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A General Method

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Renormalization of one-loop effective action on an arbitrary curved space-time: a general method

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Abstract: Using zeta-function regularization for the one-loop effective action, we carry out the renormalization of the one-loop effective Lagrangian for a self-interacting scalar field theory in an arbitrary gravitational background. We give very general expressions and recover known results as special cases.

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1 Introduction

As was first pointed out by Jona-Lasinio [1], the effective potential for a quantum field theory is very useful in studies of spontaneous symmetry breaking and a lot of works has been made in this direction in Minkowski space-time (see for example the well known paper by Coleman and Weinberg [2]), but above all in curved space-time, where the role of curvature and non trivial topology on symmetry breaking (or restoration) and mass generation has been carefully considered (see for example Refs. [3,4,5,6,7,8,9,10,11]).

The effective potential plays an important role in the dynamics of the early universe at scales bigger than the Planck one and in particular, in new inflationary models [12,13], the effective cosmological constant can be derived from the effective potential of a self-interacting scalar field. To be fully consistent, the effective potential or more generally the effective action has to be calculated for general space-times, taking into account dynamics, geometry and topology of the space-time itself.

In order to analyze the influence of the properties of space-time on the effective action, a variety of methods has been developed during the last decade. Here let us mention the quasilocal approximation scheme for slowly varying background gravitational field at zero temperature [14] (see also Refs. [15,16,17,18,19]) and non-zero temperature [20] and the renormalization-group approach especially elaborated in Refs. [21,22,23].

Some works are devoted to space-times with constant curvature [24] or non-trivial topology [6,7,8,9,10,11,25,26,27], or a combination of both [28,29,30]. Some particular attention was also

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dedicated to the anisotropy in different Bianchi type universes [5,16,31,32,33,34,35,36,37] and to space-times with conical singularities [38].

As has been stressed in Ref. [39], the functional evaluation of the effective potential by use of the path-integral formulation of quantum field theory is more convenient than the traditional perturbative approach, which consists in summing infinite series of Feynman graphs at zero-momentum. Moreover, in principle it permits to go over the one-loop approximation.

In this paper we shall use the Feynman path-integral method and ζ -function regularization in order to define the one-loop effective action on an arbitrary Riemannian manifold (Sec. 2). Using heat-kernel techniques, then we shall perform the renormalization of the one-loop effective Lagrangian for a self-interacting scalar field and we shall give a very general expression for the renormalized effective Lagrangian, valid on any Riemannian manifold (Sec. 3). Specializing our general expression to particular space-times, at the end we shall obtain some already known results as examples (Sec. 4). Finally, in the Appendices we shall collect few results on heat-kernel coefficients and ζ -function we need in the paper.

Throughout the paper, units in which $\hbar = c = 1$ shall be used.

2 Effective action and ζ -function regularization

The quantum theory of a matter field ϕ living on a given curved background N dimensional manifold with metric $g_{\mu\nu}$ can be conveniently defined by the path integral

$$Z[g] = \int \mathcal{D}\phi e^{iS[\phi,g]}, \quad (2.1)$$

where $S[\phi, g]$ is the classical action and the integral is taken over all matter fields satisfying suitable boundary or periodicity conditions. To be more precise, $Z[g]$ is defined by Eq.(2.1), a part an infinite normalization constant, which has been omitted because it is irrelevant at this level.

Since the dominant contributions to $Z[g]$ will come from fields which are near the classical solution ϕ_c , which extremize the action, for the purposes of the present paper it is sufficient to consider classical actions which are quadratic in the quantum fluctuations $\tilde{\phi} = \phi - \phi_c$. In fact, a Taylor expansion of the action around ϕ_c gives

$$S[\phi, g] = S[\phi_c, g] + \left. \frac{\delta^2 S[\phi, g]}{\delta \phi^2} \right|_{\phi=\phi_c} \frac{\tilde{\phi}^2}{2} + \text{higher order terms in } \tilde{\phi}, \quad (2.2)$$

and so the one loop approximated theory is determined by the “zero temperature partition function”

$$Z[\phi_c, g] \sim e^{-S[\phi_c, g]} \int \mathcal{D}\tilde{\phi} e^{-\frac{1}{2} \int \tilde{\phi} L \tilde{\phi} d^N x}. \quad (2.3)$$

To ensure diffeomorphism invariance of the functional measure, the functional integral dummy variables $\tilde{\phi}$ are chosen to be scalar densities of weight $-1/2$. In Eq. (2.3) a Wick rotation of the time axis in the complex plane has to be understood. In this manner, the metric $g_{\mu\nu}$ becomes Euclidean and the small disturbance operator L selfadjoint.

A part the sign, the one-loop effective action $\Gamma[\phi_c]$ is given by the logarithm of Eq. (2.3). So we have

$$\Gamma[\phi_c, g] = \int_{\mathcal{M}} \mathcal{L}(\phi_c, g) \sqrt{g} d^N x = S[\phi_c, g] + \nu \ln \det \frac{L}{\mu^2}, \quad (2.4)$$

where $\mathcal{L}(\phi_c, g)$ is the effective Lagrangian, μ an arbitrary mass parameter, which is necessary for dimensional reasons and which is fixed by renormalization, while $\nu = -1, 1, 1/2$ according

to whether one is dealing with Dirac spinor, charged or neutral scalar fields respectively [40]. Eq. (2.4) is also valid for a multiplet of matter fields. In such a case L is a matrix of differential operators. Here we limit ourselves to a neutral scalar field ($\nu = 1/2$), the extension to a charged scalar field (or a multiplet) being straightforward.

Looking at equations above, we see that in order to define the (one-loop) quantum theory via path integral, one needs a rigorous definition of determinant of (elliptic) differential operators. Here we use ζ -function regularization, but of course other regularizations work very well (for a comparison of different regularizations of determinant see for example Ref. [29]).

We indicate by $\zeta(s|A)$ the ζ -function related to the positive operator A (the massless case shall be obtained as the limit of the massive one). Then, for the one-loop quantum corrections to the effective action we have

$$\Gamma^{(1)}[\phi_c, g] = -\frac{1}{2}\zeta'(0|\frac{L}{\mu^2}), \quad (2.5)$$

the prime representing the derivative with respect to s . A similar equation is also valid for the one-loop effective Lagrangian $\mathcal{L}(\phi_c, g) = \mathcal{L}_c(\phi_c, g) + \mathcal{L}^{(1)}(\phi_c, g)$, which is implicitly defined by Eq. (2.4) ($\mathcal{L}_c$ represents the classical Lagrangian density). We have

$$\int_{\mathcal{M}} \mathcal{L}^{(1)}(\phi_c, g) \sqrt{g} d^N x = -\frac{1}{2}\zeta'(0|\frac{L}{\mu^2}). \quad (2.6)$$

It has to be noticed that $\mathcal{L}^{(1)}$ as defined by Eq. (2.9) is a smooth function on $\mathcal{M}$ only for smooth manifolds without boundary. If $\mathcal{M}$ has singular points or a boundary, then $\mathcal{L}^{(1)}$ has to be understood as a distribution (see for example Ref. [38], where the one-loop effective Lagrangian has been evaluated for a scalar field on a cone and the contribution of the singularity carefully considered).

Now we recall that by Seeley theorem [41], the kernel of ζ -function related to a second order differential operator on a smooth N -dimensional manifold is a meromorphic function in the s complex plane, with simple poles at the points $s = (N - n)/2$ ($n = 0, 1, \dots$). More precisely we have the Laurent expansion

$$\zeta(s|A) = \sum_{n=0}^{\infty} \frac{K_n(A)}{\Gamma(s)[s - \frac{N-n}{2}]} + \zeta_0(s|A), \quad (2.7)$$

where $\zeta_0(s|A)$ is an entire function and $K_n(A)$ are the spectral coefficients for the operator A . They enter the asymptotic expansion

$$\text{Tr } e^{-tA} \sim \sum_{n=0}^{\infty} K_n(A) t^{\frac{n-N}{2}}. \quad (2.8)$$

Due to the presence of $\Gamma(s)$ in Eq. (2.7), not all the points $s = (N - n)/2$ are simple poles of $\zeta(s|A)$. Moreover, in the absence of boundaries, all K_n with odd n vanishes, so in this case, for even N we have $N/2$ poles situated at the integers $1, 2, \dots, N/2$, while for odd N we have infinite poles at all half integers $\leq N/2$. In particular, $\zeta(s|A)$ is finite for $s = 0$ and the known relation $\zeta(0|A) = K_N(A)$ follows. The coefficients K_n are local invariants depending on geometrical object and some of them have been evaluated for arbitrary Riemannian manifolds (for a recent review see Ref. [42]) and also for arbitrary Riemann-Cartan manifolds [43].

It is also convenient to introduce the (diagonal) kernel $\zeta(s; x|A)$ and the heat-kernel coefficients $k_n(x|A) (4\pi)^{-N/2}$, which integrated on (compact) $\mathcal{M}$ give the ζ -function and $K_n(A)$ respectively. For more generality, we think of $\zeta(s; x|A)$ and $k_n(x|A)$ as distributions, in this manner we take into account of possible singularities or boundaries. In Appendix A we report the coefficients we need in the paper.

At this point we consider the physical interesting case of a 4-dimensional manifold and rewrite the Lagrangian in the form

$$\mathcal{L}^{(1)}(\phi_c, g) = \frac{1}{64\pi^2} \left[2k_4(x|L) \ln \frac{M^2(\phi_c, g)}{\mu^2} - \frac{3}{2} M^4(\phi_c, g) \right] + f(\phi_c, g), \quad (2.9)$$

where $M(\phi_c, g)$ is (for the moment) an arbitrary function of the fields and

$$f(\phi_c, g) = -\frac{\zeta'(0; x|L)}{2} - \frac{1}{64\pi^2} \left[2k_4(x|L) \ln M^2(\phi_c, g) - \frac{3}{2} M^4(\phi_c, g) \right], \quad (2.10)$$

does not depend on μ . The advantage of such a rewriting will be clear in the following.

3 $\lambda\phi^4$ self-interacting theory: renormalization

Now we focus our attention on a scalar field in a 4-dimensional curved space-time with a self-interacting $\lambda\phi^4$ term. Since our aim is just to illustrate the method of quantization, here we suppose $\mathcal{M}$ to be a smooth Riemannian manifold without boundary, but the difficulty in considering the general case is only due to the greater number of terms which one has to take into consideration.

We start with the classical Euclidean action (for the matter field)

$$S[\phi, g] = \int_{\mathcal{M}} \left[-\frac{1}{2} \phi \Delta \phi + V_c(\phi, g) \right] \sqrt{g} d^N x, \quad (3.1)$$

where Δ is the Laplace-Beltrami operator acting on functions in $\mathcal{M}$ and

$$V_c(\phi, g) = \frac{\lambda\phi^4}{24} + \frac{m^2\phi^2}{2} + \frac{\xi R\phi^2}{2}, \quad (3.2)$$

According to Refs. [44,45], the classical action (3.1) with a second order kinetic term is more fundamental than the usual one with a first order kinetic term. Of course, in the absence of boundary, they are equivalent. Here we adopt this point of view.

From Eq. (3.1) we obtain the small disturbance operator

$$L = -\Delta + V_c''(\phi_c, g) = -\Delta + m^2 + \xi R + \frac{\lambda\phi_c^2}{2}, \quad (3.3)$$

the prime representing the derivative with respect to ϕ . Now we set

$$M^2(\phi_c, g) = V_c''(\phi_c, g) - \frac{R}{6} = m^2 + \left(\xi - \frac{1}{6} \right) R + \frac{\lambda\phi_c^2}{2} \quad (3.4)$$

and recalling the general expression for $k_4(x|A) = a_2(x|A)$, Eq. (A.3), for this special case we obtain

$$\mathcal{L}^{(1)}(\phi_c, g) = \frac{1}{64\pi^2} \left[\left(M^4 - \frac{c_0 R^2}{2} - \sum_{q=1}^4 c_q \Psi^q \right) \ln \frac{M^2}{\mu^2} - \frac{3}{2} M^4 \right] + f(\phi_c, g), \quad (3.5)$$

where for convenience we have set $\{\Psi^q\} \equiv \{W, G, \Delta R, \Delta\phi_c^2\}$, W and G being the square of the Weyl tensor and the Gauss-Bonnet density respectively (they are defined in Appendix A). When R , W and G are independent fields, then $c_0 = 0$ and the other coefficients c_q are given by (see Eq. (A.3))

$$c_1 = -\frac{1}{60}, \quad c_2 = \frac{1}{180}, \quad c_3 = \frac{1}{3} \left(\xi - \frac{1}{5} \right), \quad c_4 = \frac{\lambda}{6}. \quad (3.6)$$

We have introduced also the unusual R^2 term in Eq. (3.5) in order to take into account of the possibility that W or/and G are proportional to R^2 , as happens for example on S^4 and H^4 . In such cases, $c_0 \neq 0$, while c_1 or/and c_2 will be vanishing. This is not strictly necessary if one chooses arbitrary values for all fields in order to define the renormalized coupling constants.

We see that the one loop quantum corrections to the classical Lagrangian generate quadratic terms in the curvature, so, for gravitation we take the general expression [46]

$$\mathcal{L}_g = \Lambda + \alpha_1 R + \frac{\alpha_2 R^2}{2} + \sum_{q=1}^3 \varepsilon_q \Psi^q, \quad (3.7)$$

$\Lambda \sim 0$, α and ε being the cosmological and coupling constants respectively. The total (matter plus gravitation) classical Lagrangian is a function of ϕ_c , R and the three independent geometric objects Ψ^q . When convenient we indicate by $(\eta_p, \varepsilon_q) \equiv (\lambda, m^2, \xi, \alpha_1, \alpha_2, \varepsilon_q)$ all the coupling constants and by $\Phi \equiv (\phi_c, R, \Psi^q)$ the whole set of all independent fields.

All physical coupling constants are related to the classical Lagrangian $\mathcal{L}_c = \mathcal{L}_m + \mathcal{L}_g$ (matter plus gravitation) by

$$\begin{aligned} \eta_1 &= \frac{\partial^4 \mathcal{L}_c(\Phi_1)}{\partial \phi_c^4}, & \eta_4 &= \frac{\partial \mathcal{L}_c(\Phi_4)}{\partial R}, \\ \eta_2 &= \frac{\partial^2 \mathcal{L}_c(\Phi_2)}{\partial \phi_c^2}, & \eta_5 &= \frac{\partial^2 \mathcal{L}_c(\Phi_5)}{\partial R^2}, \\ \eta_3 &= \frac{\partial^3 \mathcal{L}_c(\Phi_3)}{\partial R \partial \phi_c^2}, & \varepsilon_q &= \frac{\partial \mathcal{L}_c(\Phi_{5+q})}{\partial \Psi^q}, \end{aligned} \quad (3.8)$$

the values $\Phi_n \equiv (\phi_n, R_n, \Psi_n^q)$ determining the scales at which the different coupling constants are measured. At this level they are arbitrary, but $\Phi_2 \equiv (0, 0, \Psi_2^q)$ and $\Phi_4 \equiv (0, 0, \Psi_4^q)$, which define mass and Newton constant respectively. The behaviour of coupling constants with respect to a change of scale is determined by the renormalization group equations (see for example Refs. [24,14]) and so the particular values Φ_q are not really important. Following Refs. [14,37], we choose $\Psi_n^q = 0$ for any q , and arbitrary values for ϕ_c and R , but $\Phi_2 = \Phi_4 \equiv (0, \dots, 0)$.

In order to preserve Eqs. (3.8) also to the one-loop level, we have to add to the effective Lagrangian $\mathcal{L} = \mathcal{L}_c + \mathcal{L}^{(1)}$ a counterterm $\delta\mathcal{L}$ of the kind

$$\begin{aligned} \delta\mathcal{L} = & \delta\Lambda + \frac{\delta\eta_1 \phi^4}{24} + \frac{\delta\eta_2 \phi^2}{2} + \frac{\delta\eta_3 R \phi^2}{2} \\ & + \delta\eta_4 R + \frac{\delta\eta_5 R^2}{2} + \sum_q \delta\varepsilon_q \Psi^q. \end{aligned} \quad (3.9)$$

Note that q runs from 1 to 4 since the (μ -dependent) unrenormalized one-loop effective Lagrangian contains also the term $\Psi^4 = \Delta\phi_c^2$. On a static and homogeneous space-time, M is a constant, and so this last term, as well as the terms proportional to Ψ^2 and Ψ^3 can be disregarded. In the presence of boundary or singularities, other terms (distributions with support on the boundary or on the singularities) are present in the one-loop effective Lagrangian and for each of such terms we have to add a corresponding counterterm in $\delta\mathcal{L}$.

In order to get the conditions which determine the explicit form of the counterterms, we have to impose Eqs. (3.8) to the whole Lagrangian density

$$\mathcal{L}_{eff} = \mathcal{L}_c + \mathcal{L}^{(1)} + \delta\mathcal{L} = \mathcal{L}_c + \mathcal{L}_{eff}^{(1)}. \quad (3.10)$$

This is equivalent to impose the conditions

$$\begin{aligned}
\frac{\partial^4 \mathcal{L}_{eff}^{(1)}(\Phi_1)}{\partial \phi_c^4} &= 0, & \frac{\partial \mathcal{L}_{eff}^{(1)}(\Phi_4)}{\partial R} &= 0, \\
\frac{\partial^2 \mathcal{L}_{eff}^{(1)}(\Phi_2)}{\partial \phi_c^2} &= 0, & \frac{\partial^2 \mathcal{L}_{eff}^{(1)}(\Phi_5)}{\partial R^2} &= 0, \\
= 0 \frac{\partial^3 \mathcal{L}_{eff}^{(1)}(\Phi_3)}{\partial R \partial \phi_c^2}, & & \frac{\partial \mathcal{L}_{eff}^{(1)}(\Phi_{5+q})}{\partial \Psi^q} &= 0.
\end{aligned} \tag{3.11}$$

We have also to impose $\mathcal{L}_{eff}^{(1)}(\Phi_0) = 0$, which is equivalent to say that the cosmological constant is equal to $\mathcal{L}_{eff}$ valuated at the point $\Phi_0 \equiv (\phi_0, 0, \dots, 0)$, $\phi_0 = \langle \phi_c \rangle$ being the mean value of the field.

From such conditions, after straightforward calculations, we get the counterterms

$$\begin{aligned}
64\pi^2 (\delta\Lambda + f(\Phi_0)) &= 64\pi^2 \left[\left(\frac{\partial^2 f(\Phi_2)}{\partial \phi_c^2} - m^2 \right) \frac{\phi_0^2}{2} + \left(\frac{\partial^4 f(\Phi_1)}{\partial \phi_c^4} - \lambda \right) \frac{\phi_0^4}{24} \right] \\
&\quad - \lambda m^2 \phi_0^2 \left(1 - \ln \frac{m^2}{\mu^2} \right) + M_0^4 \left(\frac{3}{2} - \ln \frac{M_0^2}{\mu^2} \right) \\
&\quad + \frac{\lambda^2 \phi_0^4}{4} \ln \frac{M_1^2}{\mu^2} - \frac{\lambda^4 \phi_0^4 \phi_1^4}{12 M_1^4} + \frac{\lambda^3 \phi_0^4 \phi_1^2}{2 M_1^2}, \\
64\pi^2 \left[\delta\lambda + \frac{\partial^4 f(\Phi_1)}{\partial \phi_c^4} \right] &= -6\lambda^2 \ln \frac{M_1^2}{\mu^2} + \frac{2\lambda^4 \phi_1^4}{M_1^4} - \frac{12\phi_1^2 \lambda^3}{M_1^2}, \\
64\pi^2 \left[\delta m^2 + \frac{\partial^2 f(\Phi_2)}{\partial \phi_c^2} \right] &= 2\lambda m^2 \left(1 - \ln \frac{m^2}{\mu^2} \right), \\
64\pi^2 \left[\delta\xi + \frac{\partial^3 f(\Phi_3)}{\partial R \partial \phi_c^2} \right] &= -2\lambda \left(\xi - \frac{1}{6} \right) \ln \frac{M_3^2}{\mu^2} - \frac{2\lambda^2 \left(\xi - \frac{1}{6} \right) \phi_3^2}{M_3^2} \\
&\quad + \frac{c_0 \lambda R_3^2}{M_3^2} \left(1 - \frac{\lambda \phi_3^2}{M_3^2} \right), \\
64\pi^2 \left[\delta\eta_4 + \frac{\partial f(\Phi_4)}{\partial R} \right] &= 2m^2 \left(\xi - \frac{1}{6} \right) \left(1 - \ln \frac{m^2}{\mu^2} \right), \\
64\pi^2 \left[\delta\eta_5 + \frac{\partial^2 f(\Phi_5)}{\partial R^2} \right] &= - \left[2 \left(\xi - \frac{1}{6} \right)^2 - c_0 \right] \ln \frac{M_5^2}{\mu^2} + \frac{2 \left(\xi - \frac{1}{6} \right) c_0 R_5^2}{M_5^2}, \\
64\pi^2 \left[\delta\varepsilon_q + \frac{\partial f(\Phi_{5+q})}{\partial \Psi^q} \right] &= c_q \ln \frac{M_{5+q}^2}{\mu^2},
\end{aligned} \tag{3.12}$$

where we introduced the constants $M_n^2 = m^2 + \left(\xi - \frac{1}{6} \right) R_n + \frac{\lambda}{2} \phi_n^2$.

The renormalized one-loop contribution to the effective Lagrangian looks very complicated. We write it in the form

$$\begin{aligned}
64\pi^2 \mathcal{L}_{eff}^{(1)} = & -32\pi^2 m^2 \phi_0^2 - \frac{8\pi^2 \lambda \phi_0^4}{3} + \lambda m^2 \phi_0^2 \left(\ln \frac{m^2}{M_0^2} + \frac{1}{2} \right) \\
& - \frac{\lambda^2 \phi_0^4}{4} \left[\ln \frac{M_0^2}{M_1^2} - \frac{3}{2} - \frac{4(M_1^2 - m^2)(2M_1^2 + m^2)}{3M_1^4} \right] \\
& + m^4 \ln \frac{M^2}{M_0^2} + 2m^2 \left(\xi - \frac{1}{6} \right) R \left(\ln \frac{M^2}{m^2} - \frac{1}{2} \right) + \left(\xi - \frac{1}{6} \right)^2 R^2 \left(\ln \frac{M^2}{M_5^2} - \frac{3}{2} \right) \\
& + \left\{ \left(\xi - \frac{1}{6} \right) R \left[\ln \frac{M^2}{M_3^2} - \frac{3}{2} - \frac{\lambda \phi_3^2}{M_3^2} \right] + m^2 \left[\ln \frac{M^2}{m^2} - \frac{1}{2} \right] \right\} \lambda \phi_c^2 \\
& + \left\{ \left(\ln \frac{M^2}{M_1^2} - \frac{25}{6} \right) + \frac{4m^2(m^2 + M_1^2)}{3M_1^4} \right\} \frac{\lambda^2 \phi_c^4}{4} \\
& - c_0 \left\{ \frac{\lambda R_3^2}{2M_3^2} \left(\frac{\lambda \phi_3^2}{M_3^2} - 1 \right) R + \left[\ln \frac{M^2}{M_5^2} - \frac{2 \left(\xi - \frac{1}{6} \right) R_5^2}{M_5^2} \right] \frac{R^2}{2} \right\} \\
& - \sum_q c_q \ln \frac{M^2}{M_{5+q}^2} \Psi^q - F(f), \tag{3.13}
\end{aligned}$$

where $F(f)$ contains all terms depending on $f(\Phi)$. It reads

$$\begin{aligned}
F(f) = & 64\pi^2 \left[f(\Phi_0) - \frac{\partial^4 f(\Phi_1) \varphi_0^4}{\partial \phi_c^4} \frac{1}{24} - \frac{\partial^2 f(\Phi_2) \varphi_0^2}{\partial \phi_c^2} \frac{1}{2} \right. \\
& - f(\Phi) + \frac{\partial^4 f(\Phi_1) \phi_c^4}{\partial \phi_c^4} \frac{1}{24} + \frac{\partial^2 f(\Phi_2) \phi_c^2}{\partial \phi_c^2} \frac{1}{2} + \frac{\partial^3 f(\Phi_3) R \phi_c^2}{\partial \phi_c^2 \partial R} \frac{1}{2} \\
& \left. + \frac{\partial f(\Phi_4)}{\partial R} R + \frac{\partial^2 f(\Phi_5) R^2}{\partial R^2} \frac{1}{2} + \sum_q \frac{\partial f(\Phi_{5+q}) \Psi^q}{\partial \Psi^q} \right]. \tag{3.14}
\end{aligned}$$

We see that the computation of the one-loop effective Lagrangian reduces to the determination of the function $f(\phi_c, g)$ on the arbitrary manifold $\mathcal{M}$.

What is relevant for the discussion of the phase transition of the system, is the second derivative of $\mathcal{L}_{eff}$ with respect to the background field ϕ_c . In this way one defines

$$\mathcal{L}_{eff} = \Lambda_{eff}(g) + \frac{m_{eff}^2(g) \phi_c^2}{2} + O(\phi_c^4), \tag{3.15}$$

where $\Lambda_{eff}(g)$ and $m_{eff}^2(g)$ are complicated expressions not depending on ϕ_c . Of course, they are equal to the cosmological constant Λ (actually ~ 0) and the mass m^2 respectively, when evaluated at $R = \Psi^q = 0$. By a straightforward computation we get

$$\begin{aligned}
m_{eff}^2 = & m^2 + \xi R + \frac{\lambda}{32\pi^2} \left\{ m^2 \ln \frac{m^2 + \left(\xi - \frac{1}{6} \right) R}{m^2} \right. \\
& \left. + \left(\xi - \frac{1}{6} \right) R \left[\ln \frac{m^2 + \left(\xi - \frac{1}{6} \right) R}{M_3^2} - \frac{\lambda \phi_3^2}{M_3^2} - 1 \right] \right\} \\
& - \frac{\lambda c_0 R^2}{128\pi^2 \left[m^2 + \left(\xi - \frac{1}{6} \right) R \right]} - \sum_{q=1}^4 \frac{\lambda c_q \Psi^q}{64\pi^2 \left[m^2 + \left(\xi - \frac{1}{6} \right) R \right]} \\
& + \frac{\partial^2 f(0, R, \Psi^q)}{\partial \phi_c^2} - \frac{\partial^2 f(0, \dots, 0)}{\partial \phi_c^2} - \frac{\partial^3 f(0, R_3, 0)}{\partial \phi_c^2 \partial R} R. \tag{3.16}
\end{aligned}$$

Effective potential. Very often, the Lagrangian density in Eq. (2.4) is expanded in terms of derivatives of the background field ϕ_c , defining in this way the one-loop effective potential $V(\phi_c, g)$, which is identified with the zero-derivative term. One has (for the matter field)

$$\Gamma[\phi_c, g] = \int_{\mathcal{M}} \left[V(\phi_c, g) + \frac{1}{2} Z(\phi_c, g) g^{ij} \partial_i \phi_c \partial_j \phi_c + \dots \right] \sqrt{g} d^N x \quad (3.17)$$

and so the renormalized one-loop contribution to the effective potential can be obtained from Eq. (3.13) disregarding all derivative terms of the field. If we consider a flat space-time, take a constant background field ϕ_c and the limit $m \rightarrow 0$, of course we get the Coleman-Weinberg result [2]

$$V_{eff} = \frac{\lambda \phi_c^4}{24} + \frac{\lambda^2 \phi_c^4}{256\pi^2} \left(\ln \frac{\lambda \phi_c^2}{2M_1^2} - \frac{25}{6} \right). \quad (3.18)$$

4 Some examples

As simple applications of the general method we have described, here we just report some results already appeared in the literature. It will be sufficient to evaluate the coefficients c_q and the function $F(f)$, the rest of Eq. (3.13) being common to all examples (of course, in a flat manifold, one has to put $R = 0$ everywhere in all formulae). For any example, here we shall evaluate the function $f(\phi_c, g)$, leaving the explicit computation of $F(f)$ to the interested reader.

To begin with, we consider ultrastatic 4-dimensional space-times of the form $\mathbb{R} \times \mathcal{M}^3$. We observe that on an ultrastatic manifold the field operator is of the kind $L = -\partial_\tau^2 + L_3$, L_3 acting on functions in $\mathcal{M}^3$. Then one has the factorization property

$$\frac{\zeta(s|L)}{\Omega_1} = \frac{\Gamma(s - \frac{1}{2})}{(4\pi)^{\frac{1}{2}} \Gamma(s)} \zeta(s - \frac{1}{2}|L_3), \quad (4.1)$$

$$\frac{\zeta'(0|L)}{\Omega_1} = \frac{(1 - \ln 2) K_4(L_3)}{\sqrt{\pi}} - \text{PP}(\zeta(s|L_3); s = -\frac{1}{2}), \quad (4.2)$$

Ω_1 being a large volume in $\mathbb{R}$ and PP represents the principal part of the function at the specified point.

Space-time with maximally symmetric spatial section. As the (quite trivial) simplest examples, we suppose $\mathcal{M}^3$ to be a maximally symmetric 3-dimensional manifold, that is $\mathbb{R}^3$, H^3 or S^3 . Such manifolds are homogeneous and have zero, negative and positive constant curvature respectively. For all these cases we have $c_q = 0$ ($q = 0, 1, 2, 3$, see Appendix B, where we recall some useful expressions concerning constant curvature manifolds) and moreover we can choose $\phi_c = \text{const}$, then also $c_4 = 0$ and $k_4 = M^4/2$. Of course, for $\mathbb{R}^3$ we have $f(\phi_c, g) = 0$, but one can directly check using Eq. (B.3) that this is also true for H^3 and as a consequence, on such two manifolds $F(f) = 0$.

The situation is a bit more complicated on S^3 . In fact, using Eq. (B.5) with $M = \alpha$ we get

$$f(\phi_c, g) = \frac{M^4}{2\pi^2} \int_1^\infty \frac{t^2 \sqrt{t^2 - 1}}{e^{2\pi M |\kappa|^{-1/2} t} - 1} dt, \quad (4.3)$$

and $F(f)$ can be obtained from definition, Eq. (3.16), by observing that f depends only on ϕ_c and R through M and the constant curvature $\kappa = R/6$.

Space-time of the form $\mathbb{R}^2 \times \mathcal{M}^2$. Now we suppose $\mathcal{M}^3$ to be the product of $\mathbb{R}$ and a maximally symmetric 2-dimensional manifold, that is $\mathcal{M}^2 = H^2$ or $\mathcal{M}^2 = S^2$. Also for these cases we have $c_q = 0$ for $q = 1, 2, 3, 4$, but now $c_0 = 1/180$. Using Eq. (4.1) we get

$$\frac{\zeta(s|L)}{\Omega_2} = \frac{\zeta(s-1|L_2)}{4\pi(s-1)}, \quad (4.4)$$

Ω_2 being a large volume in $\mathbb{R}^2$. Now, from Eqs. (B.4) and (B.6) with $\alpha^2 = M^2 + \kappa/12$, κ being the constant curvature, negative for H^2 and positive for S^2 , we obtain

$$\begin{aligned} f(\phi_c, g) &= \frac{1}{64\pi^2} \left\{ -\frac{\kappa^2}{30} + \pi M^4 \int_0^\infty \left[1 - \frac{\kappa(t^2 - 1/12)}{M^2} \right]^2 \right. \\ &\quad \times \ln \left| 1 - \frac{\kappa(t^2 - 1/12)}{M^2} \right| \frac{dt}{\cosh^2 \pi t} \Big\} \\ &= O(\kappa/M^2). \end{aligned} \quad (4.5)$$

As before, f depends only on ϕ_c and R through M and κ , which now is equal to $R/2$.

Constant curvature manifolds with non trivial topology. As examples of manifolds with non trivial topology we consider two cases which looks very similar to the sphere S^3 . These are T^3 , the 3-dimensional torus and H^3/Γ , Γ being a discrete group of isometries of H^3 containing only hyperbolic elements (see Appendix B).

We consider an equilateral torus T^3 with radius r . $\mathcal{M} = \mathbb{R} \times T^3$ is a flat manifold and so we have $R = \Psi^q = 0$. Using Eq. (B.7), for $f(\phi_c, g)$ we get

$$f(\phi_c, g) = \frac{2M^4}{3\pi^2} \sum_{\vec{n} \in \mathbb{Z}^3, \vec{n} \neq 0} \int_1^\infty \sqrt{t^2 - 1} e^{-2\pi M r |\vec{n}| t} dt. \quad (4.6)$$

For the manifold $\mathcal{M} = \mathbb{R} \times H^3/\Gamma$ we again have $k_4 = M^4/2$ (note that this is true if the group Γ contains only hyperbolic elements), but now the topological contributions give (see Eq. (B.9))

$$f(\phi_c, g) = -\frac{M^2 |\kappa|^{-1/2}}{2\pi \Omega(\mathcal{F}_3)} \int_1^\infty \sqrt{t^2 - 1} \frac{Z'_3}{Z_3} (1 + tM|\kappa|^{-1/2}) dt, \quad (4.7)$$

$Z_3(s)$ being the Selberg zeta-function on H^3/Γ [47,48] and $\Omega(\mathcal{F}_3)$ the volume of the fundamental domain.

Maximally symmetric space-time. As last examples in which the ζ -function is known, we consider the space-times H^4 and S^4 . Using Eqs. (B.4), (B.6) and the recurrence relation, Eq. (B.8), we easily get

$$\begin{aligned} f(\phi_c, g) &= \frac{1}{64\pi^2} \left\{ \frac{\kappa^2}{15} - \frac{\kappa M^2}{3} + \pi M^4 \int_0^\infty \left[1 - \frac{\kappa^2(t^2 + 1/4)}{M^4} \right] \right. \\ &\quad \times \ln \left| 1 - \frac{\kappa(t^2 + 1/4)}{M^2} \right| \frac{dt}{\cosh^2 \pi t} \Big\} \\ &= O(\kappa/M^2). \end{aligned} \quad (4.8)$$

Also for these two cases we have $c_q = 0$ ($q = 1, 2, 3, 4$), but $c_0 = 1/1080$.

Arbitrary space-time: approximation scheme. In an arbitrary (4-dimensional) space-time, the ζ -function and as a consequence the function $f(\phi_c, g)$ is not known. One can obtain an approximated result by expanding it in inverse powers of M^2 (see for example Refs. [14,18,19] and references cited therein). Such an expansion is sensible if the effective mass M^2 is large enough compared to the magnitude of some typical curvature, say $M^2 \gg |R|$. Using the partially summed form of the Schwinger-DeWitt asymptotic expansion as introduced in Ref. [49,50] (see Appendix A), we obtain

$$f(\phi_c, g) \sim -\frac{1}{32\pi^2 M^2} \sum_{n=0}^{\infty} \frac{n! b_{3+n}}{M^{2n}}, \quad (4.9)$$

where the coefficients b_n depend on the derivatives of ϕ_c but not on ϕ_c itself. Such an expansion is therefore useful in evaluating the one-loop effective potential. Note that in deriving Eq. (4.9) one must suppose the heat expansion to be valid for any t . For example, the first contribution to $f(\phi_c, g)$ in this approximation (which derive from b_3) is $-\lambda^2 \phi_c^2 g^{ij} \partial_i \phi_c \partial_j \phi_c / 384 M^2$ and so

$$F(f) = \frac{\lambda^2 \phi_c^2 g^{ij} \partial_i \phi_c \partial_j \phi_c}{6 M^2}. \quad (4.10)$$

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A Appendix: Heat kernel coefficients

In this appendix we report on some results concerning the heat-kernel expansion of a non negative, second order elliptic differential operator. The operator of interest in the given considerations is $A = -\Delta + X(x)$, defined on a smooth N -dimensional Riemannian manifold without boundary.

The kernel of $\exp(-tA)$, which satisfies a heat type equation, has an asymptotic expansion valid for small t given by

$$K_t(x, x|A) \sim \frac{1}{(4\pi t)^{N/2}} \sum_{n=0}^{\infty} a_n(x|A) t^n, \quad (A.1)$$

$a_n(x|A) = k_{2n}(x|A)$ being the spectral (Minakshisundaram-Seeley-DeWitt-Gilkey) coefficients, which are invariant quantities built up with curvature and their derivatives. Some of such coefficients was computed by many authors (for a recent review see Ref. [42]). In particular we have

$$a_0(x|A) = 1, \quad a_1(x|A) = \frac{R}{6} - X, \quad (A.2)$$

$$\begin{aligned} a_2(x|A) &= \frac{1}{2} a_1^2 + \frac{1}{6} \Delta a_1 + \frac{1}{180} [R^{ijrs} R_{ijrs} - R^{ij} R_{ij} + \Delta R] \\ &= \frac{1}{2} a_1^2 - \frac{1}{6} \Delta a_1 + \frac{1}{180} \left[\frac{3}{2} W - \frac{1}{2} G + \Delta R \right], \end{aligned} \quad (A.3)$$

where W and G are related to curvature tensors by

$$W = R^{ijrs} R_{ijrs} - 2R^{ij} R_{ij} + \frac{1}{3} R^2, \quad G = R^{ijrs} R_{ijrs} - 4R^{ij} R_{ij} + R^2. \quad (A.4)$$

G is the well known Gauss-Bonnet density while W becomes the square of the Weyl tensor but only in 4-dimensions.

Sometimes it may be convenient to factorize the exponential $\exp(ta_1)$ and consider an expansion very closely related to Eq. (A.1), that is (for details see Refs. [49,50,51])

$$K_t(x, x|A) \sim \frac{e^{-t(X-R/6)}}{(4\pi t)^{N/2}} \sum_{n=0}^{\infty} b_n(x|A) t^n, \quad (\text{A.5})$$

with $b_0 = 1$, $b_1 = 0$ and more generally

$$b_n(x|A) = \sum_{l=0}^n \frac{(-1)^l a_{n-l} a_1^l}{l!}. \quad (\text{A.6})$$

In this way, all coefficients b_n do not depend on a_1 [51].

B Appendix: Constant curvature manifolds

Here we just report some representations for the ζ -function of the Laplacian for scalar fields on manifolds $\mathcal{M}^N$ with constant curvature ($\mathbb{R}^N$, hyperbolic manifold H^N , sphere S^N and torus T^N). For details, we refer the reader to the literature. We consider the operator $L_N = -\Delta_N + \alpha^2 + \kappa \varrho_N^2$, where κ is the constant curvature of the manifold, which is zero for $\mathbb{R}^N$ and T^N , negative for H^N and positive for S^N and $\varrho_N = (N-1)/2$. We recall that for maximally symmetric spaces (H^N and S^N in our case) the Riemann tensor reads

$$R_{ijrs} = \kappa(g_{ir}g_{js} - g_{is}g_{jr}), \quad (\text{B.1})$$

from which $G = N(N-1)(N-2)(N-3)\kappa^2$ and $R = \kappa/N(N-1)$, R being the scalar curvature of $\mathcal{M}^N$.

We have the following representations (to avoid volume divergences, we consider the densities $\tilde{\zeta}(s|L_N) = \zeta(s; x|L_N)$)

$$\tilde{\zeta}_{\mathbb{R}^N}(s|L_N) = \frac{\Gamma(s - \frac{N}{2})\alpha^{N-2s}}{(4\pi)^{\frac{N}{2}}\Gamma(s)}, \quad (\text{B.2})$$

$$\tilde{\zeta}_{H^3}(s|L_3) = \frac{\Gamma(s - \frac{3}{2})\alpha^{3-2s}}{(4\pi)^{\frac{3}{2}}\Gamma(s)}, \quad (\text{B.3})$$

$$\tilde{\zeta}_{H^2}(s|L_2) = \frac{\alpha^{2-2s}}{4(s-1)} \int_0^\infty \frac{(1+t^2)^{1-s}}{\cosh^2 \pi \alpha |\kappa|^{-1/2} t} \alpha |\kappa|^{-1/2} dt, \quad (\text{B.4})$$

$$\tilde{\zeta}_{S^3}(s|L_3) = \frac{\Gamma(s - \frac{3}{2})\alpha^{3-2s}}{(4\pi)^{\frac{3}{2}}\Gamma(s)} - \frac{\alpha^{3-2s} \sin \pi s}{\pi^2} \int_1^\infty \frac{t^2(t^2-1)^{-s}}{e^{2\pi \alpha \kappa^{-1/2} t} - 1} dt, \quad (\text{B.5})$$

$$\begin{aligned} \tilde{\zeta}_{S^2}(s|L_2) = \frac{\alpha^{2-2s}}{4(s-1)} & \left[\int_0^1 \frac{(1-t^2)^{1-s}}{\cosh^2 \pi \alpha |\kappa|^{-1/2} t} \alpha |\kappa|^{-1/2} dt \right. \\ & \left. - \cos \pi s \int_1^\infty \frac{(t^2-1)^{1-s}}{\cosh^2 \pi \alpha |\kappa|^{-1/2} t} \alpha |\kappa|^{-1/2} dt \right], \end{aligned} \quad (\text{B.6})$$

$$\begin{aligned} \tilde{\zeta}_{T^N}(s|L_N) = \frac{\Gamma(s - \frac{N}{2})\alpha^{N-2s}}{(4\pi)^{\frac{N}{2}}\Gamma(s)} & + \frac{2\alpha^{N-2s}}{(4\pi)^{\frac{N-1}{2}}\Gamma(\frac{N+1}{2} - s)} \\ & \times \sum_{\vec{n} \neq 0} \int_1^\infty (t^2-1)^{\frac{N-1}{2}-s} e^{-2\pi \alpha r |\vec{n}| t} dt, \end{aligned} \quad (\text{B.7})$$

where $\vec{n} \in \mathbb{Z}^N$. In Eq. (B.7), in order to simplify the notation, we have considered an equilateral torus with radius r . The representation (B.6) is valid only for $\alpha > 0$.

For the ζ -function on H^N and S^N we also have the recurrence relation

$$\tilde{\zeta}(s|L_{N+2}) = -\frac{1}{2\pi N} \left[(\alpha^2 + \kappa \varrho_N^2) \tilde{\zeta}(s|L_N) - \tilde{\zeta}(s-1|L_N) \right], \quad (\text{B.8})$$

which permits to get the ζ -function on any H^N or S^N starting from Eqs. (B.3-B.6).

To finish this section, we report also the ζ -function in the case in which the manifold is H^N/Γ , Γ being a discrete group of isometries with only hyperbolic elements. In this case the ζ -function has a contribution which is equal to the one which one has for H^N (due to the identity element of Γ , Eqs. (B.3)) and (B.4) in 3 and 2 dimensions respectively) and a topological (analytic) contribution ζ_{top} , which has the integral representation [29]

$$\zeta_{top}(s|L_N) = \frac{\alpha^{-2s} \sin \pi s}{\pi} \int_1^\infty (t^2 - 1)^{-s} \frac{Z'_N}{Z_N} (\varrho_N + \alpha |\kappa|^{-1/2} t) \alpha |\kappa|^{-1/2} dt, \quad (\text{B.9})$$

where $Z_N(s)$ is the Selberg zeta function on H^N/Γ .

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