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**CONSERVATION LAWS IN NORMAL METALS:
LUTTINGER LIQUID VS. FERMI LIQUID**

CONSERVATION LAWS IN NORMAL METALS: LUTTINGER LIQUID VS. FERMİ LIQUID

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Abstract

The structure of Fermi and Luttinger liquids is compared, emphasizing the crucial role of asymptotic conservation laws. The crossover between both liquids as a function of continuous dimensionality is analyzed. Exploiting the conservation laws, we sum up perturbation theory to all orders. Above one dimension, the Fermi liquid phase turns out to be stable with respect to regular residual interactions.

1. Introduction

The unusual and unexpected properties of high temperature superconductors have led to a profound reinvestigation of basic concepts in the theory of metals. The high superconducting transition temperature is hard to obtain from phonon-induced pairing, and the coherence length is too short for applying simple BCS mean field theory, whatever the pairing mechanism may be. Even more striking is the behavior of the normal metallic phase in these materials, which does not fit in a simple Fermi liquid type description. Although photoemission experiments yield convincing evidence for the existence of a large Fermi surface in these materials, many transport properties are incompatible with a Fermi liquid picture of low-lying excitations. The most prominent among many other quantities showing non-Fermi-liquid behavior is the electrical resistivity, whose temperature dependence deviates significantly from the T^2 law predicted by Fermi liquid theory. The layer structure and the giant anisotropy observed in various properties like the resistivity indicate that the normal phase is governed by electrons confined essentially to two dimensions. The vicinity to an antiferromagnetic

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Mott insulating phase reveals the importance of electron-electron interactions. These observations have stimulated speculations on the existence of non-Fermi-liquid metallic phases in two-dimensional interacting electron systems.[1, 2]

The problem of the normal phase is intricate and still highly controversial for several reasons. Experimentally entering into the "strange" metallic phase by lowering the temperature or varying the carrier concentration, there is no sharp phase transition or singularity revealing the appearance of a new type of state. Theoretically the Fermi liquid phase is stable within perturbation theory in 2D as in 3D, and the quasiparticle scattering rate in 2D acquires only a logarithmic correction to the quadratic energy dependence valid in 3D.[3] Needless (and easy) to say, these theoretical results may be of course invalidated by hitherto unknown non-perturbative effects.

In this context much interest has been attracted by one-dimensional systems,[4, 5] where Fermi liquid theory is known to break down, and is replaced by a different class of metallic low-energy behavior, the so-called Luttinger liquid.[6] Both Fermi and Luttinger liquids can be understood within the renormalization group approach, which is probably the most suitable language for discussing also other types of low energy behavior. In this paper we will review the structure of Fermi and Luttinger liquids within this common framework, emphasizing the role of asymptotic conservation laws. We show that the related Ward identities restrict and help classify possible new scenarios for low-energy behavior of interacting electrons, as long as it is governed by excitations close to a Fermi surface. It is due to conservation laws that in both Fermi and 1D Luttinger liquids a few basic assumptions determine the entire low-energy behavior in terms of a relatively small set of parameters.[7] Conservation laws also make possible a detailed study of the dimensional crossover between both liquids,[8] showing the stability of the Fermi liquid phase with respect to regular interactions in dimensions above one.

2. Fermi liquid

Luttinger and Fermi liquid theory [9] are in many respects similar. Both describe the low-energy asymptotics of a large class of metals in terms of a small set of parameters, yielding non-trivial relations between different quantities for interacting systems, without being confined to weak coupling. A common framework embracing both theories is provided by Wilson's renormalization group in momentum space.[10, 11] Integrating out states which are far away from the Fermi surface, one ends up with an effective low energy theory defined on a thin shell in k -space. Neglecting irrelevant terms, the leading low energy behavior is then described by an effective Hamiltonian of the form

$H = H_0 + H_I$, defined on the states within the shell, where

$$H_0 = \sum_{\mathbf{k}, \sigma} v_{\mathbf{k}} k_r a_{\sigma}^{\dagger}(\mathbf{k}) a_{\sigma}(\mathbf{k}) \quad (1a)$$

and

$$H_I = V^{-1} \sum_{\mathbf{k}\mathbf{k}'\mathbf{q}} \sum_{\sigma\sigma'} g_{\mathbf{k}\mathbf{k}'}^{\sigma\sigma'}(\mathbf{q}) a_{\sigma}^{\dagger}(\mathbf{k} + \mathbf{q}) a_{\sigma}(\mathbf{k}) a_{\sigma'}^{\dagger}(\mathbf{k}' - \mathbf{q}) a_{\sigma'}(\mathbf{k}') \quad (1b)$$

Here k_r is the distance of $\mathbf{k}$ from the Fermi surface, defined positively outside and negatively inside; V is the volume of the system. All momentum sums are restricted by the condition that the particles lie in a thin shell of width $\Lambda \ll k_F$ around the Fermi surface. We assume spherical symmetry for simplicity, i.e. we have $k_r = |\mathbf{k}| - k_F$, and the Fermi velocity $v_{\mathbf{k}} = v_F$ is constant all over the Fermi surface.

In a *Fermi liquid* all momentum transferring scattering processes become irrelevant at low energy, i.e. only Hartree interactions survive. The fixed point Hamiltonian describing the scale invariant regime of a Fermi liquid is therefore given by Landau's energy functional

$$H_F = H_0 + V^{-1} \sum_{\mathbf{k}\mathbf{k}'} \sum_{\sigma\sigma'} f_{\mathbf{k}\mathbf{k}'}^{\sigma\sigma'} \delta n_{\mathbf{k}\sigma} \delta n_{\mathbf{k}'\sigma'} \quad (2)$$

This implies the existence of asymptotically stable fermionic excitations, the quasiparticles, which are subject only to Hartree interactions, parametrized by the Landau function $f_{\mathbf{k}\mathbf{k}'}^{\sigma\sigma'}$. The wave function renormalization Z has a finite limit, and the single particle propagator acquires the polar, scale-invariant asymptotic form $G(\mathbf{k}, \omega) = Z/(\omega - v_F k_r)$, which implies a finite discontinuity Z in the momentum distribution function at the Fermi surface, a finite density of single-particle excitations, and a peak of the form $Z\delta(\omega - v_F k_r)$ in the spectral function. The charge/spin density-density response for small momenta and frequencies is given exactly by a random phase approximation with the Landau function $f_{\mathbf{k}\mathbf{k}'}^{\sigma\sigma'}$ replacing bare couplings or, equivalently, by the Boltzmann-Landau kinetic equation where a slow time and space variation in the Hartree terms in equ. (2) is allowed. In particular, Fermi liquids are stable metals with a finite interaction dependent compressibility and spin susceptibility, and a Drude peak (not renormalized by Z) in the electrical conductivity.

Note that the fixed point Hamiltonian H_F describes only the *leading* low-energy behavior, which is scale invariant. Several important properties, such as the life-time of quasiparticles in Fermi liquids, are associated with subleading corrections to scaling, generated by irrelevant residual interactions which scale to zero only in the zero energy limit. In particular, the temperature dependence of the electrical conductivity is

obtained only on a subleading level (at the fixed point the dc conductivity is infinite in pure metals). In a Fermi liquid the temperature dependence of the dc-conductivity can usually be related to the life-time of quasiparticles. In general one must keep in mind that the conductivity is a two-particle correlation function, related to the charge density response, which may be completely independent from single particle properties, and have different renormalizations.

Strictly speaking, a Fermi liquid fixed point is actually never reached. As shown by Kohn and Luttinger [12], at sufficiently low energy the couplings generating Cooper processes always develop some attractive channel in $d > 1$, and thus diverge, even if the bare interactions are all repulsive. Hence, if not anticipated by another instability, a Cooper instability occurs. However, in the absence of an additional direct pairing mechanism, the corresponding energy scale T_c is usually very small, orders of magnitude below the Fermi energy.[12] In these cases, for low energies above T_c , one really observes scaling behavior governed by a Fermi liquid fixed point.

In $d \geq 2$ dimensions a breakdown of Fermi liquid behavior can occur only if the effective couplings $g_{\mathbf{k}\mathbf{k}'}^{ss'}(\mathbf{q})$ in (1b) have sufficiently strong singularities, which may be either due to singularities in the bare couplings, or generated in the process of integrating out high energy degrees of freedom. The latter possibility would for instance be realized if a broken symmetry ground state would substitute the Fermi liquid. For regular or weakly singular $g_{\mathbf{k}\mathbf{k}'}^{ss'}(\mathbf{q})$ phase space restrictions due to the Pauli principle near the Fermi surface make scattering processes irrelevant at low energy. The situation is different in $d = 1$.

3. One-dimensional Luttinger liquid

The breakdown of Fermi liquid theory in *one-dimensional* interacting Fermi systems is already signalled by second-order perturbation theory: the perturbative contributions to the quasiparticle weight diverge logarithmically at the Fermi surface. The problem of treating these divergences has first been solved by a perturbative renormalization group approach, in the so-called "g-ology".[4] This model incorporates the generic low-energy dynamics of a 1D Fermi system. Its kinetic energy term, H_0 , describes a linear band, with a cutoff Λ . The interaction H_I generates various types of scattering processes, as listed in Figure 1: small momentum transfer processes (H_2 and H_4), backscattering (H_1) and umklapp scattering (H_3). The corresponding coupling constants $g_i^{ss'}$, $i = 1, \dots, 4$, which led to the term "g-ology", may depend on the relative spin projection of the scattering particles, but not on their distance from the Fermi surface. Due to

momentum conservation, umklapp processes (H_3) can be important only if $4k_F$ is equal to a reciprocal lattice vector, as is the case, for example, for a one-band lattice model at half-filling.

Figure 1.
Interaction terms
in the g-ology
model; "-" and
"+" indicate the
left and right
Fermi point,
respectively.

A g-ology Hamiltonian is the general outcome of a partial Wilson renormalization procedure applied to a one dimensional microscopic model, such as the 1D Hubbard model. It is a 1D version of the low-energy Hamiltonian (1), extended by allowing for umklapp scattering. A simple dimensional analysis indicates that non-linear dependences in the band structure and k-dependences in the couplings $g_1 \dots g_4$ scale rapidly to zero under renormalization.

The infrared (low energy) divergences encountered in perturbative calculations of correlation functions for the g-ology model require a wave function renormalization Z and renormalized coupling constants $\tilde{g}_i$. Depending on the values of the bare couplings, the renormalized couplings flow either to infinity, signalling the opening of a gap for charge or spin excitations, or to the *Luttinger model*, [13] i.e. $\tilde{g}_1, \tilde{g}_3 \rightarrow 0$, with a line of fixed points for $\tilde{g}_2$ and $\tilde{g}_4$. The leading asymptotic low energy behavior of one-dimensional metals with gapless charge and spin excitations is therefore generically described by the Luttinger model $H_L = H_0 + H_I$, where H_0, H_I are of the form (1), with

$$g_{\mathbf{k}\mathbf{k}'}^{\sigma\sigma'}(\mathbf{q}) = \begin{cases} g_1^{\sigma\sigma'} & \text{for } |\mathbf{q}| < \Lambda \text{ and } \mathbf{k}, \mathbf{k}' \text{ parallel} \\ g_2^{\sigma\sigma'} & \text{for } |\mathbf{q}| < \Lambda \text{ and } \mathbf{k}, \mathbf{k}' \text{ anti-parallel} \\ 0 & \text{elsewhere} \end{cases} \quad (3)$$

In addition to the usual conservation of charge and spin, the Luttinger model satisfies another peculiar conservation law: due to the absence of large momentum transfers, charge and spin are conserved separately on each Fermi point. This property makes possible an exact calculation of all correlation functions, and has important physical consequences, in particular the cancellation of infrared divergences in the density response functions, and the marginality of scattering processes with small momentum transfer.[7, 14] Technically speaking, the Ward identities associated with the various conservation laws provide a complete system of equations for the correlation functions of the Luttinger model.[15, 16] Alternatively the model may be solved by the bosonization technique, where the fermionic operators are represented in terms of (bosonic) density-fluctuation operators.[6, 17]

The resulting "Luttinger liquid" behavior can be parametrized in terms of four interaction-dependent parameters, namely the velocities v_c and v_s associated with the conserved currents corresponding to total charge and spin conservation, and the velocities $\tilde{v}_c$ and $\tilde{v}_s$ corresponding to conservation of the charge and spin difference between the two Fermi points.[6, 7]

The charge density response function has the form

$$C(\mathbf{q}, \omega) = \frac{2}{\pi} v_c \frac{q^2}{\omega^2 - (u_c q)^2} \quad (4)$$

for small $\mathbf{q}$ and ω , where the velocity u_c is related to v_c and $\tilde{v}_c$ by

$$u_c = (v_c \tilde{v}_c)^{1/2} \quad (5)$$

The poles in $\omega = \pm u_c |\mathbf{q}|$ imply the existence of gapless charge-density modes with velocity u_c in the system. An analogous result holds for the spin-density response. The compressibility is thus obtained as $\kappa = 2/\pi \tilde{v}_c$, the spin susceptibility as $\chi = 2/\pi \tilde{v}_s$, and the electrical conductivity has a Drude peak $\text{Re}\sigma(\omega) = 2v_c \delta(\omega)$. Hence, the Luttinger liquid also (as the Fermi liquid) is a *stable metal* with finite interaction-dependent compressibility and (Pauli-type) spin susceptibility. The (bosonic) charge and spin density modes are the only elementary excitations of the system, and lead to a T-linear specific heat $c_V = \frac{\pi}{6}(u_c^{-1} + u_s^{-1})T$.

So far, the interaction has led merely to a finite renormalization of parameters with respect to the non-interacting case, as in the Fermi liquid picture. By contrast, properties associated with single particle excitations are drastically altered. The small- $\mathbf{q}$ scattering processes generate two distinct singular effects, namely i) anomalous wave function renormalization and ii) spin-charge separation. The former destroys the jump in the momentum distribution function at the Fermi surface, giving rise to a power law behavior

$$n_{\mathbf{k}} - n_{\mathbf{k}_F} \propto (|\mathbf{k}| - k_F)^\eta \quad (6)$$

where $\eta > 0$ is a non-universal (i.e. interaction dependent) exponent; in addition, the density of single particle excitations vanishes at low energy as

$$N(\omega) \propto \omega^\eta \quad (7)$$

In spin-rotationally invariant systems, the exponent η is uniquely determined by the velocities v_c and $\tilde{v}_c$ via

$$4\eta = (v_c/\tilde{v}_c)^{1/2} + (\tilde{v}_c/v_c)^{1/2} - 2 \quad (8)$$

Spin-charge separation means that spin and charge associated with an electron inserted in the system propagate with two different velocities, u_s and u_c , respectively. Correspondingly, the single particle spectral function has two peaks for each momentum $\mathbf{k}$, one in $\omega = u_s k_x$, the other in $\omega = u_c k_x$, [18, 19] indicating the strong mixing of single particles with the collective modes. Within the Ward identity approach, this is obtained by summing the scattering processes with collective modes, which appear in the effective (RPA-like) interaction, to all orders.

Although the jump in $n_{\mathbf{k}}$ has disappeared, the low energy behavior is still determined by excitations close to the Fermi surface of the non-interacting system, and the single-particle properties are singular near k_F . In this sense the interacting system has still a Fermi surface obeying Luttinger's theorem, i.e. with a volume unaltered by the interactions.

Luttinger liquid behavior is not confined to special weak-coupling models but may also govern strongly interacting systems. Although in the latter case a perturbative calculation of the correlation functions is not adequate, the low-energy properties are still uniquely characterized by a small number of parameters, which are directly related to simple physical quantities. This leap beyond weak coupling, which follows the spirit of Fermi liquid theory, has been pioneered by Haldane [6], who also introduced the suggestive term "Luttinger liquid". Recently it has been found that the 1D Hubbard

model, which is exactly soluble by the Bethe ansatz method [20], is probably a Luttinger liquid for any coupling strength, except for half-filling.[21, 22, 23, 24] This conjecture is well established for weak coupling, while for strong coupling it is supported by the validity of Luttinger liquid relations,[21], the absence of non-analyticities in the exact ground state energy at finite coupling,[22], by studies of the infinite coupling limit,[25], as well as by numerical evaluations of finite systems.[26]

4. Dimensional Crossover

Above 1D, but below two dimensions, the small momentum transfer scattering processes maintain their dominance, suggesting that a summation procedure to all orders in the couplings may still be performed. The terms producing a Luttinger liquid in 1D would thus be automatically included. Below two dimensions the dominant low energy scattering processes indeed obey an asymptotic conservation law, which extends to higher dimensions the separate conservation of charge and spin on each point of the Fermi surface, discussed for 1D systems in the preceding section. As in 1D, the leading scattering processes can therefore be summed to all orders in the coupling constants. We are thus able to evaluate the crossover from 1D Luttinger liquid behavior to Fermi liquid behavior in higher dimensions as a function of continuous dimensionality.[8] In the case of *regular* interactions we find that the mixing with collective modes is not strong enough to suppress single particle excitations at the Fermi surface, and the quasi-particles are asymptotically well defined in any dimension above one, but the life-time and other subleading corrections deviate from the conventional Fermi liquid behavior known from 3D systems in any dimension below two.

As before, we assume that high energy states have already been integrated out, such that we end up with the effective Hamiltonian (1) describing low-lying excitations close to the Fermi surface. We consider a rotationally invariant system with a spherical Fermi surface and a constant Fermi velocity v_F for simplicity.

The continuation of the theory to non-integer dimensions is obtained as usual by analytic continuation of Feynman diagrams, defined for general integer d , in the complex d -plane. For our purposes it will be sufficient to continue momentum integrals of functions $f(k)$ which depend on k only via $|k|$ and an angle θ between k and another momentum which is fixed. In these cases one can use,

$$\int d^d k \dots = S_{d-1} \int_0^\infty |k|^{d-1} \int_0^\pi d\theta (\sin \theta)^{d-2} \dots \quad (9)$$

where $S_d = 2\pi^{d/2}/\Gamma(d/2)$ is the surface of the d -dimensional unit sphere. In the limit $d \rightarrow 1$ one has $S_{d-1} \sim d-1$, and thus $S_{d-1}(\sin \theta)^{d-2} \rightarrow \delta(\theta) + \delta(\theta - \pi)$ as expected.

We begin our analysis by calculating the self energy to second order in a constant coupling g acting between opposite spins. Typical results for the imaginary part $\text{Im}\Sigma(p, \xi)$ are shown in Fig. 2. In 3D we observe the well-known quadratic energy dependence, $\text{Im}\Sigma \propto \xi^2$, without any special feature at $\xi \sim \xi_p$. For $1 < d \leq 2$ contributions of order ξ^2 are superposed by larger p -dependent terms, which are generated by forward (small momentum transfer q) and Cooper scattering processes. In $d < 2$, for small $p_r = |p| - k_F$ and ξ , the self energy scales as $\text{Im}\Sigma(p, \xi) = p_r^d \text{Im}\tilde{\Sigma}(\xi/p_r)$, and diverges in $\xi = \xi_p$ as

$$|\text{Im}\Sigma(p, \xi)| \sim C_d g^2 k_F^{d-1} v_F^{-(d+1)} \xi^2 |\xi - \xi_p|^{d-2} \quad (10)$$

where C_d is a constant depending only on dimensionality. This singularity is exclusively due to forward scattering of particles with opposite spin and almost parallel momenta. Forward scattering of particles with antiparallel momenta yields a contribution proportional to $(\xi - \xi_p)^d \theta(\xi - \xi_p)$ in $d < 2$.

Figure 2 $\text{Im}\Sigma(p, \xi)$ from second order perturbation theory as function of ξ for fixed $p_r = 0.1k_F$ in dimensions $d = 1.5, 2$ and 3 ($k_F = v_F = 1$, arbitrary units on y-axis).

In $d < 2$, the single particle propagator G is drastically affected by the contributions of forward scattering to the second order self-energy. Let us consider the 1D case first. In $d = 1$, forward scattering between particles with opposite momenta yields

a contribution proportional to $(\xi - \xi_p)\theta(\xi - \xi_p)$ to $\text{Im}\Sigma$, which, via Kramers-Kronig, yields a real part proportional to $(\xi - \xi_p)\log(\xi - \xi_p)$. This leads to a wave function renormalization $Z_p = (1 - \partial\Sigma/\partial\xi)^{-1} \sim 1/\log(\xi - \xi_p) \rightarrow 0$ as $\xi \rightarrow \xi_p$, which is the well-known perturbative signal for the breakdown of Fermi liquid theory in one dimension.[1] Forward scattering between particles with parallel momenta does not contribute to the wave function renormalization, but nevertheless destroys the quasiparticle pole in the propagator, leading to separation of spin and charge degrees of freedom.[1] In contrast to common wisdom this latter effect also has a clear perturbative signal: In $d \rightarrow 1$, (10) reduces to $|\text{Im}\Sigma| \sim (g^2/8\pi v_F^2)\xi^2\delta(\xi - v_F p_r)$, yielding, by Kramers-Kronig, a real part $\text{Re}\Sigma \sim (g^2/8\pi)(p_r^2/(\xi - v_F p_r))$. Inserting this into $G = (\xi - \xi_p - \Sigma)^{-1}$ one obtains a propagator which has two poles instead of one, i.e. the spectral function becomes a sum of two δ -functions with weight 1/2 each.

In $1 < d < 2$ the perturbatively calculated wave function renormalization is finite, but the forward scattering processes of particles with almost parallel momenta still have dramatic consequences: $\Sigma(p, \xi)$ has an algebraic divergence proportional to $(\xi - \xi_p)^{d-2}$ for $\xi \rightarrow \xi_p$, leading to two well-separated peaks of comparable weight in the spectral function. Their width is finite in $d > 1$, but smaller than the distance between the two. Hence, second order perturbation theory seems to indicate destruction of the quasiparticle pole due to forward scattering of particles with almost parallel momenta in any dimension below two! However, the divergence found in Σ clearly forces us to go beyond perturbation theory even for weak coupling constants.

As a second step, we calculate the self-energy within the random phase approximation (RPA), expecting a smoothing of singularities. The RPA self-energy is obtained as usual from convoluting the bare electron propagator with the effective interaction D between parallel spins, which for a constant coupling g between opposite spins is given by $D(q) = g^2\Pi_0(q)/[1 - g^2\Pi_0^2(q)]$, $q = (\mathbf{q}, \omega)$, where $\Pi_0(q)$ is the bare particle-hole bubble. For small $\mathbf{q}$ and ω , $D(q)$ depends only via $\omega/|\mathbf{q}|$ on q , in any dimension, and has an undamped pole associated to a charge density mode (zero sound) at $\omega = u_c|\mathbf{q}|$, where u_c is a velocity larger than v_F . Close to one dimension, D also has a damped pole at $\omega = u_s|\mathbf{q}|$, $u_s < v_F$, which becomes sharp only in $d = 1$.

In $d < 2$ the leading contributions to $\text{Im}\Sigma$ still scale as $\text{Im}\Sigma(p, \xi) = p_r^d \text{Im}\tilde{\Sigma}(\xi/p_r)$. However the divergence in $\xi = \xi_p$ has now disappeared, and is substituted by two finite peaks below and above ξ_p , which are due to low energy charge and spin density fluctuations. In Fig. 3 we show the contributions from small momentum transfer to $\text{Im}\Sigma$ calculated for $g = 2$ in $d = 1.5$. In contrast to the perturbative result, $\tilde{\Sigma}$ is now a

bounded function. Hence, in $d > 1$ and for p sufficiently close to the Fermi surface, the RPA self-energy does not destroy the quasiparticle pole, but will give it only a width of order p_r^d .

Figure 3. $\text{Im}\Sigma(p, \xi)$ from RPA as function of ξ for fixed $p_r = 0.1k_F$ in $d = 1.5$ ($k_F = v_F = 1$, coupling $g = 2$).

A priori RPA is not more reliable than perturbation theory is, and is known to be insufficient in $d \rightarrow 1$. Hence we will now try to sum scattering processes with small momentum transfers to all orders in the couplings. As discussed in Sec. 3, in 1D this problem is exactly solvable, as a consequence of conservation of charge and spin separately on each Fermi point.[7, 15] In $d > 1$ this conservation law is not exact, but the asymptotic dominance of forward scattering observed in the perturbative results indicates that in $d < 2$ it still holds asymptotically, with increasing accuracy as the Fermi surface is approached. We will now exploit this property via Ward identities.

We will calculate the effect of residual scattering on the single particle propagator near a (tentative) Fermi liquid fixed point in $1 < d < 2$, using the effective low energy model (1), and keeping only couplings $g_{\mathbf{k}\mathbf{k}'}^{\sigma\sigma'}(\mathbf{q})$ with small momentum transfer $|\mathbf{q}| < \Lambda \ll k_F$; this includes the processes leading to the singularity in the perturbatively calculated self energy, and reduces to the Luttinger model in $d = 1$. We further assume

that $g_{\mathbf{k}\mathbf{k}'}^{\sigma\sigma'}(\mathbf{q})$ is a slowly varying function of $\mathbf{k}$ and $\mathbf{k}'$ on the scale set by Λ .

The self energy correction due to small- $\mathbf{q}$ residual scattering obeys the Dyson equation

$$\Sigma(p) = i \int_{\mathbf{q}} D_{\mathbf{p}}(q) G(p - q) \Lambda^0(p - q/2; q) \quad (11)$$

where $\int_{\mathbf{q}}$ is a shorthand notation for $(2\pi)^{-(d+1)} \int d q_0 \int d^d \mathbf{q}$, $D_{\mathbf{p}}(q) = D_{\mathbf{p}\mathbf{p}}^{\sigma\sigma}(q)$ is the effective interaction between particles with parallel spin and momenta, and Λ^0 is the density component of the irreducible current vertex Λ^μ . The latter is defined by

$$\Lambda^\mu(p; q) = -(j_\sigma^\mu(q) a_\sigma(p - q/2) a_\sigma^\dagger(p + q/2))_{\text{irr}} \quad (12)$$

where

$$j_\sigma^\mu(\mathbf{q}) = (\rho_\sigma, \mathbf{j}_\sigma) = \sum_{\mathbf{k}} (1, \mathbf{v}_{\mathbf{k}}) a_\sigma^\dagger(\mathbf{k} - \mathbf{q}/2) a_\sigma(\mathbf{k} + \mathbf{q}/2) \quad (13)$$

is the current operator associated with H_0 . The index "irr" indicates truncation of external fermion lines and omission of diagrams which can be split in two pieces by cutting a single interaction line. The restriction to small momentum transfers $|\mathbf{q}| < \Lambda \ll k_F$ implies that, when inserted in $\Lambda^\mu(p; q)$, (12), the $\mathbf{k}$ -sum in (13) is effectively reduced to momenta $\mathbf{k} \sim \mathbf{p}$. This justifies the simplified form (11) of the Dyson equation, involving only $D_{\mathbf{p}\mathbf{p}}^{\sigma\sigma}$ instead of $D_{\mathbf{p}\mathbf{p}'}^{\sigma\sigma'}$. For small $\mathbf{q}$, one can prove the Ward identity

$$q_0 \Lambda^0(p; q) - \mathbf{q} \cdot \Lambda(p; q) = G^{-1}(p + q/2) - G^{-1}(p - q/2) \quad (14)$$

which is valid at least to order q^2 . This identity, which reflects charge and spin conservation, is exact only asymptotically because we have neglected irrelevant terms in $\mathbf{j}_\sigma$. Even in 1D, the identity is exact only for models with linear dispersion, but asymptotically exact in general. The Ward identity (14) allows one to express the $\mathbf{q} = 0$ limit of the density vertex and the $q_0 = 0$ limit of the divergence of the current vertex in terms of the propagator G .

Using the above Ward identity, one may write the density vertex Λ^0 in the form

$$\Lambda^0(p; q) = \frac{G^{-1}(p + q/2) - G^{-1}(p - q/2)}{q_0 - \mathbf{q} \cdot \mathbf{v}_{\mathbf{p}} - Y(p; q)} \quad (15)$$

where

$$Y(p; q) = \mathbf{q} \cdot [\Lambda(p; q) - \mathbf{v}_{\mathbf{p}} \Lambda^0(p; q)] / \Lambda^0(p; q) \quad (16)$$

In $d = 1$, Y vanishes identically and (15) reduces to the well-known Ward identity following from separate charge and spin conservation on each Fermi point,[7, 15] which expresses the density vertex in terms of G for general (small) $\mathbf{q}$ and q_0 . For small momentum transfers, Y is generally very small even in $d > 1$, since the velocities $\mathbf{v}_{\mathbf{k}}$

contributing to $\Lambda(p; q)$ are almost parallel to $\mathbf{v}_{\mathbf{p}}$; nevertheless Y may become important for $q_0 \sim \mathbf{q} \cdot \mathbf{v}_{\mathbf{p}}$, since it cuts off the pole in (15).

The Ward identity (15) justifies the construction of the effective interaction $D_{\mathbf{p}}(q)$ with bare bubbles instead of dressed ones: as in 1D,[7] vertex and self-energy corrections cancel each other in bubbles if Y is small. We now insert (15) with Y approximated by zero in the Dyson equation (11), obtaining thus a closed equation for G :

$$(p_0 - \xi_{\mathbf{p}}) G(p) = 1 + \int_{\mathbf{p}'} \frac{i D_{\mathbf{p}}(p - p')}{p_0 - p'_0 - \mathbf{v}_{(\mathbf{p}+\mathbf{p}')/2} \cdot (\mathbf{p} - \mathbf{p}')} G(p') \quad (17)$$

where a term of the form $-\delta\mu G(p)$ on the right hand side has been absorbed by shifting the chemical potential, keeping thus the density fixed. For a rotationally invariant system, $G(p) = G(p_r, p_0)$ depends only on the distance of $\mathbf{p}$ from the Fermi surface; further $v_{\mathbf{k}} = v_F$ is a constant, i.e. $\mathbf{v}_{(\mathbf{p}+\mathbf{p}')/2} \cdot (\mathbf{p} - \mathbf{p}') = v_F(p_r - p'_r) + \mathcal{O}(q^2)$. The only angular dependence is now in $D_{\mathbf{p}}(p - p')$. For small $q = (\mathbf{q}, q_0)$ and regular couplings, $D_{\mathbf{p}}(q)$ depends only on the ratios q_r/q_0 and q_t/q_0 , i.e. $D_{\mathbf{p}}(q) = D(q_r/q_0, q_t/q_0)$, where $q_r = \mathbf{q} \cdot \hat{\mathbf{p}}$ and $q_t = (q^2 - q_r^2)^{1/2}$ are the radial and tangential components of $\mathbf{q}$. One thus derives from (17)

$$(p_0 - v_F p_r) G(p_r, p_0) = 1 + \int_{p'_0, p'_r} \frac{i \bar{D}_\Lambda(p_r - p'_r, p_0 - p'_0)}{p_0 - p'_0 - v_F(p_r - p'_r)} G(p'_r, p'_0) \quad (18)$$

where

$$\bar{D}_\Lambda(q_r, q_0) = \frac{S_{d-1}}{(2\pi)^{d-1}} \int_0^{(\Lambda^2 - q_r^2)^{1/2}} dq_t q_t^{d-2} D(q_r/q_0, q_t/q_0) \quad (19)$$

In the limit $d \rightarrow 1$ one has $\bar{D} = D$ and one thus recovers the exact equation for the propagator of the 1D Luttinger model.[15] In $1 < d < 2$, q_t is asymptotically peaked at small values, thus justifying the neglect of Y in $\Lambda^0(p; q)$ for $q_0 \neq v_F q_r$. For $q_0 \rightarrow v_F q_r$ the vertex is drastically changed by Y ; however, since q_r and q_0 are integrated independently in (18), this error carries over to $G(p_r, p_0)$ only if contributions from $q_0 \sim v_F q_r$ are peaked, which is the case only for $p_0 \sim v_F p_r \rightarrow 0$. In this latter limit (18) is not reliable in $d > 1$ and corrections due to $Y \neq 0$ should be considered.

For non-spherical systems we would have obtained an equation similar to (18) with D and G containing the position on the Fermi surface as a parameter. We have thus arrived at a description of the low energy behavior of G in terms of a "tomographic Luttinger model" of the sort introduced by Anderson as an effective model dealing with singular forward scattering in 2D systems.[1] In our case (regular couplings, $1 < d < 2$) the effective interaction scales however to zero at low energy, since (19) implies

$$\bar{D}_{\Lambda\Lambda}(\lambda q_r, \lambda q_0) = \lambda^{d-1} \bar{D}_\Lambda(q_r, q_0) \quad (20)$$

This result clearly displays the marginality of small- q scattering processes in $d = 1$, and their irrelevance in any higher dimension.

The solution of (18) proceeds as in 1D,[15] by transforming to real 1+1 dimensional space-time $x = (r, t)$. The result for $i = 0$ is particularly simple. In $d = 1$, one recovers the well-known anomalous scaling $G(r, 0) \propto 1/r^{1+\eta}$, $\eta > 0$, while $G(r, 0) \propto 1/r$ for $r \rightarrow \infty$ in $d > 1$. Fourier transforming $G(r, 0)$ yields the momentum distribution n_k near the Fermi surface. In any $d > 1$, a finite discontinuity Δn_k is obtained at the Fermi surface, which vanishes only for $d \rightarrow 1$.

The substitution of Y by zero in (15) is too crude to determine the asymptotic behavior of $G(p_r, p_0)$ in the limit $p_r, p_0 \rightarrow 0$ with $p_0/v_F p_r \rightarrow 1$, where the solution of (18) develops unphysical singularities in $d > 1$, which we expect to be eliminated by Y . However, the structure of the solution (18) for $p_0 \neq v_F p_r$ indicates that the limit $p_r, p_0 \rightarrow 0$ is already well described by RPA in $d > 1$, i.e. the singularity (10) is obviously an artefact of standard perturbation theory.

5. Conclusion

The basic feature distinguishing Fermi from Luttinger liquids is the effect of residual scattering between fermionic excitations near the Fermi surface. In the Fermi liquid, all momentum transferring scattering processes are irrelevant, and yield only subleading corrections to the asymptotic behavior characterized by fermionic quasiparticles. At the fixed point the momentum of each quasiparticle is conserved separately, and only Hartree interactions, parametrized by the Landau function, survive. In the Luttinger liquid scattering processes with small momentum transfer remain marginal, and destroy the fermionic quasiparticle completely. The fixed point is governed by a peculiar conservation law, namely conservation of charge and spin separately on each point of the Fermi surface. Both liquids are stable metals with a Drude peak and finite interaction dependent compressibility and spin susceptibility. In the Luttinger liquid spin-charge separation and anomalous wave function renormalization (resumming singular perturbative terms) are present. Singular contributions in the response functions are instead cancelled at any order in perturbation theory as a consequence of the separate charge and spin conservation on each Fermi point, to give rise to a stable metallic phase.[7] Any metallic phase with anomalous single particle properties must undergo a similar delicate cancellation of corresponding anomalous terms in the response functions and be associated to a line of fixed points, to have a finite interaction-dependent compressibility and spin susceptibility. To achieve this, an additional conservation law is required,

as in the 1D case.

The existence of an asymptotic conservation law (and a singularity in the second order self-energy) for interacting Fermi systems in $1 < d < 2$ suggested the possibility of metallic non-Fermi liquid behavior. On the other hand, the same conservation law has allowed us to sum small momentum transferring scattering processes to all orders in the coupling, and thus perform a quantitative analysis of the dimensional crossover from 1D to 2D. This crossover can be described by a tomographic Luttinger model with scale-dependent effective interactions. It turned out that a normal Fermi system with regular asymptotic interactions has two "critical dimensions" where the low energy behavior undergoes drastic changes. Below two dimensions, long wavelength spin and charge density fluctuations yield the dominant contribution to the self energy, and make the inverse life time scale as $(k - k_F)^d$ instead of the square law valid above 2D. These scattering corrections do however not destroy the Fermi liquid fixed point until $d = 1$ is reached.

In contrast to the situation in one dimension, in $d > 1$ a breakdown of Fermi liquid theory is not obtained within perturbation theory, even if summed to all orders in the couplings, at least not in a one-band model with short range interactions. Additional degrees of freedom or non-perturbative phenomena leading to singular low-energy interactions would be required to accomplish this task.

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