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Technique for Deriving Feynman Integrals in Axial-type Gauges

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Abstract

Employing the n_μ^* -prescription for axial-type gauges, and working in the unified-gauge formalism defined by n_μ and the null vector F_μ , we develop a simple technique for evaluating higher-rank integrals from known lower-rank integrals. Based on partial differentiation and implementation of an auxiliary constraint condition, the technique is applicable to a wide variety of integrals containing either single, double or triple spurious poles. We illustrate our procedure by several examples. There are strong indications that the new technique is also capable of generating the finite parts of axial-type integrals.

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1 INTRODUCTION

The quantization of gauge theories in noncovariant gauges continues to tax the imagination and patience of theorists. Despite progress during the past ten years in both the Hamiltonian and the path-integral formalism, it seems that most people's practical "know-how" of noncovariant gauges is greater in the path-integral formalism than it is in the canonical formulation. And for good reasons too, since both the decoupling of the ghosts from the gauge field and the implementation of the gauge constraint $F^b[A_\mu^a] = 0$ via the gauge-breaking term $\mathcal{L}_{GB} = -(2\alpha)^{-1}(F^b[A_\mu^a(x)])^2$ are more translucent in the path-integral approach. Here F^b is a local functional of the Yang-Mills field $A_\mu^a(x)$ ($a, b = 1, \dots, N^2 - 1$), N being the dimension of the group and α being the gauge parameter. These comments apply to all noncovariant gauges, in particular to the Coulomb gauge which is characterized by $\vec{\nabla} \cdot \vec{A}(x) = 0$, and to the axial gauges defined by $n \cdot A^a(x) = 0$, n^μ labeling a fixed axis in space-time ($n^2 = 0$ or $n^2 \gtrless 0$).

Of course, as everyone knows, formal implementation of a gauge constraint like $n \cdot A^a(x) = 0$ does not *per se* guarantee sensible results via path-integral quantization unless the poles in the Feynman integrals can be handled with a sound prescription. The latter is absolutely essential if the path-integral formalism is to succeed at all. Now, in the case of covariant gauges, there are no "prescription difficulties": Feynman's $i\epsilon$ -prescription, which is causal and permits a Wick rotation, works consistently in both Abelian and non-Abelian models.

But for noncovariant gauges, good prescriptions are not so easy to come by. For instance, we still have no prescription for the Coulomb gauge that gives consistent results in non-Abelian theories beyond two loops[1, 2]. And it took several years before the n_μ^* -prescription for the light-cone gauge was proposed[3, 4], and even longer before a uniform prescription for all the axial-type gauges was developed by one of the authors[5, 6, 7] (see also Gaigg[8, 9]).

This uniform prescription, an extension of the n_μ^* -prescription for the light-cone gauge (commonly referred to as the Mandelstam-Leibbrandt (ML) prescription), permits a Wick rotation to Euclidean space and is known to yield consistent results for one-loop and two-loop integrals. These integrals are usually evaluated from first principles which is a safe, albeit

tedious and time-consuming procedure. Due to the complexity of some of the calculations, however, and the large number of *new* integrals emerging already at two loops[6] , it is desirable to simplify the derivation of integrals in the n_μ^* -prescription. The purpose of this article is, therefore, as follows: given a set of basic integrals, such as

$$\mathcal{A} \equiv \left\{ \int dq \frac{1}{(q-p)^2 q \cdot n}, \int dq \frac{1}{(q-p)^2 (q \cdot n)^2}, \right. \\ \left. \int dq \frac{q^\mu}{(q-p)^2 q \cdot n}, \int dq \frac{q^\mu q^\nu}{q^2 (q-p)^2 (q \cdot n)}, \dots \right\}, \quad (1.1)$$

we wish to generate from $\mathcal{A}$ other, more complicated integrals, such as

$$\mathcal{B} \equiv \left\{ \int dq \frac{q^\mu}{(q-p)^2 (q \cdot n)^2}, \int dq \frac{q^\mu}{(q-p)^2 (q \cdot n)^3}, \right. \\ \left. \int dq \frac{q^\mu q^\nu}{(q-p)^2 (q \cdot n)^2}, \int dq \frac{q^\mu q^\nu q^\rho}{q^2 (q-p)^2 (q \cdot n)^2}, \dots \right\}. \quad (1.2)$$

All the integrals are regularized dimensionally in 2ω dimensions, and dq is a shorthand for $d^{2\omega}q$. The analysis of these and related integrals is nontrivial, since it involves the dual vector n_μ^* . The latter arises naturally in the n_μ^* -prescription[4] for the spurious poles of $(q \cdot n)^{-\beta}$, $\beta \in N$:

$$\frac{1}{(q \cdot n)^\beta} \equiv \frac{1}{(q \cdot n)^\beta} \Big|_{ML} = \lim_{\epsilon \rightarrow 0^+} \left(\frac{q \cdot n^*}{q \cdot n q \cdot n^* + i\epsilon} \right)^\beta, \quad \epsilon > 0, \quad (1.3)$$

where $n_\mu = (n_0, \vec{n})$ and $n_\mu^* = (n_0, -\vec{n})$ in a given Lorentz frame. Application of prescription (1.3) enables us to evaluate any axial-type integral from first principles, and to express its value either as a function of n_μ and n_μ^* , or in terms of n_μ and a new, axial vector F_μ [5], where F_μ is a null vector: $F^2 = 0$. We shall discuss the properties of F_μ in greater detail in Section 2, but suffice it to say for now that working with n_μ and F_μ is considerably more convenient than working with n_μ and n_μ^* . In fact, we believe that the $\{n, F\}$ -formalism is superior to the $\{n, n^*\}$ -formalism on several counts:

- (i) Since $F^2 = 0$, computations are easier and final results more compact in the $\{n, F\}$ -formalism, particularly in the case of higher-rank tensor integrals, i.e. for integrals involving $q_\mu q_\nu$, $q_\mu q_\nu q_\rho$, ..., in the integrand.

- (ii) Contraction of tensor integrals with F_μ , $F_\mu F_\nu$, etc. yields scalar integrals which are either zero or possess a simple form. These scalar integrals occur in large numbers in two-loop diagrams[6].
- (iii) The $\{n, F\}$ -formalism is particularly well suited for the unified-gauge formalism[5] which incorporates the axial gauge ($n^2 < 0$) and the temporal gauge ($n^2 > 0$) in a systematic way.

In view of these advantages of the $\{n, F\}$ -formalism, we shall henceforth assume that all our integrals in the original set $\mathcal{A}$, Eq. (1.1), are expressed in terms of n_μ and F_μ . With this *proviso* in mind, we may rephrase the aim of our program thus: given a set of basic integrals $\mathcal{A}$, Eq. (1.1), our objective is to employ partial differentiation of the integrand,

$$\frac{\partial}{\partial n^\mu} \left(\frac{1}{q \cdot n} \right)^\beta \Big|_{ML} = - \beta q_\mu \left(\frac{1}{q \cdot n} \right)^{\beta+1} \Big|_{ML}, \quad \beta \in N, \quad (1.4)$$

to generate from $\mathcal{A}$ new, higher-rank tensor integrals. We shall see that this technique, though infinitely faster than derivation from first principles, involves nevertheless a series of delicate steps because the values of the integrals are now expressed in terms of $\{n, F\}$. Before elaborating on F_μ and its derivatives, we remind the reader that identity (1.4) is characteristic of the n_μ^* -prescription and holds in the sense of distributions.

2 The vector F_μ and its derivatives

Initially introduced by one of the authors[5], the vector F_μ may be defined as

$$F_\mu = \frac{1}{2\sigma\rho} \left[(\sigma + \rho) n_\mu^* - (\sigma - \rho) n_\mu \right], \quad (2.1)$$

where σ, ρ are functions of n_μ and n_μ^* , defined implicitly by the equations:

$$\sigma^2 + \rho^2 = n \cdot n^* ; \quad \sigma^2 - \rho^2 = n^2 = n^{*2}. \quad (2.2)$$

Note that definition (2.1) makes sense only in gauges where $\sigma\rho \neq 0$.

A list of integrals in the $\{n, F\}$ -system may be found in refs. [5, 10, 6] (For a corresponding summary of integrals in the $\{n, n^*\}$ -formalism, see for instance ref. [11].) Since F_μ is a function of n_μ and n_μ^* , we need to find an expression for the derivative of F_σ with respect to n^μ . We proceed as follows.

As already alluded to in the Introduction, identity (1.4) is central to deriving new integrals from known ones, and gives correct answers as long as n_μ and n_μ^* are treated as formally independent vectors. Clearly, there is no problem with this condition in the $\{n, n^*\}$ -system, because there n_μ and n_μ^* are automatically independent. But in the $\{n, F\}$ -formalism, where F_σ is defined[5] on the shell $n^2 = n^{*2}$, we have to be more careful. In order to calculate the derivative of F_σ with respect to n^μ , one must first extend the definition of F_σ to $n^2 \neq n^{*2}$, i. e. off the shell $n^2 = n^{*2}$. This extension may be achieved by expressing the scalars σ and ρ in the form,

$$\sigma \equiv \sqrt{\frac{n \cdot n^* + |n||n^*|}{2}}, \quad \rho \equiv \sqrt{\frac{n \cdot n^* - |n||n^*|}{2}}, \quad |n| \equiv \sqrt{n_\mu n^\mu}, \quad (2.3)$$

where n_μ and n_μ^* can now be regarded as independent vectors, so that Eq.(2.1) defines F_μ also off the shell. The above definitions (2.3) lead to the relationships

$$\sigma^2 - \rho^2 = |n||n^*| = \sqrt{n^2 n^{*2}}, \quad (2.4)$$

$$\sigma^2 + \rho^2 = n \cdot n^*, \quad (2.5)$$

which generalize Eqs.(2.2) off the shell.

It is sometimes convenient to work with the scalar quantity D , first defined in [11] as:

$$D \equiv \sqrt{(n \cdot n^*)^2 - n^2 n^{*2}}, \quad (2.6)$$

the relationship $D = 2\sigma\rho$ then follows readily from Eqs. (2.4)–(2.5). On the shell $n^2 = n^{*2}$, σ and ρ satisfy Eq.(2.2), (cf. [5], p. 417); moreover,

$$\sigma + \rho = n \cdot F \quad (2.7)$$

$$\sigma - \rho = \frac{n^2}{n \cdot F} \quad (2.8)$$

$$D = \frac{(n \cdot F)^4 - n^4}{2(n \cdot F)^2}. \quad (2.9)$$

$$\rho = \frac{1}{2} \left(n \cdot F - \frac{n^2}{n \cdot F} \right), \quad (2.22)$$

$$D = \frac{(n \cdot F)^4 - n^4}{2(n \cdot F)^2}. \quad (2.23)$$

This analysis completes the first step in our program: Eq. (2.17) enables us to take the derivative of an integral with respect to n^μ directly in the $\{n, F\}$ -formalism.

3 General formula for the derivative of an integral containing a single spurious pole

Let us consider a Feynman integral with a simple spurious pole, such as

$$I(m, 1)_{\mu_1 \dots \mu_k} \equiv \int dq \frac{q_{\mu_1} \dots q_{\mu_k}}{q^2(q-p_1)^2 \dots (q-p_{m-1})^2} \frac{1}{q \cdot n}; \quad (3.1)$$

in the $\{n, F\}$ -formalism, we may think of $I(m, 1)_{\mu_1 \dots \mu_k}$ as a known function $k(n, F)_{\mu_1 \dots \mu_k}$ of the vectors n_μ and F_μ , i.e.

$$I(m, 1)_{\mu_1 \dots \mu_k} = k(n, F)_{\mu_1 \dots \mu_k} Y(p), \quad (3.2)$$

where

$$Y(p) \equiv \int dq [q^2(p-q)^2]^{-1} = i\pi^2 \frac{\Gamma(2-\omega) [\Gamma(\omega-1)]^2}{\Gamma(2\omega-2)} (p^2)^{\omega-2}, \quad (3.3)$$

and

$$\text{divergent part of } Y(p) \equiv \bar{I} = i \frac{\pi^2}{(2-\omega)}, \quad \omega \rightarrow 2^+. \quad (3.4)$$

To simplify the notation we shall replace the tensor indices $\mu_1 \dots \mu_k$ by the collective index $\mu(k)$, so that $I(m, 1)_{\mu_1 \dots \mu_k} \equiv I(m, 1)_{\mu(k)}$. Suppose we know the value of $I(m, 1)_{\mu(k)}$ in Eq. (3.1) explicitly. Our aim is to exploit this value of $I(m, 1)_{\mu(k)}$ to deduce swiftly and with minimum effort, the values of other, more complicated integrals having double, triple, and quadruple spurious poles. To achieve this goal we proceed as follows.

We first notice that rescaling n_μ and F_μ by λ and β , namely $n_\mu \rightarrow \lambda n_\mu$, and $F_\mu \rightarrow \beta F_\mu$, implies that $k(\lambda n, \beta F)_{\mu(k)} = \lambda^{-1} \beta^0 k(n, F)_{\mu(k)}$. Euler's theorem on homogeneous functions

then yields the identities

$$n^\sigma \frac{\partial}{\partial n^\sigma} k_{\mu(k)} = -k_{\mu(k)} ; \quad F^\sigma \frac{\partial}{\partial F^\sigma} k_{\mu(k)} = 0 , \quad (3.5)$$

which will be very helpful in simplifying the derivative of $I(m, 1)_{\mu(k)}$. Our next step consists of using the formula

$$\frac{\partial}{\partial n^\rho} \frac{1}{q \cdot n} \Big|_{ML} = -q_\rho \frac{1}{(q \cdot n)^2} \Big|_{ML} \quad (3.6)$$

to find a value for $I(m, 2)_{\mu(k)\rho}$:

$$I(m, 2)_{\mu(k)\rho} \equiv \int dq \frac{q_{\mu_1} \cdots q_{\mu_k}}{q^2 (q - p_1)^2 \cdots (q - p_{m-1})^2} \frac{q_\rho}{(q \cdot n)^2} , \quad (3.7)$$

$$= - \frac{\partial}{\partial n^\rho} I(m, 1)_{\mu(k)} . \quad (3.8)$$

The next step is the most delicate one and involves taking the partial derivative in Eq. (3.8). The reason for this difficulty stems from the fact that Eq. (3.6) is only valid if we consider n_μ and n_μ^* as formally independent vectors. Since the expression (3.2) for $I(m, 1)_{\mu(k)}$ is in general evaluated on the shell $n^2 = n^{*2}$, we somehow need to extend (3.2) off the shell before taking the partial derivative¹ with respect to n^ρ . We do this by adding to $k_{\mu(k)}$ a piece proportional to F^2 (notice that F^2 does not vanish off-shell):

$$I(m, 1)_{\mu(k)} = \left[k(n, F)_{\mu(k)} + h(n, F)_{\mu(k)} F^2 \right] Y(p) , \quad (3.9)$$

where the function $h(n, F)_{\mu(k)}$ has the same tensorial indices as $k(n, F)_{\mu(k)}$.

Differentiating $I(m, 1)_{\mu(k)}$ partially with respect to n^ρ , we obtain

$$\partial_\rho I(m, 1)_{\mu(k)} = \left[d_\rho k_{\mu(k)} + 2 h_{\mu(k)} F^\sigma \partial_\rho F_\sigma + (d_\rho h_{\mu(k)}) F^2 \right] Y(p) , \quad (3.10)$$

where d_ρ denotes the total derivative with respect to n^ρ ,

$$d_\rho = \partial_\rho + (\partial_\rho F^\tau) \frac{\partial}{\partial F^\tau} . \quad (3.11)$$

The term $(d_\rho h_{\mu(k)}) F^2$ may be dropped from here on, since we are only interested in the value of the derivative on-shell, where $F^2 = 0$; hence

$$I(m, 2)_{\mu(k)\rho} = - \left[d_\rho k_{\mu(k)} + 2 h_{\mu(k)} (F^\sigma \partial_\rho F_\sigma) \right] Y(p) , \quad (3.12)$$

¹A similar problem was encountered in Sec. 2 when we had to calculate the derivative of F_μ with respect to n^ρ .

and it remains to find an explicit expression for the function $h(n, F)_{\mu(k)}$. Noting that

$$n^\rho I(m, 2)_{\mu(k)\rho} = I(m, 1)_{\mu(k)}, \quad (3.13)$$

we readily deduce from Eqs.(3.9) and (3.12), and with the help of Eq. (3.5), that

$$h(n, F)_{\mu(k)} = - \left[2n^\rho F^\lambda \partial_\rho F_\lambda \right]^{-1} \frac{\partial k_{\mu(k)}}{\partial F^\sigma} (n^\tau \partial_\tau F^\sigma). \quad (3.14)$$

Moreover, Eqs. (2.17)–(2.20) imply the contractions

$$F^\sigma \partial_\rho F_\sigma = B_\rho (n \cdot F) + C F_\rho = - \frac{n_\rho}{D}, \quad (3.15)$$

$$n^\rho F^\sigma \partial_\rho F_\sigma = C (n \cdot F) = - \frac{n^2}{D}, \quad (3.16)$$

$$n^\tau \partial_\tau F_\sigma = - \frac{1}{2} F_\sigma - \frac{\sigma - \rho}{D} n_\sigma, \quad (3.17)$$

so that $h(n, F)_{\mu(k)}$ in Eq. (3.14) eventually simplifies to

$$h(n, F)_{\mu(k)} = - \frac{\sigma - \rho}{2n^2} n^\sigma \frac{\partial k_{\mu(k)}}{\partial F^\sigma}. \quad (3.18)$$

Substitution of $h(n, F)_{\mu(k)}$ into Eq.(3.12) yields the following value for the integral $I(m, 2)_{\mu(k)\rho}$:

$$I(m, 2)_{\mu(k)\rho} = - \left\{ \partial_\rho k_{\mu(k)} + \left[g^{\sigma\tau} - \frac{n^\sigma F^\tau}{n \cdot F} \right] (\partial_\rho F_\tau) \frac{\partial k_{\mu(k)}}{\partial F^\sigma} \right\} Y(p). \quad (3.19)$$

This result may be further simplified by noting that

$$n_\tau \left[g^{\sigma\tau} - \frac{n^\sigma F^\tau}{n \cdot F} \right] = 0, \quad (3.20)$$

and from (3.5), that

$$F_\tau \left[g^{\sigma\tau} - \frac{n^\sigma F^\tau}{n \cdot F} \right] \frac{\partial k_{\mu(k)}}{\partial F^\sigma} = F^\sigma \frac{\partial k_{\mu(k)}}{\partial F^\sigma} = 0. \quad (3.21)$$

Therefore, only the $C g_{\rho\tau}$ part of $(\partial_\rho F_\tau)$ in Eq.(2.17) contributes to $I(m, 2)_{\mu(k)\rho}$:

$$I(m, 2)_{\mu(k)\rho} = - \frac{\partial k_{\mu(k)}}{\partial n^\rho} Y(p) + \left(\frac{\sigma - \rho}{D} \right) Y(p) \left[g_\rho^\sigma - \frac{n^\sigma F_\rho}{n \cdot F} \right] \frac{\partial k_{\mu(k)}}{\partial F^\sigma}, \quad (3.22)$$

or

$$I(m, 2)_{\mu_1 \dots \mu_k \rho} = - \frac{\partial k_{\mu_1 \dots \mu_k}}{\partial n^\rho} Y(p) + \frac{1}{2} (n \cdot F) X(p) \left[g_\rho^\sigma - \frac{n^\sigma F_\rho}{n \cdot F} \right] \frac{\partial k_{\mu_1 \dots \mu_k}}{\partial F^\sigma}; \quad (3.23)$$

here $X(p)$ is defined-as[6]

$$X(p) \equiv \int dq \frac{1}{(p-q)^2 (q \cdot n)^2}, \quad (3.24)$$

while the divergent parts of X and Y are related by[6, 12]

$$\text{div } X(p) = \frac{2}{n \cdot F} \frac{\sigma - \rho}{D} \text{div } Y(p). \quad (3.25)$$

Formula (3.23) is a powerful result. Not only does it hold for any one of the axial-type gauges, namely the light-cone gauge ($n^2 = 0$), the pure axial gauge ($n^2 < 0$) and the temporal gauge ($n^2 > 0$), but it also allows us to deduce, by contraction with F^ρ , integrals of the form

$$F^\rho I(m, 2)_{\mu_1 \dots \mu_k \rho} = \int dq \frac{q_{\mu_1} \dots q_{\mu_k}}{q^2 (q - p_1)^2 \dots (q - p_{m-1})^2} \frac{q_\rho F^\rho}{(q \cdot n)^2}. \quad (3.26)$$

Noting that the coefficient of the term proportional to $X(p)$ vanishes when contracted with F^ρ (cf. Eq. (3.21)), we easily find that

$$F^\rho I(m, 2)_{\mu(k) \rho} = - F^\rho \frac{\partial k_{\mu(k)}}{\partial n^\rho} Y(p). \quad (3.27)$$

4 Applications

In this section we shall apply the result in Eq. (3.23) to three specific integrals.

(a) $I(1, 1) \rightarrow I(1, 2)_\mu$

In this first example, we shall start with the scalar integral $I(1, 1)$,

$$I(1, 1) = \int dq \frac{1}{(q-p)^2 q \cdot n}, \quad (4.1)$$

to determine the value of

$$I(1, 2)_\mu = \int dq \frac{q_\mu}{(q-p)^2 (q \cdot n)^2}. \quad (4.2)$$

Only the divergent parts of these integrals will be considered. Since we know[6] the value of $I(1, 1)$, namely

$$I(1, 1) = 2 \frac{p \cdot F}{n \cdot F} Y(p) \equiv k(n, F) Y(p), \quad (4.3)$$

where $k(n, F) = 2 \frac{p \cdot F}{n \cdot F}$, we can readily deduce from formula (3.23) the following expression for $I(1, 2)_\mu$:

$$I(1, 2)_\mu = - \frac{\partial k}{\partial n^\mu} Y(p) + \frac{1}{2} (n \cdot F) X(p) \left[g_\mu^\tau - \frac{F^\mu n_\tau}{F \cdot n} \right] \frac{\partial k}{\partial F^\tau}. \quad (4.4)$$

Computing the derivatives $\partial_\mu k$ and $\partial k / \partial F^\tau$,

$$\frac{\partial k}{\partial n^\mu} = -2 \frac{p \cdot F}{(n \cdot F)^2} F_\mu, \quad (4.5)$$

$$\frac{\partial k}{\partial F^\tau} = \frac{2}{n \cdot F} \left[p_\tau - \frac{p \cdot F}{n \cdot F} n_\tau \right], \quad (4.6)$$

and substituting them into Eq.(4.4), we obtain

$$I(1, 2)_\mu = 2 \frac{p \cdot F}{(n \cdot F)^2} F_\mu Y(p) \quad (4.7)$$

$$+ \left[p_\mu - \frac{p \cdot F}{n \cdot F} n_\mu - \frac{1}{n \cdot F} \left(p \cdot n - n^2 \frac{p \cdot F}{n \cdot F} \right) F_\mu \right] X(p). \quad (4.8)$$

This expression is in agreement with the integral listed in Appendix B of Ref. [6].

(b) $I(2, 1)_\mu \rightarrow I(2, 2)_{\mu\nu}$

In this case we start from the integral

$$I(2, 1)_\mu \equiv \text{div} \int dq \frac{q_\mu}{q^2(q-p)^2 q \cdot n} = Y(p) \frac{F_\mu}{n \cdot F}, \quad (4.9)$$

and identify $k(n, F)_\mu$ to be $k(n, F)_\mu = F_\mu / n \cdot F$; the corresponding derivatives read

$$\partial_\rho k_\mu = - \frac{F_\mu F_\rho}{(n \cdot F)^2}, \quad (4.10)$$

and

$$\frac{\partial k_\mu}{\partial F^\rho} = \frac{1}{n \cdot F} \left[g_{\mu\rho} - \frac{F_\mu n_\rho}{F \cdot n} \right]. \quad (4.11)$$

Our aim is to find the divergent part of $\int dq q_\mu q_\nu [(q-p)^2 (q \cdot n)^2]^{-1}$, namely

$$I(2, 2)_{\mu\nu} \equiv \text{div} \int dq \frac{q_\mu q_\nu}{(q-p)^2 (q \cdot n)^2}, \quad (4.12)$$

which is seen to possess a double spurious pole. Substituting Eqs.(4.10) and (4.11) into the general formula, Eq. (3.23), we again obtain the expected answer[6]:

$$I(2, 2)_{\mu\nu} = \frac{F_\mu F_\nu}{(n \cdot F)^2} Y(p) + \frac{1}{2} X(p) \left[g_{\mu\nu} - \frac{F_\mu n_\nu + F_\nu n_\mu}{F \cdot n} + \frac{n^2}{(F \cdot n)^2} F_\mu F_\nu \right]. \quad (4.13)$$

Notice the ease with which this result for the integral in Eq. (4.12) was derived.

$$(c) \quad I(1,1)_\mu \rightarrow I(1,2)_{\mu\nu}$$

In our third example, which is somewhat lengthier, we shall employ the integral $I(1,1)_\mu$,

$$I(1,1)_\mu \equiv \text{div} \int dq \frac{q_\mu}{(q-p)^2 q \cdot n} \quad (4.14)$$

$$= 2 \frac{p \cdot F}{n \cdot F} \left[p_\mu - \frac{p \cdot F}{2 n \cdot F} n_\mu - \frac{1}{2 F \cdot n} \left(2 p_\sigma - \frac{p \cdot F}{n \cdot F} n_\sigma \right) n_\sigma F_\mu \right] Y(p), \quad (4.15)$$

to evaluate

$$I(1,2)_{\mu\rho} \equiv \text{div} \int dq \frac{q_\mu q_\rho}{(q-p)^2 (q \cdot n)^2} \quad (4.16)$$

$$= \frac{\partial k_\mu}{\partial n^\rho} Y(p) + \frac{1}{2} (n \cdot F) X(p) \left[g_\rho^\tau - \frac{n^\tau F_\rho}{n \cdot F} \right] \frac{\partial k_\mu}{\partial F^\tau}. \quad (4.17)$$

We see from Eq.(4.15) that the crucial function $k(n, F)_\mu$ is given by

$$k(n, F)_\mu = 2 L \left[R_\mu + \frac{1}{2} L n_\mu - \frac{1}{2 F \cdot n} (R + p) \cdot n F_\mu \right], \quad (4.18)$$

where

$$L \equiv \frac{p \cdot F}{n \cdot F} \quad \text{and} \quad R_\mu \equiv p_\mu - L n_\mu, \quad \text{with } F \cdot R = 0. \quad (4.19)$$

In order to apply formula (4.17), we need to calculate, from Eq. (4.18), the derivatives $\partial_\rho k_\mu$ and $\partial k_\mu / \partial F^\rho$, i.e. we require the derivatives $\partial_\mu L$, $\partial_\nu R_\mu$, etc. This intermediate computation yields

$$\partial_\mu L = - \frac{L}{n \cdot F} F_\mu; \quad \frac{\partial L}{\partial F^\mu} = \frac{R_\mu}{n \cdot F}; \quad F_\mu \frac{\partial L}{\partial F^\mu} = 0, \quad (4.20)$$

and

$$\partial_\nu R_\mu = - L \left(g_{\mu\nu} - \frac{n_\mu F_\nu}{n \cdot F} \right); \quad \frac{\partial R_\mu}{\partial F^\nu} = - \frac{n_\mu R_\nu}{n \cdot F}, \quad (4.21)$$

so that

$$n^\nu \partial_\nu R_\mu = 0; \quad F^\mu \partial_\nu R_\mu = 0; \quad F^\nu \frac{\partial R_\mu}{\partial F^\nu} = 0. \quad (4.22)$$

The derivatives of k_μ needed in Eq. (4.17) can be readily computed from (4.18) with the help of Eqs.(4.20)-(4.22) :

$$\begin{aligned} \partial_\nu k_\mu &= \frac{2L}{n \cdot F} \left\{ -\frac{1}{2} (p \cdot F) g_{\mu\nu} - (p_\mu F_\nu + p_\nu F_\mu) \right. \\ &\quad \left. + \frac{1}{n \cdot F} \left[2p \cdot n - \frac{3}{2} n^2 L \right] F_\mu F_\nu + L (n_\mu F_\nu + n_\nu F_\mu) \right\}, \end{aligned} \quad (4.23)$$

and

$$\begin{aligned} \frac{\partial k_\mu}{\partial F^\nu} &= \frac{2}{n \cdot F} \left\{ p_\mu p_\nu + L^2 n_\mu n_\nu - L(p_\mu n_\nu + p_\nu n_\mu) \frac{R \cdot n}{F \cdot n} p_\nu F_\mu \right. \\ &\quad \left. - \frac{L}{2} (2 p \cdot n - n^2 L) g_{\mu\nu} + \frac{L}{n \cdot F} \left(2 p \cdot n - \frac{3}{2} n^2 L \right) n_\nu F_\mu \right\} \quad (4.24) \end{aligned}$$

The final expression for the integral $I(1,2)_{\mu\nu}$ is, therefore, given by

$$\begin{aligned} \text{div} \int dq \frac{q_\mu q_\nu}{(q-p)^2 (q \cdot n)^2} &= L \left\{ g_{\mu\nu} \left[L Y(p) + X(p) \left(\frac{1}{2} n^2 L - p \cdot n \right) \right] \right. \\ &\quad + (p_\mu F_\nu + p_\nu F_\mu) \left[\frac{2}{n \cdot F} Y(p) + X(p) \left(\frac{n^2}{n \cdot F} - \frac{p \cdot n}{p \cdot F} \right) \right] \\ &\quad + (n_\mu F_\nu + n_\nu F_\mu) \frac{1}{n \cdot F} \left[-2LY(p) + \left(2p \cdot n - \frac{3}{2} n^2 L \right) X(p) \right] \\ &\quad + F_\mu F_\nu \frac{1}{(n \cdot F)^2} \left[(3n^2 L - 4p \cdot n) Y(p) + X(p) \left((p \cdot n)^2 \frac{1}{L} + \frac{3}{2} n^4 L - 3n^2 p \cdot n \right) \right] \\ &\quad \left. - (n_\mu p_\nu + n_\nu p_\mu) X(p) + n_\mu n_\nu L X(p) + p_\mu p_\nu \frac{1}{L} X(p) \right\} . \quad (4.25) \end{aligned}$$

This nontrivial result is valid both for $n^2 \gtrless 0$ and $n^2 = 0$. For instance, in the light-cone gauge, where $X(p) = 0$, formula (4.25) reduces to

$$\begin{aligned} \text{div} \int dq \frac{q_\mu q_\nu}{(q-p)^2 (q \cdot n)^2} &= \frac{p \cdot F}{(n \cdot F)^2} [g_{\mu\nu} p \cdot F + 2 (p_\mu F_\nu + p_\nu F_\mu) \\ &\quad - 2 \frac{p \cdot F}{n \cdot F} (n_\mu F_\nu + n_\nu F_\mu) + F_\mu F_\nu \frac{1}{n \cdot F} (3n^2 \frac{p \cdot F}{n \cdot F} - 4p \cdot n)] , \quad (4.26) \end{aligned}$$

which agrees with the value of the integral derived from first principles. A fourth, more complicated, application of Eq. (3.23) has been summarized in App. A.

5 Integrals with triple spurious poles

We can now apply the procedure used in Section 3 to derive integrals with a triple spurious pole $I(m,3)_{\mu(k)\rho}$ from integrals $I(m,2)_{\mu(k)}$ with a double spurious pole. Suppose that we have a value for

$$I(m,2)_{\mu_1 \dots \mu_k} \equiv I(m,2)_{\mu(k)} = \int dq \frac{q_{\mu_1} \dots q_{\mu_k}}{q^2 (q-p_1)^2 \dots (q-p_{m-1})^2} \frac{1}{(q \cdot n)^2} \quad (5.1)$$

$$= k(n, F)_{\mu_1 \dots \mu_k}, \quad (5.2)$$

where $k(n, F)_{\mu_1 \dots \mu_k}$ is a known function of the vectors n_μ and F_μ . The main difference between the integral in (5.2) and the integrals studied in Section 3 is that the function $k(n, F)_{\mu_1 \dots \mu_k}$ is *no longer a homogeneous function of F_μ* , so that there is only one identity for $k_{\mu(k)}$ (cf. Eq. (3.5)), namely

$$n^\sigma \partial_\sigma k_{\mu(k)} = - 2 k_{\mu(k)} ; \quad (5.3)$$

as before, we have replaced the tensor indices $\mu_1 \dots \mu_k$ by the collective index $\mu(k)$. Repeating the procedure outlined in Section 3, we add to $I(m, 2)_{\mu(k)}$ an off-shell term proportional to F^2 , so that

$$I(m, 2)_{\mu(k)} = k(n, F)_{\mu(k)} + h(n, F)_{\mu(k)} F^2. \quad (5.4)$$

Differentiating $I(m, 2)_{\mu(k)}$ with respect to n^ρ ,

$$\partial_\rho I(m, 2)_{\mu(k)} = d_\rho k_{\mu(k)} + 2 h_{\mu(k)} (F^\sigma \partial_\rho F_\sigma) , \quad (5.5)$$

and solving for $h_{\mu(k)}$ by exploiting the identity,

$$[n^\rho \partial_\rho + 2] I(m, 2)_{\mu(k)} = 0, \quad (5.6)$$

we easily find

$$h_{\mu(k)} = \frac{1}{2} \frac{D}{n^2} \left[-\frac{1}{2} F^\sigma - \frac{\sigma - \rho}{D} n^\sigma \right] \frac{\partial k_{\mu(k)}}{\partial F^\sigma}. \quad (5.7)$$

Notice that $h_{\mu(k)}$ differs from expression (3.18) by the extra term

$$- \frac{1}{4} \frac{D}{n^2} F^\sigma \frac{\partial k_{\mu(k)}}{\partial F^\sigma} , \quad D = 2 \sigma \rho. \quad (5.8)$$

Employing partial differentiation of the integrand in Eq. (5.2) (cf. formula (1.4)), we have

$$I(m, 3)_{\mu(k)\rho} = - \frac{1}{2} \partial_\rho I(m, 2)_{\mu(k)} ; \quad (5.9)$$

the right hand side of (5.9) can be rewritten, by using the expression for $\partial_\mu I(m, 2)_{\mu(k)}$ in Eq. (5.5) together with the value for h in Eq.(5.7), to yield

$$I(m, 3)_\rho = - \frac{1}{2} \partial_\rho k_{\mu(k)} - \frac{1}{2} \frac{\partial k_{\mu(k)}}{\partial F^\sigma} \left[\partial_\rho F^\sigma + \frac{n_\rho F^\sigma}{2n^2} + \frac{\sigma - \rho}{D} \frac{n_\rho n^\sigma}{n^2} \right]. \quad (5.10)$$

With the help of $\partial_\rho F_\sigma$ in (2.17), we find a general formula for the integral $\tilde{I}(m, 3)_\rho$ in terms of the derivatives of $k(n, F)_{\mu(k)}$:

$$I(m, 3)_{\mu_1 \dots \mu_k \rho} = -\frac{1}{2} (\partial_\rho k_{\mu_1 \dots \mu_k}) - \frac{1}{2} \left(\frac{\partial k_{\mu_1 \dots \mu_k}}{\partial F^\sigma} \right) T_\rho^\sigma, \quad (5.11)$$

where the tensor T_ρ^σ reads

$$T(n, F)_\rho^\sigma = -\frac{\sigma - \rho}{D} g_\rho^\sigma + \frac{\sigma - \rho}{\sigma + \rho} \frac{1}{D} F_\rho n^\sigma \quad (5.12)$$

$$+ \frac{2(\sigma^2 + \rho^2) - D}{2Dn^2} n_\rho F^\sigma - \frac{\sigma^2 + \rho^2 - \sigma\rho}{D(\sigma + \rho)} F_\rho F^\sigma. \quad (5.13)$$

To summarize, once we know the value of an integral with a double pole, such as

$$I(m, 2)_{\mu(k)} = \int dq \frac{q_{\mu_1} \dots q_{\mu_k}}{q^2(q - p_1)^2 \dots (q - p_{m-1})^2} \frac{1}{(q \cdot n)^2} = k(n, F)_{\mu_1 \dots \mu_k}, \quad (5.14)$$

we can readily deduce the value of the corresponding integral with a triple spurious pole, namely:

$$\begin{aligned} I(m, 3)_{\mu(k)\rho} &= \int dq \frac{q_{\mu_1} \dots q_{\mu_k} q_\rho}{q^2(q - p_1)^2 \dots (q - p_{m-1})^2} \frac{1}{(q \cdot n)^3} \\ &= -\frac{1}{2} (\partial_\rho k_{\mu_1 \dots \mu_k}) - \frac{1}{2} \left(\frac{\partial k_{\mu_1 \dots \mu_k}}{\partial F^\sigma} \right) T_\rho^\sigma. \end{aligned} \quad (5.15)$$

It is also easy to see that the contraction of $I(m, 3)_{\mu_1 \dots \mu_k \rho}$ with F^ρ leads to

$$F^\rho I(m, 3)_{\mu_1 \dots \mu_k \rho} = \int dq \frac{q_{\mu_1} \dots q_{\mu_k}}{q^2(q - p_1)^2 \dots (q - p_{m-1})^2} \frac{q \cdot F}{(q \cdot n)^2} \quad (5.16)$$

$$- \frac{1}{2} (\partial_\tau k_{\mu_1 \dots \mu_k}) F_\tau - \frac{1}{4(\sigma - \rho)} \left(F_\tau \frac{\partial k_{\mu_1 \dots \mu_k}}{\partial F^\tau} \right), \quad (5.17)$$

since $T_\rho^\sigma F^\rho = F^\sigma / (2(\sigma - \rho))$.

Formula (5.11) assumes a simple form when the function $k_{\mu(k)}$ is a scalar, i.e. when there are no q_{μ_r} 's in the numerator of the integral (5.2). In that case $\partial k / \partial F^\sigma$ has the structure

$$\frac{\partial k}{\partial F^\sigma} = a(n, F) F_\sigma + b(n, F) n_\sigma \quad (5.18)$$

where $a(n, F)$ and $b(n, F)$ are undetermined coefficients. Contracting the derivative (5.18) with the tensor $T(n, F)_\mu^\sigma$ in Eq.(5.13), we find

$$\frac{\partial k}{\partial F^\sigma} T_{\mu\sigma} = -\frac{1}{2} b \left[F_\mu - \frac{1}{\sigma - \rho} n_\mu \right], \quad (5.19)$$

so that the general equation (5.11) reduces to the simple form

$$I(m, 3)_\rho = -\frac{1}{2} (\partial_\rho k) + \frac{1}{4} b \left[F_\rho - \frac{n_\rho}{\sigma - \rho} \right]. \quad (5.20)$$

As an illustration of this “scalar case” we apply formula (5.20) to derive the value of the integral

$$I(1, 3)_\mu \equiv \int dq \frac{q_\mu}{(q-p)^2(q \cdot n)^3} \quad (5.21)$$

from the divergent part of $I(1, 2)$:

$$I(1, 2) = X(p) = \int dq \frac{1}{(q-p)^2(q \cdot n)^2} = \frac{2}{D} \frac{n^2}{(n \cdot F)^2} \bar{I}. \quad (5.22)$$

Differentiation of $k(n, F) = 2n^2/(D(n \cdot F)^2) \bar{I}$ with respect to n_μ and F_σ yields, respectively,

$$\partial_\mu k = \frac{4}{D^2} \left\{ -(\sigma - \rho) F_\mu + \frac{\sigma^2 + \rho^2}{(\sigma + \rho)^2} n_\mu \right\} \bar{I} \quad (5.23)$$

and

$$\frac{\partial k}{\partial F^\sigma} = -\frac{4}{D} (\sigma - \rho) n_\sigma \bar{I}, \quad (5.24)$$

so that $b = -4(\sigma - \rho)\bar{I}/D$. Using these derivatives we immediately get the following answer for $I(1, 3)_\mu$:

$$\int dq \frac{q_\mu}{(q-p)^2(q \cdot n)^3} = \frac{2n^2}{D^2 (n \cdot F)} \left[F_\mu - \frac{n^2}{(n \cdot F)^3} n_\mu \right] \bar{I}. \quad (5.25)$$

6 Concluding Remarks

It is a well known fact that calculations at higher and higher loops involve, progressively, newer and more complicated integrals. A perfect example is the sunset diagram in Yang-Mills theory: the amplitude for this two-loop diagram[6] contains new types of integrals not present at the one-loop level. Since computation of these integrals from first principles is both tedious and time-consuming, we decided to search for a swifter, more efficient means of evaluation.

In this paper we have discussed a simple mechanism/procedure for deriving higher-rank integrals from certain lower-rank integrals, whose values are known to us explicitly. Since the

derivation is carried out in the framework of the unified-gauge formalism, our new integrals hold in all axial-type gauges, namely in the light-cone gauge, the pure axial gauge, the planar gauge and the temporal gauge. Probably the simplest way of explaining the new technique is to consider the divergent part of a specific integral, such as

$$I(1,1) = \int \frac{dq}{(q-p)^2 q \cdot n}, \quad (6.1)$$

whose value is known, and to derive from it the divergent part of $I(1,2)_\mu$,

$$I(1,2)_\mu = \int \frac{dq q_\mu}{(q-p)^2 (q \cdot n)^2}. \quad (6.2)$$

The main steps of our technique may then be summarized as follows.

1. Express the value of $I(1,1)$ in terms of n_μ and F_μ , where F_μ is a null vector:

$$\int \frac{dq}{(q-p)^2 q \cdot n} = \frac{2 p \cdot F}{n \cdot F} Y(p) = k(n, F) Y(p). \quad (6.3)$$

Here $k(n, F) = 2 p \cdot F / n \cdot F$; the pole part $Y(p)$ is given in Eq. (3.3). At this point, differentiation with respect to n^μ is allowed on the left hand side of (6.3), but not on the right hand side, since $k(n, F)$ is only defined on the shell $n^2 = (n^*)^2$. To remedy the situation we:

2. Augment the right hand side of (6.3) by an off-shell term $h(n, F) F^2$, keeping $F^2 \neq 0$ for the time being. Thus

$$\int \frac{dq}{(q-p)^2 q \cdot n} = k(n, F) Y(p) + h(n, F) F^2, \quad F^2 \neq 0. \quad (6.4)$$

3. Differentiate the left hand side of (6.4) with respect to n^μ , and the right side with respect to both n^μ and F^μ (Cf. Eq. (3.10) in text) to obtain:

$$\int \frac{dq q_\mu}{(q-p)^2 (q \cdot n)^2} = - \left[d_\mu k(n, F) + 2 h(n, F) F^\sigma \partial_\mu F_\sigma + F^2 d_\mu h(n, F) \right] Y(p), \quad (6.5)$$

where

$$d_\mu \equiv \frac{d}{dn^\mu} = \frac{\partial}{\partial n^\mu} + \frac{\partial F_\sigma}{\partial n^\mu} \frac{\partial}{\partial F_\sigma}. \quad (6.6)$$

Equate $F^2 d_\mu h(n, F)$ to zero, as we are only interested in the on-shell value of $I(1, 2)_\mu$, where $F^2 = 0$. Hence (6.5) reduces to:

$$\int \frac{dq q_\mu}{(q-p)^2 (q \cdot n)^2} = - [d_\mu k(n, F) + 2 h(n, F) F^\sigma \partial_\mu F_\sigma] Y(p). \quad (6.7)$$

4. Employ the identity

$$n^\mu I(1, 2)_\mu = I(1, 1) \quad (6.8)$$

to solve for the unknown function $h(n, F)$, namely

$$h(n, F) = - \frac{\sigma - \rho}{2n^2} n^\sigma \frac{\partial k(n, F)}{\partial F^\sigma}, \quad (6.9)$$

and then substitute this value back into Eq. (6.7):

$$\int \frac{dq q_\mu}{(q-p)^2 (q \cdot n)^2} = - \frac{\partial k(n, F)}{\partial n^\mu} Y(p) + \frac{1}{2} (n \cdot F) X(p) \left[g_\mu^\sigma - \frac{n^\sigma F_\mu}{n \cdot F} \right] \frac{\partial k(n, F)}{\partial F^\sigma} \quad (6.10)$$

with $X(p)$ defined as in (3.24).

5. Compute the indicated derivatives in (6.10), using $k(n, F) = 2 p \cdot F / n \cdot F$. The final result reads

$$\begin{aligned} I(1, 2)_\mu &= \int \frac{dq q_\mu}{(q-p)^2 (q \cdot n)^2} \\ &= 2 \frac{p \cdot F}{(n \cdot F)^2} F_\mu Y(p) \\ &+ \left[p_\mu - \frac{p \cdot F}{n \cdot F} n_\mu - \frac{1}{n \cdot F} \left(p \cdot n - n^2 \frac{p \cdot F}{n \cdot F} \right) F_\mu \right] X(p). \end{aligned} \quad (6.11)$$

The procedure outlined in this paper permits us to derive not only the divergent parts of massless noncovariant-gauge Feynman integrals, but also the corresponding massive Feynman integrals. Moreover, it appears from preliminary calculations that the technique also yields the finite parts of the new integrals which, as everyone knows, can be notoriously difficult to obtain. A sample calculation of the finite part of a noncovariant-gauge Feynman integral is given in Appendix B.

A Appendix

Here we shall apply the procedure developed in Sec. 3 to derive the value of the divergent rank 3 integral

$$I(2,2)_{\mu\nu\rho} = \int dq \frac{q_\mu q_\nu q_\rho}{q^2 (q-p)^2 (q \cdot n)^2}. \quad (\text{A.1})$$

We know from Eq.(3.23) that we can determine the value of $I(2,2)_{\mu\nu\rho}$ from the value of the rank 2 integral[6]

$$I(2,1)_{\mu\nu} = \int dq \frac{q_\mu q_\nu}{q^2 (q-p)^2} \frac{1}{(q \cdot n)} \quad (\text{A.2})$$

$$= \frac{1}{2} \frac{1}{n \cdot F} \left\{ p \cdot F g_{\mu\nu} + (p_\mu F_\nu + p_\nu F_\mu) - \frac{p \cdot F}{n \cdot F} (n_\mu F_\nu + n_\nu F_\mu) - \frac{p \cdot F}{n \cdot F} (n_\mu F_\nu + n_\nu F_\mu) - \frac{1}{n \cdot F} \left(p_\sigma - \frac{p \cdot F}{n \cdot F} n_\sigma \right) n^\sigma F_\mu F_\nu \right\}, \quad (\text{A.3})$$

namely

$$I(2,2)_{\mu\nu\rho} = - \frac{\partial k_{\mu\nu}}{\partial n^\rho} Y(p) + \frac{1}{2} (n \cdot F) X(p) \left[g_\rho^\sigma - \frac{n^\sigma F_\rho}{n \cdot F} \right] \frac{\partial k_{\mu\nu}}{\partial F^\sigma}; \quad (\text{A.4})$$

according to the right hand side of Eq. (A.3), the required function $k(n, F)_{\mu\nu}$ is given by

$$k(n, F)_{\mu\nu} \equiv \frac{1}{2} \left\{ L g_{\mu\nu} + \frac{1}{n \cdot F} (p_\mu F_\nu + p_\nu F_\mu) - \frac{L}{n \cdot F} (n_\mu F_\nu + n_\nu F_\mu) - \frac{1}{(n \cdot F)^2} (R \cdot n) F_\mu F_\nu \right\}, \quad (\text{A.5})$$

with L and R_μ defined in Sec. 4.

By virtue of Eqs. (4.20)-(4.22) we can compute the needed derivatives

$$\begin{aligned} \frac{\partial k_{\mu\nu}}{\partial n^\rho} &= \frac{1}{2} \left\{ - \frac{L}{n \cdot F} (F_\mu g_{\nu\rho} + F_\nu g_{\rho\mu} + F_\rho g_{\mu\nu}) - \frac{1}{(n \cdot F)^2} (p_\mu F_\nu F_\rho + p_\nu F_\rho F_\mu + p_\rho F_\mu F_\nu) \right. \\ &\quad \left. + \frac{2L}{(n \cdot F)^2} (n_\mu F_\nu F_\rho + n_\nu F_\rho F_\mu + n_\rho F_\mu F_\nu) + \frac{2p \cdot n - 3n^2 L}{(n \cdot F)^3} F_\mu F_\nu F_\rho \right\}, \quad (\text{A.6}) \end{aligned}$$

and

$$\begin{aligned} \frac{\partial k_{\mu\nu}}{\partial F^\rho} &= \frac{1}{2} \left\{ \frac{1}{n \cdot F} (p_\mu g_{\nu\rho} + p_\nu g_{\rho\mu} + p_\rho g_{\mu\nu}) - \frac{L}{n \cdot F} (n_\mu g_{\nu\rho} + n_\nu g_{\rho\mu} + n_\rho g_{\mu\nu}) \right. \\ &\quad - \frac{R \cdot n}{(n \cdot F)^2} (F_\mu g_{\nu\rho} + F_\nu g_{\rho\mu}) - \frac{1}{(n \cdot F)^2} [p_\rho (n_\mu F_\nu + n_\nu F_\mu) + n_\rho (p_\mu F_\nu + p_\nu F_\mu)] \\ &\quad \left. + \frac{2L}{(n \cdot F)^2} (n_\mu F_\nu + n_\nu F_\mu) n_\rho + \frac{1}{(n \cdot F)^3} (2p \cdot n - 3n^2 L) F_\mu F_\nu n_\rho + \frac{n^2}{(n \cdot F)^3} F_\mu F_\nu p_\rho \right\}, \quad (\text{A.7}) \end{aligned}$$

and evaluate the contraction

$$\begin{aligned}
n^\tau \frac{\partial k_{\mu\nu}}{\partial F^\tau} &= \frac{1}{2(n \cdot F)} \{ (n \cdot R) g_{\mu\nu} - 2L n_\mu n_\nu + (p_\mu n_\nu + p_\nu n_\mu) \\
&- \frac{n^2}{n \cdot F} (p_\mu F_\nu + p_\nu F_\mu) + \frac{n^2}{(n \cdot F)^2} 3[p \cdot n - n^2 L] F_\mu F_\nu \\
&+ \frac{1}{n \cdot F} [-2p \cdot n + 3n^2 L] (F_\mu n_\nu + F_\nu n_\mu) \} , \tag{A.8}
\end{aligned}$$

so that Eq.(A.4) finally yields

$$\begin{aligned}
I(2,2)_{\mu\nu\rho} &= \frac{1}{2(n \cdot F)^2} Y(p) [(F_\mu g_{\nu\rho} + F_\nu g_{\rho\mu} + F_\rho g_{\mu\nu}) (p \cdot F) \\
&- \frac{2p \cdot F}{(n \cdot F)} (n_\mu F_\nu F_\rho + n_\nu F_\rho F_\mu + n_\rho F_\mu F_\nu) \\
&+ (p_\mu F_\nu F_\rho + p_\nu F_\rho F_\mu + p_\rho F_\mu F_\nu) - \frac{1}{n \cdot F} \left(2p \cdot n - \frac{3n^2 p \cdot F}{n \cdot F} \right) F_\mu F_\nu F_\rho] \\
&+ \frac{1}{4} X(p) \left[(p_\mu g_{\nu\rho} + p_\nu g_{\rho\mu} + p_\rho g_{\mu\nu}) - (n_\mu g_{\nu\rho} + n_\nu g_{\rho\mu} + n_\rho g_{\mu\nu}) \frac{p \cdot F}{n \cdot F} \right. \\
&+ (F_\mu g_{\nu\rho} + F_\nu g_{\rho\mu} + F_\rho g_{\mu\nu}) \left(\frac{n^2 p \cdot F}{(n \cdot F)^2} - \frac{p \cdot n}{n \cdot F} \right) \\
&+ (n_\mu F_\nu F_\rho + n_\nu F_\rho F_\mu + n_\rho F_\mu F_\nu) \frac{1}{(n \cdot F)^2} \left(p \cdot n - \frac{3n^2 p \cdot F}{(n \cdot F)} \right) \\
&- \frac{1}{n \cdot F} [p_\mu n_\nu F_\rho + p_\mu n_\rho F_\nu + p_\nu n_\rho F_\mu + p_\nu n_\mu F_\rho + p_\rho n_\mu F_\nu + p_\rho n_\nu F_\mu] \\
&+ \frac{2p \cdot F}{(n \cdot F)^2} (n_\mu n_\nu F_\rho + n_\nu n_\rho F_\mu + n_\rho n_\mu F_\nu) \\
&\left. - \frac{3}{(n \cdot F)^3} \left(n^2 p \cdot n - \frac{n^4 p \cdot F}{n \cdot F} \right) F_\mu F_\nu F_\rho + \frac{n^2}{(n \cdot F)^2} (p_\mu F_\nu F_\rho + p_\nu F_\rho F_\mu + p_\rho F_\mu F_\nu) \right] . \tag{A.9}
\end{aligned}$$

B Appendix

We show how one can use a generalization of Eq. (3.23) to calculate the finite part of a Feynman integral. We start from the integral

$$\int dq \frac{1}{(q-p)^2 q \cdot n} = I^{div} + I^{finite} , \tag{B.1}$$

whose value, in the light-cone gauge, is known[13] to be

$$I^{div}(1,1) = 2 \frac{p \cdot n^*}{n \cdot n^*} Y(p), \quad (B.2)$$

$$I^{finite}(1,1) = -2 i \pi^2 \frac{p \cdot n^*}{n \cdot n^*} \left\{ \ln \pi + \gamma_E - 1 + \ln \left(-2 p \cdot n \frac{p \cdot n^*}{n \cdot n^*} \right) \right\}, \quad (B.3)$$

where γ_E is Euler's constant. The finite part I^{finite} can be easily rewritten in the $\{n, F\}$ -formalism, since

$$\frac{p \cdot n^*}{n \cdot n^*} = \frac{p \cdot F}{n \cdot F}, \quad \text{for } n^2 = 0, \quad (B.4)$$

so that

$$I^{finite}(1,1) = -2 i \pi^2 \frac{p \cdot F}{n \cdot F} \left\{ \ln \pi + \gamma_E - 1 + \ln \left(-2 p \cdot n \frac{p \cdot F}{n \cdot F} \right) \right\}. \quad (B.5)$$

Eq. (3.22) can be readily generalized to include the finite part of an integral:

$$\begin{aligned} I^{finite}(m,2)_{\mu(k)\rho} &= - \frac{\partial I^{finite}(m,1)_{\mu(k)}}{\partial n^\rho} \\ &+ \left(\frac{\sigma - \rho}{D} \right) \left[g_\rho^\sigma - \frac{n^\sigma F_\rho}{n \cdot F} \right] \frac{\partial I^{finite}(m,1)_{\mu(k)}}{\partial F^\sigma}, \end{aligned} \quad (B.6)$$

which reduces in the light-cone gauge, where $\sigma = \rho$, to

$$I(m,2)_{\mu_1 \dots \mu_k \rho}^{finite} = - \frac{\partial I^{finite}(m,1)_{\mu_1 \dots \mu_k}}{\partial n^\rho}. \quad (B.7)$$

The finite part of $I(1,2)_\mu$ may then be found by taking the derivative of $I^{finite}(1,1)$ in (B.5) with respect to n^μ , i.e.

$$\begin{aligned} \text{finite part of } \int dq \frac{q_\mu}{(q-p)^2 (q \cdot n)^2} &= - \frac{\partial}{\partial n^\mu} I^{finite}(1,1) \\ &= -2 i \pi^2 \frac{p \cdot F}{(n \cdot F)^2} \left\{ F_\mu \left[\ln \pi + \gamma_E - \frac{3}{2} \right. \right. \\ &\quad \left. \left. + \ln \left(-2 p \cdot n \frac{p \cdot F}{n \cdot F} \right) \right] - \frac{1}{2} \frac{n \cdot F}{n \cdot p} p_\mu \right\}. \end{aligned} \quad (B.8)$$

The reader will appreciate the ease with which we were able to derive the finite part of the integral $\int dq q_\mu [(q-p)^2 (q \cdot n)^2]$ in the light-cone gauge.

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