Axial Symmetries in Lattice QCD with Kaplan Fermions

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ABSTRACT

The chiral limit of lattice QCD with Kaplan fermions is defined. This limit corresponds to infinitely many heavy “regulator” fields, realized through the introduction of an unphysical fifth dimension. It is proved that in the full quantum theory, all non-singlet axial Ward identities are restored in the chiral limit. The magnitude of anomalous effects on a finite lattice is estimated.
1. Introduction

Because of the axial anomaly [1], consistent regularization methods which preserve gauge invariance must break axial symmetries explicitly. However, flavour non-singlet axial symmetries (NSAS for short) are recovered in renormalized correlation functions to all orders in perturbation theory [2, 3]. (For recent progress and references to earlier literature see ref. [4]).

Going beyond perturbation theory, the rigorous definition of QCD relies on the lattice regulator. The most popular method to avoid the fermion doubling problem employs Wilson’s prescription for the fermion action [5]. The advantage of this method is in its simplicity. But the ensuing breaking of axial symmetries is hard, in the sense that perturbative corrections to quark masses are $O(1/a)$ where $a$ is the lattice spacing. Hence, one has to fine tune the bare quark masses in order to recover the correct renormalized masses in the continuum limit.

Using Wilson fermions, it was shown that weak coupling perturbation theory (WCPT) on the lattice reproduces the axial anomaly [6], and that NSAS are recovered to all orders in WCPT in the continuum limit [6, 7]. These results have in fact some validity beyond the scope of WCPT, and one can discuss the renormalization of gauge invariant composite operators. But, because of the severe fine tuning problem inherent to Wilson fermions, one cannot give a completely general non-perturbative proof of the restoration of NSAS.

Assuming the existence of the chiral limit in the full quantum theory, what one can do in numerical simulations with Wilson fermions is to determine the correct finely tuned values of the bare masses by measuring some correlation functions. Fixing the bare parameters this way, one hopes that NSAS will be recovered in all other correlation functions, thus reproducing for example the results of current algebra [8].

This situation is unsatisfactory for several reasons. On the theoretical level, one would like to have a true non-perturbative proof of the restoration of NSAS. Moreover, the construction of a lattice model of QCD where the existence of the chiral limit can (a) be proved and (b) does not require any fine tuning, is important for practical reasons. To elucidate the importance of such framework, we can mention for example the problems involved in measuring weak matrix elements on the lattice with Wilson fermions (for a recent review see [9]). The fine tuning problem is not over when the bare masses have been fixed. Because of the hard breaking of the axial symmetries, the definition of renormalized four fermion operators which are necessary for the computation of weak decays, involves additional fine-tuning. Moreover, some of the relevant operator mixings receive genuinely non-perturbative contributions,
which cannot be determined even in principle by short distance expansions such as the OPE.

In this paper we discuss a lattice formulation of QCD with a very mild breaking of axial symmetries. The formulation is based on the introduction of many (in the chiral limit infinitely many) heavy “regulator” fields [10, 11, 12]. More specifically, we use a variant [13] of Kaplan’s proposal [11] to realize light ordinary fermions as zero modes bound to four dimensional defects in a theory of massive five dimensional Dirac fermions [11-20]. In the present model, the right-handed (RH) and left-handed (LH) components of the physical quark arise as surface modes bound to opposite boundaries of a five dimensional slab with free boundary conditions in the fifth direction. One five dimensional fermion field is needed for every physical quark. The boundary fermions scheme has also been discussed recently in ref. [14].

While Kaplan intended to propose a solution to the long-standing problem of defining chiral gauge theories on the lattice, the feasibility of reaching this goal is still unclear [15-18,21]. But the advantages of using chiral defect fermions for lattice QCD are obvious. In particular, one can easily show that, in the limit where the width of the five dimensional slab tends to infinity, quark masses undergo only multiplicative renormalization to all orders in perturbation theory [13].

The reason for the absence of additive $O(1/a)$ corrections to quark masses, is the vanishing of the overlap between the RH and LH components of the quark’s wave function. This happens because, as mentioned above, the RH and LH components are surface modes localized on opposite boundaries of the five dimensional slab. At tree level, the tail of the RH wave function goes like

$$(1 - Ma)^{-s}.$$  

(1.1)

Here $s$ is the fifth coordinate which takes the values $s = 1, \ldots, 2N$. The Dirac mass $M$ that appears in the five dimensional fermion action obeys $0 < Ma < 1$. A similar expression applies to the LH component, with $s$ in eq. (1.1) replaced by $2N - s$. Thus, the perturbative overlap vanishes very rapidly with increasing $N$.

One can also introduce direct couplings between the LH and RH components of every quark, by adding links that couple the layers $s = 1$ and $s = 2N$. These couplings are controlled by dimensionful parameters $m_i$, where $i$ is a flavour index. As shown in ref. [13], the $m_i$ play the role of multiplicatively renormalized quark masses.

Taking advantage of the special properties of the model, one can define axial currents whose divergences, for $m_i = 0$, are completely localized on the two middle layers $s = N$ and $s = N + 1$. As a result, the anomalous term in the Ward identities
of NSAS is governed by the small tail (1.1) of the quark’s wave function at the center of the five dimensional slab. On the other hand, the divergence of the singlet axial current can couple to two gluons, giving rise in the limit $N \to \infty$ to the expected axial anomaly [19, 17, 15, 14].

In this paper we investigate to what extent the perturbative behaviour is modified by non-perturbative effects. We find that in the full quantum theory, all anomalous terms in the Ward identities of NSAS vanish in the limit $N \to \infty$. To our knowledge, this is the first non-perturbative proof of the restoration of NSAS. We also estimate the magnitude of anomalous effects for finite values of the parameters of the model.

We begin in sect. 2 with the definition of the model. Apart from the obvious gauge and fermion fields, the model includes a set of massive five dimensional scalar fields needed to compensate for the contribution of massive fermions to the Dirac sea. If the number of massive four dimensional fermions was kept finite, these fields would have decoupled anyway in the continuum limit. The need to cancel their contribution arises because one is interested in taking the limit $N \to \infty$ before the continuum limit $g_0 \to 0$. Such scalar fields have often been called Pauli-Villars (PV) fields in the literature, and we will continue to use this terminology here. But it should be stressed that the action of the scalar fields is positive, and so the partition function of the PV fields is well-defined. The peculiar property of the PV fields is that their action is essentially the square of the fermionic action.

In sect. 3 we develop a transfer matrix formalism to represent the fermionic partition function as well as correlation functions. The transfer matrix technique was introduced in the context of chiral defect fermions by Narayanan and Neuberger [15] and relies on the work of Lüscher [22]. It proves particularly convenient for the investigation of the model. Formulae are given both for finite $N$ and for the limiting case $N \to \infty$.

Sect. 4 contains a discussion of some physical properties of the effective action $S_{eff}^\infty$ obtained by integrating out both fermion and PV fields and taking the limit $N \to \infty$. An interesting result is that $\exp\{-S_{eff}^\infty\}$, while being always real, is not necessarily positive. This behaviour can be explained on the basis of familiar instanton results.

In sect. 5 we turn to the main subject of this paper. We begin by developing a criterion to identify gauge field configurations which allow for the propagation of light fermionic excitations from one boundary to the other across the five dimensional balk. The existence of light fermion states inside the five dimensional balk is related to the absence of Vafa-Witten bounds in the model. Because of the close relation between the fermionic and PV actions, the same gauge field configurations support also light PV states. These anomalously light balk states are responsible
for the anomalous term in non-singlet axial Ward identities. A particularly simple configuration which supports anomalously light balk states is the *dynamical domain wall*.

These results are then used in sect. 6 to prove the restoration of NSAS in the *chiral limit* $N \to \infty$. The vanishing of anomalous effects in the chiral limit, is a simple consequence of the vanishing of the phase space for gauge field configurations which support balk states whose energy is exactly zero.

In order to estimate the magnitude of anomalous effects on a finite lattice we first observe that, in certain important Ward identities, the anomalous term is positive definite. Hence, one must bound the absolute value of the anomalous term configuration by configuration. We prove that anomalously light balk state can exist only for configurations whose field strength is singular in the continuum limit. Additional suppression arises from the fact that the correlation length in the fifth direction is finite. Our final result for the magnitude of anomalous effects is eq. (6.15).

Our conclusions are given in sect. 7. In numerical simulations one can choose between two options. One possibility is to work with a finite five dimensional lattice. Eq. (6.15) then provides an estimate of the expected magnitude of anomalous effects. Moreover, the anomalous contribution to any Ward identity can be *measured* directly and unambiguously. Thus one has full control over the magnitude of lattice artifacts. Alternatively, one can work on a four dimensional lattice using the "$N = \infty$" formulae. Unlike the case of chiral gauge theories [17], no subtleties are involved in taking the limit $N \to \infty$ in the vector model. The usefulness of this method depends on one's ability to calculate the relevant second quantized matrix elements efficiently. We comment on possible investigations of spontaneous chiral symmetry breaking in the present model at both strong and weak coupling. Some technical points are discussed in two appendices.

### 2. Definition of the model

In this section we give the definition of lattice QCD with the surface mode’s variant of chiral defect fermions [13]. We also define physical quark operators and axial currents [17] appropriate for the model. Most of the ingredients of the model have been introduced previously, and we give them here to make the present exposition self-contained.

For definiteness, we take the physical case of four dimensions. This means that the fermion and Pauli-Villars (PV) fields live on five dimensional lattices, whereas the gluon fields are four dimensional. The ordinary four coordinates, labeled $x_\mu$, range
from 1 to \( L \), whereas the extra coordinate takes the values \( s = 1, \ldots, 2N \) for the fermionic lattice. The PV lattice is only half as big, with \( s \) ranging from 1 to \( N \). The preferred boundary conditions in the four ordinary dimensions are periodic or antiperiodic. Free boundary condition in these directions would result in extra unwanted light states which can propagate along the spatial boundaries.

The above scheme is realized by requiring that the link variables in the fermion and PV action obey \( U_s(x, s) = 1 \) and \( U_{\mu}(x, s) = U_{\mu}(x) \) independently of \( s \). The topology of the fifth dimension is taken to be a circle, but the couplings which reside on the links connecting the layers \( s = 2N \) and \( s = 1 \) are proportional to a parameter \(-m_i\) (\( i \) is a flavour index. We henceforth set the lattice spacing to \( a = 1 \)). The case \( m_i = 1 \) corresponds to antiperiodic boundary conditions, where the model supports no light fermionic state. The case \( m_i = 0 \) corresponds to open boundaries, and it should give rise to the physics of QCD with massless quarks by taking first the limit \( N \to \infty \) and then \( g_0 \to 0 \).

The action is given by:

\[
S = S_G(U) + S_F(\bar{\psi}, \psi, U) + S_{PV}(\phi^I, \phi, U). \tag{2.1}
\]

The pure gauge part of the action \( S_G(U) \) contains the dependence on the bare coupling \( g_0 \). It may be the usual sum over plaquettes, or any other four dimensional lattice action which reduces to the standard Yang-Mills action in the classical continuum limit.

The fermion and PV actions contain a sum over all flavours \( i = 1, \ldots, N_f \). The only difference between various flavours can be in the mass parameter \( m_i \). We give below the one flavour action. The fermionic part has the following form

\[
S_F(\bar{\psi}, \psi, U) = \sum_{x, y, s, s'} \bar{\psi}(x, s) D_F(x, s; y, s') \psi(y, s'), \tag{2.2}
\]

where the fermionic matrix is defined by

\[
D_F(x, s; y, s') = \delta_{s,s'} D^\parallel(x, y) + \delta_{x,y} D^\perp(s, s'), \tag{2.3}
\]

\[
D^\parallel(x, y) = \frac{1}{2} \sum_{\mu} \left( (1 + \gamma_\mu) U_{\mu}(x) \delta_{x+\hat{\mu}, y} + (1 - \gamma_\mu) U^{-1}_{\mu}(y) \delta_{x-\hat{\mu}, y} \right) + (M - 4) \delta_{x,y}, \tag{2.4}
\]

\[
D^\perp(s, s') = P_R \delta_{s+1, s'} + P_L \delta_{s-1, s'} - \delta_{s,s'} - m \left( P_R \delta_{s,2N} \delta_{s',1} + P_L \delta_{s,1} \delta_{s',2N} \right), \tag{2.5}
\]
and
\[ P_{R,L} = \frac{1}{2}(1 \pm \gamma_5). \]  

Notice that \( D^\pm(s, s') \) is independent of the gauge field. Also, apart from the unconventional sign of the mass term, \( D^\parallel(x, y) \) is the usual four dimensional gauge covariant Dirac operator for Wilson fermions.

For a given flavour, when \( m_i = 0 \) the spectrum of surface states describes a RH Weyl fermion near the boundary \( s = 1 \) and a LH Weyl fermion near the other boundary. These four dimensional fermion fields have the same coupling to the gauge field, and so they in fact describe \( N_f \) quarks. If we ignore the exponentially small overlap between the tails of the LH and RH surface states, then these states describe massless quarks. Switching \( m_i \) on, we now mix the RH and LH components of each quark and provide it with a Dirac mass \( \overline{m}_i \) which is proportional to \( m_i \). At tree level one has [13]
\[ \overline{m}_i = M(2 - M)m_i. \]  

As we indicated above, as long as the \( s \)-direction is finite there is actually a small, but finite, mixing between the RH and LH modes. The energy of the zero momentum state is therefore slightly greater than zero. An interesting question is whether one can interpret that effect as the inclusion of a current quark mass. As we discuss in the last section, we suspect that this interpretation may not be consistent. In our approach, we account for non-zero current masses by directly coupling the boundary layers, whereas any effect resulting from the tail of the quark’s wave function deep inside the five dimensional bulk is interpreted as an anomalous effect.

The PV fields are intended to cancel the contribution of the heavy fermion modes to the effective gauge field action, which is obtained by integrating out the matter fields. This contribution, while formally being local in the continuum limit, is proportional to \( N \). If one does not subtract it out by hand, that lattice artifact will dominate the effective action in the limit \( N \to \infty \).

Let us denote the dependence of the Dirac operator in eq. (2.3) on \( m_i \) and on the number of sites in the \( s \)-direction by \( D_F = D_F(2N, m_i) \). The PV fields live on a five dimensional lattice with \( N \) sites in the \( s \)-direction, and using the above notation, the PV action is
\[ S_{PV}(\phi^\dagger, \phi, U) = \sum_{x,y,z,s,s',s''} \phi^\dagger(x, s) D^\parallel_F(N, 1; x, s; z, s') \times D_F(N, 1; z, s'; y, s') \phi(y, s'). \]

The second order operator in eq. (2.8) is the square of the Dirac operator on the smaller lattice. The choice \( m_i = 1 \) for the PV fields prevents the appearance of light
scalar modes on the layers \( s = 1 \) and \( s = N \). As will be shown below, this choice of \( S_{PV} \) ensures that the effective action will remain finite in the limit \( N \to \infty \).

Let us denote by \( \mathcal{R} \) the reflection relative to the hyper-plane \( s = N + 1/2 \)

\[
\mathcal{R} \psi(x, s) = \psi(x, 2N + 1 - s).
\]

(2.9)

The Dirac operator \( D_F \) of eq. (2.3) satisfies the following identity

\[
\gamma_5 \mathcal{R} D_F \gamma_5 = D_F^1.
\]

(2.10)

This identity is a generalization of a similar relation obeyed by the four dimensional Dirac operator for Wilson fermion\(^1\). As in the case of Wilson fermions, eq. (2.10) plays an important role in establishing the positivity of pion correlators (see App. B).

Eq. (2.10) also implies that the operator \( \gamma_5 \mathcal{R} D_F \) is hermitian. One can use this hermitian operator in the definition of the fermionic action instead of \( D_F \). The reason is that, considered as a matrix operator, one has \( \det (\gamma_5 \mathcal{R}) = 1 \) trivially, and so \( \det (D_F) = \det (\gamma_5 \mathcal{R} D_F) \). As a result, the fermionic determinant is real. However, one cannot conclude that the fermionic determinant is necessarily positive. The fermionic determinant can in fact change sign, due to level crossing of an odd number of states (see below).

For \( m_i = m \), the five dimensional fermion action is invariant under a global \( U(N_f) \) symmetry. The conserved five dimensional current has the following components. For \( \mu = 1, \ldots, 4 \)

\[
j_\mu^a(x, s) = -\frac{1}{2} \left( \overline{\psi}(x, s)(1 + \gamma_\mu)U_\mu(x)\lambda^a\psi(x + \hat{\mu}, s) - \overline{\psi}(x + \hat{\mu}, s)(1 - \gamma_\mu)U_\mu^\dagger(x)\lambda^a\psi(x, s) \right), \quad 1 \leq s \leq 2N.
\]

(2.11)

As for the fifth component, we define

\[
j_5^a(x, s) = \begin{cases} 
\overline{\psi}(x, s)P_R \lambda^a\psi(x, s + 1) - \overline{\psi}(x, s + 1)P_L \lambda^a\psi(x, s), & 1 \leq s < 2N, \\
\overline{\psi}(x, 2N)P_R \lambda^a\psi(x, 1) - \overline{\psi}(x, 1)P_L \lambda^a\psi(x, 2N), & s = 2N.
\end{cases}
\]

(2.12)

This five dimensional current satisfies the continuity equation

\[
\sum_\mu \Delta_\mu j_\mu^a(x, s) = \begin{cases} 
-j_5^a(x, 1) - mj_5^a(x, 2N), & s = 1, \\
-\Delta_5 j_5^a(x, s), & 1 < s < 2N, \\
j_5^a(x, 2N - 1) + mj_5^a(x, 2N), & s = 2N.
\end{cases}
\]

(2.13)

Here

\[
\Delta_\mu f(x, s) = f(x, s) - f(x - \hat{\mu}, s).
\]

(2.14)

\(^1\)In our notation, that relation is \( \gamma_5 D^\parallel \gamma_5 = (D^\parallel)^1 \).
\[ \Delta_\delta f(x, s) = f(x, s) - f(x, s - 1). \quad (2.15) \]

\( \lambda^a \) is a flavour symmetry generator. Notice the special form of the boundary terms in the continuity equation.

We now give the definitions of four dimensional vector and axial currents [17]. There is a unique set of conserved currents, given by

\[ V_\mu^a(x) = \sum_{s=1}^{2N} j_\mu^a(x, s). \quad (2.16) \]

Conservation of the vector current \( V_\mu^a \) follows from eq. (2.13).

There is a lot of arbitrariness in defining axial transformations in the model. Any transformation which assigns opposite charges to the LH and RH chiral modes will reduce to the familiar axial transformation in the continuum limit.

Here we can take advantage of the global separation of the LH and RH modes in the \( s \)-direction. We define our axial transformation to act \textit{vectorially} on a given four dimensional layer, but we assign opposite charges to fermions in the two half-spaces

\[ \delta^a_A \psi_{x,s} = +iq(s)\lambda^a\psi_{x,s}, \quad (2.17) \]
\[ \delta^a_A \bar{\psi}_{x,s} = -iq(s)\bar{\psi}_{x,s}\lambda^a, \quad (2.18) \]

where

\[ q(s) = \begin{cases} 
1, & 1 \leq s \leq N, \\
-1, & N < s \leq 2N. 
\end{cases} \quad (2.19) \]

The corresponding axial currents are

\[ A_\mu^a(x) = -\sum_{s=1}^{2N} \text{sign}(N - s + \frac{1}{2})j_\mu^a(x, s). \quad (2.20) \]

For \( m = 0 \), the non-invariance of the action under the transformation (2.17) resides entirely in the coupling between the layers \( s = N \) and \( s = N + 1 \). For \( m \neq 0 \), there is an additional contribution coming from the direct coupling between the layers \( s = 1 \) and \( s = 2N \). As a result, the axial currents satisfy the following divergence equations

\[ \Delta_\mu A_\mu^a(x) = 2mJ_5^a(x) + 2J_{5q}^a(x), \quad (2.21) \]

where

\[ J_5^a(x) = j_5^a(x, 2N), \quad (2.22) \]
\[ J_{5q}^a(x) = j_5^a(x, N). \quad (2.23) \]

In order to understand the physical content of eq. (2.21) let us define quark operators as follows

\[ q(x) = P_R \psi(x, 1) + P_L \psi(x, 2N) \]
\[ \bar{q}(x) = \bar{\psi}(x, 2N)P_R + \bar{\psi}(x, 1)P_L. \quad (2.24) \]
These operators have a finite overlap with the zero modes in the chiral limit $N \to \infty$, and so they can play the role of bare quark fields. There are of course many other operators which are localized near the boundaries and, hence, can interpolate quark states. Our choice eq. (2.24) is the simplest possible one, and it leads to considerable simplification of the expressions for correlation functions.

In terms of the quark fields, $J_5^a$ takes the familiar form

$$J_5^a = \bar{q} \gamma_5 \lambda^a q.$$ (2.25)

Thus, up to the finite normalization factor in eq. (2.7), the $J_5^a$ term on the r.h.s. of eq. (2.21) is the expected contribution from a classical mass term. $J_5^a$ represents an additional, quantum breaking term. Below we will be interested in axial Ward identities of the general form

$$\Delta_{\mu} \left\langle A_{\mu}^a(x) O(y_1, y_2, \ldots) \right\rangle = 2m \left\langle J_5^a(x) O(y_1, y_2, \ldots) \right\rangle + 2 \left\langle J_{\gamma}^a(x) O(y_1, y_2, \ldots) \right\rangle + i \left\langle \delta_A^a O(y_1, y_2, \ldots) \right\rangle,$$ (2.26)

where $A_{\mu}^a$ is a non-singlet axial current. Our aim is to show that, if $O(y_1, y_2, \ldots)$ involves only the quark operators of eq. (2.24) then the anomalous term

$$\left\langle J_{\gamma}^a(x) O(y_1, y_2, \ldots) \right\rangle$$ (2.27)

vanishes in the limit $N \to \infty$.

3. Transfer matrix formalism

It is convenient to discuss non-perturbative effects using the transfer matrix formalism. This technique was adapted to domain wall fermions in ref. [15]. Here we in fact have a constant five dimensional mass $M$, and the only deviation compared to ref. [22] is in the $m_t$-dependent value of the couplings on the links connecting the layers $s = 1$ and $s = 2N$.

A simple generalization of the result of ref. [22] gives rise to the following second quantized expression for the Grassmann path integral

$$\det D_F(2N, m) = \int D\bar{\psi} D\psi e^{S_F(\bar{\psi}, \psi, U)} = (\det B)^{2N} \text{Tr} T^{2N} O(m).$$ (3.1)

The second quantized transfer matrix $T$ acts in a Fock space spanned by the action of fermionic creation operators $\hat{a}_x^\dagger$ on a vacuum state $|0\rangle$ annihilated by $\hat{a}_x$. 
The operators $\hat{a}_x$ and $\hat{a}_x^\dagger$ satisfy canonical anticommutation relations. They live on the sites $x_\mu$ of a four dimensional lattice of size $L^4$, and they also carry Dirac, colour and flavour indices which we have suppressed. The decomposition of $\hat{a}_x$ into RH and LH components is
\[
\hat{a} = \begin{pmatrix}
\hat{c} \\
\hat{d}^\dagger
\end{pmatrix}
\quad (3.2)
\]
The transfer matrix is defined by
\[
T = e^{-\hat{a}^\dagger H\hat{a}},
\quad (3.3)
\]
where
\[
e^{-H} = \begin{pmatrix}
B^{-1} & B^{-1} C \\
C^+ B^{-1} & C^+ B^{-1} C + B
\end{pmatrix},
\quad (3.4)
\]
\[
B_{x,y} = (5 - M)\delta_{x,y} - \frac{1}{2} \sum_\mu \left[ \delta_{x+\mu,y} U_{x,\mu} + \delta_{x-\mu,y} U_{y,\mu}^\dagger \right], \quad (3.5)
\]
\[
C_{x,y} = \frac{1}{2} \sum_\mu \left[ \delta_{x+\mu,y} U_{x,\mu} - \delta_{x-\mu,y} U_{y,\mu}^\dagger \right] \sigma_\mu, \quad (3.6)
\]
and $\sigma_\mu = (i, \sigma)$. Notice that $D^\parallel$ can be expressed in terms of $B$ and $C$ as follows
\[
D^\parallel = \begin{pmatrix}
1 - B & C \\
-C^\dagger & 1 - B
\end{pmatrix}. \quad (3.7)
\]
The operator $\mathcal{O}(m)$ contains all the $m$-dependence, and it is given by
\[
\mathcal{O}(m) = \prod_n (\hat{c}_n \hat{c}_n^\dagger + mc_n \hat{c}_n^\dagger) (\hat{d}_n \hat{d}_n^\dagger + md_n \hat{d}_n^\dagger). \quad (3.8)
\]
In this equation, $n$ is a generic name for all indices. The special cases $m = 0$ and $m = 1$ deserve special attention. For $m = 1$, $\mathcal{O}(m)$ becomes the identity operator, whereas for $m = 0$ it is a projection operator on a different ground state $|0\rangle$ annihilated by all the $\hat{c}$-s and $\hat{d}$-s. The relation between $|0\rangle$ and $|0\rangle'$ is
\[
|0\rangle = \prod_n \hat{d}_n^\dagger |0\rangle'. \quad (3.9)
\]
Both $|0\rangle$ and $|0\rangle'$ are “kinematical” ground states which can serve as convenient reference states in the construction of the Fock space. But none of them play a special role dynamically. Since we use creation and annihilation operators in the coordinate basis, both $|0\rangle$ and $|0\rangle'$ are very different from the filled Dirac sea even in the case of free fermions.

Like $D^\parallel$, the hermitian operator $H$ is a $N_i \times N_i$ matrix, where $N_i = 4N_cN_fL^4$. Let $R$ be the unitary matrix which diagonalizes $H$
\[
\sum_i H_{mi} R_{ni} = E_n R_{mn}. \quad (3.10)
\]
Corresponding to the splitting into two-by-two matrices in Dirac space in eq. (3.4), we write \( R \) as two \( \frac{1}{2} N_t \times N_t \) matrices \( P \) and \( Q \). We will assume that the columns of \( R \) are ordered according to their eigenvalues, with the negative eigenvalues on the left. Accordingly, we further decomposed \( P \) and \( Q \) into submatrices which contain the positive and negative eigenvectors

\[
R = \begin{pmatrix} P^- & P^+ \\ Q^- & Q^+ \end{pmatrix}.
\]

(3.11)

\( P^\pm \) and \( Q^\pm \) are \( \frac{1}{2} N_t \times N_t \) matrices, where \( N^+ + N^- = N_t \).

The ground state of the second quantized operator \( \hat{a}^\dagger H \hat{a} \) is obtained by filling all negative energy states

\[
\left| 0_H \right> = \prod_{i=1}^{N^-} \left( \hat{c}_{i,\alpha} P_{i,\alpha}^- + \hat{\tilde{c}}_{i,\alpha} Q_{i,\alpha}^- \right) \left| 0 \right>.
\]

(3.12)

In the limit \( N \rightarrow \infty \), \( T^{2N} \) is proportional to a projector on the ground state of \( \hat{a}^\dagger H \hat{a} \)

\[
T^{2N} \rightarrow \left| 0_H \right> \lambda_{\text{max}}^{2N} \left< 0_H \right|, \quad N \rightarrow \infty,
\]

(3.13)

where

\[
\lambda_{\text{max}} = \exp \left\{ - \sum_{i=1}^{N^-} E_i \right\}.
\]

(3.14)

We now turn to the scalar PV fields. The action is bilinear in these fields, and so

\[
\int \mathcal{D}\phi \mathcal{D}\phi^\dagger e^{-s_{\text{PV}}(\phi^\dagger,\phi,U)} = \text{det}^{-1} \left( D_F(N,1) D_F(N,1) \right) = \left( \text{det} B \right)^{-2N} \left( T \right)^{-2}.
\]

(3.15)

In going from the first to the second row we have used eq. (3.1).

The effective action \( S_{\text{eff}}(U) \) is defined by integrating out both fermion and PV fields. Using eqs. (3.1) and (3.15) we find

\[
\exp \{- S_{\text{eff}}\} = \frac{T \left( T \right)^{2N} \mathcal{O}(m)}{(T \left( T \right)^{N})^2}.
\]

(3.16)

In the limit \( N \rightarrow \infty \) the effective action becomes

\[
\exp \{- S_{\text{eff}}^\infty\} \equiv \lim_{N \rightarrow \infty} \exp \{- S_{\text{eff}}\} = \left< 0_H \mathcal{O}(m) \left| 0_H \right> \right>.
\]

(3.17)

Eq. (3.17) defines the \( N \rightarrow \infty \) limit of the model. This limiting case is completely well defined here, and it is free of any subtleties of the kind encountered in trying to
define chiral gauge theories on the lattice using chiral defect fermions [17]. The explicit expression eq. (3.17) together with similar expressions for correlation functions (some examples are given in App. A) provide a framework for both analytical and numerical investigations.

4. The effective action

As a first step, we want to study the behaviour of \( \langle 0_H | \mathcal{O}(m) | 0_H \rangle \) under various conditions. The physically interesting case is \( m \ll 1 \). In the case \( m = 1 \) one has \( \langle 0_H | \mathcal{O}(1) | 0_H \rangle = 1 \). The reason for this trivial result is the subtraction of the bulk effect through the PV fields. By contrast, in the opposite limit \( m = 0 \) one has

\[
\exp \{-S_{eff}\} = \left| \left\langle 0_H | 0^1 \right\rangle \right|^2, \quad m = 0. \tag{4.1}
\]

Thus, in a model of massless chiral defect fermions, the physical information is not in the trace of the transfer matrix but, rather, in the overlap of its ground state with some other state. This was first found by Narayanan and Neuberger for domain wall fermions, where the overlap formula reads [15]

\[
\exp \{-S_{eff}(\text{domain wall})\} = \left| \left\langle 0_{H+} | 0_{H-} \right\rangle \right|^2. \tag{4.2}
\]

Here \( H_{\pm} \) are the hamiltonians corresponding to the two sides of the domain wall. Hence, one has to compare two dynamical ground states. But all the non-trivial dynamics of the model is contained in the \( H_+ \) hamiltonian. In the continuum limit, eqs. (4.1) and (4.2) should describe the same physics. It is therefore advantageous to work with the boundary fermion scheme, where one has to calculate the overlap of \( |0_H\rangle \) with a fixed reference state \( |0^0\rangle \).

An explicit expression for the \( m = 0 \) overlap eq. (4.1) can be easily written down [15]. We first comment that for free fermions, as well as for perturbative gauge field configurations, the numbers of positive and negative eigenvalues of \( H \) are equal \( N^\pm = \frac{1}{2} N_t \). In this case \( P^\pm \) and \( Q^\pm \) are square matrices. Using eq. (3.9) it follows that only the \( \hat{d} \)-dependent terms in eq. (3.12) contribute to the overlap eq. (4.1). Taking into account the anticommuting character of the fermionic operators one arrives at

\[
\left\langle 0_H | 0^1 \right\rangle = \det Q^- . \tag{4.3}
\]

The phase ambiguity in defining the columns of \( Q^- \) is irrelevant, because only the modulus of \( \det Q^- \) enters eq. (4.1).

For non-perturbative configurations there may be level crossing, resulting in \( N^\pm \neq \frac{1}{2} N_t \). In this case the \( m = 0 \) overlap vanishes identically. As discussed in
ref. [15] this is a welcomed phenomenon, which signals that the chiral defect fermion model can reproduce instanton effects.

We now want to generalize the explicit expression for \( \langle 0_H | \mathcal{O}(m) | 0_H \rangle \) to \( m \neq 0 \). We first notice that \( \mathcal{O}(m) \) can be expanded as

\[
\mathcal{O}(m) = \sum_{k=0}^{N_v} m^k \sum_{l+n=k} \frac{1}{l! n!} \hat{c}_{i_1}^l \hat{d}_{j_1}^l \cdots \hat{c}_{i_n}^l \hat{d}_{j_n}^l \langle 0^e | \langle 0^l | \hat{c}_{i_1} \hat{c}_{i_2} \cdots \hat{c}_{i_n} \hat{d}_{j_1} \hat{d}_{j_2} \cdots \hat{d}_{j_n} | \langle 0^l | | 0^e \rangle \rangle. \tag{4.4}
\]

In eq. (4.4), \( m^k \) multiplies a sum over orthogonal projection operators whose number grows like \( V^k \) where \( V = L^4 \) is the four-volume. In calculating the \( m \)-expansion of the effective action, at the \( k \)-th order we will therefore encounter \( O(V^k) \) terms, where the magnitude of each term is bounded by one. This is in agreement with the anticipated behaviour of a system which undergoes spontaneous symmetry breaking. The finite volume partition function should be analytic in \( m \), but in the infinite volume limit singularities may appear because the product \( m V \) diverges.

Since we have taken the limit \( N \to \infty \), the analyticity in \( m \) of the partition function is not completely trivial. We first notice that \( \langle 0_H | \mathcal{O}(m) | 0_H \rangle \) is analytic in \( m \) and bounded. Analyticity in \( m \) follows from the definition of \( \mathcal{O}(m) \) and from the fact that the four dimensional lattice is kept finite, and so the fermionic Fock space over it is finite dimensional too. Notice that, because of the possibility of level crossing, \( \langle 0_H | \mathcal{O}(m) | 0_H \rangle \) is a non-analytic function of the group variables \( U_\mu(x) \). The analyticity in \( m \) of the \( N \to \infty \) partition function now follows from the fact that the gauge field configuration space is compact.

Returning to the calculation of \( \langle 0_H | \mathcal{O}(m) | 0_H \rangle \), let us denote

\[
\Delta = \left| \frac{1}{2} N_+ - N_- \right|. \tag{4.5}
\]

The first term in the expansion eq. (4.4) which contributes to \( \langle 0_H | \mathcal{O}(m) | 0_H \rangle \) is the term with \( k = \Delta \). This immediately implies that

\[
\langle 0_H | \mathcal{O}(m) | 0_H \rangle = O(m^\Delta). \tag{4.6}
\]

Moreover, for sufficiently small \( m \), the sign of \( \langle 0_H | \mathcal{O}(m) | 0_H \rangle \) is determined by the sign of \( m^\Delta \). Hence, \( \langle 0_H | \mathcal{O}(m) | 0_H \rangle \) will be negative for \( m < 0 \) and odd \( \Delta \). For example, in the one flavour case \( \langle 0_H | \mathcal{O}(m) | 0_H \rangle \) is negative for \( m < 0 \) if an odd number of level crossing takes place.

This behaviour is not unexpected and, in fact, it is in agreement with the familiar instanton result [24]. In the one flavour case, the fermionic determinant in an instanton background is proportional to \( m \), and so it changes sign if \( m \) does. The gauge field’s effective measure \( \exp \{ S_G + S_{\gamma j} \} \) is therefore real, but not always positive.
We still expect the partition function to be strictly positive when one approaches the continuum limit, because configurations with a non-zero topological charge are rare. But we are unable to prove the positivity of the partition function in a completely general way. Special cases where \( \exp \{ S_{eff} \} \) is strictly positive include \( m > 0 \), or \( m \neq 0 \) and even \( N_f \).

An explicit expression for \( \langle 0_H | O(m) | 0_H \rangle \) is more easily obtained using the definition eq. (3.8). Straightforward application of the canonical anticommutation relations gives rise to

\[
O(m)|0_H\rangle = m^{\frac{1}{2}} N_t \prod_{i=1}^{N_t} \left( m \delta_{i,1} P_{i,i}^+ - m^{-1} \delta_{i,1} Q_{i,i}^- \right) |0\rangle. \tag{4.7}
\]

Introducing the \( N_t \times N^- \) matrix

\[
R^- (m) = m^{\frac{1}{2}} N_t \begin{pmatrix} m P^- \\ m^{-1} Q^- \end{pmatrix}, \tag{4.8}
\]

we find

\[
\langle 0_H | O(m) | 0_H \rangle = \det R^\dagger(1) R^-(m). \tag{4.9}
\]

In the case \( m > 0 \) one can use the relation \( O(m) = O(m^{\frac{1}{2}}) O(m^{\frac{1}{2}}) \) to write

\[
\langle 0_H | O(m) | 0_H \rangle = \det R^\dagger(m^{\frac{1}{2}}) R^-(m^{\frac{1}{2}}), \tag{4.10}
\]

which is manifestly positive.

Using the transfer matrix formalism one can write down expressions for correlation functions too. The correlation functions for quark operators (2.24) take a particularly simple form. In Appendix A we give the expressions for the quark condensate and for a pion correlator.

5. The dynamical domain wall

In the previous section we discussed some properties of \( \exp S_{eff} \), and in particular we noted that its behaviour for topologically non-trivial configurations agrees with what one knows about instanton effects in the continuum. We now turn to the main issue of this paper, namely, the existence of the chiral limit for lattice QCD with chiral defect fermions.

Our aim is to show that the anomalous term (2.27) vanishes in the limit \( N \to \infty \) in all Ward identities of NSAS. But first we have to develop a criterion to identify those gauge field configurations which make the dominant contribution to the anomalous term.
Unsuppressed fermionic propagation in the $s$-direction occurs if the matrix $e^{-H}$ has a unit eigenvalue
\[ e^{-H} \psi_0 = \psi_0, \] (5.1)
where $\psi_0$ is the corresponding eigenvector. The matrix operator $e^{-H}$ is non-local, and to understand better the physical content of eq. (5.1) we want to find a simpler equation which $\psi_0$ satisfies. To proceed, we use the identity
\[ e^{-H} = KK^\dagger, \] (5.2)
where
\[ K = \left( \begin{array}{cc} B^{-1/2} & 0 \\ C^\dagger B^{-1/2} & B^{1/2} \end{array} \right). \] (5.3)
Taking into account eq. (3.7) it is now straightforward to show that
\[ 0 = (K^\dagger - K^{-1})\psi_0 = B^{-1/2}\gamma_5 D^\parallel \psi_0. \] (5.4)
Since $B$ is a positive definite operator, we conclude that $\psi_0$ is a zero mode of the hermitian operator $\gamma_5 D^\parallel$. Eq. (5.4) allows us to characterize those gauge field configurations which support unsuppressed fermionic propagation across the five dimensional slab. As we will see, the relevant configurations and the related fermionic zero modes exist for finite values of $g_0$, but they disappear in the continuum limit $g_0 \to 0$.

Since $D^\parallel(M)$ looks like a massive Dirac operator, one could ask whether eq. (5.4) has any non-trivial solutions at all. An indirect way to argue that such solutions should exist, is to observe that this is a necessary condition for $\langle 0_H | O(m) | 0_H \rangle$ to reproduce the instanton results. The vanishing of this expectation value for $m = 0$ requires that level crossing should occur in the spectrum of $H$. At the crossing point one has a solution of eq. (5.1). This observation was made by Narayanan and Neuberger [15], who also found numerically solutions of eq. (5.1).

The reason why solutions of eq. (5.1) exist in spite of the mass term present in $D^\parallel(M)$, is the unconventional sign of that mass term. For comparison, the conventional Dirac operator for massive Wilson fermions is $D^\parallel(-M)$ in our notation. The “wrong” sign of the mass term relative to the Wilson term, implies that the sum of these two terms is not a positive definite operator. As a result, Vafa-Witten bounds [23] cannot be established here. On the other hand, if one were to choose the conventional relative sign, then a Vafa-Witten bound could be establish for the propagation of fermions in all directions. This, in turn, would imply the absence of any light states in the model. Indeed, one can easily check that the massless surface modes disappear for $M < 0$. We comment that in the domain wall case, too, all
the non-trivial dynamics occurs on that side of the wall where one has the “wrong” relative sign.

In the absence of gauge fields (and in perturbation theory) the only zero modes of the complete five dimensional Dirac operator are the ones discussed in refs. [11, 20, 13]. But with dynamical gauge fields, $D^\parallel(M)$ can have other zero modes. Since all gauge field configurations are $s$-independent, any anomalously light state of $D^\parallel(M)$ will be able to propagate in the fifth direction.

An interesting example is provided by the dynamical domain wall. For simplicity we consider a $U(1)$ gauge group. (The same configuration exists also for an $SU(2)$ gauge group. In the case of general $SU(N)$ one can simply embed the below configuration in some $SU(2)$ subgroup). We consider the following configuration of link variables. We let $U_\mu(x) = 1$ for $\mu = 1, 3, 4$. For $U_2(x)$ we take

$$U_2(x) = \begin{cases} 
1, & x_1 < x_1^0, \\
-1, & x_1 \geq x_1^0. 
\end{cases} \tag{5.5}$$

Notice that this is essentially a two dimensional configuration. Eq. (5.5) describes a wall of magnetic flux, with one unit of flux going through each plaquette to the left of the line $x_1 = x_1^0$.

We now consider the ansatz

$$\psi_0 = \frac{1}{2}(1 - \gamma_1)\Psi_0(x_1). \tag{5.6}$$

One can easily check that the following is a zero mode of $D^\parallel(M)$

$$\Psi_0(x_1) = \begin{cases} 
(1 - M)^{x_1^0 - x_1}, & x_1 < x_1^0, \\
(3 - M)^{x_1 - x_1^0}, & x_1 \geq x_1^0. 
\end{cases} \tag{5.7}$$

This is the simplest case of a dynamically generated zero mode. Other gauge field configurations with a topological character should also support zero modes. These include for example lattice analogs of the static string-like singularity discussed in ref. [25]. The zero modes observed in ref. [15] are actually of that type.

Finally, we recall the close relation between the fermionic and PV action. Whenever $D^\parallel(M)$ has light excitations, the same will be true for the PV fields.

6. The chiral limit

The configuration eq. (5.5) defines a dynamical domain wall located at $x_1 = x_1^0$. It allows for massless fermionic propagation in all other directions, including in the fifth direction. A similar statement applies to the propagation of the PV field. If we
consider that gauge field configuration as a fixed background, then the anomalous term (2.27) will not be suppressed.

In the Ward identity (2.26), the anomalous term is obtained by averaging over all gauge field configurations. We first observe that configurations which support exact zero modes, have zero measure under the functional integration. This is true because, in order to have a zero mode, the Dirac operator must satisfy the constraint $\text{det} \ D^{\parallel} (M) = 0$. Moreover, on the finite four dimensional lattice, the space of gauge field configurations is finite dimensional and compact. If we keep all other parameters finite and let $N$ tend to infinity, the contribution to the anomalous term will come from a smaller and smaller region around the manifold defined by the above constraint. In the limit, the anomalous term will vanish because the volume of the relevant region in configuration space tends to zero. This proves the restoration of all Ward identities of NSAS in the chiral limit $N \to \infty$.

In practice, it is important to know how close to the chiral limit are we for finite values of the parameters of the model. We now address ourselves to the problem of estimating the magnitude of the anomalous term.

We first notice that in certain important Ward identities, such as the one which determines the pion mass, the anomalous term is positive definite for all gauge field configurations. In that case the anomalous term is the correlator of two pseudoscalar densities. As a generalization of the Wilson fermion’s case, in the present model one can prove the positivity of a whole family of correlators of pseudoscalar densities. The details can be found in Appendix B.

Since the anomalous term in the pion correlator is the integral of a positive definite quantity, the only way to put an upper bound on its magnitude is to show that there is a lower bound on the action of any gauge field configurations for which $D^{\parallel}$ has an approximate zero mode. In fact, since the configurations in question are typically extended, the relevant quantity which should be bounded is the action density $S_x$ in the region which supports the extended configuration. By definition,

$$S_G = \frac{1}{g^2} \sum_x S_x .$$  \hspace{1cm} (6.1)

As an example, for the Wilson action one has

$$S_x = \sum_{\mu \neq \nu} \text{Re} \ tr (I - U_{\mu \nu}) .$$  \hspace{1cm} (6.2)

Here $U_{\mu \nu}$ is the plaquette variable. While our reasoning will apply most directly to the Wilson action eq. (6.2), our arguments are rather general and should apply to any lattice action which reduces to the standard Yang-Mills action in the classical continuum limit.
The plaquette variable has the following expansion

\[ I - U_{\mu\nu} = i a^2 F_{\mu\nu} + \cdots \]  \hfill (6.3)

where \( F_{\mu\nu} \) is the covariant field strength and we have reinstated the lattice spacing \( a \). The dots stand for higher order terms in an expansion in powers of \( a \). Any acceptable lattice action should have the following expansion

\[ S = \frac{1}{4} a^4 F_{\mu\nu}^2 + \cdots \]  \hfill (6.4)

If the action density is very small everywhere, the gauge field configuration should be smooth in some sense. (We will see below in precisely what sense this is true). Let us now assume that the action density obeys the bound

\[ |S_x| \leq \epsilon^4, \]  \hfill (6.5)

for all \( x \) and \( \epsilon \ll 1 \). In view of eqs. (6.3) and (6.4) this is equivalent to the bound

\[ |I - U_{\mu\nu}| \leq \epsilon^2. \]  \hfill (6.6)

We will now show that if the bound (6.6) holds, then all eigenvalues of \( \gamma_5 D^\parallel \) are \( O(1) \). Our reasoning will also implies that in order to have an arbitrarily small eigenvalue, the action density must be \( O(1) \) somewhere. As a result, we obtain the lower bound

\[ S_{\text{min}} = \frac{c_0}{g_0^2}, \]  \hfill (6.7)

where \( c_0 \) is some \( O(1) \) constant, for any gauge field configuration which gives rise to an (approximate) zero mode of \( D^\parallel \). These configurations will therefore be suppressed in the continuum limit.

We begin by writing \( D^\parallel \) as

\[ D^\parallel = M - W + i D_K. \]  \hfill (6.8)

Here \( W \) stands for the four dimensional Wilson term. Comparison to eq. (2.4) implies that \( W \) as defined above is a non-negative operator. \( D_K \) is the massless hermitian Dirac operator, containing the \( \gamma_\mu \)-dependent terms in eq. (2.4).

Consider now the non-negative second order operator

\[ (D^\parallel)^\dagger D^\parallel = (M - W)^2 + D_K^2 - i [ W, D_K]. \]  \hfill (6.9)

The reader can easily verify that each term in the commutator \([ W, D_K]\) is the product of a factor of the form \( I - U_{\mu\nu} \), times a two step covariant translation. The bound (6.6) therefore implies an identical bound on the commutator term in eq. (6.9).
The other two terms on the r.h.s. of eq. (6.9) are non-negative. As a result, in order to have an approximate zero mode \( \psi_0 \) of \( D^\parallel \), \( \psi_0 \) must be simultaneously an approximate zero mode of \( D_K \) and of \( M - W \). This, however, is impossible. Physically, the reason is that any low energy eigenstate of \( D_K \) on the lattice corresponds (in a sufficiently regular gauge) to an eigenstate of the massless \textit{continuum} Dirac operator. But any state with this property will be annihilated by the Wilson term. Consequently, any such state will be an approximate eigenstate of \((M - W)^2\) whose eigenvalue is equal to \( M^2 \).

In more detail, let \( \psi_0 \) be an (approximate) eigenstate of \( D_K^2 \) whose eigenvalue is \( O(\epsilon^2) \). In addition, assume that the \textit{link variables} obey the bound

\[
|I - U_\mu(x)| \leq c \epsilon^2 , \tag{6.10}
\]

everywhere inside some domain \( \mathcal{D} \). Denote by \( W_D \) the restriction of the Wilson term to \( \mathcal{D} \), defined by summing the relevant terms in eq. (2.4) over all \( x \in \mathcal{D} \). It then follows that

\[
\| W_D \psi_0 \| \leq O(\epsilon^2) \| \psi_D' \| , \tag{6.11}
\]

where \( \psi_D' \) denotes the restriction of \( \psi_0 \) to a domain \( \mathcal{D}' \) which is bigger than the domain \( \mathcal{D} \) by two lattice sites in each direction. Inequality (6.11) follows because \( D_K^2 \) is the sum of a term proportional to \( I - U_\mu \) and a Wilson-like term \( W' \) which involves next to nearest neighbours. One can therefore bound the l.h.s. of eq. (6.11) by writing \( W = (W - W') + W' \), and applying the bound (6.10) to \( W - W' \). The fact that the domain \( \mathcal{D}' \) can be chosen to be only slightly bigger than \( \mathcal{D} \) reflects the locality of the Wilson term.

The bound (6.6) has a gauge invariant significance. On the other hand, \( I - U_\mu(x) \) does not transform homogeneously, and so a non-trivial bound on \( I - U_\mu(x) \) can exist only for some definite gauge choice. Assuming the bound (6.6) holds, we can impose inequality (6.10) \textit{locally} and \textit{uniformly} in the following sense. We first divide the entire lattice into many small domains of a maximal size \( l \) in lattice units. In every domain it is then possible to choose a gauge where the bound (6.10) applies, with the \textit{same} constant \( c \) in all domains and where \( c \) is linearly related to \( l \). In a given domain this is achieved by first setting all link variables to the identity on a maximal tree, and then adjusting the remaining link variables so as to obtain the desired value for each plaquette. (An interesting question is whether one can prove the existence of a \textit{globally} regular gauge, where the vector potential satisfies \( aA_\mu \approx I - U_\mu(x) = O(\epsilon) \) and \( \epsilon/a \) is some physical scale. While we have some circumstantial evidence that this may be true, such a stronger result is not necessary for our purposes).

Using the uniform bound (6.10) in every domain and summing over all domains
we conclude that, for a normalized state, the norm of \( W \psi_0 \) is \( O(\epsilon^2) \) too. As a result,

\[
(M - W)^2 \psi_0 = M^2 \psi_0,
\]

(6.12)

up to negligible corrections. We have thus shown that the bound (6.5) implies the absence of common (approximate) zero modes of \( D^2_K \) and \( (M - W)^2 \), and, hence, that all eigenvalues of \( (D^\parallel)^{\downarrow}D^\parallel \) are \( O(1) \).

Actually, we have to prove a slightly different statement, namely, we have to show that given the bound (6.5), the operator \( e^{-H} \) cannot have an eigenvalue which is arbitrarily close to unity. But this is a straightforward corollary of what we already know. Suppose that \( \psi_0 \) satisfies

\[
e^{-H} \psi_0 = (1 + \epsilon') \psi_0.
\]

(6.13)

This can be rewritten as

\[
(\gamma_5 D_F - \epsilon' B^\parallel K^{-1}) \psi_0 = 0.
\]

(6.14)

The operator \( B^\parallel K^{-1} \) is local and bounded, and so we can proceed by taking the absolute square of \( \gamma_5 D_F - \epsilon' B^\parallel K^{-1} \), and we arrive at the same contradiction as before.

The above discussion implies that the anomalous term in the pion correlator is proportional to \( \exp(-\alpha_0/g_0^2) \) times some function of \( N \). For finite values of \( N \) and \( g_0 \), we expect the correlation length in the \( s \)-direction to be equal to the confinement length \( \Lambda^{-1}(g_0) \). Let \( U_\mu(x) \) be a configuration of link variables which supports a zero mode of \( D^\parallel \). Recall that the field strength of this configuration must be singular in the continuum limit. Now consider a new configuration given by \( U'_\mu(x) = U_{\mu}(x)U_\mu(x) \), where \( U_\mu(x) \) is some typical configuration which contributes to the QCD vacuum. The class of configurations \( U'_\mu(x) \) has the following properties. First, to leading order in \( a^2 \), the action of all such configurations is equal to the action of the singular configuration \( U_{\mu}(x) \). Second, apart from the overall factor of \( \exp(-\alpha_0/g_0^2) \), such configurations have a finite phase space in the continuum limit. Finally, going from \( U_{\mu}(x) \) to \( U'_\mu(x) \) typically changes the eigenvalue of the zero mode by an amount which is \( O(\alpha) \).

The dominant contribution to the anomalous term eq. (2.27) comes from the subspace of configurations \( U'_\mu(x) \) described above. These configurations give rise to a correlation length in the \( s \)-direction which is \( O(\Lambda^{-1}) \) in lattice units. Taking into account eq. (6.7) we thus arrive at the following estimate

\[
\exp(-\alpha_0/g_0^2) \exp(-\Lambda N),
\]

(6.15)

for the magnitude of the anomalous term.
7. Conclusions

In this paper we considered the formulation of lattice QCD using the boundary fermions scheme. We proved the vanishing of all anomalous terms in the Ward identities of non-singlet axial symmetries in the limit \( N \to \infty \) while taking into account all quantum effects (both perturbative and non-perturbative). To our knowledge, this is the first proof of this kind.

We also gave an estimate of the magnitude of anomalous effects for finite values of the parameters of the model. In numerical simulations one should make sure that the product in eq. (6.15) is sufficiently small, in order that lattice artifacts will be suppressed in physical observables which are sensitive to the approximate axial symmetries of the QCD lagrangian. Moreover, in the present formulation the anomalous term is defined unambiguously by eq. (2.27) which is free of any (explicit or implicit) fine tuning. Thus, one can always measure the anomalous term directly. This is an important advantage, because it allows one to have an accurate knowledge of the error induced by the finite lattice artifacts.

Alternatively, one can work directly in the “\( N = \infty \)” formalism. This is a consistent framework, whose expressions for the effective action as well as for quark correlators are developed in this paper. The application of this formalism depends on the availability of efficient methods for calculating fermionic expectation values of the kind that occur on the r.h.s. of eq. (3.17).

Apart from its potential use in numerical simulations, the present model is also a convenient tool for analytical studies of chiral symmetry breaking in lattice QCD. An interesting observation which follows from the presence of the factor \( \exp(-\Lambda N) \) in eq. (6.15), is that the anomalous terms should vanish in the limit \( N \to \infty \) regardless of the value of the bare coupling \( g_0 \). If eq. (6.15) remains true when extrapolated to strong coupling, it should be possible to obtain a massless pion in the strong coupling limit as well. The study of the strong coupling behaviour of the model may therefore provide further insight into the relation between chiral symmetry breaking and confinement [26].

finally, we wish to comment here on an alternative approach, in which one keeps \( N \) finite and interprets the small tree level overlap between the LH and RH components of the quark’s wave function as a classical mass term. Since the perturbative overlap is proportional to \( (1 - M)^2N \), one is lead to a relatively small fifth dimension \( N \approx -\log m \). This is of course advantageous from the point of view of numerical simulations. However, we see serious difficulties with the interpretation of the perturbative overlap as a current mass.
One problem with this approach is the big uncertainty in determining what the precise current mass really is. One would have to calculate perturbative corrections to the tail of the quark’s wave function in order to determine the renormalized current mass.

A related difficulty is that one no longer has a clear separation between “anomalous” and “non-anomalous” effects. Using the definition (2.17) for axial transformations, we see that the very same terms in the action (those terms that couple the layers \( s = N \) and \( s = N + 1 \)) should give rise to an anomalous effect in the case of the singlet current, while in the case of non-singlet currents we wish to interpret them as describing a classical mass term. (This problem is generic, and it will arise for any acceptable definition of axial transformations). At the non-perturbative level it may therefore be very difficult, if not impossible, to disentangle spontaneous breaking effects from anomalous ones. Since different observables will in general have different sensitivity to the details of the tail of the quark’s wave function, we may end up with mutually incompatible values for the current mass if we try to impose current algebra relations in several different channels.

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A. Correlation functions in the \( N = \infty \) limit

Using the transfer matrix formalism one can write down expressions for correlation functions too. The correlation functions for quark operators (2.24) take a particularly simple form in the limit \( N = \infty \). For example, the expression for the quark condensate is

\[
\langle \bar{q}q \rangle = -\left\langle 0_H \left| \hat{c}^\dagger \mathcal{O}(m) \hat{c} \right| 0_H \right\rangle - \left\langle 0_H \left| \hat{d}^\dagger \mathcal{O}(m) \hat{d} \right| 0_H \right\rangle. \tag{A.1}
\]

An example of a pion correlator in the case \( N_f = 2 \) is

\[
\langle \pi^- (x) \pi^+ (y) \rangle = -\left\langle 0_H \left| \hat{c}_{x \downarrow}^\dagger \hat{c}_{y \uparrow} \mathcal{O}(m) \hat{c}_{x \uparrow} \hat{c}_{y \downarrow} \right| 0_H \right\rangle + \left\langle 0_H \left| \hat{d}_{x \downarrow}^\dagger \hat{d}_{y \uparrow} \mathcal{O}(m) \hat{c}_{x \uparrow} \hat{c}_{y \downarrow} \right| 0_H \right\rangle + \left\langle 0_H \left| \hat{d}_{x \downarrow}^\dagger \hat{c}_{y \uparrow} \mathcal{O}(m) \hat{d}_{x \uparrow} \hat{c}_{y \downarrow} \right| 0_H \right\rangle - \left\langle 0_H \left| \hat{d}_{x \downarrow}^\dagger \hat{d}_{y \uparrow} \mathcal{O}(m) \hat{d}_{x \uparrow} \hat{d}_{y \downarrow} \right| 0_H \right\rangle. \tag{A.2}
\]

Here

\[
\pi^- = \bar{q} \gamma_5 q, \quad \pi^+ = \bar{q} \gamma_0 q.
\]
\[
\pi^+ = \bar{q}_1 \gamma_5 q_1. \tag{A.3}
\]

The arrows denote isospin. Notice that the pion operators are special cases of the pseudoscalar densities eq. (2.22).

The general prescription for correlation functions of the quark operators eq. (2.24) is the following. Considering \( q_{R,L} \) and \( \bar{q}_{R,L} \) as Grassmann variables, one first reorder each product of quark operators such that \( q_L \) and \( \bar{q}_L \) occur to the left of all \( q_R \) and \( \bar{q}_R \). This step may result in a minus sign. One then translates the result into a matrix element of the form
\[
\left\langle 0_H \cdots \mathcal{O}(m) \cdots | 0_H \right\rangle. \tag{A.4}
\]

The operators to the left of \( \mathcal{O}(m) \) are obtained from the ordered product of \( q_L \)-s and \( \bar{q}_L \)-s by the substitution
\[
\begin{align*}
\bar{q}_L & \to \gamma^4, \\
q_L & \to \gamma^4.
\end{align*} \tag{A.5}
\]

Similarly, on the right of \( \mathcal{O}(m) \) one makes the substitution
\[
\begin{align*}
\bar{q}_R & \to \gamma^5, \\
q_R & \to -\gamma^5.
\end{align*} \tag{A.6}
\]

All indices are left unchanged in this substitution. (The transition from the Grassmann path integral to operator language involves a non-local transformation at an intermediate step [22, 15]. But this non-locality cancels out in the final expression).

**B. Inequalities**

Eq. (2.10) implies an analogous identity for fermion propagator, considered as a matrix. Writing the coordinates explicitly one has
\[
\gamma_5 G(x,s; y, s') \gamma_5 = G^\dagger(y, 2N + 1 - s'; x, 2N + 1 - s). \tag{B.1}
\]

Here the dagger refers only to the the suppressed internal indices. We will use this identity to prove the positivity of correlators of the following pseudoscalar densities
\[
K^a(x, z) = \bar{\psi}(x, N + z) P_R \lambda^a \psi(x, N + 1 - z) \\
- \bar{\psi}(x, N + 1 - z) P_L \lambda^a \psi(x, N + z). \tag{B.2}
\]

Notice the special cases
\[
K^a(x, 0) = J^a_{\gamma_5}(x), \tag{B.3}
\]
\[ K^a(x, N) = J^a_\alpha(x). \tag{B.4} \]

We assume \( m_i = m \), \( i = 1, \ldots, N_f \). Integrating out the fermion fields and using eq. (B.1) we obtain

\[ \langle K^a(x, z) K^a(y, z') \rangle = \delta_{a, a'} \int DU e^{-S(U)} (\det D_F(U))^N f I(U; x, z; y, z'), \tag{B.5} \]

\[ I = tr \left\{ P_R G(y, N + z'; x, N + 1 - z) P_R G^\dagger(y, N + z'; x, N + 1 - z) \right. \]
\[ + P_R G(y, N + 1 - z'; x, N + 1 - z) P_R G^\dagger(y, N + 1 - z'; x, N + 1 - z) \]
\[ + P_L G(y, N + z'; x, N + z) P_L G^\dagger(y, N + z'; x, N + z) \]
\[ + P_L G(y, N + 1 - z'; x, N + z) P_L G^\dagger(y, N + 1 - z'; x, N + z) \right\}. \tag{B.6} \]

For even \( N_f \) or for \( m > 0 \), the factor \((\det D_F(U))^N\) is positive. We assume that one of these conditions is satisfied. (Otherwise one cannot prove the positivity of pseudoscalar correlators). It is now straightforward to prove the positivity of each term in eq. (B.6). Consider the second row as an example. Independently of the values of \( x \) and \( y \), it has the generic form

\[ tr \left\{ P_R A P_L A^\dagger \right\} = tr \left\{ P_R^2 A P_L^2 A^\dagger \right\} \]
\[ = tr \left\{ (P_R A P_L^\dagger)(P_R A P_L) \right\}. \tag{B.7} \]

The last row is manifestly positive.

Consider now the Ward identity (2.26) for the special case \( O(y) = \pi^a(y) \). This Ward identity determines the pion mass. Notice that

\[ \pi^a(y) = K^a(x, N). \tag{B.8} \]

Using eq. (B.3) the anomalous term in this Ward identity can be written as

\[ \langle K^a(x, 0) K^a(x, N) \rangle, \tag{B.9} \]

which is positive according to the above discussion. The same reasoning proves the positivity of the two-pion correlator.

The positivity of the two-pion correlator can also be established directly from the \( N = \infty \) formula eq. (A.2). One should notice that the Fock space is a direct product of the Up and Down Fock spaces. Thanks to factorization of \( O(m) = O_1(m) O_2(m) \), each expectation value in eq. (A.2) becomes the product of two complex conjugate expectation values, one in the Up Fock space and one in the Down Fock space.

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References


