THE SOURCES OF THE A AND B DEGENERATE STATIC VACUUM FIELDS

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Abstract- We attempt to a physical interpretation of some known static vacuum solutions of Einstein’s equations, namely, the A and B metrics of Ehlers and Kundt. All of them have axial symmetry, so they can be transformed to the Weyl form. In Weyl coordinates \( \ln \sqrt{-g_{44}} \) obeys a Laplace equation, and from this a source, called \emph{The Newtonian image source} can be identified. We use the image sources to interpret the metrics. The procedure is successful in some cases. In others it fails because the Weyl transform does not have reasonable properties at infinity.


1. Introduction

For many exact solutions of Einstein’s equations the physical interpretation is difficult and uncertain. On the simplest non-trivial vacuum solution, that the Schwarzschild, numerous papers have been written and it seems to the author of this work that there are still aspects of it which are puzzling. According to Bonnor [1], general relativity cannot be understood unless one can find the physics behind the exact solutions we know and the essence of the interpretation of their physical meaning is to understand their sources.

In this short paper we consider the physical interpretation of certain static and axially symmetric vacuum solutions of Einstein’s equations well known for a long time, namely, the A and B metrics of Ehlers and Kundt. An outstanding problem with these solutions is that, using their original coordinate, one cannot see the nature and the and location of the sources which could give rise to these gravitational fields. The only exception is the A1 case that represents the Schwarzschild solution. Hence in order to make proposals for the material sources, one should seek a more convenient representation for them.

Following an approach presented by Bonnor [2] and also by Bonnor and Martins [3] these solutions are transformed into the Weyl metric. The field equations for each of the transformed metrics include a Laplace’s equation from which one can, of course, identify a Newtonian gravitational potential. The sources of this potential have been called [4] Newtonian image sources and we shall use them to interpret the corresponding vacuum metrics. However, as pointed out in [2] and [3] this procedure has to be carried out with caution because the Newtonian potential can be misleading as a guide to the physical meaning of relativistic sources. In all cases considered we have found that the Newtonian potential refers to semi-infinite line masses (slm). Having converted the A and B metrics to the Weyl form we give the Newtonian image sources and these are depicted in a Figure. However in the A2 and B2 cases the interpretation of the vacuum space-times is incomplete, as we shall show by a study of the algebraic invariants of the Riemann tensor.

For sake of completeness, in section 2 we shall summarize some results concerning to Weyl metrics and semi infinite line masses as presented in [3]. In section 3 we give the Weyl transform of each of the A and B metrics and at that stage we are ready to attempt the interpretation of these metrics. The paper ends with a short conclusion.
2. Weyl Metrics and Semi Infinite Line Masses

Weyl Metrics

The metrics for a static, axially symmetric vacuum fields can be expressed in the Weyl form [5]

\[ ds^2 = e^{2\omega} \left( dz^2 + dr^2 \right) + r^2 e^{-2\mu} \, d\phi^2 - e^{2\mu} \, dt^2 \]  \hspace{2cm} (2.1)

where \( \omega \) and \( \mu \) are functions of \( z \) and \( r \).

The ranges of the coordinates will be assumed to be

\[-\infty < z < \infty, \quad r \geq 0, \quad 0 \leq \phi \leq 2\pi, \quad -\infty < t < \infty \]  \hspace{2cm} (2.2)

and we shall number them

\[ x^1 = z, \quad x^2 = r, \quad x^3 = \phi, \quad x^4 = t \]

points on \( \phi = 0 \) and \( \phi = 2\pi \) will be indetified.

Imposing the vacuum field equations \( R_{ij} = 0 \) we find that \( \mu = \sqrt{-g_{44}} \) satisfies a Laplace’s equation in cylindrical polar coordinates

\[ \mu_{,11} + \mu_{,22} + \frac{1}{r} \mu_{,2} = 0 \]  \hspace{2cm} (2.3)

where colon denotes ordinary partial derivative.

Once \( \mu \) is prescribed, \( \omega \) is determined (up to an arbitrary constant) by the remaining field equations

\[ \omega_{,1} + \mu_{,1} = 2r \mu_{,1} \mu_{,2} \quad \text{and} \quad \omega_{,2} + \mu_{,2} = \frac{1}{r} \left( \mu_{,2}^2 - \mu_{,1}^2 \right) \]  \hspace{2cm} (2.4)

From the non-zero components of the Riemann tensor, for the metric (2.1) one can form two distinct algebraic invariants which does not vanish identically in the case of a vacuum solution, namely

\[ L = R^{ijkl} R_{ijkl} = 8e^{-4\omega} \left( P^2 + Q^2 + S^2 + 2T^2 \right) \]  \hspace{2cm} (2.5)

\[ N = R^{ijkl} R_{klmn} R^{mn}_{\, ij} = 48e^{-6\omega} \left( QPS - T^2 \right) \]  \hspace{2cm} (2.6)
where

\[ P = \mu_{22} + \mu_{12}^2 + \mu_{11} - \mu_{21} \]

(2.7.a)

\[ Q = \mu_{11}^2 + \mu_{12}^2 - \frac{1}{r} \mu_{12} \]

(2.7.b)

\[ S = \mu_{11} + \mu_{12}^2 - \mu_{21} \]

(2.7.c)

\[ T = \mu_{21} + \mu_{11} - \mu_{12} - \mu_{21} \]

(2.7.d)

We now look for stress singularities (sometimes called *conical singularities*) on the z-axis. Such a singularity is not revealed by an examination of L and N. A sufficient condition for the absence of a conical singularity on the axis \( r = 0 \) is

\[ \lim_{r \to 0} (\mu + \omega) = 0 \]

(2.8)

Notice that the invariants L and N will be singular along the parts of the z-axis occupied by the material sources. However the Riemann tensor components (and so L and N, as mentioned before) may be well behaved even along those parts of the z-axis where (2.8) is not satisfied.

*Semi-Infinite Line Masses*

For a semi-infinite line mass of density \( \sigma \) per unity length, lying on the z-axis between \( z = z_i \) and infinity, the Newtonian potential is

\[ \psi = \sigma \log[R_i + \epsilon_i (z - z_i)] + \log C \]

(2.9)

where the gravitational constant is taken to unity, C is an arbitrary constant, and

\[ R_i = \sqrt{(z - z_i)^2 + r^2} \]

\[ \epsilon_i = +1, \text{ if the silm extends to } -\infty \]

\[ \epsilon_i = -1, \text{ if the silm extends to } +\infty \]

The general relativistic metric corresponding to this source is

\[ ds^2 = X_{i}^{-2} \left[ \left( \frac{X_i}{2R_i} \right)^4 (dz^2 + dr^2) + r^2 d\omega^2 \right] - X_i^2 \sigma dt^2 \]

(2.10)
where

\[ X_i = R_i + \epsilon_i(z - z_i) \]

The quadratic invariant \( L \) and the cubic invariant \( N \) of the Riemann tensor for the metric (2.10) are

\[
L = 12 G^2 \left( 1 - \frac{2}{3Y} \right) \quad \text{(2.12)}
\]

\[
N = -12 G^3 \left( 1 - Y \right) \quad \text{(2.13)}
\]

where

\[
G = 2^{2\sigma} C^2 \sigma (2\sigma - 1) R^{-2(1-\sigma)} \left[ \csc(\theta_i/2) \right]^{2(4\sigma^2 - 2\sigma + 1)}
\]

\[
Y = (1 - \sigma)(1 + 2\sigma) \cos^2(\theta_i/2)
\]

and \( \theta_i \) is the angle between the end of the silm \( z_i \) and an arbitrary point in the background Euclidean space with coordinates \((z, r, \phi)\). Notice that the length of \( z_iP \) is \( R_i \).

We should make some remarks about this Weyl metric before ending this section. As it is known, in weak static fields, \( \log \sqrt{-g_{tt}} \) can be interpreted as an approximate Newtonian potential of the gravitational field. Thus it seems reasonable to assume that (2.10) gives the space-time of a silm of line density \( \sigma \), if \( \sigma \) is small. However: i) For \( \sigma = \frac{1}{2} \) or \( \sigma = 0 \), (2.10) represents a flat space-time. ii) For \( \sigma \geq 1 \) it is misleading as a metric referring to a space-time containing a single silm. If \( \sigma > 1 \) it does not have reasonable properties at infinity (because \( L \) and \( N \) diverge as \( R_i \to \infty \)), indicating the presence of addditional sources. If \( \sigma = 1 \) it acquires an extra arbitrary constant \( C \) and assumes the form

\[
ds^2 = (C X_i)^{-2} \left\{ (X_i/2R_i)^4 \left( dz^2 + dr^2 \right) + r^2 d\phi^2 \right\} + (C X_i)^2 dt^2 \quad \text{(2.11)}
\]

In [3] Bonnor interpreted (2.11) as an infinite hollow cylinder with an applied gravitational field parallel to its axis. iii) For \( \sigma < 0 \) , (2.10) apparently refers to a silm with a negative mass density. However when \( \sigma = -1/2 \) it is a transform of Taub's plane metric.

### 3. The Sources of the A and B Metrics

The metrics we consider in this paper and the respective ranges of their coordinates were given by Ehlers and Kundt [6]. We shall list them below:

**A1: \( ds^2 = u^2 (dv^2 + \sin^2 v \, d\phi^2) + (1 - 2m/u)^{-1} \, du^2 - (1 - 2m/u) \, dt^2 \quad (3.1)\)**
where

\[ 0 \leq v \leq 2\pi, \quad 0 < 2m < u < \infty \quad \text{or} \quad 2m < 0 < u < \infty, \]  
\[ \text{(3.2)} \]

**A2**: 
\[ ds^2 = u^2(dv^2 + \sinh^2 v \, d\phi^2) + (2m/u - 1)^{-1}du^2 - (2m/u - 1)dt^2 \]  
\[ \text{(3.3)} \]

where

\[ 0 \leq v < \infty, \quad 0 < u < 2m, \]  
\[ \text{(3.4)} \]

**A3**: 
\[ ds^2 = u^2(dv^2 + v^2 \, d\phi^2) + u \, du^2 - u^{-1} \, dt^2 \]  
\[ \text{(3.5)} \]

where

\[ 0 \leq v < \infty, \quad 0 < u < \infty, \]  
\[ \text{(3.6)} \]

**B1**: 
\[ ds^2 = (1 - 2m/u)^{-1} \, du^2 + (1 - 2m/u) \, d\phi^2 + u^2 \, (dv^2 - \sin^2 v \, dt^2) \]  
\[ \text{(3.7)} \]

where

\[ 0 < v < \pi, \quad 0 < 2m < u < \infty \quad \text{or} \quad 2m < 0 < u < \infty, \]  
\[ \text{(3.8)} \]

**B2**: 
\[ ds^2 = (2m/u - 1)^{-1} \, du^2 + (2m/u - 1) \, d\phi^2 + u^2 \, (dv^2 - \sinh^2 v \, dt^2) \]  
\[ \text{(3.9)} \]

where

\[ 0 < v < \infty, \quad 0 < u < 2m, \]  
\[ \text{(3.10)} \]

**B3**: 
\[ ds^2 = u \, du^2 + u^{-1} \, d\phi^2 + u^2 \, (dv^2 - v^2 \, dt^2) \]  
\[ \text{(3.11)} \]

where

\[ 0 < u < \infty, \quad 0 < v < \infty. \]  
\[ \text{(3.12)} \]

In all cases \( 0 \leq \phi \leq 2\pi \) and 0 and 2\( \pi \) are inden ti/n0ced/; the range of \( t \) is unrestricted.

It will be noticed that metrics B1, B2, B3 arise from A1, A2, A3 respectively by the complex tranformation

\[ \phi \rightarrow i \, t, \quad t \rightarrow i \, \phi \]  
\[ \text{(3.13)} \]

The transformation of these metrics to hte Weyl form is straightforward and we outline just the case A1 as an example. It will take the form

\[ z = z(u, v), \quad r = r(u, v), \quad \phi = \phi, \quad t = t. \]  
\[ \text{(3.14)} \]
Thus, the product $g_{33}g_{44}$ is invariant and so (2.1) gives at once

$$r = u \left(1 - \frac{2m}{u}\right)^{1/2} \sin v$$

(3.15)

The transformation for $z$ is got by using the fact that the Weyl sub-metric for $z$ and $r$ is conformally euclidean. By inspection one finds

$$z = (u - m) \cos v$$

(3.16)

The ranges (3.2) of $u$ and $v$, specified for $A1$ yield ranges (2.2) for $z$ and $r$.

The use of (3.15) and (3.16) gives the the Weyl of $A1$, namely

$$ds^2 = \frac{(R_1 + R_2 + 2m)^2}{4R_1 R_2} (dz^2 + dr^2) + r^2 \frac{R_1 + R_2 + 2m}{R_1 + R_2 - 2m} d\phi^2 - \frac{R_1 + R_2 - 2m}{R_1 + R_2 + 2m} dt^2$$

(3.17)

where

$$R_1 = \sqrt{(z + m)^2 + r^2} \text{ and } R_2 = \sqrt{(z - m)^2 + r^2}$$

(3.18)

This result is well known [7]: $A1$ is the Schwarzschild solution and (3.17) is its Weyl transform.

By means of some trigonometry we can show

$$\left[ \frac{R_1 - (z - z_1)}{R_2 - (z - z_2)} \right] = \left[ \frac{R_1 + R_2 - (z_2 - z_1)}{R_1 + R_2 + (z_2 - z_1)} \right]$$

(3.19)

and thus

$$\mu = 1/2 \log \left[ R_1 - (z + m) \right] - 1/2 \log \left[ R_2 - (z - m) \right]$$

(3.20)

In view of (2.9) we see that the Newtonian sources of the $A1$ metric are two overlapping semi-infinite rods on the $z$-axis, one of line density $\sigma = +1/2$ extending from $z_1 = -m$ to $+\infty$ and the other of line density $\sigma = -1/2$ extending from $z_2 = +m$ to $+\infty$. They make up a finite rod with ends at $z_1$ and $z_2$ and density $\sigma = +1/2$ if $m > 0$ and density $\sigma = -1/2$ if $m < 0$. The total mass of the rod is therefore $m$. There is no other physical singularities on (3.17). The fact that the Newtonian image of the sources - represented by the potential $\mu = \log \sqrt{-g_{44}}$ in (3.17) - is a finite rod of length $2m$ (as shown in the Figure) whereas the relativistic source of $A1$ is of course, a spherical particle, serves as a warning against too facile use of this Newtonian image.
Let us turn now to the B1 metric. It is obtained from A1 by the transformation (3.13) and it is clear that the Weyl form of B1 will be obtained from the Weyl form of A1 by the same transformation. We therefore transform (3.17) by (3.13) with result

\[
\begin{align*}
    ds^2 &= \frac{(R_1 + R_2 + 2m)^2}{4 R_1 R_2} (dz^2 + dr^2) + \frac{R_1 + R_2 - 2m}{R_1 + R_2 + 2m} \phi^2 - r^2 \left( \frac{R_1 + R_2 + 2m}{R_1 + R_2 - 2m} \right) dt^2 \quad (3.21)
\end{align*}
\]

which, with the use of (3.19) and the identity

\[
    r^2 = [R_j + (z - z_j)] [R_j - (z - z_j)]
\]

becomes

\[
\begin{align*}
    ds^2 &= \frac{(R_1 + R_2 + 2m)^2}{4 R_1 R_2} (dz^2 + dr^2) + \frac{r^2 d\phi^2}{[R_1 + (z + m)][R_2 - (z - m)]} - \\
    &\quad - [R_1 + (z + m)][R_2 - (z - m)] dt^2.
\end{align*}
\]

(3.23)

Thus, for the B1 metric

\[
    \mu = \frac{1}{2} \log [R_1 + (z + m)] + \frac{1}{2} \log [R_2 - (z - m)]
\]

(3.24)

The Newtonian image of the sources in this case is seen to consist of two semi infinite line masses, each of line density \(\sigma = 1/2\), one extending from \(z_1 = -m\) to \(-\infty\) and the other from \(z_2 = +m\) to \(+\infty\).

If \(m > 0\) there is a empty section of the z-axis in \(-m < z < m\) but as it stands the metric (3.23) does not satisfy the regularity condition (2.8) there. This can be remedied if we introduce the transformation

\[
    \tilde{\phi} = (4m)^{-1} \phi, \quad \tilde{t} = (4m) t
\]

(3.25)

Doing this and dropping the bar over \(\phi\) and \(t\) we shall have the Weyl transform of the B1 metric

\[
\begin{align*}
    ds^2 &= \frac{(R_1 + R_2 + 2m)^2}{4 R_1 R_2} (dz^2 + dr^2) + \frac{16m^2 r^2 d\phi^2}{[R_1 + (z + m)][R_2 - (z - m)]} - \\
    &\quad - \frac{1}{16m^2} [R_1 + (z + m)][R_2 - (z - m)] dt^2.
\end{align*}
\]

(3.26)
the ranges of the coordinates being as in (2.2). It applies whether \( m \) is positive or negative, but if \( n = 0 \) there is no regular part of the \( z \)-axis. (See Figure below.)

In both \( A1 \) and \( B1 \) metrics the Riemann tensor invariants \( L \) and \( N \) are infinite on the line sources and nowhere else. Moreover \( L \) and \( N \) tend to zero at coordinate infinity away from the line sources, and the proper distances corresponding to coordinate infinity are infinite. Thus the semi infinite line masses shown in the Figure are the only sources of the \( A1 \) and \( B1 \) metrics. (For a contrasting case see below.)

The Weyl transforms of the remaining \( A \) and \( B \) metrics listed in the beginning of this Section can be obtained by a simple adaption of the foregoing method. We shall give them bellow, fulfilling the regularity condition (2.8) by a transformation of the \( \phi \) and \( t \) coordinates wherever possible. \( R_1 \) and \( R_2 \) are as defined in (3.18). In each case we state the transformation of the coordinates required to cast each metric into its Weyl form. We shall also state the Newtonian sources, and these are depicted in Figure mentioned above.

\[ A2 \text{ metric } (m > 0) \]

Weyl Form:

\[
ds^2 = \frac{[R_2 - R_1 + 2m]^2}{4 R_1 R_2} (dz^2 + dr^2) + \frac{[R_1 + (z + m)]}{[R_2 - (z - m)]} r^2 d\phi^2 - \frac{[R_2 - (z - m)]}{[R_1 + (z - m)]} dt^2 \]

Transformation required:

\[
r = u(2m/u - 1)^{1/2} \sinh v, \quad z = (u - m) \cosh v, \quad \phi = \psi, \quad t = t
\]

Sources: semi infinite line mass with \( \sigma = +1/2 \), from \( z = +m \) to +\( \infty \); semi infinite line mass with \( \sigma = -1/2 \), from \( z = -m \) to \( -\infty \).

The \( z \)-axis is regular on \( -m < z < m \).

\[ B2 \text{ metric } (m > 0) \]

Weyl Form:

\[
ds^2 = \frac{[R_2 - R_1 + 2m]^2}{4 R_1 R_2} (dz^2 + dr^2) + \frac{16 m^2 r^2}{[R_1 + (z + m)][R_2 - (z - m)]} d\phi^2 - \frac{1}{16 m^2 [R_1 + (z + m)][R_2 + (z - m)]} dt^2.
\]

\[ 3.29 \]
\begin{align}
    r &= u(2m/u - 1)^{1/2} \sinh v, \quad z = (u - m) \cosh v, \quad \phi = \phi, \quad t = t \quad (3.30)
\end{align}

Sources: semi infinite line mass with \( \sigma = 1/2 \) from \( z = m \) to \(-\infty\); semi infinite line mass with \( \sigma = +1/2 \) from \( z = -m \) to \(-\infty\).

In the A2 and B2 metrics the Kretschmann scalar \( L \) is infinite at the location of the line sources as expected. However, one can show that \( L \) does not tend to zero at some other regions of the coordinate infinity (e.g. as \( r \to 0 \) on \( z = 0 \)). Hence other sources (or fields at infinity) must be present. Thus the Newtonian image sources are not the only determinant of the physics of these metrics, and the interpretation of them by means of the Weyl transform is unsatisfactory, or at best incomplete. In the B2 metric the image sources together constitute in the region \( z < -m \) a semi infinite line mass with \( \sigma = 1 \) so this conclusion is not surprising in the light of Section 2. However, it is not clear why the A2 metric should not be determined by the Newtonian image sources.

**A3 metric**

Weyl form:
\begin{align}
    ds^2 &= \frac{(r + z)^2}{4R} (dz^2 + dr^2) + \frac{1}{2} r^2 (R + z) \, d\phi^2 - 2 (R + z)^{-1} \, dt^2 \quad (3.31)
\end{align}

where
\begin{align}
    R &= (z^2 + r^2)^{1/2} \quad (3.32)
\end{align}

Transformation required:
\begin{align}
    r &= v \sqrt{u}, \quad z = u - \frac{v^2}{4}, \quad \phi = \phi, \quad t = t \quad (3.33)
\end{align}

Source: semi infinite line mass with \( \sigma = -1/2 \) on \( z < 0 \).

The positive \( z \)-axis is regular. (3.31) is, but for a trivial scale transformation the same as (2.10) with \( \sigma = -1/2, \epsilon = +1, z_i = 0 \). It is also isometric with Taub’s general plane symmetric vacuum metric, as given in [7], eqn. (13.30).
Weyl form:

\[ ds^2 = \frac{(R + z)^2}{4R} (dz^2 + dr^2) + \frac{2r^2}{(R + z)^2(R - z)} d\phi^2 - \frac{1}{2} \frac{(R + z)^2(R - z)}{n_28} dt^2 \]  

(3.34)

Transformation required:

\[ r = v\sqrt{u}, \quad z = u - \frac{v^2}{4}, \quad \phi = \phi, \quad t = t \]  

(3.35)

Sources: semi infinite line mass with \( \sigma = 1/2 \) on \( z > 0 \); semi infinite line mass with \( \sigma = 1 \) on \( z < 0 \).

The entire \( z \)-axis is singular. The Riemann tensor invariants L and N tend to zero at infinity except at the ends of the \( z \)-axis. Hence the semi infinite line masses mentioned seem to be the only sources.

**Conclusion**

Our attempt to interpret The A and B metrics by means of the corresponding Weyl transforms has met with mixed success. It seems adequate for the interpretation of A1, B1, A3 and B3 cases. A2 and B2 are not satisfactorily explained by their Weyl transforms because the latter do not have reasonable asymptotic properties.

One of the problems in interpretations metrics arises from the coordinate freedom inherent in General Relativity. A singularity interpreted as a semi infinite line mass in one system of coordinate may become quite different in another (e.g. A3 metric (3.31) is isometric with Taub’s plane-symmetric vacuum metric, as mentioned before). Thus we should try to find another system of coordinate in which physical interpretation for both A2 and B2 might arise in a natural way. The existence of this possible physical interpretation will be presented in a future paper.

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References

FIGURE CAPTION

The disposition of line sources generating the A and B metrics, according to their Weyl transforms. In the A2 and B2 solutions additional sources seem to be present (see text).