, including $\mathcal{A}$).

4. Analytical investigation of the detection of tails

The gravitational-wave signal $h(t)$ is now explicitly supposed to be constituted by the wave front $h_0(t)$ and by the wave tail (the latter having in factor the total mass M of the source). It is convenient to assume that M itself is one of the parameters λ_α of the signal. We shall keep the value 0 of the greek label α for the total mass M , so that $\lambda_0 = M$, and shall introduce latin labels $i, j = 1, \dots, N$ for the parameters of the wave front. Thus we denote

$$\lambda_\alpha = \{M, \lambda_i\} \quad (4.1)$$

and write the Fourier transform (2.6) of the signal (with our choice (2.10) for the constant b') in the form

$$\tilde{h}(\omega; \mathcal{A}, \lambda_\alpha) = \tilde{h}_0(\omega; \mathcal{A}, \lambda_i) \left\{ 1 + \frac{2GM}{c^3} \left[\frac{\pi}{2} |\omega| + i\omega \ln(|\omega|/\omega_s) \right] \right\} \quad (4.2a)$$

where

$$\tilde{h}_0(\omega; \mathcal{A}, \lambda_i) = \mathcal{A} \tilde{k}_0(\omega; \lambda_i) . \quad (4.2b)$$

The total mass M is not independent (in the real signal) of the parameters λ_i of the wave front. This is for instance the case of the radiation generated by an inspiralling binary system where the parameters λ_i include the usual chirp mass $\mathcal{M} = \mu^{3/5} M^{2/5}$ (where μ is the reduced mass of the binary), and also, if we consider first post-Newtonian corrections, the parameter $\nu = \mu/M$ separately (see [39]–[41]). Thus, M is given as $M = \mathcal{M}\nu^{-3/5}$ in terms of parameters of the wave front. However, in the search of the tail effect, it is important to correlate the output of the detector with a family of matched filters (3.11) depending on test parameters ${}_t\lambda_\alpha = \{{}_tM, {}_t\lambda_i\}$ where ${}_tM$ is *independent* of the other parameters ${}_t\lambda_i$ (i.e., in the case of an inspiralling binary, where ${}_tM$ is not necessarily equal to ${}_t\mathcal{M} {}_t\nu^{-3/5}$). If the wave tail does *not* exist, the best filter of the family (3.11) will be able to find a measured value ${}_mM$ compatible with zero, together with some realistic values of the parameters of the wave front (e.g. ${}_m\mathcal{M}$ and ${}_m\nu$). On the other hand, if the wave tail does exist, the optimal filter will find a measured value ${}_mM$ which is compatible with the other parameters (e.g., which is approximatively equal to ${}_m\mathcal{M} {}_m\nu^{-3/5}$).

4.1. Computation of the covariance matrix

The covariance matrix of the signal $\mathcal{C}_{\alpha\beta}$, inverse of the matrix $\mathcal{D}_{\alpha\beta}$ defined by (3.16), will give us the level at which it will be possible to detect the effects of tails. The differentiation of the signal (4.2a) with respect to the parameters M and λ_i is easily performed. By differentiating the logarithm of (4.2a), we obtain

$$\frac{1}{\tilde{h}} \frac{\partial \tilde{h}}{\partial M} = \frac{2G}{c^3} \left[\frac{\pi}{2} |\omega| + i\omega \ln(|\omega|/\omega_s) \right] + O(G^2) , \quad (4.3a)$$

$$\frac{1}{\tilde{h}} \frac{\partial \tilde{h}}{\partial \lambda_i} = \frac{1}{\tilde{h}_0} \frac{\partial \tilde{h}_0}{\partial \lambda_i} . \quad (4.3b)$$

For the sake of clarity in the derivation of the covariance matrix, we indicate by $O(G^n)$ some negligible terms of the order of n times the tail term (which carries the constant G in front). In terms of the post-Newtonian parameter ε we have $O(G^n) = O(\varepsilon^{3n})$. With the help of (4.3), we can work out the components of $\mathcal{D}_{\alpha\beta}$ at leading order in G (i.e., at leading order in ε^3 following our notation). We find, separating the label $\alpha = 0$ corresponding to the parameter M from the latin labels i, j corresponding to

the parameters of the wave front (and showing explicitly the dependence in G/c^3),

$$\mathcal{D}_{00} = \frac{G^2}{c^6} \mathcal{D}_{00}^{(2)} + O(G^3) , \quad (4.4a)$$

$$\mathcal{D}_{0i} = \frac{G}{c^3} \mathcal{D}_{0i}^{(1)} + O(G^2) , \quad (4.4b)$$

$$\mathcal{D}_{ij} = \mathcal{D}_{ij}^{(0)} + O(G) . \quad (4.4c)$$

In (4.4), $\mathcal{D}_{00}^{(2)}$ is given by the formula

$$\begin{aligned} \mathcal{D}_{00}^{(2)} = & 4 \int_0^{+\infty} \frac{d\omega \omega^2}{\pi S_h(\omega)} |\tilde{h}_0(\omega)|^2 \ln^2(\omega/\omega_s) \\ & + \pi^2 \left\{ \int_0^{+\infty} \frac{d\omega \omega^2}{\pi S_h(\omega)} |\tilde{h}_0(\omega)|^2 - \frac{1}{\rho_0^2} \left(\int_0^{+\infty} \frac{d\omega \omega}{\pi S_h(\omega)} |\tilde{h}_0(\omega)|^2 \right)^2 \right\} , \end{aligned} \quad (4.5)$$

$\mathcal{D}_{0i}^{(1)}$ is given by

$$\begin{aligned} \mathcal{D}_{0i}^{(1)} = & 2 \operatorname{Im} \int_0^{+\infty} \frac{d\omega \omega}{\pi S_h(\omega)} \tilde{h}_0^*(\omega) \frac{\partial \tilde{h}_0}{\partial \lambda_i} \ln(\omega/\omega_s) + \pi \operatorname{Re} \left\{ \int_0^{+\infty} \frac{d\omega \omega}{\pi S_h(\omega)} \tilde{h}_0^*(\omega) \frac{\partial \tilde{h}_0}{\partial \lambda_i} \right. \\ & \left. - \frac{1}{\rho_0^2} \left(\int_0^{+\infty} \frac{d\omega \omega}{\pi S_h(\omega)} |\tilde{h}_0(\omega)|^2 \right) \left(\int_0^{+\infty} \frac{d\omega}{\pi S_h(\omega)} \tilde{h}_0^*(\omega) \frac{\partial \tilde{h}_0}{\partial \lambda_i} \right) \right\} , \end{aligned} \quad (4.6)$$

and $\mathcal{D}_{ij}^{(0)}$ is

$$\mathcal{D}_{ij}^{(0)} = \left\langle \frac{\partial h_0}{\partial \lambda_i}, \frac{\partial h_0}{\partial \lambda_j} \right\rangle - \frac{1}{\rho_0^2} \langle h_0, \frac{\partial h_0}{\partial \lambda_i} \rangle \langle h_0, \frac{\partial h_0}{\partial \lambda_j} \rangle . \quad (4.7)$$

As we see, $\mathcal{D}_{ij}^{(0)}$ is the information matrix of the wave front h_0 alone, whose inverse, $\mathcal{C}_{ij}^{(0)}$ say, is the covariance matrix of the wave front. In (4.5)-(4.7) we denote by Re and Im the real and imaginary parts, and by ρ_0 the optimal SNR associated with the wave front, i.e.

$$\rho_0 = \langle h_0, h_0 \rangle^{1/2} . \quad (4.8)$$

The matrix (4.4), in which we have (4.5)-(4.8), must now be inverted. We perform the computation at leading order in G/c^3 and express the result in terms of the coefficients $\mathcal{D}_{00}^{(2)}$ and $\mathcal{D}_{0i}^{(1)}$ given by (4.5) and (4.6), and in terms of the components of the covariance matrix $\mathcal{C}_{ij}^{(0)}$ of the wave front, satisfying $\mathcal{C}_{ij}^{(0)} \mathcal{D}_{jk}^{(0)} = \delta_{ik}$ where $\mathcal{D}_{ij}^{(0)}$ is given by (4.7). A short computation gives

$$\mathcal{C}_{00} = \frac{c^6}{G^2} \left\{ \frac{1}{\mathcal{D}_{00}^{(2)} - \mathcal{D}_{0k}^{(1)} \mathcal{D}_{0l}^{(1)} \mathcal{C}_{kl}^{(0)}} + O(G) \right\} , \quad (4.9a)$$

$$\mathcal{C}_{0i} = \frac{c^3}{G} \left\{ \frac{-\mathcal{D}_{0j}^{(1)} \mathcal{C}_{ij}^{(0)}}{\mathcal{D}_{00}^{(2)} - \mathcal{D}_{0k}^{(1)} \mathcal{D}_{0l}^{(1)} \mathcal{C}_{kl}^{(0)}} + O(G) \right\} , \quad (4.9b)$$

$$\mathcal{C}_{ij} = \mathcal{C}_{ij}^{(0)} + \frac{\mathcal{D}_{0m}^{(1)} \mathcal{D}_{0n}^{(1)} \mathcal{C}_{im}^{(0)} \mathcal{C}_{jn}^{(0)}}{\mathcal{D}_{00}^{(2)} - \mathcal{D}_{0k}^{(1)} \mathcal{D}_{0l}^{(1)} \mathcal{C}_{kl}^{(0)}} + O(G) . \quad (4.9c)$$

We can then express this result in terms of $\mathcal{D}_{00}$ and $\mathcal{D}_{0i}$ (and still $\mathcal{C}_{ij}^{(0)}$) rather than in terms of $\mathcal{D}_{00}^{(2)}$ and $\mathcal{D}_{0i}^{(1)}$. This reintroduces further negligible terms in G/c^3 . However, now that the derivation of the result is done, we simply skip all indications $O(G^n)$ of negligible terms. Hence we obtain, with the required precision, the final expressions

$$\mathcal{C}_{00} = \frac{1}{\mathcal{D}_{00} - \mathcal{D}_{0k}\mathcal{D}_{0l}\mathcal{C}_{kl}^{(0)}} , \quad (4.10a)$$

$$\mathcal{C}_{0i} = \frac{-\mathcal{D}_{0j}\mathcal{C}_{ij}^{(0)}}{\mathcal{D}_{00} - \mathcal{D}_{0k}\mathcal{D}_{0l}\mathcal{C}_{kl}^{(0)}} , \quad (4.10b)$$

$$\mathcal{C}_{ij} = \mathcal{C}_{ij}^{(0)} + \frac{\mathcal{D}_{0m}\mathcal{D}_{0n}\mathcal{C}_{im}^{(0)}\mathcal{C}_{jn}^{(0)}}{\mathcal{D}_{00} - \mathcal{D}_{0k}\mathcal{D}_{0l}\mathcal{C}_{kl}^{(0)}} . \quad (4.10c)$$

The expressions (4.10), together with the explicit formulas for $\mathcal{D}_{00}$ and $\mathcal{D}_{0i}$ computed from (4.5)–(4.6), and of the knowledge of the covariance matrix $\mathcal{C}_{ij}^{(0)}$ of the wave front (inverse of (4.7)), give the variances and correlation coefficients of the variables $\delta\lambda_\alpha = \lambda_\alpha - {}_m\lambda_\alpha$. In particular, by integrating the probability distribution (3.19) over the variables $\delta\lambda_i = \lambda_i - {}_m\lambda_i$ of the wave front, we obtain the probability distribution of the variable

$$\delta M = M - {}_mM , \quad (4.11)$$

difference between the actual and measured total masses parametrizing the wave tail:

$$P(\delta M) = \frac{1}{\sqrt{2\pi\sigma_M^2}} \exp \left\{ -\frac{\delta M^2}{2\sigma_M^2} \right\} , \quad (4.12)$$

where the variance σ_M^2 (square of the standard deviation σ_M) is given by

$$\sigma_M^2 = \mathcal{C}_{00} = \frac{1}{\mathcal{D}_{00} - \mathcal{D}_{0i}\mathcal{D}_{0j}\mathcal{C}_{ij}^{(0)}} . \quad (4.13)$$

4.2. Application to inspiralling binaries

The computation of the previous subsection is now specialized to the radiation generated by an inspiralling binary of compact objects like neutron stars. Recall that the wave generation formalism of [36], on which is based the expression of the wave tail in (2.1), is valid for compact objects, which can be treated as point masses. Looking for the numerical value of the standard deviation σ_M of the measured total mass M , we must insert into (4.13) the explicit expression of (the Fourier transform of) the wave front $h_0(t)$. We shall consider only the “Newtonian” part of the wave front; that

is, we shall approximate in (2.2)-(2.3) the tensor $K_{ij}(u, \mathbf{n})$ by the usual Newtonian quadrupole moment of the binary.

The wave front $h_0(t)$ generated by a binary system of two point masses moving on an inspiralling circular orbit is given, at the Newtonian approximation, by (see e.g. [2],[53])

$$h_0(t) = \frac{3}{16} \mathcal{A} \eta^{-1} [\Omega(t)]^{2/3} \cos \left[\int_T^t \Omega(t') dt' + \varphi \right]. \quad (4.14)$$

Our notation is as follows. The amplitude parameter $\mathcal{A}$ is given by

$$\mathcal{A} = \frac{c\Theta}{r}, \quad \Theta^2 = \frac{F_+^2(1 + \cos^2 \iota)^2 + 4F_\times^2 \cos^2 \iota}{4}, \quad (4.15)$$

where r is the distance to the source (in radiative coordinates), where F_+ and $F_\times$ are the detector beam-pattern functions appearing in (2.2), and where ι is the angle between the line of sight from the source to the detector and the normal to the orbital plane. Note that Θ^2 defined in (4.15) satisfies $\Theta^2 \leq 1$. The parameter η is related to the chirp mass $\mathcal{M} = \mu^{3/5} M^{2/5}$ of the binary by the formula

$$\eta = \frac{3}{4} \left(\frac{4G\mathcal{M}}{c^3} \right)^{-5/3}. \quad (4.16)$$

This particular choice of parameter simplifies the computation of the covariance matrix $\mathcal{C}_{ij}^{(0)}$ below. (In the numerical simulations of section 5 we shall choose instead of η the coalescence time τ left till coalescence starting from the initial frequency ω_s .) The frequency $\Omega(t)$ in (4.14) is equal to two times the orbital frequency of the binary, and is given, in the case where the inspiral of the orbit is driven by radiation reaction [56], by

$$\Omega(t) = \left[\frac{3}{5\eta} (T - t) \right]^{-3/8}, \quad (4.17)$$

where T is the instant of coalescence of the binary. Finally, φ in (4.14) denotes the final phase of the wave at the instant T . Recall that (4.14) is valid only in the “adiabatic” regime of the inspiral of the orbit, where the relative changes of the frequency in (half) the orbital period are small, i.e. $2\pi\dot{\Omega}/\Omega^2 \ll 1$. (The adiabatic regime of the inspiral of the orbit is the one during which most observations of inspiralling binaries will take place [2].)

To compute the various quantities entering the expression (4.13) of the variance, we need the Fourier transform of the wave (4.14). The latter can be computed in the stationary phase approximation (valid in the adiabatic regime of the inspiral of the orbit), and reads as

$$\tilde{h}_0(\omega) = \frac{\sqrt{5\pi}}{8} \mathcal{A} \eta^{-1/2} \omega^{-7/6} \exp \left\{ i \left[\omega T + \eta \omega^{-5/3} - \varphi - \frac{\pi}{4} \right] \right\}. \quad (4.18)$$

The expression (4.18) is written for $\omega > 0$; for $\omega < 0$, we use $\tilde{h}_0(\omega) = \tilde{h}_0^*(-\omega)$. (See [2],[41],[46],[10] for the derivation of (4.18).) The SNR ρ_0 is readily obtained by inserting (4.18) into (4.8). We find

$$\rho_0^2 = \langle h_0, h_0 \rangle = \frac{5}{64} \mathcal{A}^2 \eta^{-1} f_{7/3} , \quad (4.19)$$

where we use the definition, for any α , of the moments

$$f_\alpha = \int_0^{+\infty} \frac{d\omega \omega^{-\alpha}}{S_h(\omega)} . \quad (4.20)$$

The covariance matrix $C_{ij}^{(0)}$ of the wave front (4.18), inverse of (4.7), is now computed. Three parameters relevant to the construction of filters have to be considered, which are $\lambda_i = \{\eta, \varphi, T\}$. (The amplitude parameter $\mathcal{A}$ can be treated, knowing $C_{ij}^{(0)}$, by equations similar to (3.21)–(3.23).) The logarithmic derivatives of (4.18) with respect to the λ_i 's are given, for $\omega > 0$, by

$$\frac{1}{\tilde{h}_0} \frac{\partial \tilde{h}_0}{\partial \eta} = -\frac{1}{2\eta} + i\omega^{-5/3} , \quad (4.21a)$$

$$\frac{1}{\tilde{h}_0} \frac{\partial \tilde{h}_0}{\partial \varphi} = -i , \quad (4.21b)$$

$$\frac{1}{\tilde{h}_0} \frac{\partial \tilde{h}_0}{\partial T} = i\omega . \quad (4.21c)$$

By substituting these expressions into (4.7), we obtain the covariance matrix in the form

$$C_{ij}^{(0)} = \frac{f_{7/3}}{\rho_0^2} F_{ij} , \quad (4.22)$$

where ρ_0^2 is given by (4.19), and where the matrix F_{ij} is the inverse matrix of the matrix E_{ij} defined in terms of the moments (4.20) by

$$E_{ij} = \begin{pmatrix} f_{17/3} & -f_4 & f_3 \\ -f_4 & f_{7/3} & -f_{4/3} \\ f_3 & -f_{4/3} & f_{1/3} \end{pmatrix} , \quad (4.23)$$

where the indices i, j range on the parameters η, φ, T in that order (we have $F_{ij}E_{jk} = \delta_{ik}$). Finally, we need to compute the quantities $\mathcal{D}_{00}$ and $\mathcal{D}_{0i}$ entering the expression (4.13). These quantities are given with the required precision by (4.4)–(4.6). They are readily evaluated with the help of (4.18) and (4.21), and we find

$$\mathcal{D}_{00} = \frac{G^2}{c^6} \frac{\rho_0^2}{f_{7/3}} \left[\pi^2 \left(f_{1/3} - \frac{f_{4/3}^2}{f_{7/3}} \right) + 4h_{1/3} \right] , \quad (4.24a)$$

$$\mathcal{D}_{0i} = \frac{2G}{c^3} \frac{\rho_0^2}{f_{7/3}} G_i , \quad G_i = \begin{pmatrix} g_3 \\ -g_{4/3} \\ g_{1/3} \end{pmatrix} . \quad (4.24b)$$

In these expressions, the new moments g_α and h_α are defined by

$$g_\alpha = \int_0^{+\infty} \frac{d\omega \omega^{-\alpha}}{S_h(\omega)} \ln(\omega/\omega_s) , \quad h_\alpha = \int_0^{+\infty} \frac{d\omega \omega^{-\alpha}}{S_h(\omega)} \ln^2(\omega/\omega_s) . \quad (4.25)$$

By inserting (4.24) and (4.22) into the expression (4.13) of the variance σ_M^2 , we then obtain our looked-for formula

$$\sigma_M^2 = \frac{c^6}{G^2} \frac{f_{7/3}}{\rho_0^2} \frac{1}{\pi^2 \left(f_{1/3} - \frac{f_{4/3}^2}{f_{7/3}} \right) + 4(h_{1/3} - F_{ij} G_i G_j)} , \quad (4.26)$$

(where we sum on repeated indices in $F_{ij} G_i G_j$). As a check of the correctness of this formula, we can show that σ_M^2 is independent of the particular choice of radiative coordinate system we have made in (2.10), i.e. is independent of the value of the frequency ω_s . This frequency appears only in the second term in the denominator of (4.26). From (4.20) and (4.25), we have $\partial h_\alpha / \partial \omega_s = -2g_\alpha / \omega_s$ and $\partial g_\alpha / \partial \omega_s = -f_\alpha / \omega_s$, from which we deduce $\partial[h_{1/3} - F_{ij} G_i G_j] / \partial \omega_s = 0$. Thus, we have shown $\partial(\sigma_M^2) / \partial \omega_s = 0$.

The variance σ_M^2 obtained in (4.26) is valid for any spectral density of noise $S_h(\omega)$. In order to compute it, we use some simplified models of noise in the detector. First we suppose that $S_h(\omega)$ is infinite below the seismic cutoff ω_s , and above some upper cutoff ω_u ,

$$S_h(\omega) = \infty , \quad \omega < \omega_s \quad \text{or} \quad \omega > \omega_u . \quad (4.27)$$

Then we choose, within the bandwidth $[\omega_s, \omega_u]$, either a model of white noise, i.e.

$$S_h(\omega) = S , \quad \omega_s \leq \omega \leq \omega_u , \quad (4.28)$$

where S is some constant, or a model of “coloured” noise appropriate for shot noise in laser interferometric detectors that employ Fabry-Perot cavities and use standard recycling, i.e.

$$S_h(\omega) = K \omega_k \left[1 + \left(\frac{\omega}{\omega_k} \right)^2 \right] , \quad \omega_s \leq \omega \leq \omega_u , \quad (4.29a)$$

(see e.g. [57],[9],[2],[53]). The constant K in (4.29a) depends notably on the laser power and on the reflectivities of the end mirrors. (We shall not need the expression of K nor the value of S .) The frequency ω_k in (4.29a) is the so-called “knee” frequency, and can be chosen by the experimenters at will. We adopt here the value of ω_k that maximizes (all other parameters remaining fixed) the SNR (4.19), i.e. that maximizes the moment $f_{7/3}$. This value is known [41],[2],[53] to be

$$\omega_k = 1.44 \omega_s . \quad (4.29b)$$

The relevant moments f_α , g_α and h_α are computed using the two models of noise (4.28) and (4.29), and substituted into (4.26). It is immediately seen that the standard deviation σ_M will take the form

$$\sigma_M = a \frac{c^3}{G\rho_0\omega_s} \quad (4.30)$$

where a is a pure dimensionless number, depending only on the type of noise in the detector (white or coloured) and on the ratio ω_u/ω_s . The number a is composed of many residual dimensionless integrals resulting from the various moments f_α , g_α and h_α . We have computed its numerical value in the case $\omega_u/\omega_s = 10$ (for instance $\omega_s/2\pi = 100\text{Hz}$ and $\omega_u/2\pi = 1000\text{Hz}$ as will be used in the simulations of section 5), and obtain

$$a = 0.20 \quad (\text{white noise}) , \quad (4.31a)$$

$$a = 0.41 \quad (\text{coloured noise}) . \quad (4.31b)$$

The standard deviation σ_M as given by (4.30) is inversely proportional to the SNR, and also inversely proportional to the lower cutoff ω_s . Note that when we improve the detection by lowering ω_s , the standard deviation σ_M *decreases*, contrarily to what could be expected from (4.30), and thus the measurement of M becomes more precise. This is of course due to the fact that the SNR ρ_0 also depends on ω_s . For instance, in the case of an inspiralling binary and of the model of noise (4.29) (with fixed ratio ω_u/ω_s), ρ_0 depends on ω_s like $\omega_s^{-7/6}$ and so $\rho_0\omega_s$ varies like $\omega_s^{-1/6}$ which increases when ω_s decreases.

For a high enough value of the SNR (or for a low enough value of ω_s), the precision σ_M in the measurement of M becomes smaller than the value of M itself. When this occurs, we are able to attribute to the total mass M a definite non-zero value at the one-sigma confidence level. That is, we are able to detect the presence of the wave tail in the signal. Thus, we adopt the condition $\sigma_M \leq M$ as a criterion to decide that the wave tail is detected (at the 1- σ level). By (4.30) this condition reads as

$$\rho_0 \geq a \frac{c^3}{GM\omega_s} . \quad (4.32)$$

Note that (4.32) can be fulfilled for reasons owing both to the source and to the detector. For instance, it can be true because the mass of the source is very large or because the source is located very near. It can also be true because the seismic cutoff of the detector is very low. In the case of an inspiralling binary made of two neutron

stars with total mass $2.8M_\odot$, with $\omega_s = 2\pi \times 100\text{Hz}$ and the numerical values (4.31) of a , we obtain

$$\rho_0 \geq 23 \quad (\text{white noise}) , \quad (4.33a)$$

$$\rho_0 \geq 47 \quad (\text{coloured noise}) . \quad (4.33b)$$

In the case of a binary with total mass $10M_\odot$, we obtain

$$\rho_0 \geq 6.4 \quad (\text{white noise}) , \quad (4.34a)$$

$$\rho_0 \geq 13 \quad (\text{coloured noise}) . \quad (4.34b)$$

Note that the SNR ρ_0 in (4.32)–(4.34) can equally well be replaced by the optimal SNR ρ ; see (A.13) in the appendix A.

The prospects, open by the estimates (4.33)–(4.34), of detecting the effects of tails have been discussed in the introduction. We now turn to a numerical simulation of the detection of tails which, as we shall see, will confirm these estimates.

5. Numerical investigation of the detection of tails

5.1. Construction of a lattice of filters

The question of the construction of a lattice of matched filters (or “templates”) for the detection of tails needs to be settled before the simulation. The templates are given by (3.11), i.e.

$$\tilde{q}(\omega; {}_t\lambda_\alpha) = \gamma' \frac{\tilde{k}(\omega; {}_t\lambda_\alpha)}{S_h(\omega)} . \quad (5.1)$$

They depend on a set of test parameters ${}_t\lambda_\alpha$, where the label α ranges from 0 to N , which are assumed to be all independent of each other. The test parameters include all the parameters of the signal except the amplitude parameter $\mathcal{A}$, containing the distance to the source, which is not relevant to the construction of filters. Following [46],[47], we choose the arbitrary constant γ' in (5.1) to be such that the noise power of the filtered output of the detector is unity, i.e.

$$\int_{-\infty}^{\infty} \frac{d\omega}{2\pi} S_h(\omega) |\tilde{q}(\omega; {}_t\lambda)|^2 = 1 \quad \implies \quad \gamma' = \langle k({}_t\lambda), k({}_t\lambda) \rangle^{-1/2} . \quad (5.2)$$

With this choice of normalization, we can write the family of filters (5.1) in the form

$$\tilde{q}(\omega; {}_t\lambda_\alpha) = \frac{\tilde{K}(\omega; {}_t\lambda_\alpha)}{S_h(\omega)} \quad (5.3)$$

where $\tilde{K}(\omega; {}_t\lambda_\alpha)$ denotes a family of signals of unit SNR (3.8) or, using the terminology of [46],[47], of unit strength, i.e.

$$\tilde{K}(\omega; {}_t\lambda) = \frac{\tilde{k}(\omega; {}_t\lambda)}{\langle k({}_t\lambda), k({}_t\lambda) \rangle^{1/2}} \implies \langle K({}_t\lambda), K({}_t\lambda) \rangle = 1 . \quad (5.4)$$

Let us first note that, in practice, it is impossible to use the family of filters (5.1) or (5.3) in which the test parameters ${}_t\lambda_\alpha$ take on all possible values. Thus, one must consider only a finite (discrete) lattice of filters. Since it is unlikely that any member of the family will have its parameters exactly matching those of a signal present in the data train, we will not reach, in general, the optimal SNR (3.8). We denote by κ the maximal relative drop ($\kappa > 1$) in the optimal SNR that is due to the discrete nature of the lattice of filters. Thus, if ρ is the optimal SNR, we assume that $\kappa^{-1}\rho$ is the *smallest* SNR obtained by filtering an arbitrary signal through the discrete lattice of filters. In an on-line analysis of data, a compromise is clearly in order. On one hand, we want to minimize κ in order to reach as high a SNR as possible, and on the other hand we want to minimize the number of filters in the lattice (and thus to increase κ) in order to save on computing time. Here, the problem is simpler because the detection of tails will not be done in an on-line analysis of data, but rather in a further analysis later (see the appendix A). Thus, we shall not optimize the value of κ , but simply calculate the spacing between filters, in all directions of the parameter space, for some good value of κ , say $\kappa^{-1} = 0.99$.

In order to decide upon the spacing between filters, we consider the behaviour of the correlation function

$$C(\lambda_\alpha, {}_t\lambda_\alpha) = \langle K(\lambda_\alpha), K({}_t\lambda_\alpha) \rangle . \quad (5.5)$$

This correlation function is equal to (the value at $t = 0$ of) the correlation between the filter (5.3), parametrized by test parameters ${}_t\lambda_\alpha$, and a signal of unit strength $\tilde{K}(\lambda_\alpha)$, parametrized by signal parameters λ_α . In the notation of (3.8) and (3.12), we have $C(\lambda_\alpha, {}_t\lambda_\alpha) = \overline{\sigma(\lambda_\alpha, {}_t\lambda_\alpha)} / \rho(\lambda_\alpha)$, where the dependence on the signal parameters λ_α is explicitly indicated. By the matched filtering theorem, the correlation (5.5) attains its maximum *one* when all the test parameters exactly match those of the signal, ${}_t\lambda_\alpha = \lambda_\alpha$. Looking for the spacing between filters in the particular direction

$\xi = \lambda_{\alpha_0}$ (where $0 \leq \alpha_0 \leq N$) of the parameter space, we maximize (5.5) over all the test parameters except ${}_t\xi = {}_t\lambda_{\alpha_0}$. That is, we consider the maximum correlation

$$C_{max}(\xi, {}_t\xi) = \max_{\substack{{}_t\lambda_{\alpha} \\ \alpha \neq \alpha_0}} C(\lambda_{\alpha}, {}_t\lambda_{\alpha}), \quad (5.6)$$

as a function of the mismatch between ξ and ${}_t\xi$ [46],[47]. (A priori, (5.6) still depends on the other signals parameters λ_{α} , $\alpha \neq \alpha_0$, but we do not mention this for simplicity.) Note that, due to the existence of covariance amongst the various parameters, the maximum correlation (5.6) does *not* reach its maximum when the test parameters ${}_t\lambda_{\alpha}$, $\alpha \neq \alpha_0$, are equal to the signal parameters λ_{α} , $\alpha \neq \alpha_0$. Indeed, the mismatch between ξ and ${}_t\xi$ induces in (5.6) a mismatch on the other parameters as well.

Let $[\xi_L, \xi_U]$ be an interval over which the test parameters ${}_t\xi$ are chosen. This interval represents for instance an astrophysically interesting range of values of ξ , or an interval surrounding some preliminary measured value ${}_m\xi^{(0)}$ obtained in an on-line analysis of data. We denote by ξ_k the k th value of the test parameter ${}_t\xi$ in the lattice of filters, some values of the other parameters ${}_t\lambda_{\alpha}$, $\alpha \neq \alpha_0$, being given. We begin at the lower end of the interval by choosing a filter with $\xi_1 = \xi_L$. The correlation (5.6) between this filter and a signal having exactly $\xi = \xi_1$ is one, and the SNR is equal to its optimal value ρ . As ξ increases, the correlation with the first filter decreases and for some value of ξ , say $\xi = \xi_1 + \Delta_+\xi_1$, it reaches the level κ^{-1} ,

$$C_{max}(\xi_1 + \Delta_+\xi_1, \xi_1) = \kappa^{-1}. \quad (5.11)$$

The quantity $\Delta_+\xi_1$, solution of (5.11), is the increment from ξ_1 at which the SNR drops to $\kappa^{-1}\rho$. Further mismatch between ξ and ξ_1 reduces the correlation with the first filter below κ^{-1} , and another filter is needed. We choose $\xi_2 > \xi_1$ so that for $\xi = \xi_1 + \Delta_+\xi_1$ the filter with ξ_2 also reaches the level κ^{-1} ,

$$C_{max}(\xi_1 + \Delta_+\xi_1, \xi_2) = \kappa^{-1}. \quad (5.12)$$

Since the correlation $C_{max}(\xi, {}_t\xi)$ is a monotonically decreasing function of $|\xi - {}_t\xi|$ (because it reaches its maximum when ${}_t\xi = \xi$), it is clear that, by construction, we have $\xi_1 < \xi_1 + \Delta_+\xi_1 < \xi_2$. We pose

$$\xi_1 + \Delta_+\xi_1 = \xi_2 - \Delta_-\xi_2, \quad (5.13a)$$

and define the distance between the first two filters by

$$\Delta\xi_{1,2} = \Delta_+\xi_1 + \Delta_-\xi_2. \quad (5.13b)$$

Proceeding iteratively along the same lines, the m th filter of the lattice is given by

$$\xi_m = \xi_L + \sum_{k=2}^m \Delta \xi_{k-1,k}, \quad (5.14)$$

where $\Delta \xi_{k-1,k} = \Delta_+ \xi_{k-1} + \Delta_- \xi_k$ is the distance between the $(k-1)$ th and the k th filters. The process terminates at the n th filter such that $\xi_n + \Delta_+ \xi_n > \xi_U$. The lattice of filters so constructed is such that the k th filter yields a correlation larger than κ^{-1} (and hence a SNR larger than $\kappa^{-1} \rho$) for all signals having $\xi \in [\xi_k - \Delta_- \xi_k, \xi_k + \Delta_+ \xi_k]$.

We now consider specifically the waveform of an inspiralling binary. As in section 4, we neglect all post-Newtonian corrections in the waveform except the one which is associated with the tail (a more complete study will be left to future work). The waveform depends on four parameters (besides the amplitude parameter $\mathcal{A}$): three “Newtonian” parameters $\lambda_i = \{\eta, \varphi, T\}$, and the total mass M which is in front of the tail. For the purpose of the simulation, it is convenient to use, instead of the parameter η , the time τ left till coalescence starting from the “arrival” time t_s at which the frequency reaches the seismic cutoff ω_s . From (4.17) we have

$$\tau = \frac{5\eta}{3} \omega_s^{-8/3}. \quad (5.15)$$

It is also convenient to use, instead of T and φ , the arrival time $t_s = T - \tau$ and the initial phase ψ of the signal at t_s . We discuss each of these parameters separately.

The coalescing time τ . Figure 1 shows the behaviour of the correlation function $C_{max}(\tau, \iota\tau)$ defined in (5.6), as a function of the mismatch $\iota\tau - \tau$. The function is a monotonically decreasing function of $|\iota\tau - \tau|$ taking its maximum one when $\iota\tau = \tau$, and which is symmetric about the vertical axis because, in a signal of unit strength, the parameter τ arises only *linearly* in the phase of the signal [46],[47]. $C_{max}(\tau, \iota\tau)$ is plotted for both the white noise and the coloured noise defined in (4.28) and (4.29). The curves fall off much more steeply when the detector noise is white than when it is coloured. This is to be expected because the detector is relatively narrowband in the coloured noise case as compared to the white noise case. Thus, the signal is cut off more prematurely in the former case, and the precision on the measurement of its parameters is lower [47]. Since the curves depend solely, for a given type of noise, on the difference $\iota\tau - \tau$, apart from a slight dependence on the total mass M , we can choose the distance (5.13b) between filters to be a constant $\Delta\tau$. From figure 1, we see that $\Delta\tau = 1\text{ms}$ is sufficient to ensure in any case a value of κ^{-1} better than 0.99.

The total mass M . The function $C_{max}(M, \iota M)$, shown in figure 2, is a monotonically decreasing function of $|\iota M - M|$ which, contrary to $C_{max}(\tau, \iota\tau)$, is

not symmetric about the vertical axis. Indeed M appears not only linearly in the phase of the signal, but appears also in the amplitude of the signal (see (2.8)–(2.9)). For the same reason as for τ , the curves fall off more steeply in the white noise case than in the coloured noise case. They depend also slightly on the value of M separately. With $\kappa^{-1} = 0.99$, we see from figure 2 that the distance between filters should be approximatively $\Delta M = 10M_{\odot}$ in the white noise case and $\Delta M = 40M_{\odot}$ in the coloured noise case. Thus, very few filters are needed in the M -direction of the parameter space to ensure a good value of κ^{-1} . This reflects, of course, the fact that the tail effect is a sub-dominant effect, of relative order ε^3 with respect to the dominant quadrupole radiation. Thus, the number of filters in the M -direction will be determined only by the precision at which one wants to measure M .

The time of arrival t_s . The unknown time of arrival of the signal is characterized by the lag j of the filter relatively to the output of the detector in the discrete correlation (5.20) below. The number of filters for the time of arrival is determined by the sampling rate Δ^{-1} at which output and filters are sampled. The sampling rate has to be, by Nyquist's theorem, larger than twice the bandwidth (for instance $\Delta^{-1} = 2000\text{Hz}$ for $\omega_u = 1000\text{Hz}$). The precision in the arrival time of the signal is thus limited to Δ (say 0.5ms).

The initial phase ψ . Just two filters in the ψ -direction of the parameter space are sufficient, one corresponding to ${}_t\psi = 0$ and one corresponding to ${}_t\psi = \pi/2$ (for instance). This results from the fact that the correlation (5.5) between a filter with initial phase ${}_t\psi$ and a signal of unit strength with initial phase ψ can be expressed as the linear combination

$$C(\psi, {}_t\psi) = \cos {}_t\psi C(\psi, 0) + \sin {}_t\psi C(\psi, \pi/2). \quad (5.17)$$

The maximum of $C(\psi, {}_t\psi)$ can then be obtained analytically. It is reached for ${}_t\psi = \psi_{max}$ where

$$\psi_{max} = \tan^{-1} \left[\frac{C(\psi, \pi/2)}{C(\psi, 0)} \right], \quad (5.18)$$

and the maximum $C(\psi, \psi_{max})$ is given by

$$C(\psi, \psi_{max}) = [C^2(\psi, 0) + C^2(\psi, \pi/2)]^{1/2}. \quad (5.19)$$

This way of proceeding is equivalent to having an infinite number of filters in the ψ -direction.

5.2. Numerical simulation

The waveform $h(t)$ of an inspiralling binary, composed of the usual quadrupole radiation and of the wave tail, is added to simulated Gaussian noise $n(t)$. The parameters of the waveform are $\mathcal{A}, \mathcal{M}, \tau, \psi, t_s$. The noise is chosen to be either white or coloured (see (4.28) and (4.29)). The data $o(t) = h(t) + n(t)$ is our simulated noisy detector output. It is filtered through a family of templates $q(t)$ constructed using the algorithm developed above.

For numerical implementation of the filtering process we need to consider the discrete version of the correlation (3.5) between the detector output $o(t)$ and a filter $q(t)$ satisfying the normalization condition (5.2). Let o_k and q_k , with $k = 0, \dots, K-1$, be the K samples of the detector output and filter, respectively, sampled at the rate Δ : $o_k \equiv o(t_k)$, $q_k \equiv q(t_k)$ and $t_k = k\Delta$. The discrete correlation $\sigma_j \equiv \sigma(t_j)$ of the samples o_k and q_k is then given by

$$\frac{\sigma_j}{\Delta} = \sum_{k=0}^{K-1} o_k q_{k+j} = \frac{1}{K} \sum_{l=0}^{K-1} \tilde{o}_l \tilde{q}_l^* e^{i2\pi l j/K}, \quad (5.20)$$

where j is the lag, where $\tilde{o}_l = \sum_{k=0}^{K-1} o_k \exp(i2\pi l k/K)$ and $\tilde{q}_l = \sum_{k=0}^{K-1} q_k \exp(i2\pi l k/K)$ are the discrete Fourier coefficients of the detector output and filter, and where the second equality follows from the discrete correlation theorem. Due to the availability of fast Fourier transforms it is numerically less expensive to compute the correlation in the Fourier domain (second equality in (5.20)). Note that the use of the discrete correlation (5.20) implicitly assumes that the filter q_k is periodic with period K , i.e. $q_{k+K} = q_k$. This periodicity introduces spurious contributions in the correlation, and we solve this problem by “padding” the filter with zeros so that the correct correlation is nevertheless obtained for some subset of values of the lag (see [53]).

The family of templates we use in the simulation has the following characteristics. The bandwidth is [100 Hz, 1000 Hz] and the sampling rate is $\Delta^{-1} = 2048$ Hz. The range of values of ${}_t\tau$ is $[\tau - 0.01s, \tau + 0.01s]$, and the distance between filters is $\Delta\tau = 1\text{ms}$. We use a rather small range of values of ${}_t\tau$ because it is assumed that an approximate value of τ is already known from a preliminary data analysis. However this range is big enough to take care of the covariance between τ and the other parameters (notably M). The range of values of ${}_tM$ is $[-10M_\odot, +30M_\odot]$, where some negative values of ${}_tM$ are tried. The distance between filters ΔM is adjusted so that it is smaller than the precision in the measurement of M . Thus, in the computation below, the precision σ_M is meaningful only as long as $\Delta M < \sigma_M$. If a computation yields a value of σ_M smaller than ΔM , we have to redo the computation with a finer lattice of filters.

By maximizing the correlation (5.20) over the family of templates, for a given realization of noise, we obtain a first set of measured parameters. The same signal is then added to a different realization of noise and the resultant data are again filtered through the same family of templates, giving us another set of measured parameters. The whole procedure is repeated for a large number of realizations of noise (~ 50). The averages of the distribution of the measured parameters, together with the standard deviations around the averages, are then computed. This yields notably the average $\overline{mM}$ and the standard deviation σ_M of the measured total mass. (By a slight abuse of notation, we denote by the same symbols $\overline{mM}$ and σ_M as in sections 3 and 4 what corresponds here only to a *finite* number of realizations of noise; in particular, $\overline{mM}$ here needs not be exactly equal to M .) The results of the computation of σ_M and $\overline{mM}$ are shown in figures 3 and 4, where these quantities are plotted against the optimal SNR (3.8) of the signal, for various values of the total mass M of the signal, and for both the white and coloured noises.

Based on these figures we can conclude the following. (i) The standard deviation σ_M in figures 3 and 4 is a decreasing function of the SNR which depends on the type of noise and is only slightly sensitive on the total mass M of the signal. This behaviour is in rather good agreement with the theoretical result (4.30) (which we recall was computed in the idealized cases of high SNR and of an infinite number of noise realizations). (ii) The value of σ_M is significantly higher in the coloured noise case than in the white noise case. This is to be expected for the same reason as noticed earlier: the detector is relatively narrowband in the former case as compared to the latter case, and thus the precision on the measurement of M is lower. (iii) From figure 3, we see that in order to detect the tail effect at the one-sigma level (i.e. to have $\sigma_M \leq M$) in the case of a binary with total mass $M = 2.8M_\odot$, we need a SNR larger than about 40-50 with coloured noise, and larger than about 25-30 with white noise. If the total mass is $M = 10M_\odot$, the minimal SNR is about 15 with coloured noise and less than 10 with white noise. These numerical values are also (approximatively) in agreement with the analytical computations (4.33) and (4.34).

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Appendix A. Non-optimal filtering

The aim of this appendix is to investigate the consequences of using a family of filters matched on the wave front $h_0(t)$ of the signal. That is, we assume that the experimenters do not believe that the wave tail exists, while it is really present in the signal, and correlate the output of the detector (3.1) not with the optimal family of filters (3.11), but with the non-optimal family of filters

$$\tilde{q}_0(\omega; {}_t\lambda_i) = \gamma' \frac{\tilde{k}_0(\omega; {}_t\lambda_i)}{S_h(\omega)}, \quad (\text{A.1})$$

where $\tilde{k}_0$ is the Fourier transform of the wave front (divided by the amplitude parameter) as defined in (4.2b), and where ${}_t\lambda_i$ is a family of test parameters of the wave front, the latin index ranging from $i = 1$ to N . By substituting (A.1) into (3.5) and choosing $t = 0$, we obtain, instead of (3.12),

$$\sigma_0({}_t\lambda_i) = \frac{\langle o, k_0({}_t\lambda_i) \rangle}{\langle k_0({}_t\lambda_i), k_0({}_t\lambda_i) \rangle^{1/2}}. \quad (\text{A.2})$$

The experimenters then choose as their best estimate of the parameters of the source the measured values ${}_m\lambda_i^{(0)}$ that maximize the ratio (A.2), i.e. such that

$$\frac{\partial \sigma_0}{\partial {}_t\lambda_i} ({}_m\lambda^{(0)}) = 0 \quad , \quad i = 1, 2, \dots, N. \quad (\text{A.3})$$

We denote the difference between the source parameters and the non-optimally measured ones by

$$\delta\lambda_i^{(0)} = \lambda_i - {}_m\lambda_i^{(0)}. \quad (\text{A.4})$$

The expression of $\delta\lambda_i^{(0)}$ can be immediately deduced, in the high SNR approximation, from (3.15) by simple replacement of h by h_0 and of n by $n + \delta h$, where $\delta h = h - h_0$ denotes the wave tail, difference between the signal and its wave front. We obtain

$$\delta\lambda_i^{(0)} = C_{ij}^{(0)} \left\{ -\langle n + \delta h, \frac{\partial h_0}{\partial \lambda_j} \rangle + \frac{\langle n + \delta h, h_0 \rangle}{\langle h_0, h_0 \rangle} \langle h_0, \frac{\partial h_0}{\partial \lambda_j} \rangle \right\} \quad (\text{A.5})$$

where $C_{ij}^{(0)}$ is the covariance matrix of the wave front, inverse of the matrix (4.7). The matrix $C_{ij}^{(0)}$ is given by (4.22)–(4.23) in the case of inspiralling binaries. As we see from (A.5), the variables $\delta\lambda_i^{(0)}$ are Gaussian (for Gaussian noise). But, contrarily to the variables $\delta\lambda_\alpha = \{\delta M, \delta\lambda_i\}$ associated with the optimal filtering of section 3, their expectation values are not zero but given by

$$\overline{\delta\lambda_i^{(0)}} = C_{ij}^{(0)} \left\{ -\langle \delta h, \frac{\partial h_0}{\partial \lambda_j} \rangle + \frac{\langle \delta h, h_0 \rangle}{\langle h_0, h_0 \rangle} \langle h_0, \frac{\partial h_0}{\partial \lambda_j} \rangle \right\}. \quad (\text{A.6})$$

As for their second-order moments, they are

$$\overline{(\delta\lambda_i^{(0)} - \overline{\delta\lambda_i^{(0)}})(\delta\lambda_j^{(0)} - \overline{\delta\lambda_j^{(0)}})} = \mathcal{C}_{ij}^{(0)} . \quad (\text{A.7})$$

Equation (A.6) means that by using the non-optimal filtering (A.1) we make *errors* (on average) in the determination of the parameters of the wave front. These errors are of course of the same order $O(G) = O(\varepsilon^3)$ as the wave tail; they have explicitly in factor the total mass M of the source. Using $\delta h = M\partial h/\partial M$ and the expression (3.16) of the matrix $\mathcal{D}_{\alpha\beta}$, together with (4.10), we can express (A.6) in the simple form

$$\overline{\delta\lambda_i^{(0)}} = -M \mathcal{D}_{0j} \mathcal{C}_{ij}^{(0)} = M \frac{\mathcal{C}_{0i}}{\mathcal{C}_{00}} \quad (\text{A.8})$$

(where higher-order terms are neglected). Similarly to (A.6), one can also show that the difference $\delta\mathcal{A}^{(0)} = \mathcal{A} - {}_m\mathcal{A}^{(0)}$ between the actual value of the amplitude parameter and the non-optimally measured one (defined, similarly to (3.20), as the ratio between $\sigma_0({}_m\lambda_i^{(0)})$ and $\langle k_0({}_m\lambda_i^{(0)}), k_0({}_m\lambda_i^{(0)}) \rangle^{1/2}$), is a Gaussian variable with non-zero expectation value given by

$$\frac{\overline{\delta\mathcal{A}^{(0)}}}{\mathcal{A}} = -\frac{\langle \delta h, h_0 \rangle}{\langle h_0, h_0 \rangle} - \frac{\overline{\delta\lambda_i^{(0)}}}{\langle h_0, h_0 \rangle} \langle h_0, \frac{\partial h_0}{\partial \lambda_i} \rangle , \quad (\text{A.9})$$

where $\overline{\delta\lambda_i^{(0)}}$ is given by (A.8).

The errors (A.8) and (A.9) made on average in the determination of the parameters λ_i and $\mathcal{A}$ can be readily computed in the case of an inspiralling binary, where we have $\lambda_i = \{\eta, \varphi, T\}$. Inserting (4.22) and (4.24b) into (A.8) yields

$$\overline{\delta\lambda_i^{(0)}} = -\frac{2GM}{c^3} F_{ij} G_j . \quad (\text{A.10})$$

Similarly, we can compute (A.9). In this case, it is easy to show that the second term in the right-hand-side of (A.9), containing $\overline{\delta\lambda_i^{(0)}}$, is in fact of order $O(\dot{\Omega}/\Omega^2)$ relatively to the first term, where Ω is the frequency (4.17) and where $\dot{\Omega}/\Omega^2 \ll 1$ in the “adiabatic” regime of the decay of the orbit, and can be neglected. We then obtain

$$\frac{\overline{\delta\mathcal{A}^{(0)}}}{\mathcal{A}} = -\frac{\pi GM}{c^3} \frac{f_{4/3}}{f_{7/3}} . \quad (\text{A.11})$$

Note that the errors (A.10) on the determination of the parameters $\lambda_i = \{\eta, \varphi, T\}$ depend only on the tail-induced modification of the *phase* of the wave front, i.e. $\theta(\omega)$ given by (2.9), while the error (A.11) on the determination of the amplitude $\mathcal{A}$ depends only on the tail-induced modification of its *amplitude*, i.e. the square brackets of (2.8).

Finally, let us check that the maximal value ρ_0 of the SNR obtained by means of the non-optimal filtering based on the wave front is equal, modulo negligible higher-order $O(G^2)$ terms (using the same notation as in (4.3)-(4.4)), to the optimal value ρ of the SNR obtained by means of the optimal filtering of section 3. Indeed, from (A.2) and (A.3), the non-optimal SNR is

$$\rho_0 = \overline{\sigma_0(\lambda_i^{(0)})} = \overline{\sigma_0(\lambda_i)} + O(G^2) = \frac{\langle h, h_0 \rangle}{\langle h_0, h_0 \rangle^{1/2}} + O(G^2) . \quad (\text{A.12})$$

Inserting $h_0 = h - \delta h$ with $\delta h = O(G)$, we then obtain

$$\rho_0 = \langle h, h \rangle^{1/2} + O(G^2) = \rho + O(G^2) \quad (\text{A.13})$$

where ρ denotes the optimal SNR (3.8). Thus, it is not possible to detect the effects of tails by simply comparing the values of the SNRs obtained by means of the optimal filtering of section 3 and of the non-optimal filtering of this appendix. A contrario, this means that it is not necessary to include the effects of tails in the filters of a preliminary on-line analysis of data aiming at *searching* the signal.

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Figure Captions

Figure 1: The correlation function $C_{max}(\tau, {}_t\tau)$ plotted against ${}_t\tau - \tau$.

Figure 2: The correlation function $C_{max}(M, {}_tM)$ plotted against ${}_tM - M$.

Figure 3: The standard deviation σ_M in the measurement of M plotted against the optimal SNR ρ .

Figure 4: The expectation value of the measured total mass ${}_mM$, with error bars σ_M , plotted against the optimal SNR ρ .

Fig 1

Fig 2

Fig 3

Fig 4