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**The measurement of the decorrelation function
in underground muon pairs as a probe of
primary Cosmic Ray interactions.**

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ABSTRACT

We discuss the measurement of the average angle within pairs of muons detected in underground experiments versus their relative distance, as a tool for studying primary Cosmic Ray interactions. We propose to call $\langle\phi(x)\rangle$ "decorrelation" function. Under simplifying approximation, the decorrelation function can be computed analytically to show its dependence on details of Cosmic Ray interactions as well as on the propagation in the rock. We argue that this new measurement is useful in a real experiment, complementing traditional ones like the decoherence function.

1. INTRODUCTION.

Modern underground Cosmic Ray experiments are characterized by large acceptance and good accuracy in track reconstruction; in particular they are suitable to make delicate measurements of the quantities related to underground muons.

These measurements allow in principle to extract detailed information on primary Cosmic Ray flux (for instance energy behaviour and composition), on high energy hadronic and nuclear interactions (total and differential cross sections) and also on the physical mechanisms related to the propagation of TeV muons through the rock [1,2].

In this paper we propose to exploit the good space and angle resolution of modern muon detectors to measure the relative angle between pairs of muons in multiple muon events respect to their distance. This new measurement, which we call "decorrelation" function, reflects the distribution of the interaction height of the hadrons (either primaries or high energy products in the cascade) and hence depends on the cross section and on the structure of the high atmosphere; it reflects also the distribution of transverse momenta of the muon parents and, finally, is sensitive to the details of the muon propagation into the rock.

While a quantitative result will need a comparison of experimental data with detailed Monte Carlo codes, in this paper we follow as much as possible an analytical approach in order to gain a qualitative understanding of the physical processes involved and of their relevance in determining the numerical value of the angle. In order to do this we are obliged to introduce several approximations, which will be discussed in the following.

Let us describe the results obtained.

The functional form of $\langle\phi(x)\rangle$ contains a "geometrical" term and a term sensitive to the multiple scattering in the propagation in the rock. The geometrical term is just the angle the muon would have in absence of effects of propagation, and increases linearly with the distance between muons. Concerning the multiple scattering contribution, one can identify three different regimes: at small distances the decorrelation is increasing and its form depends on the numerical values of the parameters of both propagation and transverse momentum distribution; at intermediate distances the decorrelation is *qualitatively* sensitive to the form of the p_t distribution, in the sense that it increases (decreases) with the distance if the lateral distribution outside the mountain decreases faster (slower) than exponential; for an exponential it tends to a constant whose value depends on $\langle p_t \rangle$. At large distances the geometrical term is dominant, implying a linear increase with slope equal to the inverse of the distance of interaction.

With appropriate definitions of angles and distances the average angle

is always positive (i.e. there are at the detection level more diverging than converging muon tracks). This is obvious for the geometrical term; for the term depending on multiple scattering, it derives from the fact that displacements in angle and distance due to the propagation are correlated in such a way that (for a lateral distribution in the atmosphere decreasing with the distance) muons arrive underground at a certain distance with positive angles more likely than negative.

We have tested these ideas comparing the analytical results with the outputs of a simplified (but not unrealistic) simulation. In this simulation we do not follow the shower in the atmosphere, but instead generate muons by drawing their production height, energy and p_t from given distributions and then propagate them into the mountain using a realistic propagation code.

The comparison shows a very good agreement between the simulation (in which most of the approximations are not enforced) and the analytical results, implying that the approximations introduced do not spoil the usefulness of the derivations.

To summarize: the decorrelation function is in principle a powerful tool for studying details of hadronic interactions of primaries in the atmosphere. From a single measurement it is possible (with the statistics made possible by the large acceptance of modern underground muon detectors) to derive information on the (average) production height, and hence total cross section, and on the form and numerical parameters (namely $\langle p_t/E_\mu \rangle$) of the transverse momentum distribution.

2. PHYSICAL MECHANISMS.

Muons detected deep underground are mainly produced in the first stages of the cascade generated by the interaction of primary Cosmic Ray nuclei in the atmosphere. Their distributions reflect the physical mechanisms in their production and passage through the overburden rock, as well as the statistical and systematical errors in the measurements.

Let us schematically describe the processes involved. Underground muons are generated mostly by the decay of π, K produced by the interaction of primary cosmic rays in the high atmosphere.

At surface level muons can be characterized by distributions in energy (E_μ), and angle and distance from the shower axis.

These distributions reflect those in energy, atomic number and height of interaction of the primaries, as well as the distributions in transverse momenta and decay probability of the parent meson(s).

The propagation in the atmosphere, at the high energies relevant for underground muons, does not introduce essentially any effect, with the

important exception of the interaction of the muons with the geomagnetic field.

In the propagation through the rock the muons lose energy, and their lateral distributions are modified by displacements in angle and distance due to electromagnetic and nuclear interactions in the surrounding rock.

In general a detailed Monte Carlo simulation is needed to describe these processes. To proceed analytically we make several simplifying assumptions:

- a) We will consider production and propagation only on a plane, *i.e.* a projection of the real processes.
- b) Muons are produced with fixed energy (assumed the same of the parent meson) and with transverse momenta distributed as those of the parent.
- c) They originate at a fixed distance (h_p) from the surface, including the path of the decaying meson.

With these assumptions, muons will arrive to the mountain with energy E_μ and a distribution of distances ($y = \alpha h_p$) from the shower axis according to

$$Y(y) = \frac{dN}{dy} = \frac{E_\mu}{h_p} \frac{dN}{dp_t} \quad (1)$$

Before arriving to the detector, muons must pass through X g/cm² of rock, in which they lose energy and suffer displacements in angle θ and distance r due to multiple scattering.

In general,

$$\frac{dN}{drd\theta} = F(r^2, \theta^2, r\theta).$$

We will only consider the gaussian approximation to these distributions [3], *i.e.*

$$\frac{dN}{drd\theta} = (\pi\sigma\mu)^{-1} \exp\left(-2\frac{\theta^2}{\sigma^2} - 2\frac{r^2}{\mu^2} + 2\sqrt{3}\frac{\theta r}{\mu\sigma}\right) \quad (2)$$

around their original direction.

The widths σ, μ are in general functions of both the initial and final energy of the muons, if energy losses are treated stochastically (see Appendix). So, even for the case we are considering here, *i.e.* fixed energy, there will be a spread in the values of σ, μ due to the fluctuations in the energy loss. As a further essential approximation, we consider here their values fixed.

To simplify our derivations we measure distances and angles relative to the shower axis. Having in mind a real experiment, in which the axis position is not known, and relative projected distances are chosen positive, we use here the same convention.

Let x, ϕ be the distance and angle of the muon relative to the shower axis. For a given x, ϕ we have two possible configurations (at fixed y , the distance at mountain level), corresponding to the definitions:

$$x = r + y; \quad \phi = \theta \quad (3.a)$$

$$x = -y - r'; \quad \phi = -\theta' \quad (3.b)$$

(see Fig. 1). Notice that, in accordance with the way the experimentally measured quantities are defined, $x \geq 0$.

After folding with the lateral distribution outside the mountain, we have

$$\frac{dN}{dx d\phi} = \frac{1}{2} \int dy Y(y) [G((x - y - \alpha h_r), (\phi - \alpha)) + (x \rightarrow -x, \phi \rightarrow -\phi)] \quad (4)$$

where h_r is the distance traveled in the mountain and $\alpha = y/h_p$ is the angle with which muons enter in the mountain, having been produced at a height h_p in the atmosphere.

This is (apart from apparatus effects) the experimental distribution. With the limited statistics of a real experiment it is in general impossible to reconstruct the full distribution, so we will limit ourselves to the measurement of its first few moments.

In particular,

$$D(x) = \int d\phi \frac{dN}{dx d\phi} \quad (5)$$

is the number of muons as a function of the distance from the axis. The analogous quantity computed for pairs of muons within muon bundles enters in the definition of the decoherence function [2,4].

The first moment is the average angle:

$$\langle \phi(x) \rangle = \frac{\int d\phi \phi \frac{dN}{dx d\phi}}{D(x)} = \quad (6)$$

$$= \frac{\sqrt{3}}{2} \frac{\sigma}{\mu} \left[x + \left(\frac{h_p + h_r}{h_p} \right) \frac{\int dy y Y(y) D_-(x, \bar{y})}{\int dy Y(y) D(x, \bar{y})} \right] - \frac{1}{h_p} \frac{\int dy y Y(y) D_-(x, \bar{y})}{\int dy Y(y) D(x, \bar{y})}$$

In Eq. (6) the terms in square brackets would be present even for muons penetrating the mountain parallel to the shower axis; the last term carries the memory of the angle of the muon in the atmosphere.

The functions $D(x, \bar{y})$, $D_-(x, \bar{y})$, ($\bar{y} = (1 + h_r/h_p)y$) are simple combinations of gaussians, defined in the Appendix.

Notice that the first contribution to the average angle, although depending on $Y(y)$, the lateral distribution at mountain level, is

proportional to the correlation term in the propagation. Moreover the term in square brackets in (6) does not depend on σ , which enters only in the normalization.

Of course this fact depends on the approximations made, which essentially amount to neglect the fluctuations of σ, μ : however for this it is only necessary that the ratio σ/μ suffers small fluctuations, and this is physically plausible since, at least for a thin layer, σ and μ are proportional (see Appendix).

This is as far as one can go without specifying the form of $Y(y)$. In general the integrals above cannot be done analytically, if not in few (extreme) cases. However, the form of $\langle\phi(x)\rangle$ can be explicitly derived both at small and large distances.

The form of $Y(y)$ reflects the transverse momentum distribution of the mesons which decay into muons. In the following we will examine some simple, pedagogical cases.

Let first assume that $Y(y) = \text{const.}$ Then (at all distances)

$$\langle\phi(x)\rangle = \frac{x}{(h_p + h_r)} \quad (7)$$

It is illuminating to discuss the origin of this behaviour. Equation (7) expresses the geometrical relation between angle and distance in absence of multiple scattering.

The term in (6) due to multiple scattering is zero in this case for the following reasons: let consider the muons that arrive to the detector at a fixed distance x (from the axis; remember $x \geq 0$); multiple scattering introduces a correlation such that those detected with a positive angle are more likely to come from $y < x$, the contrary for those at negative angles; a flat $Y(y)$ then implies that positive and negative angle muons arriving at x have the same probability, hence $\langle\phi(x)\rangle_{ms} = 0$ and the only contribution is the geometrical one.

It is then obvious that for realistic, decreasing p_t distributions, i.e. decreasing $Y(y)$, both contributions to $\langle\phi(x)\rangle$ are positive, i.e. the average angle is positive (Fig. 2).

The next simple example is a gaussian distribution of transverse momenta:

$$Y(y) = \frac{1}{\rho} \exp\left(-\frac{y^2}{2\rho^2}\right), \quad \rho = h_p \frac{p_t^0}{E_\mu} \quad (8)$$

notice that ρ , the width of the distribution, depends on the product $h_p p_t^0$. Also in this case the integrals can be performed analytically for every distance x , with the result

$$\langle\phi(x)\rangle = \frac{h_p x}{\mu^2 h_p^2 + \rho^2 (h_p + h_r)^2} \left[\frac{\sqrt{3}}{2} \sigma \mu h_p + \rho^2 \right] \quad (9)$$

i.e. the average angle increases linearly with the distance, with a slope depending both on the parameters of the propagation through the rock and on the width of the lateral distribution outside the mountain.

Lastly, for an exponential transverse momentum distribution, we have

$$Y(y) = \frac{1}{\rho} \exp\left(-\frac{|y|}{\rho}\right), \quad (10)$$

with the same definition of ρ .

In this case the integrals cannot be done analytically for every distance. However, at large distances ($x \gg \sqrt{2}\mu \sim$ a few meters, see Appendix for details)

$$\langle\phi(x)\rangle \sim \frac{\sqrt{3}}{2} \frac{\sigma\mu h_p}{\rho(h_r + h_p)} + \frac{x}{(h_r + h_p)} \quad (11)$$

The geometrical contribution to $\langle\phi\rangle$ has the same form for the cases considered above, and in general for any $Y(y)$; what differs qualitatively for different lateral distributions is the first term in (6).

We have derived its form only in a few examples; however it can easily be shown that its general behaviour can be described in the following way:

1. if the lateral distribution $Y(y)$ decreases exponentially, then $\langle\phi(x)\rangle_{ms} \rightarrow \text{const.}$ at large distances;
2. if it decreases *faster* than exponentially, then $\langle\phi(x)\rangle_{ms}$ grows with distance;
3. if the decrease is *slower* than exponential, then $\langle\phi(x)\rangle_{ms} \rightarrow 0$.

The constants in the functional form of $\langle\phi(x)\rangle$ are related to the parameters of the propagation and of the lateral distribution.

The behaviour of $\langle\phi(x)\rangle$ can be analytically evaluated for $x \rightarrow 0$ too. In general $Y(y)$ is a function symmetric around zero, and decreasing, so that it can always be approximated by a gaussian for y small enough, and hence (see Eq. (9))

$$\text{Lim}_{x \rightarrow 0} \langle\phi(x)\rangle \sim \frac{\sqrt{3}}{2} \frac{\sigma\mu x}{\mu^2 + \rho^2} \quad (12)$$

The fact that this is justified can be seen in Fig. 3, where we report a projection of the p_t distribution used in the MonteCarlo program HEMAS [5], superposed with a gaussian fit around zero.

Notice that, at least for an exponential distribution,

$$\frac{\langle\phi(x)\rangle}{\langle\phi\rangle'(0)} \rightarrow \frac{\rho^2 + \mu^2}{\rho} \sim \rho = h_p \frac{p_t^0}{E_\mu} \quad (13)$$

It is interesting to give an idea of the order of magnitude of the measured quantities. Taking $h_p = 30$ Km, the geometrical term is of the order of 2×10^{-3} degrees/m. For an exponential transverse momentum

distribution with $\langle p_t \rangle = 0.4 \text{ GeV}$, $\rho \sim 6m$ at $E_\mu = 2 \text{ TeV}$; typical parameters for the transport through the rock are $\sigma = 0.4^\circ$, $\mu = 1.5m$ so that $\langle \phi \rangle_{ms} \sim 0.1^\circ$.

With the same parameters, near $x = 0$ one has $\langle \phi \rangle_{ms} \sim 0.014^\circ \times x$. It is therefore clear that only experiments with a good intrinsic angular resolution and a large acceptance can accumulate enough statistics to measure average angles so small.

Up to this point we had in mind a measurement feasible in an underground muon experiment, in which the shower axis is not accessible. It is however possible to perform joint measurements between underground apparata and Extensive Air Showers experiments at the surface.

In this case the coordinates of the shower axis are known, with some errors in angle σ_E and position μ_E at the measurement level, which propagate to the underground experiment and, even in absence of multiple scattering, do introduce a correlation term analogous to the one in (2). In fact the effect of the uncertainties in the position of the shower axis at the underground level are typically of the same order (or larger) of the displacements due to multiple scattering, so that as a first approximation we will neglect the effect of propagation in the rock. We then have

$$\frac{dN}{dx d\phi} \sim \exp \left[-\frac{\phi^2}{2} \left(\frac{1}{\sigma_E^2} + \frac{h_r^2}{\mu_E^2} \right) - \frac{(x-y)^2}{2\mu_E^2} + \frac{h_r}{\mu_E^2} (x-y)\phi \right] \quad (14)$$

Again, at large distances, we have (for an exponential p_t distribution, and neglecting α):

$$\langle \phi(x) \rangle \sim \frac{\sigma_E^2 h_r}{\rho} \quad (15)$$

however now (15) holds for $x \gg \sqrt{2}\mu_E$, typically a few *tens* of meters.

At small distances, in the gaussian approximation for $Y(y)$ we get

$$\langle \phi(x) \rangle = \frac{\sigma_E^2 h_r}{\rho^2 + \mu_E^2 + \sigma_E^2 h_r^2} x \sim \frac{x}{h_r} \quad (16)$$

3. A SIMPLIFIED SIMULATION.

We have been able to obtain simple formulas for the decorrelation function as a consequence of the approximations we have introduced. To estimate the effect of these approximations and at the same time have quantitative predictions on $\langle \phi(x) \rangle$ we have performed a simplified simulation.

The main ingredients of the simulation are:

- a. We work in a projection and consider angles and distances referred to the axis of the shower.
- b. We draw single muons from *independent* energy ($N(> E_\mu) \sim E_\mu^{-2.7}$) and p_t (typically exponential) distributions. Production height is either kept fixed or drawn from a given (realistic) distribution.
- c. As for the propagation in the rock, the energy loss is treated realistically (with fluctuations) using a routine by Lipari and Stanev [5]. The rock is divided in layers (typically $\leq 10hg/cm^2$), in each layer σ and μ are computed from the average energy and angles and lateral displacements are drawn from the distribution (2).
- d. We work at fixed azimuth and zenith angles and rock depth.

This simulation is very fast, thus allowing us to vary the physical parameters, while at the same time being able to have very good statistics.

The output of the simulation is a set of decorrelation functions and, for each set of parameters, the distributions of ρ and σ, μ and $\frac{\sigma}{\mu}$.

The distributions of ρ, σ , and μ are reported in Figs. 4, 5 and 6 respectively, for a production height of $30km$, a projected exponential p_t distribution ($p_t^0 = 0.4GeV$) and a rock depth of $3700hg/cm^2$.

While in these simulations the dependence of ρ, σ, μ on the energy, as well as their fluctuations, is appropriately taken into account, the limited spread of these distributions justifies neglecting their fluctuations and the use of average values in the derivation of the analytical results.

In Fig. 7 we plot, for the same set of input parameters, the decorrelation function generated by the simulation program superposed to the analytical approximations of (9,11), using the averages of ρ, σ, μ from Figs. 4-6. In view of the approach used, the agreement is extremely good.

The continuous line in this figure is a simple analytical interpolation

$$\langle \phi(x) \rangle_{ms} = \frac{\sqrt{3}}{2} \frac{\sigma \mu x}{\mu^2 + \rho^2 + x\rho} \quad (17)$$

with the same parameters as above.

4. CONCLUSIONS

In this paper we propose a new measurement which can be performed in underground experiments with large acceptance, good intrinsic angular resolution and able to sample the muon distributions at large relative distances. This measurement, which we call decorrelation function, studies the relation between average relative angles and distances in pairs of muons within underground multiple muon events.

In our analytical approach, we have shown that from the form of the decorrelation function it is possible to extract representative values for the

parameters describing the propagation in the rock and, most importantly, those (p_t and interaction height distributions) of the interaction of primaries in the atmosphere.

We predict that, at large enough distances, the decorrelation increases linearly, the inverse slope being the distance of interaction, while the intercept is proportional to the widths of the displacements due to multiple scattering and inversely to that of the lateral distribution in the atmosphere.

At small distances the slope of the decorrelation near $x = 0$ depends on the same quantities in such a way that, combining the information at large and small distance, the width of the lateral distribution in the atmosphere can be deduced.

The comparison of the analytical results with a simplified, although partially realistic simulation is extremely encouraging. In particular, in the simulation the propagation of the muons from the surface is treated realistically, with fluctuations in the energy loss; also, the initial energy of the muons is taken from the known spectrum. In spite of the approximations, the analytical form for the decorrelation reproduces quite well the simulated data, and we are encouraged to use it for describing real data.

There are several effects that are to be taken into account before applying these ideas to a real experiment.

As we have already noticed, the measured angles are very small and in fact smaller than the typical angular resolution of an underground experiment, so implying that a large statistics is needed. The effect of the intrinsic resolution has not been included in the derivation, although it could be done easily; also even tiny systematic errors in the measurement of the angles would show up as deformations in the decorrelation function.

On the positive side, it has to be noted that the measurement of the decorrelation is relative, so normalization and systematic effects which do not affect angles are likely to affect little the results.

The classical approach to the measurements described in this paper is through the study of the decoherence function, which essentially measures the lateral distribution in the atmosphere. The decorrelation can give in principle information on the production height, and in any case produces an independent estimate on the same quantity. It is moreover less dependent on the finite size of the detector.

The aspects described above are as far as one can reach analytically. They are a convincing evidence that this is a sensible measurement. There are several other effects that have not been addressed here: for instance the dependence of the decorrelation on the composition of the Primary Cosmic rays, on the detailed structure of the hadronic showers, or on a prompt component in the muon flux. These effects, as well as the comparison with the real data, can only be evaluated through a detailed Monte Carlo simulation.

APPENDIX.

In this Appendix we will give some details on the analytical computations reported in the text.

The measured distribution is the convolution of distributions in angle and distances due to the effects of primary interactions, propagation into the rock above the apparatus and detection uncertainties.

Let consider propagation [5] in a thin layer (ΔX gcm⁻²) of rock, such that the energy of the muons in this layer is approximately constant. The displacements in angle and distance due to multiple scattering, in the gaussian approximation, are described by [3]

$$\frac{dN}{dr_x d\theta_x} = G(r_x, \theta_x) = (\pi\sigma\mu)^{-1} \exp\left(-2\frac{\theta_x^2}{\sigma^2} - 2\frac{r_x^2}{\mu^2} + 2\sqrt{3}\frac{\theta_x r_x}{\mu\sigma}\right) \quad (A.1)$$

with a similar term in an orthogonal projection, with

$$\sigma = \frac{\epsilon}{E_\mu} \sqrt{\frac{\Delta X}{\lambda_r}}, \quad \mu = \frac{1}{\sqrt{3}} \frac{\Delta X}{\bar{\rho}} \sigma \quad (A.2)$$

where $\epsilon \sim 14\text{MeV}$, λ_r is the radiation length and $\bar{\rho}$ is the density of the medium.

For a finite layer one has to take into account the energy lost by the muons in the layers above; the form of the distribution remains the same and its width is summed in quadrature for σ from each infinitesimal layer, while for μ one must take into account the displacement of the muon due to the angle accumulated in the layers above the one considered

It has been argued ([5]) that the gaussian approximation above gives excellent average results, if fluctuations due to stochastic processes are appropriately taken into account in the energy loss.

The variables θ_x, r_x i.e. the displacements from the position and direction with which muons enter into the mountain, are not directly measurable; in a real experiment what is measured is distance and relative angle between muon pairs. We will however derive our formulas for the quantities referred to the shower axis; this will make clearer our derivation, and in fact is not a loss of generality.

With this assumption, let us define x as the distance of the detected muon from the shower axis, and ϕ the corresponding angle. Let also call y the distance with which muons enter the mountain and α the corresponding angle. These quantities are related to the muon transverse momentum p_t relative to the primary, its energy E_μ and production height h_p

$$\alpha = p_t/E_\mu, \quad y = h_p \alpha \quad (A.3)$$

In the definitions above we identify the transverse momentum of the muon with that of the parent meson; also, we neglect the difference in energy between the muon and the parent.

We first assume $\alpha = 0$, but $y \neq 0$ i.e. the muons enter the mountain with a distance y from the shower axis, but parallel to it; this assumption will be later released.

Then y, x, ϕ, r_x and θ_x are related through (see Fig 1a):

$$r_x = x - y; \quad \theta_x = \phi \quad (A.4)$$

Let imagine that the shower axis is represented by a muon which has suffered no deflection both outside and inside the mountain. Then the distances and angles are relative distances between muons, measured always with some sort of ordering. If we work in a projection, as we do in these derivations, we will always keep the same ordering, *e.g.*:

$$x = (b_{right} - b_{left}) \cos(s_{av}) \quad (A.5)$$

$$\phi = \tan^{-1}(s_{right}) - \tan^{-1}(s_{left})$$

where s and b are the slopes and intercepts of the reconstructed tracks in a projection, and $s_{av} = \tan^{-1}(\frac{1}{2}(s_{right} + s_{left}))$.

With these definitions, for each pair of measured quantities there are two possible configurations which correspond to the same y , namely the one above and:

$$r_x = -x - y; \quad \theta_x = -\phi \quad (A.6)$$

(see Fig. 1b), so that the overall distribution due to the propagation, expressed in terms of the measured variables is

$$\left. \frac{dN}{dx d\phi} \right|_y = \frac{1}{2} [G((x - y), \phi) + G((-x - y), -\phi)] \quad (A.7)$$

At fixed energy and production height, and neglecting fluctuations in the energy loss due to propagation in the rock, the experimentally measured distribution is obtained by folding Eq. (A.7) with the distribution of distances outside the mountain

$$Y(y) = \frac{dN}{dy} = \frac{E_\mu}{h_p} \frac{dN}{dp_t} \quad (A.8)$$

so that

$$\frac{dN}{dx d\phi} = \frac{1}{2} \int dy Y(y) [G((x - y), \phi) + G((-x - y), -\phi)] \quad (A.9)$$

The first few moments of this distribution are

$$D(x) = \int d\phi \frac{dN}{dx d\phi} \quad (\text{A.10})$$

i.e. the number of muons as a function of the distance from the axis, and the average angle versus distance

$$\langle \phi(x) \rangle = \frac{\int d\phi \phi \frac{dN}{dx d\phi}}{D(x)} \quad (\text{A.11}).$$

We obtain:

$$D(x) = \frac{1}{2} \int dy Y(y) D(x, y) \quad (\text{A.12})$$

and

$$\begin{aligned} \langle \phi(x) \rangle &= \frac{\sqrt{3}}{2} \frac{\sigma}{\mu} \left[\frac{x \int dy Y(y) D(x, y) + \int dy y Y(y) D_-(x, y)}{\int dy Y(y) D(x, y)} \right] = \\ &= \frac{\sqrt{3}}{2} \frac{\sigma}{\mu} \left[x + \frac{\int dy y Y(y) D_-(x, y)}{\int dy Y(y) D(x, y)} \right] \end{aligned} \quad (\text{A.13})$$

where

$$\begin{aligned} D(x, y) &= (\sqrt{2\pi}\mu)^{-1} \left[\exp\left(-\frac{(x-y)^2}{2\mu^2}\right) + \exp\left(-\frac{(x+y)^2}{2\mu^2}\right) \right] \\ D_-(x, y) &= (\sqrt{2\pi}\mu)^{-1} \left[\exp\left(-\frac{(x+y)^2}{2\mu^2}\right) - \exp\left(-\frac{(x-y)^2}{2\mu^2}\right) \right] \end{aligned} \quad (\text{A.14})$$

The integrals in y can only be done analytically in few cases. Instead of considering these cases separately, we will derive them approximately for large x .

If $Y(y) = Y(-y)$ the integrals are symmetric under the exchange $y \rightarrow -y$ and writing $Y(y) = ce^{-g(y)}$ the integrands become

$$\exp\left(-g(y) - \frac{(x-y)^2}{2\mu^2}\right)$$

and

$$y \exp\left(-g(y) - \frac{(x-y)^2}{2\mu^2}\right)$$

respectively.

Using the saddle point approximation we get

$$\langle \phi(x) \rangle \sim \frac{\sqrt{3}}{2} \frac{\sigma}{\mu} [x - \tilde{y}(x)] \quad (\text{A.15})$$

where $\tilde{y}$ is a solution of

$$g'(\tilde{y}) - \frac{(x - \tilde{y})}{\mu^2} = 0 \quad (\text{A.16})$$

This result is valid for *any* realistic distribution $Y(y)$.

Let us now specialize to some cases. First consider a flat distribution, i.e. $Y(y) \sim \text{const.}$ Then $\tilde{y}(x) = x$ hence

$$\langle \phi(x) \rangle = \frac{\sqrt{3}}{2} \frac{\sigma}{\mu} [x - \tilde{y}] = \frac{\sqrt{3}}{2} \frac{\sigma}{\mu} [x - x] = 0$$

This result could have been directly obtained without any approximation. The other extreme case is a gaussian lateral distribution

$$Y(y) = \frac{1}{\rho} \exp\left(-\frac{y^2}{2\rho^2}\right)$$

for which

$$\tilde{y} = x \frac{\rho^2}{\rho^2 + \mu^2} \quad (\text{A.17})$$

and

$$\langle \phi(x) \rangle = \frac{\sqrt{3}}{2} \frac{\sigma}{\mu} \left[x - x \frac{\rho^2}{\rho^2 + \mu^2} \right] = \frac{\sqrt{3}}{2} \frac{\sigma \mu}{\mu^2 + \rho^2} x \quad (\text{A.18})$$

Also in this case the result is exact.

For an exponential distribution

$$Y(y) = \frac{1}{\rho} \exp\left(-\frac{|y|}{\rho}\right)$$

we get, at large x , ($x \gg \sqrt{2}$)

$$\tilde{y}(x) = x - \frac{\mu^2}{\rho} \quad (\text{A.19})$$

$$\langle \phi(x) \rangle = \frac{\sqrt{3}}{2} \frac{\sigma}{\mu} \left[x - x + \frac{\mu^2}{\rho} \right] = \frac{\sqrt{3}}{2} \frac{\sigma \mu}{\rho} \quad (\text{A.20})$$

The same technique can be used for a decreasing power-like distribution, giving an average angle decreasing (at large distances) like x^{-1} .

We are now in the position of releasing the approximation $\alpha = 0$. The effect on the distribution of a non zero angle when the muon enters the mountain is twofold: θ_x has to be replaced by $\theta_x - \alpha = \theta_x - y/h_p$, and r_x

by $r_x - h_r \alpha = r_x - y h_r / h_p$ where $h_r \alpha$ is the displacement the muon would have suffered without multiple Coulomb scattering.

After some manipulations we obtain, for large x :

$$\begin{aligned} \langle \phi(x) \rangle &\sim \frac{\sqrt{3}}{2} \frac{\sigma}{\mu} x - \left[\frac{\sqrt{3}}{2} \frac{\sigma}{\mu} \left(1 + \frac{h_r}{h_p} \right) - \frac{1}{h_p} \right] \times \\ &\times \frac{\int dy y Y(y) \exp \left(-\frac{(x - (1 + \frac{h_r}{h_p})y)^2}{2\mu^2} \right)}{\int dy Y(y) \exp \left(-\frac{(x - (1 + \frac{h_r}{h_p})y)^2}{2\mu^2} \right)} \sim \\ &\sim \frac{\sqrt{3}}{2} \frac{\sigma}{\mu} x - \frac{\sqrt{3}}{2} \frac{\sigma}{\mu} \left(1 + \frac{h_r}{h_p} \right) \tilde{y} + \frac{1}{h_p} \tilde{y} \end{aligned} \quad (\text{A.21})$$

(notice that the minimum equation has changed from the case $\alpha = 0$). Again, this relation is valid for any $Y(y)$. Specializing to the cases discussed above we have:

Flat distribution:

$$\begin{aligned} \tilde{y} &= \frac{x h_p}{h_r + h_p} \\ \langle \phi(x) \rangle &= \frac{x}{h_p + h_r} \end{aligned} \quad (\text{A.22})$$

Exponential distribution :

$$\begin{aligned} \tilde{y} &= \frac{h_p x}{h_p + h_r} - \frac{h_p^2 \mu^2}{\rho(h_p + h_r)^2} \\ \langle \phi(x) \rangle &\sim \frac{\sqrt{3}}{2} \frac{\sigma \mu h_p}{\rho(h_r + h_p)} + \frac{x}{(h_r + h_p)} \end{aligned} \quad (\text{A.23})$$

Gaussian distribution :

$$\begin{aligned} \tilde{y} &= \frac{x \rho^2 h_p (h_p + h_r)}{\mu^2 h_p^2 + \rho^2 (h_p + h_r)^2} \\ \langle \phi(x) \rangle &= \frac{h_p x}{\mu^2 h_p^2 + \rho^2 (h_p + h_r)^2} \left[\frac{\sqrt{3}}{2} \sigma \mu h_p + \rho^2 \right] \sim \\ &\sim \frac{\sqrt{3}}{2} \frac{\sigma \mu^2 h_p^2 x}{h_p^2 + \rho^2 (h_p + h_r)^2} \end{aligned} \quad (\text{A.24})$$

It is interesting to write down the form of $\langle \phi(x) \rangle$ for a distribution of muon energies, but still neglecting fluctuations in their energy loss. In this case σ, μ and ρ are functions of E_μ . Equation (6) thus changes in

$$\langle \phi(x) \rangle = \frac{\sqrt{3}}{2} \left[\frac{x \int dE_\mu \frac{dN}{dE_\mu} \frac{\sigma}{\mu} \int dy Y(y) D(x, y) + \int dE_\mu \frac{dN}{dE_\mu} \frac{\sigma}{\mu} \int dy y Y(y) D_-(x, y)}{\int dE_\mu \frac{dN}{dE_\mu} \int dy Y(y) D(x, y)} \right] \quad (A.25)$$

where dN/dE_μ is the muon energy spectrum and D, D_- are now function of energy through μ, ρ . Notice that, even taking into account the energy dependence, in the approximation that σ/μ remains constant, which is quite well verified in the simulation, this form simplifies somewhat into

$$\langle \phi(x) \rangle = \frac{\sqrt{3}}{2} \frac{\sigma}{\mu} \left[x + \frac{\int dE_\mu \frac{dN}{dE_\mu} \int dy y Y(y) D_-(x, y)}{\int dE_\mu \frac{dN}{dE_\mu} \int dy Y(y) D(x, y)} \right] \quad (A.26)$$

in particular, the dependence on σ factorizes.

It is particularly interesting to note that, for an exponential distribution, the dominant term in $\tilde{y}$ (A.23) has no dependence on energy (neglecting the dependence of h_p), so the linear growth at large distances is general and does not depend on the approximations introduced.

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FIGURE CAPTIONS

- FIG. 1 Configurations contributing to the same x, ϕ at fixed y .
- FIG. 2 Two possible configurations of muons at a fixed value of x . If $Y(y)$ is decreasing and taking into account the correlation term in (2), then $N(\phi > 0) > N(\phi < 0)$, hence $\langle \phi \rangle > 0$.
- FIG. 3 Transverse momentum distribution of parents of muons surviving 3700 hg/cm^2 of Standard Rock (from HEMAS).
- FIG. 4 Distribution of ρ from the simulation.
- FIG. 5 Distribution of σ from the simulation.
- FIG. 6 Distribution of μ from the simulation.
- FIG. 7 Decorrelation curve from the simulation (distances are in cm, angles in degrees, 10^6 events). Straight lines are the analytical approximation (Eqs. 9,11), the continuous line is the interpolation of Eq. 17. Parameters are the average from Figs. 4-6.

Figure 1 a

Figure 1 b

Figure 2

Figure 3

Figure 4

Figure 5

Figure 6

Figure 7