About the relevance of the Imaginary components of the effective couplings in the Asymmetry measurements.

M. Martinez
and
F. Teubert\(^1\)

Institut de Física d’Altes Energies (IFAE)
Universitat Autònoma de Barcelona
*Edifici C E-08193 Bellaterra (Barcelona), Spain*

Abstract

The effect coming from imaginary parts of effective couplings in the $\epsilon^+\epsilon^-$ asymmetries is investigated. It is shown that for the present level of experimental accuracy, in some asymmetries the imaginary parts of the effective couplings cannot be neglected and moreover that the use of different prescriptions on how to handle them in quoting just real effective couplings from the data may produce sizable differences. A definition of the real effective couplings specifying how to handle the imaginary parts is advocated.

\(^1\)Research supported by a F.P.I. grant from the Universitat Autònoma de Barcelona.
1 Introduction

The present accuracy in the measurements of asymmetries at LEP/SLC calls for a revision of the analysis presently used to extract physical parameters from them in order to assess the actual meaning and limitations of the language used (real effective $Z$ vector and axial couplings). Typically, this language, was established for the analysis of the asymmetries some years ago when the accuracy in the measurements was much lower than at present and when nobody could expect such a fast improvement of the experimental errors. One possible motivation for this detailed analysis is the observed discrepancy between different asymmetry measurements (namely LEP and SLC) which although might have just a statistical origin, could also be partly due to theoretical limitations in the language used to express the measurements and to compare them.

The use of the "improved Born" language to express the precision electroweak $e^+e^-$ data in terms of measurements of real effective vector and axial couplings $Z$ parameters has become standard [1] due to its conceptual simplicity and to the fact that this language provides an easy way of comparing and combining data from different experimental measurements. Some studies [2] show that the use of this language in the lineshape analysis provides an accuracy better than the one the experimental analysis may require. Concerning the analysis of the different asymmetries, the situation deserves still some clarification, specially now that the accuracy for some of the asymmetries measured at LEP/SLC reaches the permile level. It is well known, for instance, that for the lepton forward-backward asymmetry, the "naif" use of an "improved Born" approximation biases the results by an amount which is comparable to the present experimental accuracy. The largest piece causing this bias was already identified some time ago [3] to be the effect of the imaginary part of the photon vacuum polarization but, since the experimental accuracy has kept growing, it is justified to look to all the rest of "small" corrections (imaginary parts of effective couplings, mass effects and heavy boxes) to quantify how important is their effect in the parameters extracted from the data.

All these corrections exist since long time ago in the literature (see for instance [4]) and their consideration, does not introduce any conceptual problem. The only exception is the imaginary parts of the vector and axial effective couplings which although well known, require some conceptual clarification in the definition of the quoted effective couplings.

In particular, we study in this work what are the approximations used to define a "fitting formula" for the different asymmetries measured at LEP/SLC. We will show that the effect of the imaginary components of vector and axial couplings in the frame of an effective coupling language cannot be neglected, and we will try to modify the definition of the different "fitting formulae" in order to take into account the relevant effects coming from the fact that the actual effective couplings should be complex numbers.

The outline of this paper is as follows: first we will review in section 2 the experimental procedure used so far to analyze the asymmetries, and we will quantify
the size of the approximations used in the extraction of effective parameters from them. In section 3 we will show how to improve in an easy way the simple formulae used at present for the analysis of the asymmetries to match the expected future experimental accuracy. In section 4 we will show the importance of the precise definition of the effective parameters in what concerns the handling of imaginary parts. Finally, in section 5 we will discuss the conclusions of this analysis.

2 Analysis of the different asymmetries

In this section we will explain how the different asymmetries are analyzed to extract from them the physical information, and what are the approximations made in this process.

2.1 Analysis of lepton angular distribution

The information contained in the angular distribution of $e^+e^- \rightarrow l^+l^-$ with $l \neq e$ is extracted in two steps:

- first, the forward-backward asymmetry is extracted from the data, by computing directly

\[ A_{FB} = \frac{\sigma_F - \sigma_B}{\sigma_F + \sigma_B} \]  

(1)

or by fitting the angular distribution with a simple formula

\[ \frac{d\sigma}{d \cos \theta} = C(1 + \cos^2 \theta + \frac{8}{3} A_{FB} \cos \theta) \]  

(2)

where, $\theta$ is taken as the polar angle in the reduced center-of-mass frame. The first definition has the advantage of being completely general and, hence, applicable to any process. The second definition, assumes a given behaviour of the angular distribution but, under this assumption, allows a more accurate determination of $A_{FB}$ since the whole angular distribution is used instead of just two angular bins (as is the case in the first method). In addition, in the second procedure, if a log-likelihood method is applied to fit the data, then there is no need of knowing at all the angular response of the apparatus (provided it is charge-symmetric or forward-backward symmetric).

- second, the data for $A_{FB}(s)$ is fitted with a model-independent formula in which photonic corrections are also explicitly incorporated. This allows disentangling the pure photonic effects (the ones due to radiation as well as the ones due to the $\gamma$ exchange contribution) from the rest. It enables also taking properly into account the energy dependence of the asymmetry to add-up the information from all the energy points. The fit is done simultaneously
with the lineshape data or alternatively, by incorporating as constraints the information obtained from the lineshape analysis. Both methods are basically equivalent provided the full covariance matrix of lineshape parameters is used. In this way, information about $g_V$ and $g_A$ is obtained. These couplings can be interpreted as absorbing all sort of corrections not explicitly accounted for in the fitting formulae, namely real parts of $Z$ vacuum polarization, $\gamma - Z$ mixing, weak vertices and other small effects like weak boxes.

The imaginary part of the $\gamma$ vacuum polarization, which plays a relevant role in $A_{FB}$, is included explicitly in the fitting formulae. This is so since, like for the real part, its value can be precisely predicted in QED and therefore can be included in the photonic corrections. On the other hand, only the real components of effective couplings are considered in the fitting formulae.

In this procedure, regardless on the way used to compute $A_{FB}$ from the data, the basic assumption is that all the information of the angular distribution is just the one in $A_{FB}$.

At the Born level, the angular distribution is well known that it can be written as

$$\frac{d\sigma^0}{d\cos \theta} = C^0(s)(1 + \cos^2 \theta + \frac{8}{3}A_{FB}(s)\cos \theta)$$

being

$$C^0(s) = \frac{3}{8}\sigma^0(s) = \frac{\pi\alpha^2}{2s}\left[\frac{s^2}{|Z(s)|^2}(v^2 + a^2)^2 + \frac{s(s - M_Z^2)}{|Z(s)|^2}2v^2 + 1\right]$$

and

$$A_{FB}(s) = \frac{\sigma^0_F(s) - \sigma^0_B(s)}{\sigma^0(s)} = \frac{1}{\sigma^0(s)}\frac{\pi\alpha^2}{2s}\left[\frac{s^2}{|Z(s)|^2}8v^2a^2 + \frac{s(s - M_Z^2)}{|Z(s)|^2}4a^2\right]$$

The picture does not change at all when the leading non-photonic corrections are applied, since they can be implemented by doing just some simple replacements in the effective coupling language (see appendix), namely:

$$\alpha \rightarrow \alpha(s) \text{ Real part of } \gamma \text{ self-energy}$$

$$s(s - M_Z^2) \rightarrow s(s - M_Z^2) + s^2\frac{\Gamma_Z}{M_Z} \mathfrak{S}(\Delta \alpha) \text{ Imaginary part of } \gamma \text{ self-energy}$$

$$|Z(s)|^2 \rightarrow (s - M_Z^2)^2 + (s\frac{\Gamma_Z}{M_Z})^2 \text{ Imaginary part of } Z \text{ self energy}$$
\[ v, a \rightarrow \sqrt{F_G(s)} g_V, \sqrt{F_G(s)} g_A \] Real part of Z self energy, \( \gamma - Z \) mixing, weak vertices  

(6)

where

\[ \Im(\Delta\alpha) = \frac{\Im(\Pi^\gamma(s))}{1 + \Re(\Pi^\gamma(s))} \]

(7)

\[ F_G(s) = \frac{G_F M_Z^2}{2\sqrt{2}\pi\alpha(s)} \]

(8)

In such a way that, for arbitrary final state fermions other than electrons, we can write a fitting formula (ref.[6]) as:

\[
\sigma^0(s) = \frac{4}{3} \frac{\pi\alpha^2(s)}{s} \left( Q_e^2 Q_f^2 + 2 Q_e Q_f g_{Vf} g_{Vf} F_G \frac{s(s - M_Z^2)}{(s - M_Z^2)^2 + s^2 \frac{1}{2} M_Z^2} \right) + \]

\[ 2 \left( (g_{Vf})^2 + (g_{Af})^2 \right) \cdot \left( (g_{Vf})^2 + (g_{Af})^2 \right) \frac{s^2}{(s - M_Z^2)^2 + s^2 \frac{1}{2} M_Z^2} \]

\[ A_{FB}(s) = \frac{1}{\sigma^0(s)} \frac{\pi\alpha^2(s)}{s} \left( 2 Q_e Q_f g_{Af} g_{Af} F_G \frac{s(s - M_Z^2) + s \frac{1}{2} F_G \Re(\Delta\alpha)}{(s - M_Z^2)^2 + s^2 \frac{1}{2} M_Z^2} \right) + 4 g_{Ve} g_{Af} g_{Vf} g_{Af} F_G^2 \frac{s^2}{(s - M_Z^2)^2 + s^2 \frac{1}{2} M_Z^2} \]

(9)

where the couplings \( g_{Vf} \) and \( g_{Af} \) can also be re-written as:

\[ g_{Vf} = \rho_f^\frac{1}{2} (I_3^ f - 2Q_f (\sin^2 \theta_W)) \]

\[ g_{Af} = \rho_f^\frac{1}{2} I_3^ f \]

(10)

\( \rho_f \) and \((\sin^2 \theta_W)_f\) being flavour-dependent effective rho parameter and weak mixing angle respectively.

To write down expression 9 some approximations have been done

- Only real parts of effective couplings are considered.
- \( \alpha(s) \) is considered to be real (see appendix), i.e., the imaginary part of the photon self energy is neglected in the denominator of \( \alpha(s) \), and the relevant influence on the asymmetry is explicitly taken into account in the fitting formulae.
- Corrections to \( \gamma f f \) couplings are neglected.
The effective couplings are considered to be independent of $s$ in the fitting process.

Concerning photonic corrections, since the dominant ones are initial state radiation, it is rather easy to show that they do not affect sizably the simple angular distribution described in eq. 2. Lets take just initial state radiation and lets include it by convoluting the hard reduced differential cross section with the $O(\alpha^2)$ radiator function used in the lineshape analysis \cite{7} $H(s, x)$, where $x$ denotes the fractional energy carried out by photon radiation:

\[
\frac{d\sigma^0}{d\cos \theta}(s) = \int_0^{x_{\text{max}}} dx H(s, x) \left[ C^0(s(1 - x))(1 + \cos^2 \theta + \frac{8}{3} A_{FB}(s(1 - x)) \cos \theta) \right]
\]

\[
= C(s)(1 + \cos^2 \theta + \frac{8}{3} A_{FB}(s) \cos \theta)
\]

being then

\[
C(s) = \int_0^{x_{\text{max}}} dx H(s, x) C^0(s(1 - x)) = \frac{3}{8} \sigma(s)
\]

\[
A_{FB}(s) = \frac{1}{C(s)} \int_0^{x_{\text{max}}} dx H(s, x) C^0(s(1 - x)) A_{FB}(s(1 - x))
\]

This way of introducing the initial state radiation corrections, in spite of being just a reasonable approximation in the case of the differential cross section (eq. 11), turns out to be a very accurate approximation for the forward-backward asymmetry $A_{FB}$ if no strong detection cuts are applied in the final state \cite{3, 8} and the scattering angle used to define the forward and backward hemispheres is the one in the centre-of-mass of the hard process. If different choices are taken of the scattering polar angle taken to define the forward and backward regions, then the precise form of the radiator function to be used in the $A_{FB}$ has to be modified \cite{8}.

\section{2.2 Analysis of polarized asymmetries}

The experimental procedure used to extract the polarized asymmetries from the data is not so simple as the one used in the lepton forward-backward asymmetry and its discussion escapes from the scope of this paper. At any rate, when these asymmetries are already extracted from the data, then similar arguments as the ones explained above can be used to write fitting formulae to allow the extraction of effective parameters. For Left-Right asymmetry defined as

\[
A_{LR} = \frac{\sigma_L - \sigma_R}{\sigma_L + \sigma_R}
\]

In this case we can write
\[ A_{LR}(s) = \frac{4}{3} \frac{1}{\sigma^0(s)} \frac{\pi \alpha^2(s)}{s} + 2 Q_e Q_f g_{V_f} g_{A_e} F_G \frac{s(s - M_Z^2) + s \Gamma_Z \Im(\Delta \alpha)}{(s - M_Z^2)^2 + s^2 \frac{\Gamma_Z^2}{M_Z^2}} \]

\[ 2 \left( (g_{V_f})^2 + (g_{A_f})^2 \right) g_{V_f} g_{A_e} F_G^2 \frac{s^2}{(s - M_Z^2)^2 + s^2 \frac{\Gamma_Z^2}{M_Z^2}} \]

This formula has exactly the same approximations used to write 9. In the same way, one can write a fitting formula for the polarized Forward-Backward asymmetry defined as

\[ A^{FB}_{LR} = \frac{(\sigma_L - \sigma_R)_F - (\sigma_L - \sigma_R)_B}{\sigma^0(s)} \] (15)

by doing the replacements in formula 14

\[ g_{V_e} \rightarrow g_{V_f}, \]
\[ g_{A_e} \rightarrow g_{A_f}, \]
\[ g_{V_f} \rightarrow g_{V_e}, \]
\[ g_{A_f} \rightarrow g_{A_e} \] (16)

i.e. interchanging the roles between initial and final fermions, and removing the factor \(\frac{4}{3}\).

\[ A^{FB}_{LR}(s) = \frac{1}{\sigma^0(s)} \frac{\pi \alpha^2(s)}{s} + 2 Q_f Q_e g_{V_f} g_{A_f} F_G \frac{s(s - M_Z^2) + s \Gamma_Z \Im(\Delta \alpha)}{(s - M_Z^2)^2 + s^2 \frac{\Gamma_Z^2}{M_Z^2}} \]

\[ 2 \left( (g_{V_f})^2 + (g_{A_f})^2 \right) g_{V_f} g_{A_f} F_G^2 \frac{s^2}{(s - M_Z^2)^2 + s^2 \frac{\Gamma_Z^2}{M_Z^2}} \] (17)

2.3 Accuracy of the fitting formulae compared with SM predictions.

In this section we will compute the actual accuracy of the above expressions 9, 14, 17, by comparing their results with the ones obtained using a complete electroweak library (in our case BHM [9]). One can write a complete expression for the different asymmetries defined above, neglecting mass terms and box diagram contributions as [4]:

\[ A_{FB} = \frac{3 G_3(s)}{4 G_1(s)} \] (18)
\[ A_{LR} = \frac{H_1(s)}{G_1(s)} \]  
\[ A_{LR}^{FB} = \frac{3 H_3(s)}{4 G_1(s)} \]

where
\[ G_1 = \Re \sum_{j,k=1}^{2} (V_j^e V_k^e + A_j^e A_k^e)(V_j^f V_k^f + A_j^f A_k^f) \chi_j \chi_k^* \]
\[ G_3 = \Re \sum_{j,k=1}^{2} (V_j^e A_k^e + A_j^e V_k^e)(V_j^f A_k^f + A_j^f V_k^f) \chi_j \chi_k^* \]
\[ H_1 = \Re \sum_{j,k=1}^{2} (V_j^e A_k^e + A_j^e V_k^e)(V_j^f V_k^f + A_j^f A_k^f) \chi_j \chi_k^* \]
\[ H_3 = \Re \sum_{j,k=1}^{2} (V_j^e V_k^e + A_j^e A_k^e)(V_j^f V_k^f + A_j^f V_k^f) \chi_j \chi_k^* \]

according to the following table:

<table>
<thead>
<tr>
<th>( j )</th>
<th>( V_j^e )</th>
<th>( A_j^e )</th>
<th>( V_j^f )</th>
<th>( A_j^f )</th>
<th>( \chi_j )</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>( (Q^e + F_{\gamma e}) )</td>
<td>-( F_{\gamma e} )</td>
<td>( (Q^f + F_{\gamma f}) )</td>
<td>-( F_{\gamma f} )</td>
<td>( \chi_1 )</td>
</tr>
<tr>
<td>2</td>
<td>( gV_e )</td>
<td>( gA_e )</td>
<td>( gV_f )</td>
<td>( gA_f )</td>
<td>( \chi_2 )</td>
</tr>
</tbody>
</table>

Table 1:

\[ \chi_1 = \frac{4 \pi \alpha}{(1 + \Re \Pi^\gamma) + i \Im \Pi^\gamma} \]
\[ \chi_2 = \frac{\sqrt{2} G_F M_Z s}{(s - M_Z^2) + i \frac{s \Gamma_Z}{M_Z}} \]

By computing \( G_i \) and \( H_i \) in this way, we have made the choice of factorizing the initial and final state weak vertex corrections. This allows to preserve the Born structure of the calculation in the sense that only two “dressed” amplitudes are...
considered: the photon exchange and the $Z$ exchange one. This choice is different from the one in reference [4] in which the calculation (forgetting weak boxes) involves as much as 8 amplitudes. The numerical differences of both approaches referring to asymmetries predictions are of the order of $10^{-4}$, and this is of the same level than the theoretical uncertainties from different treatments of higher order radiative corrections.

We have included$^2$ also in the calculation of these expressions the contributions due to heavy box diagrams and mass terms following reference [4]. The effect due to these two sources of corrections can be seen in table 2 in which one can also see the effect of switching off completely the imaginary part of the vector and axial couplings of table 1.

<table>
<thead>
<tr>
<th></th>
<th>Complete</th>
<th>No Box.</th>
<th>No Mass</th>
<th>No Imag.</th>
<th>Exper. acc.</th>
</tr>
</thead>
<tbody>
<tr>
<td>$A_{FB}(M_{Z}^2)$</td>
<td>0.01534</td>
<td>$\leq 1$</td>
<td>-1</td>
<td>+42</td>
<td>160</td>
</tr>
<tr>
<td>$A_{LR}(M_{Z}^2)$</td>
<td>0.13348</td>
<td>$\leq 1$</td>
<td>-20</td>
<td>+187</td>
<td>600</td>
</tr>
<tr>
<td>$A_{FB}^{LR}(M_{Z}^2)$</td>
<td>0.10004</td>
<td>-1</td>
<td>-8</td>
<td>+139</td>
<td>1200</td>
</tr>
</tbody>
</table>

Table 2: Effect of switching off different “small corrections”: heavy boxes, mass terms and imaginary parts of effective couplings. The mass effects are quoted for the tau lepton. Present experimental errors are quoted in the last column as a reference (ref.[5]).

In table 3 we compare the complete result (including boxes, mass corrections and all imaginary parts) with the results using the simple formulae 9,14, 17. The differences ($\Delta$) are shown for leptons, $c$-quarks, $s$-quarks and $b$-quarks. Also some experimental errors are quoted in parenthesis for some asymmetries (ref.[5]) to have a reference for the importance of the differences. The dependence of these differences with energy can be seen in fig.1 for leptons.

<table>
<thead>
<tr>
<th></th>
<th>$\Delta_{\text{leptons}}$</th>
<th>$\Delta_{c\text{-quarks}}$</th>
<th>$\Delta_{s\text{-quarks}}$</th>
<th>$\Delta_{b\text{-quarks}}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$A_{FB}(M_{Z}^2)$</td>
<td>-0.00043 (0.0016)</td>
<td>-0.00023 (0.011)</td>
<td>+0.00000</td>
<td>+0.00006 (0.0043)</td>
</tr>
<tr>
<td>$A_{LR}(M_{Z}^2)$</td>
<td>-0.00165 (0.006)</td>
<td>-0.00058</td>
<td>+0.00009</td>
<td>+0.00096</td>
</tr>
<tr>
<td>$A_{FB}^{LR}(M_{Z}^2)$</td>
<td>-0.00131 (0.012)</td>
<td>-0.00051</td>
<td>-0.00010</td>
<td>-0.00002</td>
</tr>
</tbody>
</table>

Table 3: Differences between fitting formulae with real effective couplings and SM predictions for the three asymmetries. Experimental errors are quoted in parenthesis for the asymmetries with existing experimental determinations (ref.[5]).

What we can learn from the results in table 3 is that at least for $A_{FB}^{l}$ and $A_{LR}^{l}$ the difference is not negligible compared with the present experimental error. So at

$^2$All the numerical evaluations in this paper have been done for $M_{Z} = 91.187$ GeV, $\alpha_s = 0.12$, $M_t = 150$ GeV and $M_h = 300$ GeV and NOT taking into account QED and final state QCD corrections.
this point it’s apparent the necessity to improve expressions 9,14,17 in order to have theoretical uncertainties far away from experimental errors.

3 Improved analysis taking into account the relevant imaginary components.

By far, of the approximations listed in section 2.1, the fact that we are neglecting the imaginary parts of the effective couplings is the cause of differences shown in table 2. In this section we are going to see how we can keep the simplicity of these formulae and, at the same time take into account the relevant imaginary components in an approximated but very accurate way.

3.1 Modifications to fitting formulae.

A numerical investigation shows that essentially all the difference comes from the interference between $\gamma - Z$ channels in the numerator of the different asymmetries. This is something we should expect, because the interference term is the only place where the imaginary components, which are small corrections, appear linearly (in the rest they appear quadratically).

On the other hand, it’s well known that $\Re(g_A)$ is larger than $\Re(g_V)$ \(^3\). So, with this in mind it’s easy to understand that the main contributions come from imaginary terms that multiply $\Re(g_A)$.

In this way we can improve formulae 9,14,17 by simply doing the replacement:

$$\Im(\Delta \alpha) \rightarrow \Im(\Delta \alpha + \Delta g)$$

where $\Im(\Delta g)$ is respectively \(^4\):

$$\Im(\Delta g)_{FB} = \frac{\Im(g_{A_e})}{\Re(g_{A_e})} + \frac{\Im(g_{A_f})}{\Re(g_{A_f})} + \frac{\Im(F_{V\gamma_e})}{Q_e} + \frac{\Im(F_{V\gamma_f})}{Q_f}$$
$$\Im(\Delta g)_{LR} = \frac{\Im(g_{V_f})}{\Re(g_{V_f})}$$
$$\Im(\Delta g)_{FB}^{LR} = \frac{\Im(g_{V_e})}{\Re(g_{V_e})}$$

\(^3\)This is true also for b quarks, although the differences between $\Re(g_A)$ and $\Re(g_V)$ are smaller.

\(^4\)In the case of the forward-backward asymmetry we have considered also the effect coming from $\Im(F_v)$ because it corresponds to 20% of the difference quoted in table 2. In addition, in this case the effects coming from the imaginary components of the photon formfactors are more important than the ones originated by the real components. This is due to the structure of the interference terms, where real components of $g_A$ are merged with imaginary components of $F$. 

9
As a numerical example if one evaluates this modifications for leptons the results are:

\[ \delta(\Delta_0) \sim +0.017732 \]
\[ \delta(\Delta g)_{FB} \sim -0.004667 + 0.000714 \sim -0.003953 \]
\[ \delta(\Delta g)_{LR} \sim \delta(\Delta g)_{LR}^{FB} \sim -0.181936 \]

One can see from the above numbers that the main contributions come from \( \delta(g_{A_f}) \) and \( \delta(g_{V_f}) \). Following the definitions in the appendix, one can see that \( \delta(g_{A_f}) \) comes from \( \delta(F_{AZ_f}) \) (i.e., the formfactor of the Z\( f_f \) vertex that incorporates weak vertex corrections and external fermion self energies), and thus it is flavour dependent. Reversely, \( \delta(g_{V_f}) \) comes essentially from the universal corrections to \( s_W^2 \), namely from \( \delta(f_{\Pi^+Z}) \) (i.e. the self energy of \( \gamma - Z \) mixing), but also in this case the flavour dependent part is not negligible.

<table>
<thead>
<tr>
<th></th>
<th>( \Delta_{\text{leptons}} )</th>
<th>( \Delta_{e-\text{quarks}} )</th>
<th>( \Delta_{\mu-\text{quarks}} )</th>
<th>( \Delta_{\tau-\text{quarks}} )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( A_{FB}(M_Z^2) )</td>
<td>+0.00003</td>
<td>+0.00004</td>
<td>+0.00013</td>
<td>+0.00015</td>
</tr>
<tr>
<td>( A_{LR}(M_Z^2) )</td>
<td>+0.00024</td>
<td>+0.00018</td>
<td>+0.00005</td>
<td>+0.00105</td>
</tr>
<tr>
<td>( A_{LR}^{FB}(M_Z^2) )</td>
<td>+0.00011</td>
<td>+0.00033</td>
<td>+0.00023</td>
<td>+0.00034</td>
</tr>
</tbody>
</table>

Table 4: Differences between fitting formulae after the modifications explained in the text and SM predictions for the three asymmetries.

After the implementation of the above modifications (27,28) the level of precision is one order of magnitude higher for the asymmetries were the experimental accuracy is better as one can see in table 3 and therefore copes with the expected accuracy. We can also see in fig.1 the precision of the approximation before and after modifications as a function of the energy in the case of leptons.

4 Precise definition of fitting parameters

The precision that asymmetry measurements are presently achieving requires a more precise definition of the actual meaning of the parameters extracted from these measurements. For instance, differences in the peak lepton forward-backward asymmetry at the level of 0.0008 have been observed when comparing the results obtained using two of the most popular electroweak libraries at LEP, (BHM and ZFITTER) [10]. This is about 40% of the one sigma error at LEP, and it propagates to a difference in the effective mixing angle of \( \Delta \sin^2(\theta)_{eff} \sim 0.0004 \). We will show in this section that, in fact, the part of this difference which does not come from differences in the treatment of QED corrections ( \( \sim 0.00035 \) ), can be explained by the different
treatment of the imaginary components of the effective couplings ($\sim 0.00025$) plus differences in the predictions of the electroweak libraries related to the treatment of higher orders ($\sim 0.0001$).

From the theoretical point of view there are some conceptual differences in the definitions of the effective couplings between the two aforementioned libraries [11]. In ZFITTER the small formfactors $F_{Vf}$ and $F_{Af}$ (due to weak vertex corrections to the photon coupling) and the bosonic loops to the photon propagator have been implemented in the $Z$ boson amplitude to have a gauge invariant splitting between the photon and the $Z^0$ amplitude [12]. The price to pay is that a vector coupling $g_{Vf}$ which depends on both initial and final state in a non-factorizable way shows up in the $Z$ amplitude. In BHM, these small contributions are kept in the photon amplitude for simplicity and, in spite that, then the splitting is not gauge invariant, numerically this does not have any consequence. This makes the two definitions for $g_V$ and $g_A$ “in principle” different. “In practice”, the effects due to the different treatment of $F_{Vf}$ and $F_{Af}$ are numerically irrelevant: of the order of $\Delta g_A \sim 10^{-9}$ and $\Delta g_V \sim 10^{-5}$ but as we shall show, this fact introduces conceptual complications in the definition of which are the actual parameters extracted by the experiments.

A part from this, the fact that the two libraries use different calculational schemes to implement the calculations, and that there are some differences in the treatment of higher order corrections, originates numerically differences in $g_V$ and $g_A$ of the order of $\Delta g \sim 10^{-4}$. This propagates to a difference in the prediction of the leptons forward-backward asymmetries of the order of 0.0001.

We have shown in the previous section the importance of the imaginary components of the effective couplings for some asymmetries. Given the fact that the data is not sensitive enough to measure the effective couplings as complex numbers, just real effective coupling are extracted but then the common procedure followed in the above libraries is borrowing the imaginary parts of the effective couplings from a complete explicit Standard Model calculation.

This procedure guarantees the same accuracy as the complete calculation for the Standard Model interpretation of the parameters. Given the fact that the imaginary parts are small and that, due to their absorptive origin, they depend only mildly on the model, this procedure spoils only very softly the “model independence” of the method.

At any rate, the practical implementation of this procedure introduces some ambiguities in the actual meaning of the extracted parameters since they may be defined as:

- the modulus of the complex vector and axial effective couplings, the phase being taken from the Standard Model calculation or,
- the real parts of the vector and axial effective couplings, their imaginary parts being taken from the Standard Model calculation or even
- the real parts of the vector and axial effective couplings computed from the
real part of $\rho$ and $\sin^2 \theta_{\text{eff}}$, the imaginary parts of $\rho$ and $\sin^2 \theta_{\text{eff}}$ being taken from the Standard Model.

Given the small relative size of the imaginary components with respect to the real ones, the first two approaches are numerically equivalent and, in fact the second one is the BHM choice. In the ZFITTER package one also the option to fit $\Re(g_V)$ and $\Re(g_{A})$ as defined in equation 1, but in this case the imaginary components of the vector and axial effective couplings are not fixed to the Standard Model predictions, but instead what is fixed are the imaginary components of the $\kappa$ and $\rho$ parameters because these are the "natural" parameters that the program uses internally. This, of course, gives different results but still the effect is much below the differences we were quoting.

In fact, what produces the bulk of that difference in the lepton forward-backward asymmetry is whether one understands the extracted vector parameters as the real part of the vector couplings for each flavour

$$g_{V_i} g_{V_j}(\text{experimental}) = \Re(g_{V_j}) \Re(g_{V_i})$$

or rather as the real part of the initial-final state vector coupling

$$g_{V_i} g_{V_j}(\text{experimental}) = \Re(g_{V_{i,f}})$$

At $\sqrt{s} = M_Z$, in very good approximation $g_{V_{i,f}} \cong g_{V_j} g_{V_i}$ [12] but then,

$$\Re(g_{V_{i,f}}) = \Re(g_{V_j}) \Re(g_{V_i}) - \Im(g_{V_j}) \Im(g_{V_i})$$

so that both definitions differ when the imaginary parts are not neglected. It turns out that this product of imaginary parts produces a difference in the predicted lepton forward-backward asymmetry at the level of $\sim 0.00025$ which is what was missing to explain the difference between BHM and ZFITTER. So, in order to be consistent, one has to choose whether the fitting parameters are $(\Re(g_{V_j}), \Re(g_{A_j}))$ or any of the different alternative options described above 5, but it is not more true that at the level of accuracy we want to achieve all these approaches are still equivalent.

\section{Conclusions}

From the present study, the main conclusion is that at the present level of experimental accuracy in the asymmetry measurements, some of the approximations used in the past to define a "fitting formula" need to be reviewed. In particular we have seen how we can take into account the relevant effects coming from the imaginary

\footnote{In our opinion, using $(\Re(g_{V_j}), \Re(g_{A_j}))$ matches best the language already used by the experiments and has the simplest connection with the Born formulae and therefore this is the definition that we advocate.}
components of effective couplings with minimal changes in the definition of a "fitting formula". We have also shown that the use of different prescriptions on how to handle the imaginary parts when quoting just real effective couplings from the data may produce sizable differences. A definition of the real effective couplings specifying how to handle the imaginary parts has been advocated.

Appendix: Writing the amplitudes for $e^+e^- \rightarrow f^+f^-$ in an effective coupling language.

It has been shown by several groups that in four fermion processes, the radiatively corrected matrix element squared can be rewritten keeping a Born-like structure by defining running effective complex parameters $\beta_1, \beta_2, \beta_4$. This fact is specially transparent for neutral current processes in which the non-photonic corrections at one-loop level separate naturally from the photonic ones forming a gauge-invariant subset.

Initial-final state factorizable corrections such as self-energies and vertex corrections can be easily absorbed by redefining the Born couplings as we shall see. Concerning non-factorizable corrections such as boxes, two different approaches exist:

- Absorbing them also in the definition of the effective parameters $\beta_1$. The price to pay is that some effective parameters become not just a function of $s$ but also of $\cos \theta$ and, in addition, the Born-like structure is somewhat spoiled by the presence of effective parameters which do not show up in the pure Born approach.

- Keeping them out from the definition of the effective parameters $\beta_4$. They must be included as explicit corrections afterwards. This approach has the advantage of being very simple and producing a set of effective parameters which depend only on $s$ and have a clear Born interpretation. Nevertheless, the price to pay is that, in this case, the effective parameters are defined in a gauge non-invariant way so that attention should be paid to the gauge choice.\footnote{For a while, this observation prevented the theoretical community from accepting the usefulness of the effective parameter approach which, at the end, has been the one chosen by the experimental community to perform the measurements. In fact, if the predictions are computed in a gauge in which the non-absorbed corrections are numerically irrelevant (as is the case for the 't Hooft-Feynman gauge) the calculations using these effective parameters produce, in fact, numerically gauge-invariant results.}

For reasons of simplicity we will follow the second approach to define the meaning of the effective couplings.
Universal effective parameters.

In the 't Hooft-Feynman gauge the dominant corrections are by far, the vector boson vacuum polarizations. Since these corrections do not depend of the species of the external fermions they are, in fact, universal (process independent). The three dressed self energies showing up in $Z$ processes, can be absorbed in three universal parameters $[4]$:

- the photon self-energy $\Pi^\gamma(s)$ is absorbed in an effective coupling constant $\alpha(s)$, defined as:

$$\tilde{\alpha}(s) = \frac{\alpha_0}{1 + \Pi^\gamma(s)}$$

It is important to stress that, since $\Pi^\gamma(s)$ is a complex function, so is $\tilde{\alpha}(s)$. Nevertheless, since the imaginary part of $\tilde{\alpha}(s)$ is small compared with the real one, its main effect happens in observables which are sensitive to phase differences between photon and $Z$ exchange diagrams such as the forward-backward charge asymmetry.

- the way the $\gamma Z$ mixing is treated is slightly more complicated. Since the $\gamma Z$ mixing does not show up at tree level, to keep the Born structure, it must be absorbed in the neutral current coupling parameters. Then the neutral current is redefined as

$$C^Z_{\mu} = \gamma_{\mu}(v_f - a_f \gamma_5) + \gamma_{\mu}Q_f \frac{\Pi^\gamma Z(s)}{1 + \Pi^\gamma(s)} = \frac{1}{2s_W c_W} \left[ \gamma_{\mu}(I_3^f - 2Q_f s^2_W(s)) - I_3^f \gamma_{\mu} \gamma_5 \right]$$

being

$$s^2_W(s) = s^2_W(1 + \Delta \kappa(s))$$

the complex universal effective weak mixing angle, where

$$\Delta \kappa(s) = \frac{c_W}{s_W} \frac{\Pi^\gamma Z(s)}{1 + \Pi^\gamma(s)}$$

- finally, the $Z$ self energy has to be absorbed into a third parameter. The problem is that we have already used the only two tree level parameter ($\alpha$ and $s_W$) to absorb vacuum polarization corrections. The way out is the following: let's first consider the corrected $Z$ propagator with the overall coupling constant that will come from the initial and final state particle couplings

$$\frac{\epsilon^2}{4s_W^2 c_W^2 s - M^2_Z + (3\Sigma_2(s)) + i\Sigma_2(s)} = \frac{\epsilon^2}{4s_W^2 c_W^2 \frac{1}{1 + \Pi^Z(s)} s - M^2_Z + \frac{3\Sigma_2(s)}{1 + \Pi^\gamma(s)}}$$
where

$$\Pi^Z(s) = \frac{\Re(\Sigma_Z(s))}{s - M_Z^2}$$

Therefore, the factor \(\frac{e^2}{4s^2 W^2 c_W^4} \frac{1}{1 + \Pi^Z(s)}\) which multiplies the Breit-Wigner like propagator, can be considered as the effective strength of the purely weak interactions and an appropriate way of introducing a running parameter to account for it is recalling the tree level \(G_F\) relation:

$$\frac{e^2}{4s^2 W^2 c_W^4} \frac{1}{1 + \Pi^Z(s)} = \sqrt{2} G_F M_Z^2 (1 - \Delta r) \rho_0$$

where \(\rho_0\) is the tree level \(\rho\) parameter which in the Minimal Standard Model is exactly 1. Therefore we can write

$$\frac{e^2}{4s^2 W^2 c_W^4} \frac{1}{1 + \Pi^Z(s)} = \sqrt{2} G_F M_Z^2 \rho(s)$$

being

$$\bar{\rho}(s) = \rho_0(1 + \Delta \rho(s))$$

the universal effective \(\rho\) parameter, where

$$\Delta \rho(s) = \frac{1 - \Delta r}{1 + \Pi^Z(s)} - 1$$

\(\Delta \rho(s)\) and hence \(\bar{\rho}(s)\) are real quantities by definition, since the imaginary part of the \(Z\) vacuum polarization will be treated separately. It is important to stress that this \(\Delta \rho\) is numerically and conceptually different from the one introduced when discussing \(\Delta r\). The one introduced there accounted for the ratio of \(W\) to \(Z\) vacuum polarization corrections at \(q^2 \approx 0\), whereas the one introduced now is more complex since accounts for the ratio of the whole \(\Delta r\) correction (which in spite that is dominated by the \(W\) vacuum polarization at \(q^2 \approx 0\), includes also sizable QED corrections) to the real part of the derivative of the \(Z\) vacuum polarization at \(q^2 = s\). Nevertheless, the coefficient of the dominant \(m_t\) terms is the same.

Finally, it can be shown [4] that the imaginary part of the \(Z\) self energy can be interpreted, through the use of the optical theorem as

$$\frac{\Im(\Sigma_Z(s))}{1 + \Pi^Z(s)} = \Gamma_Z(s) \equiv \frac{s}{M_Z} \Gamma_Z(M_Z^2)$$

where \(\Gamma_Z(s)\) stands for the Born total \(Z\) decay width in terms of effective couplings.
Summarizing, for $Z$ physics, the vacuum polarization corrections can be properly included by writing the Born amplitudes in terms of the effective running couplings $\alpha(s)$, $\rho(s)$ and $\bar{z}_W(s)$ as:

$$\bar{A} = Q_e Q_f \frac{4 \pi \bar{\alpha}(s)}{s} [\gamma_\mu \otimes \gamma^\mu] + \sqrt{2} G_F M_Z^2 \bar{\rho}(s) \frac{1}{s - M_Z^2 + i s \frac{\Gamma}{M_Z}}$$

$$[\gamma_\mu (I_3^f - 2 Q_e \bar{z}_W^2(s)) - \gamma_\nu \gamma_5 I_3^s] \otimes [\gamma^\mu (I_3^f - 2 Q_f \bar{z}_W^2(s)) - \gamma^\nu \gamma_5 I_3^s]$$

This representation of the amplitudes is accurate for the calculation of any observable \(^7\) at the percent level. It is worth mentioning also that at this level, the definition of the effective amplitudes in the most popular electroweak libraries [9, 12] conceptually agree.

**Flavour-dependent effective parameters.**

The fact that the accuracy for some $Z$ observables reaches the permile level requires the consideration of the next level of corrections, namely the weak vertex ones. These corrections, unlike the vacuum polarization ones, depend explicitly on the species of the external fermions and therefore are flavour dependent. Hence, they can be absorbed into effective parameters at the price of making them flavour dependent, that is, having a set of effective parameters for every fermion species.

As of the photon exchange amplitude, the current including vertex corrections at one loop can be written as:

$$C_\mu^\gamma = [\gamma_\mu(Q_f + F_{V\gamma f}(s) - F_{A\gamma f}(s)\gamma_5)]$$

where $F_{V\gamma f}$ and $F_{A\gamma f}$ are the complex vector and axial photon formfactors (see for instance [4]).

In the case of the weak vertex corrections for the $Z$ amplitudes, at one loop we can introduce them through the use of complex form factors modifying the Born axial and vector couplings in the currents [4]:

$$C_\mu^Z = [\gamma_\mu(v_f + F_{VZ f}(s) - (a_f + F_{AZ f}(s))\gamma_5) + \gamma_\mu Q_f \frac{\Pi^Z(s)}{(1 + \Pi^Z(s))}]$$

where $F_{V\gamma Z}$ and $F_{A\gamma Z}$ are the complex vector and axial $Z$ formfactors and the $\gamma Z$ mixing correction has also been included explicitly. This can be rewritten as

$$\left(1 + \frac{F_{AZ f}(s)}{a_f}\right) \frac{1}{2 s_W c_W} [\gamma_\mu (I_3^f - 2 Q_f \bar{s}_W^2(s)) - I_3^f \gamma_5 \gamma_\nu \gamma_5]$$

\(^7\)Exception made of the observables for $b$ quarks for which, as pointed out before, vertex corrections play an important role.
being

\[ s_{Wf}^2(s) = s_W^2(1 + \Delta \kappa(s) + \Delta \kappa_f(s)) \]

the flavour-dependent complex effective weak mixing angle, where

\[
\Delta \kappa_f(s) = -\frac{1}{Q_f s_W} c_W \left( F_{VZf}'(s) - \frac{v_f}{a_f} F_{AZf}(s) \right)
\]

is the flavour-dependent vertex correction.

After this algebra, the effective strength of the purely weak interactions becomes

\[
\frac{\epsilon^2}{4 s_W^2 c_W^2} \frac{1}{1 + V^2(s)} \left( 1 + \frac{F_{AZ\epsilon}(s)}{a_\epsilon} \right) \left( 1 + \frac{F_{AZf}(s)}{a_f} \right) = \sqrt{2} G_F M_Z^2 (\rho_\epsilon(s) \rho_f(s))^{\frac{1}{2}}
\]

being

\[
\rho_f(s) = \rho_0(1 + \Delta \rho(s) + \Delta \rho_f(s))
\]

the flavour-dependent complex effective \( \rho \) parameter, where

\[
\Delta \rho_f(s) = \left( 1 + \frac{F_{AZf}(s)}{a_f} \right)^2 - 1
\]

is the flavour-dependent complex vertex correction.

With the introduction of these complex flavour-dependent effective parameters, the \( Z \) exchange amplitude can simply be written as:

\[
A_Z = \sqrt{2} G_F M_Z^2 \frac{1}{s - M_Z^2 + i s \frac{1}{M_Z^2}} \left[ (\gamma_{\mu}(g_{V\epsilon}(s) - g_{A\epsilon}(s))\gamma_\mu) \right] \otimes \left[ (\gamma^\mu(g_{Vf}(s) - g_{Af}(s))\gamma^\mu) \right]
\]

where the complex effective vector and axial couplings are defined as

\[
\begin{align*}
g_{Vf}(s) &= \sqrt{\rho_f(I_3^f - 2 Q_f s_{Wf})} \\
g_{Af}(s) &= \sqrt{\rho_f I_3^f}
\end{align*}
\]

**Acknowledgments**

We want to thank W.Hollik and D.Bardin for fruitful discussions about this topic. We are also indebted to W.Chen and Z.Feng for their early help in the comparison between BHM and ZFITTER.
Figure 1: Absolute value of differences between the SM predictions for the different asymmetries and the fitting formulae before and after modifications explained in the text.
References


