I. Initial Data

In the general case of two holes of arbitrary size and with arbitrary spins, such orbits are explicitly determined for the case of two equal-sized nonrotating black holes. An effective potential approach for locating quasi-circular orbits is outlined for the case of two nonrotating holes in a quasi-circular orbit. The construction of initial-data sets representing binary black-hole configurations in quasi-circular orbits is studied in the context of the conformal-imaging formalism. An effective potential approach for locating quasi-circular orbits is outlined for the case of two equal-sized nonrotating holes, and the innermost stable quasi-circular orbit is located. The characteristics of this innermost orbit are compared to previous estimates for it, and the entire sequence of quasi-circular orbits is compared to results from the post-Newtonian approximation. Some aspects of the numerical evolution of such data sets are explored. For this simple case, we have located those initial-data sets of binary quasars which correspond to the innermost stable quasi-circular orbit. For the case of two equal-sized nonrotating holes, there is a minimum of the effective potential at a minimum separation of the holes. If we choose the momenta of the holes appropriately, then an initial-data set occurring at a minimum of the effective potential will represent the configuration in which the system can be continued with the configuration of the holes appropriately. The minimum separation of the holes. If we fix the momenta of the two holes in a Cartesian coordinate system, the effective potential has a minimum at a separation of the holes. If we choose the momenta of the holes appropriately, then an initial-data set occurring at a minimum of the effective potential will represent the configuration in which the system can be continued with the configuration of the holes appropriately.
is constructed by solving the momentum and Hamiltonian equations chosen as the origin of coordinates, results in a vanishing dipole moment. Using the definition of the ADM energy and the dipole moment, it follows that the center of energy is at $O = d/E$. Therefore, the general definition of the orbital angular momentum is

$$J \equiv \left( C_1 - \frac{d}{E} \right) \times P_1 + \left( C_2 - \frac{d}{E} \right) \times P_2. \quad (4)$$

## III. THE EFFECTIVE-POTENTIAL METHOD

The definitions of the ADM energy, total linear and angular momenta at infinity, the dipole moment, the proper separation of the holes, and the areas of the two marginally outer-trapped surfaces are rigorously defined physical quantities. To define the effective potential of a configuration, it is necessary to use the concepts of the masses and spins of the individual black holes and to have a measure of the effective binding energy between the two holes. However, none of these quantities are rigorously defined in general relativity for a strong-field nonstationary configuration.

In the limit of large separations and small linear momenta and spin on the holes, the following definitions hold. The mass of each hole can be defined via the Christodoulou formula [12]

$$M^2 = M^2_{ir} + \frac{S^2}{4M^2_{ir}}, \quad (5)$$

with the irreducible mass $M_{ir} \approx \sqrt{A/16\pi}$ and $S$ being the magnitude of the spin on the hole. As shown in the Appendix, we can approximate $S$ for each hole by the magnitude of its respective spin parameter $|S_{1,2}|$. Though it seems possible that the linear momentum on one hole could induce a spin on the other hole, this is, in fact, not the case. Thus, the quantity $J$, which we identified above as the orbital angular momentum, is not contaminated by an induced spin on the holes. We are therefore justified in defining the orbital angular momentum of the system by equation (4), and we define the spins of the individual holes via their respective spin parameters. Finally, the effective binding energy $E_b$ between the holes is defined as

$$E_b \equiv E - M_1 - M_2, \quad (6)$$

where $M_{1,2}$ are the masses of the two holes as defined above. Note that the effective binding energy contains both the gravitational binding energy between the two holes and their orbital kinetic energies, but not the rotational kinetic energy of the individual holes.

These definitions are rigorous only in the limit of infinite separation and zero momenta on either hole.
The limit of zero momenta on the holes is required because initial-data sets containing a single black hole constructed via the conformal-imaging approach necessarily contain some spurious gravitational-wave energy [13]. The same is true of multihole initial-data sets, however, Cook and Abrahams [14] have shown that the spurious radiation content for the case of two holes is quite small so long as the holes are modestly separated and the momenta are not excessively large. We will find that these constraints are satisfied for the majority of configurations of physical interest.

Henceforth, we will take the masses of the holes, their spins, mutual binding energy, and orbital angular momentum to be defined as given above.

We turn now to the definition of an effective potential useful for determining the location of quasi-circular orbits. In general, such an effective potential should be a function of the separation and sizes of the holes, the orbital angular momentum of the system, the spins of the holes, and the gravitational radiation content of the system. Within the conformal-imaging approach, one has no freedom in specifying the radiation content of the system. This is fixed by the demands that the spatial 3-metric be conformally flat and that all fields satisfy an isometry condition (cf. Ref. [7]). With this restriction, we see that the effective potential should be a function of nine physical parameters but a general two-hole initial-data set depends on fourteen initial-data parameters.

To reduce the size of this parameter space, we first demand that the configuration be in a center of momentum frame. This restriction requires

$$P_1 + P_2 = 0.$$  \hfill (7)

A configuration representing a quasi-circular orbit should satisfy

$$P_{1,2} \cdot (C_2 - C_1) = 0.$$  \hfill (8)

Together, equations (7) and (8) reduce the fourteen-dimensional initial-data parameter space to nine dimensions. These parameters are $\alpha, \beta$, the magnitude of the linear momentum on either hole $P/a_1, S_1/a_1^2$, and $S_2/a_2^2$.

The respective dimensionless physical parameters of the effective potential are $X = M_1/M_2, \ell/m, J/\mu m, S_1/M_1^2, S_2/M_2^2$ where $M_1, M_2$ are the masses of the individual holes, $m \equiv M_1 + M_2$ is the total mass, $\mu \equiv M_1 M_2/m$ is the reduced mass, and $J$ is the magnitude of the orbital angular momentum of the system. Finally, the dimensionless effective potential is given by the binding energy as $E_b/\mu$.

Finding quasi-circular orbits is now a conceptually easy task. We compute $E_b/\mu$ as a function of $\ell/m$ while holding the remaining eight physical parameters constant. A minimum in $E_b/\mu$ then corresponds to a “stable” quasi-circular orbit. In addition to locating the quasi-circular orbits, we can also estimate the orbital angular velocity $\Omega$ of the system as measured at infinity. This is given by taking the derivative of the binding energy with respect to the orbital angular momentum while holding all other parameters fixed. In dimensionless form then one obtains

$$m \Omega = \frac{\partial E_b/\mu}{\partial J/\mu m}.$$ \hfill (9)

Though conceptually straightforward, the computational task of locating quasi-circular orbits is difficult. The main difficulty arises from the fact that the physical parameters that must be held fixed are not independently correlated with their respective initial-data parameters. That is, holding eight of the nine initial-data parameters fixed while varying $\beta$ will not result in an effective-potential curve. We see then that, in general, the problem of determining one quasi-circular orbit is quite involved. It requires finding the roots of eight functions, each of which depends on eight parameters, at each value of the separation at which the effective potential is evaluated.

Fortunately, the size of this parameter space can be cut in half. The only definition we have available for the direction of the spins of the holes (see the Appendix) is the direction of their respective initial-data spin parameters. The directions of the spins are then fixed relative to the separation of the holes and the plane of the orbit, which are defined by $P_{1,2}$ and $C_2 - C_1$. This direction is independent of the numerical solution of the Hamiltonian constraint, so only the magnitude of the two initial-data spin-vector parameters needs to be varied to hold the physical parameters fixed. As a result, for a fixed value of $\beta$ the following four equations must be satisfied:

$$X_{(\alpha, \beta)}^{(a_1, a_2, S_1, a_1^2, S_2, a_2^2)} = X_0$$ \hfill (10a)

$$\left[ \frac{J}{\mu m} \right]_{(\alpha, \beta)}^{(a_1, a_2, S_1, a_1^2, S_2, a_2^2)} = \left[ \frac{J}{\mu m} \right]_0$$ \hfill (10b)

$$\left[ \frac{S_1}{M_1^2} \right]_{(\alpha, \beta)}^{(a_1, a_2, S_1, a_1^2, S_2, a_2^2)} = \left[ \frac{S_1}{M_1^2} \right]_0$$ \hfill (10c)

$$\left[ \frac{S_2}{M_2^2} \right]_{(\alpha, \beta)}^{(a_1, a_2, S_1, a_1^2, S_2, a_2^2)} = \left[ \frac{S_2}{M_2^2} \right]_0.$$ \hfill (10d)

When these four equations are satisfied, equation (6) yields a value of the effective potential $E_b/\mu$ at some value of the physical separation $\ell/m$. Changing the value of $\beta$ and resolving Eqns. (10a)-(10d) results in another value of the effective potential at a different separation.

IV. EQUAL-MASS NONROTATING HOLES

The simplest application of the effective-potential method is to the case of two equal-sized black holes with no intrinsic spin. In this problem, $S_1/a_1^2 = S_2/a_2^2 = 0$ and, because of the symmetry between the holes, we know that $a = 1 - X = 1$. Therefore, solving for the effective potential requires solving only Eqn. (10b) as a function of $P/a_1$ alone for a given $\beta$. 

3
The method used to solve equation (10b) is the following. For a given value of \( \beta \), the initial-value equations are solved at a sufficiently large number of values of \( P/a_1 \) in order to encompass all of the values of \( J/\mu m \) at which we want to evaluate the effective potential. Using interpolation, we can estimate new values of \( P/a_1 \) that will yield solutions near the desired values of \( J/\mu m \); the procedure is repeated until any errors introduced by interpolation are sufficiently small. The most difficult part of the problem is to solve the initial-value equations with sufficient accuracy. Typically, both the ADM mass \( E/a_1 \) and the areas of the marginally outer-trapped surfaces \( A_{1,2}/a_1^2 \) need to be determined to a relative error of \( \sim 10^{-3} \).

Currently, the only numerical method capable of solving the initial-value equations to this accuracy is the multigrid-based “Cadez-coordinate approach” described in detail in section IIIA of Paper I. Such high accuracy can be obtained through the use of Richardson extrapolation. As described in Paper I, the Cadez-coordinate approach used to solve the Hamiltonian constraint results in a numerical solution for the conformal factor \( \psi^{num} \), which has an asymptotic \( (h \to 0) \) expansion given by

\[
\psi^{num} = \psi + h^2 c_2 \psi + h^4 c_4 \psi + \ldots,
\]

where \( \psi \) is the analytic solution of the Hamiltonian constraint, \( h \) is the basic scale of discretization, and \( c_2, c_4, \ldots \) are \( h \)-independent functions. In addition, the numerical integrals for \( E/a_1, d/a_1^2, A_{1,2}/a_1^2 \), and \( \ell/a_1 \) have all been constructed to yield analogous error expansions that depend strictly on powers of \( h^2 \).

One final source of error which must be examined comes from the necessity of imposing an approximate outer-boundary condition (cf. Ref. [7]). In order to minimize the effects of this approximation, the outer boundary has been placed at a distance of at least 2000 \( a_1 \) from the holes.

Figure 1 displays a set of effective-potential curves for a wide range of values for \( J/\mu m \). The displayed curves are interpolated results derived from 3000 Richardson-extrapolated data points, each resulting from the extrapolation of three separate solutions of the initial-value equations at resolutions similar to those described in Paper I. All solutions were generated on an IBM-SP1 parallel computer and required a total computational time in excess of 3000 CPU hours. The value of \( J/\mu m \) is held fixed along each of the thin curves which plots the effective potential \( E_b/\mu \) as a function of the proper separation of the holes \( \ell/m \). The bold curve crossing several of the effective-potential curves represents a sequence of quasi-circular orbits. This can be seen more clearly in Fig. 2 where the region containing minima in the effective-potential curves is shown in expanded form. The bold line representing the sequence of quasi-circular orbits begins at the right at an \( \ell/m \sim 14 \). This line should, of course, extend to larger values of \( \ell/m \), but data has not been computed in this regime. Following the sequence of quasi-circular orbits to smaller values of

---

**FIG. 1.** The effective potential \( E_b/\mu \) as a function of separation \( \ell/m \) for the following values of the orbital angular momentum \( J/\mu m \): 1.5, 2, 2.5, 2.75, 2.95, 2.976, 2.985, 3, 3.05, 3.15, 3.25, 3.37, 3.5, 3.62, 3.75, 3.85, 4, 4.25, and 4.5. These values of \( J/\mu m \) label, respectively, curves from the bottom of the figure to the top. Also, plotted as a bold line is the sequence of quasi-circular orbits which cross the effective-potential curves at local minima.

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**FIG. 2.** An enlargement of the section of Fig. 1 which contains the sequence of quasi-circular orbits. This sequence begins at the innermost stable quasi-circular orbit near \( \ell/m = 5 \) on the \( J/\mu m = 2.976 \) curve and extends in the direction of larger separation. In the figure, the sequence terminates at the minimum of the \( J/\mu m = 3.85 \) effective-potential curve, although it should continue.
TABLE I. Physical and initial-data parameters characterizing certain configurations along the sequence of stable quasi-circular orbits. The data sets represented in this table have been constructed using the minus isometry condition of the conformal-imaging approach.

<table>
<thead>
<tr>
<th>(t/m)</th>
<th>(E_h/\mu)</th>
<th>(J/\mu m)</th>
<th>(m\Omega)</th>
<th>(P/\alpha_1)</th>
<th>(\beta)</th>
</tr>
</thead>
<tbody>
<tr>
<td>4.880</td>
<td>-0.69030</td>
<td>2.076</td>
<td>0.172</td>
<td>1.685</td>
<td>11.82</td>
</tr>
<tr>
<td>5.365</td>
<td>-0.68890</td>
<td>2.085</td>
<td>0.145</td>
<td>1.392</td>
<td>13.28</td>
</tr>
<tr>
<td>5.735</td>
<td>-0.68684</td>
<td>3.000</td>
<td>0.130</td>
<td>1.230</td>
<td>14.43</td>
</tr>
<tr>
<td>6.535</td>
<td>-0.68112</td>
<td>3.050</td>
<td>0.104</td>
<td>0.9868</td>
<td>16.99</td>
</tr>
<tr>
<td>7.000</td>
<td>-0.67226</td>
<td>3.150</td>
<td>0.0774</td>
<td>0.7752</td>
<td>20.94</td>
</tr>
<tr>
<td>8.695</td>
<td>-0.65344</td>
<td>3.250</td>
<td>0.0622</td>
<td>0.6613</td>
<td>24.21</td>
</tr>
<tr>
<td>9.800</td>
<td>-0.65862</td>
<td>3.370</td>
<td>0.0504</td>
<td>0.5734</td>
<td>28.04</td>
</tr>
<tr>
<td>10.96</td>
<td>-0.65270</td>
<td>3.500</td>
<td>0.0414</td>
<td>0.5066</td>
<td>32.15</td>
</tr>
<tr>
<td>12.02</td>
<td>-0.64810</td>
<td>3.620</td>
<td>0.0352</td>
<td>0.4669</td>
<td>35.93</td>
</tr>
<tr>
<td>13.16</td>
<td>-0.64388</td>
<td>3.750</td>
<td>0.0300</td>
<td>0.4218</td>
<td>40.66</td>
</tr>
<tr>
<td>14.07</td>
<td>-0.64104</td>
<td>3.850</td>
<td>0.0270</td>
<td>0.3960</td>
<td>43.38</td>
</tr>
</tbody>
</table>

In order to gauge the accuracy with which we have located the innermost stable quasi-circular orbit, note that in the limit of a test mass orbiting a Schwarzschild black hole, the proper separation between the event horizon and the test mass is found to be

\[
\frac{t}{m} = 2\ln\left(\frac{1 + \sqrt{2/3}}{1 - \sqrt{2/3}}\right) \approx 4.58. \tag{12}
\]

This should be compared with a value of \(t/m = 4.88\) obtained from Table I. The ratio of these two values is 0.94. Kidder et al. [10] obtain an analogous ratio of 0.96, where separation is measured in terms of harmonic or deDonder coordinates. Kidder et al. obtain this result via a critical point analysis of the equations of motion through (post)\(^2\)-Newtonian order. After altering the equations of motion significantly to reproduce exactly the test mass limit, Kidder et al. find that the ratio has changed to 0.83 and that the innermost stable circular orbit is characterized by the following physical parameters: \(E_h/\mu \sim -0.0378\), \(J/\mu m \sim 3.83\), and \(m\Omega \sim 0.0605\). A comparison of these values with Table I shows that Kidder et al. are finding an innermost stable circular orbit in which the holes are much farther apart than we find with the effective-potential method. In contrast to this, Blackburn and Detweiler [9] have used a variational principle together with the assumption of a periodic solution to Einstein’s equations to obtain an estimate for the innermost orbit for two equal-sized holes. Using a single trial geometry, which they call “rather unsophisticated”, they obtain an innermost orbit characterized by \(E_h/\mu \sim -0.65\), \(J/\mu m \sim 0.85\), and \(m\Omega \sim 2\). Such an orbit is much more tightly bound than seems possible from the effective-potential method. Blackburn and Detweiler point out that the assumptions of their variational principle have been violated by the time this innermost circular orbit is reached. However, they describe a less tightly bound circular orbit that should not be in violation of the underlying assumptions of their approach. This orbit has a binding energy of \(E_h/\mu \sim -0.28\), which is still three times larger than the binding energy obtained in this paper for the innermost stable quasi-circular orbit.

V. THE POST-NEWTONIAN LIMIT

Comparison with previous estimates for the innermost stable quasi-circular orbit of two equal-mass nonrotating black holes are far from yielding a consensus as to its proper value. However, we can gain some insight into the reliability of the results derived in this paper by comparing the sequence of quasi-circular orbits against the post-Newtonian description of circular orbits. Based on the (post)\(^2\)-Newtonian results of Kidder et al. [16] for the binding energy, angular momentum, and equations of motion for a binary system with a circular orbit, it is straightforward to show that the binding energy and
FIG. 3. The effective potential $E_b/\mu$ as a function of the orbital angular momentum $J/\mu m$ for quasi-circular orbits. The solid line corresponds to the sequence of quasi-circular orbits computed in this paper. The long dashed line is the result obtained from Newtonian theory. The short dashed line is the result based on (post)$^1$-Newtonian theory, and the dotted line is the result based on (post)$^2$-Newtonian theory.

The angular momentum of two compact objects in a circular orbit must satisfy

$$\frac{E_b}{\mu} = -\frac{1}{2} \left( \frac{\mu m}{J} \right)^2 \left[ 1 + \frac{1}{4} (9 + \eta) \left( \frac{\mu m}{J} \right)^2 + \cdots \right] \quad (13)$$

$$\frac{E_b}{\mu} = -\frac{1}{2} (m\Omega)^{2/3} \left[ 1 - \frac{1}{12} (9 + \eta) (m\Omega)^{2/3} - \left( \frac{27}{8} - \frac{19}{8} \eta + \frac{1}{24} \eta^2 \right) (m\Omega)^{4/3} + \cdots \right] \quad (14)$$

$$\left( \frac{J}{\mu m} \right)^2 = (m\Omega)^{-2/3} \left[ 1 + \frac{1}{3} (9 + \eta) (m\Omega)^{2/3} + \left( 9 - \frac{17}{4} \eta + \frac{1}{9} \eta^2 \right) (m\Omega)^{4/3} + \cdots \right], \quad (15)$$

where $\eta \equiv \mu/m$. The three terms inside the square brackets represent the Newtonian results for circular orbits along with the first and second post-Newtonian corrections.

Figure 3 compares the numerical results for binding energy versus orbital angular momentum for the sequence of quasi-circular orbits against equation (13) with $\eta = 1/4$. The numerical data is displayed as a bold solid line with cross marks denoting actual data points. The long dashed line corresponds to the Newtonian result, the short dashed line to the Newtonian result together with the (post)$^1$-Newtonian corrections, and the dotted line to the full (post)$^2$-Newtonian result. Notice that for large $J/\mu m$ (large separation), the post-Newtonian expansion appears to be converging quite well toward the numerical result. Figures 4 and 5 are analogous plots for the orbital angular momentum versus orbital angular frequency and binding energy versus orbital angular frequency, respectively. Again, in the limit of large separation (small $m\Omega$), the post-Newtonian expansions appear to converge well with the numerical results. This agreement at large separations, together with the general agreement in the shapes of the curves when the separation decreases, confirm that the basic premises adopted in the definition of the effective-potential method are sound. For small separations, we know that the approximations needed to define the effective potential as outlined in Sec. III must be questioned. Unfortunately, the post-Newtonian approximation also breaks down as the separation of the holes becomes small and, it is impossible to gauge the validity of the approximations based on the post-Newtonian results.

FIG. 4. The orbital angular momentum $J/\mu m$ as a function of the orbital angular velocity $m\Omega$. Lines are as indicated in Fig. 3.

VI. DISCUSSION

The most important use of the results obtained in this paper is to narrow the range of initial-data parameters which must be considered in setting up an actual numerical simulation of the inspiral and collision of two black holes. Assuming one is interested in evolving initial data that represents something similar to a quasi-circular or-
the final Kerr hole. Assuming only that the individual
holes do not rotate, we know that the irreducible mass is
bounded by
\[ J_f / M_f^2 \geq \frac{1}{1 + \frac{1}{2} (J_f / M_f^2)^2}, \]
where we now let \( M_{ir} \) denote the irreducible mass of
the final Kerr hole. Assuming only that the individual
holes do not rotate, we know that the irreducible mass is
bounded by
\[ M_{ir} \geq \sqrt{M_1^2 + M_2^2} = m\sqrt{1 - 2\eta}, \]
where \( M_1, M_2, \) etc. are defined as before. Since we know
that \( J_f \leq J \), we find that
\[ \frac{J_f}{M_f^2} \leq \frac{\eta}{1 - 2\eta \mu m}. \]

Examining equations (16) and (18) we find that, so long
as \( J_f / M_f^2 < 2 \), the Kerr ratio \( J_f / M_f^2 \) is guaranteed to
be less than unity. For equal-sized holes, this implies
that the Kerr ratio is satisfied as long as \( J/\mu m < 4 \).
If we evolve initial data representing the innermost sta-
ble quasi-circular orbit, we find that \( J_f / M_f^2 \leq 0.958 \).
If we make the severe assumptions that all of the bind-
ing energy “released” in the coalescence is recaptured by
the resulting black hole and that half of the angular mo-
mentum is, nevertheless, radiated away during the final
plunge and coalescence, we find that \( J_f / M_f^2 \sim 0.4 \). We
see then that a black hole resulting from the inspiral and
final plunge of two nonrotating black holes must certainly
be considered to be rapidly rotating, but it will not vi-
olate the Kerr limit.

Now consider the minimal requirements of a numerical
evolution of binary coalescence in the case of two equal-
sized, nonrotating black holes. If the simulation is to
remotely resemble the final plunge of two holes following
a secular inspiral, then our best guess at initial data is
that for the innermost stable quasi-circular orbit. From
the estimate of the orbital angular velocity in Table I, we
find that the orbital period is \( \tau \sim 37m \). It is reasonable
to assume that the final plunge will occur on a time scale
comparable to that of the innermost orbit. If this initial
data leads to an evolution that begins at the most one
orbit before the beginning of the final plunge and we add
to the evolution time a period sufficient to watch some of
the ring-down, then we find that the numerical simulation
must be capable of evolving for \( 90-130m \).

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APPENDIX: THE MOMENTA OF INDIVIDUAL
HOLES

A rigorous quasi-local definition of momentum requires
the presence of a Killing vector field \( \xi^i_{(k)} \). The magnitude
of the momentum associated with \( \xi^i_{(k)} \) within a given sur-
face contained in a spatial slice is denoted \( \Pi_i^{(k)} \), where

\[ \Pi_i^{(k)} \equiv \Pi_k \xi^i_{(k)} = \frac{1}{8\pi} \int \left( K_j^i - \delta_j^i K \right) \xi^j_{(k)} d^2 S_j. \]  

(A1)

Here, \( K_j^i \) is the extrinsic curvature of the slice and \( K \)
is its trace. A general spatial slice contains no Killing
vectors and only the total momenta of an asymptotically
flat spatial slice can be defined. This definition requires
integrating equation (A1) at spatial infinity using asymp-
totic Killing vectors \( \xi^i_{(k)} \), which are Killing vectors of the
flat metric to which the physical metric is asymptotic.
Note, however, that the angular momentum will only be gauge-invariant if \( \xi_{(k)} \) is an exact symmetry of the physical metric (cf York [17]).

Now consider the momenta of gravitational initial data constructed within the conformal-imaging approach. Following this approach, we conformally decompose the physical metric of a spatial slice as \( \gamma_{ij} = \psi^4 \bar{\gamma}_{ij} \), where \( \bar{\gamma}_{ij} \) is the flat conformal background metric and \( \psi \) is the conformal factor. With the trace-free conformal background extrinsic curvature defined by \( \tilde{A}_i^j = \psi^3 (K_i^j - \frac{1}{2} \delta_i^j K) \) and with \( K = 0 \), we can rewrite equation (A1) as

\[
\Pi_{(k)} = \frac{1}{8\pi} \int_{\infty}^{\infty} \tilde{A}_i^j \xi_{(k)}^j dS_i.
\]  

(A2)

This form of the equation has the advantage that it does not involve the conformal factor \( \psi \), so we can compute the momentum without having a solution of the Hamiltonian constraint. Also, I emphasize that we have not used the fact that \( \psi \rightarrow \infty \) at spatial infinity in deriving equation (A2), so that if \( \tilde{\xi}_{(k)} \) represents an exact symmetry of the physical metric, then equation (A2) can be evaluated (for that Killing vector) by integrating over any two-surface containing the support of the gravitational field.

The concept of the momenta (linear and angular) of an individual hole in the presence of other holes cannot be rigorously defined in general relativity. However, a reasonable quasi-local definition for the momenta of an individual black hole is equation (A2) integrated over a two-surface exterior to that hole. For evaluating the components of the hole’s linear momentum, we use the three translational Killing vectors of flat Euclidean 3-space, and for the angular momentum, we use the three rotational Killing vectors with the origin of rotation chosen to be the center of the hole.

Following the conformal-imaging approach, the background extrinsic curvature of a spatial slice is constructed as the linear sum of “single-hole” extrinsic curvature solutions plus image terms that maintain an isommetry condition (cf Cook [7]). The background extrinsic curvature for a single black hole (including self-image terms) is parameterized directly in terms of the physical momenta measured at infinity. Evaluation of the black hole’s momenta via the quasi-local definition is independent of the radius of the surface on which the integral is evaluated and always yields the correct physical result. Now consider evaluating the quasi-local momentum integral over a two-surface that does not contain the black hole. Using Gauss’ law, we can rewrite the integral as

\[
\Pi_{(k)} = \frac{1}{8\pi} \int \mathbf{n}_j \left( \tilde{\xi}_{(k)}^j \tilde{A}_i^j \right) d^3V = 0,
\]

(A3)

since \( \tilde{\xi}_{(k)} \) is a Killing vector of \( \tilde{\gamma} \) and \( \tilde{A}_i^j \) satisfies the vacuum momentum constraint equation in this volume. Constructing a multi-hole extrinsic curvature from single-hole extrinsic curvature solutions, including self-image terms but not including general image terms, we see that equation (A3) implies that the contributions to the extrinsic curvature from additional holes do not effect the quasi-local momenta of a given hole.

What remains is to examine the contribution of general image terms to the quasi-local momenta of a hole. It seems reasonable that these terms should have no contribution, however, I have so far been unable to prove this analytically. Fortunately, it is straightforward to show numerically that general image terms make no contribution to the quasi-local momenta of either hole in a general binary configuration. More specifically, we can construct a general solution for the background extrinsic curvature including any number of image terms. Computing the quasi-local momenta for either of the holes, we can use Richardson extrapolation to show that, up to the numerical precision of the computer, the results are identical to those obtained in the absence of any general image terms. This is independent of the sizes and separations of the holes and of the number of image terms included in the extrinsic curvature implying that each image term independently contributes nothing to any of the surface integrals. I suspect that this result holds for any number of holes, however, this has not been verified.

We see then that within the limitations of defining the momenta of an individual hole, the momenta used to parameterize a single hole are the momenta of individual holes within a multi-hole configuration. That is, linear or angular momenta on one hole do not induce any amount of linear or angular momentum on any other hole in the system. In particular, the orbital angular momentum of a system of holes is well defined and does not contain an induced spin on any of the holes due to the linear momenta of the holes.