BEAM IMPEDANCE

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1. What is beam impedance?

2. Beam impedance is modelled as a lumped impedance

3. New formula for longitudinal beam impedance

4. Panofsky-Wenzel theorem and transverse impedance

5. Lab measurements of beam impedance
What is beam impedance?

• Beam impedance is just a normal impedance.

• However, it is very difficult to understand beam impedance because it is not a lumped impedance but measured over a length.

• In addition it is defined as the difference in impedance between an accelerator equipment and a straight vacuum chamber. The straight vacuum chamber must have constant cross-section; have the same length as the accelerator equipment and have walls that are superconducting (also called perfectly conducting PEC).

• A particle moving in a straight vacuum chamber with constant cross-section and superconducting walls have no beam impedance.
What is beam impedance?

An accelerator without beam impedance does not have instabilities. Beam impedance is **not** our friend!

Beam impedance gives the beam a **kick** i.e. a disturbing force acting on the beam. The beam impedance forces will make the beam oscillate, just like a mass suspended between springs:

NB! Landau damping is not shown because it is not damping, in spite of the name!

**Synchrotron radiation**
What is beam impedance?

An example of transverse impedance, that gives the beam a transverse kick! Here measured with the beam...
What is beam impedance?

There are many types of beam impedance:

• Beam impedance from the **currents in the walls** of accelerator equipment (beam coupling impedance):
  1) Resistive wall impedance
  2) Geometric impedance

• **Space charge** beam impedance
  1) Direct space charge impedance
  2) Indirect space charge impedance

• Damping kicker impedance, Electron cloud, impedance, ...

\[ Z_{sc}(w) \propto \frac{1}{\gamma^2} \]
What is beam impedance?

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What is beam impedance?

Vacuum chamber = Faraday cage

The wall currents must oppose the beam current, so that the fields outside the vacuum chamber are zero.
What is beam impedance?

When we calculate the beam impedance for an equipment, we compare the equipment to a perfectly conducting (PEC) vacuum chamber with the same dimensions at start and end.
What is beam impedance?

Classical thick wall regime:

\[ R = w L \]

For superconducting (PEC) vacuum chamber:

\[ \delta(w) = \sqrt{\frac{2\rho}{\mu_0 w}}, \text{ where } \rho = \text{resistivity} \]

Current density estimation

This area "\( \ell \delta(w) \)" represents the difference between superconducting (PEC) vacuum chamber and one with resistance.

\[ Z(w) = R(w) + iwL(w) \]

\[ R(w) = w \cdot L(w) = \frac{\ell \cdot \rho}{2\pi b \cdot \delta(w)} \]

\[ \ell = \int_b^{b+\delta(w)} B(r) \ell \, dr \]

\[ \phi = \int_b^{b+\delta(w)} B(r) \ell \, dr \]

\[ L = \frac{\phi}{I} = \frac{\mu_0 \delta(w)}{4\pi b} \ell \]

Courtesy of M. Migliorati
What is beam impedance?

\[
Z_{\text{Theo}}[w] = \frac{(1 + \frac{1}{w}) * L}{2 \pi (g / 2) \sigma \delta[w]}
\]

Skin depth:
\[
\delta[w] = \sqrt{\frac{2}{w \mu \sigma}}
\]

Collimator:
Length: 200 mm
Width: 120 mm
Height: 60 mm
Electrical conductivity of jaws: \(\sigma = 100 \text{ S/m}\)

\[Z(\omega) \propto \sqrt{\omega}\]
In my experience, accelerator components have only resistive and inductive coupling impedance.
Beam impedance modelled as lumped impedance

Definition of beam impedance:

\[ V(s) = - \int_0^L E(s, z) \, dz = \text{Voltage over equipment} \]

\[ V(t) = V(s) , \text{ where } s = v \cdot t \]

\[ I_d(t) = q_d \cdot \delta(t) \quad \text{Drive particle act as a current.} \]

\[ \text{(It's a Dirac delta function)} \]

\[ Z(w) = \frac{V(w)}{I_d(w)} = \frac{\mathcal{F}(V(t))}{\mathcal{F}(I_d(t))} = \frac{\int_{-\infty}^{\infty} V(t) \, e^{-iwt} \, dt}{q_d} \]

\[ Z(w) = \int_{-\infty}^{\infty} W_{\parallel}(t) \, e^{-iwt} \, dt , \text{ where } W_{\parallel}(t) = \frac{V(t)}{q_d} \]

Definition of lumped impedance:

\[ Z(w) = \frac{V(w)}{I(w)} = \frac{\mathcal{F}(V(t))}{\mathcal{F}(I(t))} = \frac{\int_{-\infty}^{\infty} h(t) \, e^{-iwt} \, dt}{\int_{-\infty}^{\infty} \delta(t) \, e^{-iwt} \, dt} \]

\[ Z(w) = \int_{-\infty}^{\infty} h(t) \, e^{-iwt} \, dt \]

\[ h(t) = \text{impulse response} \]
The wake function $W_{\parallel}(t)$ is the equipment response function, i.e. the response to a Dirac delta function. The impedance is, according to normal theory, just the Fourier transform of the response function:

$$W_{\parallel}(t) = \frac{V(t)}{q_d} = \frac{V(s)}{q_d} = -\int_0^L E(s, z) \, dz$$

where $s = v \cdot t$

$$Z(w) = \int_{-\infty}^{\infty} W_{\parallel}(t) \, e^{-iwt} \, dt$$
Beam impedance modelled by lumped impedance

\[ Z(w) = \int_{-\infty}^{\infty} W_{\parallel}(t) \, e^{-i\omega t} \, dt \]

In other texts (See e.g. Ref. [6]) one will often find this definition:

\[ Z_{\parallel}(\omega) = \int_{-\infty}^{\infty} W_{\parallel}(z) \exp \left( -\frac{i\omega z}{c} \right) \frac{dz}{c} \]
Wall currents generate electro-magnetic fields i.e. photons when bend along the cavity walls.

The electro-magnetic fields stays in the cavity and generates a resonance, which will disturb i.e. kick the following bunch.

A resonance is modeled as a RLC-circuit:

R | L
C

\[ Z_{\parallel}(\omega) = \frac{R}{1 + jQ(\omega \omega_0 - \omega_0 \omega)} \]

\[ k_{\text{loss}} = \frac{1}{2} \frac{0}{0} \left\{ Z_{\parallel}(\omega) \right\} d \]

\[ 0 = \frac{1}{\sqrt{LC}}; Q = R \sqrt{\frac{C}{L}} \]

The energy lost, is equal to the loss factor “\( k_{\text{loss}} \)” multiplied with the square of the charge of the bunch:

\[ E_{\text{loss}} = k_{\text{loss}} \cdot q_{\text{bunch}}^2 \]

NB! This definition of the loss factor is only valid for a bunch that is a dirac delta function. The more general definition will be given later.
New formula for longitudinal beam impedance

The Longitudinal beam impedance is a function of the transverse position of the drive and test particles i.e. 4 variables. It can therefore be decomposed into 15 parameters ($Z_0$, $Z_{1_{xd}}$, $Z_{1_{xt}}$, etc..) that represent all combinations of the 4 variables:

$$Z[xd, xt, yd, yt] = Z0$$

$$+ Z_{1_{xd}} \cdot xd + Z_{1_{xt}} \cdot xt + Z_{1_{yd}} \cdot yd + Z_{1_{yt}} \cdot yt$$

$$+ Z_{2_{xdxd}} \cdot xdxd + Z_{2_{xtxt}} \cdot xtxt + Z_{2_{ydyd}} \cdot ydyd + Z_{2_{ytyt}} \cdot ytyt$$

$$+ Z_{2_{xdxt}} \cdot xdxt + Z_{2_{xdyd}} \cdot xdyd + Z_{2_{xdyt}} \cdot xdyt$$

$$+ Z_{2_{xtyd}} \cdot xtyd + Z_{2_{xtyt}} \cdot xtyt + Z_{2_{ydyt}} \cdot ydyt$$
New formula for longitudinal beam impedance

Holomorphic decomposition:
Any two dimensional field, and very importantly a field that can really exist (so not an artificially constructed field), can be decomposed into multipolar components. This is the same idea used in Fourier transforms. The holomorphic decomposition expands the field into normal and skew multipolar functions:

\[ f(x, y) = a_0 + a_{1,\text{normal}} \cdot x + a_{1,\text{skew}} \cdot y + a_{2,\text{normal}} \cdot \left( \frac{x^2}{2} - \frac{y^2}{2} \right) + a_{2,\text{skew}} \cdot (xy) + \ldots \]

Zero order                                first order                                second order

NB! Notice that the coefficients for x squared and y squared are same numerical value but opposite signs
New formula for longitudinal beam impedance

The normal and skew multipolar functions are well known from accelerator magnets:

\[ f(x, y) = a_0 + a_{1,\text{normal}} \cdot x + a_{1,\text{skew}} \cdot y + a_{2,\text{normal}} \cdot \left( \frac{x^2}{2} - \frac{y^2}{2} \right) + a_{2,\text{skew}} \cdot (xy) + \ldots \]

Zero order  first order  second order

Dipole  Quadrupole  Sextupole  Octupole
New formula for longitudinal beam impedance

Using the holomorphic decomposition for both the drive and test particles, knowing that the coefficients for the squared values of $xd$ & $yd$ and $xt$ & $yt$ must be of opposite sign, the formula can be reduced to 13 terms:

$$Z[xd, xt, yd, yt] = Z0$$

$$+ Z1_{xd} \cdot xd + Z1_{xt} \cdot xt + Z1_{yd} \cdot yd + Z1_{yt} \cdot yt$$

$$+ Z2_{drive} \cdot (xdxd - ydyd) + Z2_{test} \cdot (xtxt - ytyt)$$

$$+ Z2_{xdxt} \cdot xdxt + Z2_{xdyd} \cdot xdyd + Z2_{xdyt} \cdot xdyt$$

$$+ Z2_{xtyd} \cdot xtyd + Z2_{xtyt} \cdot xtyt + Z2_{ydyt} \cdot ydyt$$
Using a property, called the Lorentz reciprocity principle, which says that if we exchange the positions of the drive and test particles, the beam impedance stays unchanged, i.e. $$Z[x_d, x_t, y_d, y_t] = Z[x_t, x_d, y_t, y_d]$$.

This leads to 5 equalities:

$$Z_{1x_d} = Z_{1x_t}, \quad Z_{1y_d} = Z_{1y_t}, \quad Z_{2\text{drive}} = Z_{2\text{test}}, \quad Z_{2xyd} = Z_{2xty}, \quad Z_{2xdy} = Z_{2xtyd}$$

The new formula for longitudinal beam impedance finally has only 8 terms:

$$Z[x_d, x_t, y_d, y_t] = Z_0 + Z_{1x} \cdot (x_d + x_t) + Z_{1y} \cdot (y_d + y_t) + Z_{2A} \cdot (x_dxd - y_dyd + xtxt - ytyt) + Z_{2B} \cdot (xxyd + xtyt) + Z_{2C} \cdot (xyyt + xtyd) + Z_{2D} \cdot xdx + Z_{2E} \cdot ydyt$$
Using a property, called the Lorentz reciprocity principle, which says that if we exchange the positions of the drive and test particles, the beam impedance stays unchanged, i.e.  
\[ Z[x_d, x_t, y_d, y_t] = Z[x_t, x_d, y_t, y_d]. \]

This leads to 5 equalities:

\[ Z_{1,x_d} = Z_{1,x_t}, \quad Z_{1,y_d} = Z_{1,y_t}, \quad Z_{2,\text{drive}} = Z_{2,\text{test}}, \quad Z_{2,x_{dyd}} = Z_{2,x_{tyt}}, \quad Z_{2,x_{dty}} = Z_{2,x_{tyd}} \]

The new formula for longitudinal beam impedance finally has only 8 terms:

\[ Z[x_d, x_t, y_d, y_t] = Z_0 \]

\[ + Z_{1,x} \cdot (x_d + x_t) + Z_{1,y} \cdot (y_d + y_t) \]

\[ + Z_{2_A} \cdot (x_d x_d - y_d y_d + x_t x_t - y_t y_t) \]

\[ + Z_{2_B} \cdot (x_d y_d + x_t y_t) + Z_{2_C} \cdot (x_d y_t + x_t y_d) \]

\[ + Z_{2_D} \cdot x_d x_t + Z_{2_E} \cdot y_d y_t \]

Quadrupolar term →
Using a property, called the Lorentz reciprocity principle, which says that if we exchange the positions of the drive and test particles, the beam impedance stays unchanged, i.e. 

\[ Z[xd, xt, yd, yt] = Z[xt, xd, yt, yd] \]

This leads to 5 equalities:

\[
\begin{align*}
Z_{1xd} &= Z_{1xt}, & Z_{1yd} &= Z_{1yt}, & Z_{2\text{drive}} &= Z_{2\text{test}}, & Z_{2xdyd} &= Z_{2xtyt}, & Z_{2xxyt} &= Z_{2xtyd} 
\end{align*}
\]

The new formula for longitudinal beam impedance finally has only 8 terms:

\[
Z[xd, xt, yd, yt] = Z0 + Z1x \cdot (xd + xt) + Z1y \cdot (yd + yt) + Z2_A \cdot (xdxd - ydyd + xtxt - ytyt) + Z2_B \cdot (xdyd + xttyt) + Z2_C \cdot (xdyt + xtyd) + Z2_D \cdot xdxxt + Z2_E \cdot ydyt
\]
The longitudinal beam impedance have 8 parameters

Interchanging the drive and test particles, will give the same beam impedance.

It is caused by the Lorentz reciprocity theorem (well known to RF people as the identity $S_{21} \equiv S_{12}$):
The longitudinal beam impedance have 8 parameters

The **Lorentz reciprocity theorem** is responsible for coupling primary and secondary windings in a transformer:

\[
V_1 = -A \cdot \mu_0 \mu_r \cdot \frac{N_1 \cdot N_2}{C} \cdot \frac{dI_2}{dt} \\
V_2 = -N_2 \cdot \frac{d\phi}{dt} = -A \cdot \mu_0 \mu_r \cdot \frac{N_1 \cdot N_2}{C} \cdot \frac{dI_1}{dt}
\]

\[
\phi = B \cdot A \\
\oint_C B \, dl = \mu_0 \mu_r I_1 \cdot N_1 \\
\Rightarrow \phi = A \cdot \mu_0 \mu_r \cdot \frac{N_1}{C} \cdot I_1
\]
The longitudinal beam impedance have 8 parameters.

The Lorentz reciprocity theorem is responsible for coupling primary and secondary windings in a transformer:

$$V_1 = -A \cdot \mu_0 \mu_r \cdot \frac{N_1}{C} \frac{dI_1}{dt}$$

$$V_2 = -N_2 \frac{dI_1}{dt}$$

This is why the name of a beam impedance that is generated by the wall currents is a beam coupling impedance.

$$\phi = B \cdot A$$

$$\oint_C B \, dl = \mu_0 \mu_r I_1 \cdot N_1$$

$$\implies \phi = A \cdot \mu_0 \mu_r \cdot \frac{N_1}{C} \cdot I_1$$
The longitudinal beam impedance have 8 parameters

The new formula shows that 90 degree symmetrical structures only have dipolar impedance and that this impedance is the same in all directions.
New formula for longitudinal beam impedance

This new formula is not valid for resonances nor for non-relativistic beams $\beta < 1$, because both are spread out in 3D.

The formula is practically valid for beams with $\beta \approx 1$, even though theoretically there will always be other terms, but these terms are proportional to $\frac{1}{\gamma^2}$, so will not be important in practice:

$$Z[xd, xt, yd, yt] = Z_0$$
$$+ Z_{1x} \cdot (xd + xt) + Z_{1y} \cdot (yd + yt)$$
$$+ Z_{2A} \cdot (x^2yd - y^2yd + xt^2t - y^2yt)$$
$$+ Z_{2B} \cdot (xdyd + xt^2t) + Z_{2C} \cdot (xdyt + xt^2y)$$
$$+ Z_{2D} \cdot xdx^t + Z_{2E} \cdot ydy^t$$
The rigid bunch approximation states that the beam motion is little affected during the passage through the structure. So the beam shape is rigid and it always moves unchanged with the bunch.

The force acting on the test particle:

\[ F(x, y, z, t) = q(E(x, y, z, t) + \mathbf{v} \times \mathbf{B}(x, y, z, t)) \]

\[ \nabla \times F = \nabla \times q(E + \mathbf{v} \times \mathbf{B}) \]

Using Maxwell’s equations:

\[ \nabla \times E(x, y, z, t) = -\frac{\partial \mathbf{B}(x, y, z, t)}{\partial t} \]

\[ \nabla \cdot \mathbf{B}(x, y, z, t) = 0 \]

\[ \nabla \times F = q \left[-\frac{\partial \mathbf{B}}{\partial t} - \beta c \frac{\partial \mathbf{B}}{\partial z}\right] \]
The force acting on the test particle:

$$\nabla \times F = q \left[ -\frac{\partial B}{\partial t} - \beta c \frac{\partial B}{\partial z} \right]$$

$$\nabla \times F = \begin{vmatrix} \hat{x} & \hat{y} & \hat{z} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ F_x & F_y & F_z \end{vmatrix} = \left( \frac{\partial F_z}{\partial y} - \frac{\partial F_y}{\partial z} \right) \hat{x} + \left( \frac{\partial F_x}{\partial z} - \frac{\partial F_z}{\partial x} \right) \hat{y} + \left( \frac{\partial F_y}{\partial x} - \frac{\partial F_x}{\partial y} \right) \hat{z}$$

$$\nabla \times F = \frac{\partial F \perp}{\partial z} - q \left[ -\frac{\partial B_y}{\partial t} \hat{y} - \beta c \frac{\partial B_y}{\partial z} \hat{y} + \frac{\partial B_x}{\partial t} \hat{x} + \beta c \frac{\partial B_x}{\partial z} \hat{x} \right]$$

$$\nabla \perp F_z(x, y, z, \tau) = -\frac{\partial F \perp(x, y, z, \tau)}{\partial s}$$

*Very important:* Because the wakefield is only a function of “s” then: $B(s)$

This leads to

Position of drive particle: $v \cdot t$

Position of the test particle: $z$

When inserting the partial differentials on the right, the terms in the bracket cancels out and gives zero.

$z = vt - s \implies \begin{cases} \frac{\partial s}{\partial t} = v \\ \frac{\partial s}{\partial z} = -1 \end{cases}$
The force acting on the test particle:

\[ \nabla \times F = q \left[ -\frac{\partial B}{\partial t} - \beta c \frac{\partial B}{\partial z} \right] \]

\[ \nabla \times F = \begin{vmatrix} \hat{x} & \hat{y} & \hat{z} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ F_x & F_y & F_z \end{vmatrix} = \left( \frac{\partial F_z}{\partial y} - \frac{\partial F_y}{\partial z} \right) \hat{x} + \left( \frac{\partial F_x}{\partial z} - \frac{\partial F_z}{\partial x} \right) \hat{y} + \left( \frac{\partial F_y}{\partial x} - \frac{\partial F_x}{\partial y} \right) \hat{z} \]

Very important:
Because the wakefield is only a function of “s” then: B(s)
This leads to

Position of drive particle: \( v \cdot t \)
Position of the test particle:

\[ z = vt - s \]

When inserting the partial differentials on the right, the terms in the bracket cancels out and gives zero.

Panofsky Wenzel theorem

\[ \nabla F_z(x, y, z, \tau) = -\frac{\partial F_z(x, y, z, \tau)}{\partial s} \]
Panofsky-Wenzel and transverse impedance

\[ \nabla_{\perp} F_z(x, y, z, \tau) = -\frac{\partial F_{\perp}(x, y, z, \tau)}{\partial s} \]

To obtain the theorem in terms of impedance, one can simply start from the wake function form:

\[ \nabla_{\perp} w_{\parallel}(x, y, z, \tau) = \frac{\partial w_{\perp}}{\partial s}(x, y, z, \tau) \]

Then change the \( s \) derivative with the time derivative. Use \( \partial s = v \partial \tau = \beta c \partial \tau \):

\[ \nabla_{\perp} w_{\parallel}(x, y, z, \tau) = \frac{1}{\beta c} \frac{\partial w_{\perp}}{\partial \tau}(x, y, z, \tau) \]

Finally take the Fourier transform on both sides:

\[ \nabla_{\perp} \int_{-\infty}^{+\infty} e^{-i\omega \tau} w_z(x, y, z, \tau) d\tau = \frac{1}{\beta c} \int_{-\infty}^{+\infty} e^{-i\omega \tau} \frac{\partial w_{\perp}(x, y, z, \tau)}{\partial \tau} d\tau \]

NB! The transverse impedance is defined with a complex \( i \) factor:

\[ \nabla_{\perp} Z_{\parallel}(x, y, z, \omega) = \frac{\omega}{\beta c} Z_{\perp}(x, y, z, \omega) \]

Panofsky Wenzel theorem
Panofsky-Wenzel and transverse impedance

\[ \nabla_\perp Z_\parallel(x, y, z, \omega) = \frac{\omega}{\beta c} Z_\perp(x, y, z, \omega) \]

Using the following definitions:

\[ \nabla_\perp Z_\parallel = \frac{\partial Z_\parallel}{\partial x t} \hat{x} + \frac{\partial Z_\parallel}{\partial y t} \hat{y} \quad \text{and} \quad Z_\perp = Z_x \hat{x} + Z_y \hat{y} \]

\[ Z_{\perp, x} = \frac{\beta c}{w} \cdot \frac{\partial Z_\parallel}{\partial x t} \]

\[ Z_{\perp, y} = \frac{\beta c}{w} \cdot \frac{\partial Z_\parallel}{\partial y t} \]

\[ Z_{\perp, x}(\omega) = Z_1 x + 2Z_2 A \cdot x t + Z_2 B \cdot y t + Z_2 C \cdot y d + Z_2 D \cdot x d \]

\[ Z_{\perp, y}(\omega) = Z_1 y - 2Z_2 A \cdot y t + Z_2 B \cdot x t + Z_2 C \cdot x d + Z_2 E \cdot y d \]

\[ Z[x d, x t, y d, y t] = Z_0 + Z_1 x \cdot (x d + x t) + Z_1 y \cdot (y d + y t) \]

\[ + Z_2 A \cdot (x d x d - y d y d + x t x t - y t y t) \]

\[ + Z_2 B \cdot (x d y d + x t y t) + Z_2 C \cdot (x d y t + x t y d) \]

\[ + Z_2 D \cdot x d x t + Z_2 E \cdot y d y t \]
Panofsky-Wenzel and transverse impedance

\[ \nabla \parallel Z(x, y, z, \omega) = \frac{\omega}{\beta c} Z_{\perp}(x, y, z, \omega) \]

Using the following definitions: \( \nabla \parallel Z = \frac{\partial Z_{\parallel}}{\partial x} \hat{x} + \frac{\partial Z_{\parallel}}{\partial y} \hat{y} \) and \( Z_{\perp} = Z_x \hat{x} + Z_y \hat{y} \)

\[ Z_{\perp, x} = \frac{\beta c}{w} \cdot \frac{\partial Z_{\parallel}}{\partial x} \]
\[ Z_{\perp, y} = \frac{\beta c}{w} \cdot \frac{\partial Z_{\parallel}}{\partial y} \]

Panofsky Wenzel theorem

In differential form

\[ Z_{\perp, x}(\omega) = Z_{1x} + 2Z_{2A} \cdot x + Z_{2B} \cdot y + Z_{2C} \cdot y + Z_{2D} \cdot x \]
\[ Z_{\perp, y}(\omega) = Z_{1y} - 2Z_{2A} \cdot y + Z_{2B} \cdot x + Z_{2C} \cdot x + Z_{2D} \cdot y \]

\[ Z[xd, xt, yd, yt] = Z_0 + Z_{1x} \cdot (xd + xt) + Z_{1y} \cdot (yd + yt) + Z_{2A} \cdot (xdxd - ydyd + xtxt - ytyt) + Z_{2B} \cdot (xdy + xyt) + Z_{2C} \cdot (xdyt + xtyd) + Z_{2D} \cdot xdx + Z_{2E} \cdot ydyt \]
Because of the rigid bunch approximation, which states that the beam motion is little affected during the passage through a structure, the wake field is the same before and after the passage of an equipment.

Therefore, it is as if B is only a function of “s”. A criterion for the Panofsky-Wenzel theorem is therefore that the vacuum chamber has to have the same cross-section before and after the equipment – otherwise the B-field is not the same.
We can measure the beam impedance with wire measurements.

This is based on the assumption that a bunch interacts with an equipment in exactly the same way as a coaxial cable (i.e. a wire inside the equipment):

Ultra-relativistic beam field

\[ E_r (r, \omega) = Z_0 H_\varphi (r, \omega) = \frac{Z_0 q}{2\pi r} \exp \left( -j \frac{\omega}{c} z \right) \]

TEM mode coax waveguide

\[ E_r (r, \omega) = Z_0 H_\varphi (r, \omega) = Z_0 \frac{\text{const}}{r} \exp \left( -j \frac{\omega}{c} z \right) \]

See A. Mostacci: [http://pcaen1.ing2.uniroma1.it/mostacci/wire_method/care_impedance.ppt](http://pcaen1.ing2.uniroma1.it/mostacci/wire_method/care_impedance.ppt)
Lab measurements of beam impedance. Wire #2

Network analyzer Port 1

\[ S_{21}^{DUT} \]

Network analyzer Port 2

\[ S_{21}^{REF} \]

\[ \frac{dZ}{dl} = \text{The beam impedance per length} \]

\[ L \quad \text{The inductance per length} \]

\[ C \quad \text{The capacity per length} \]

REF = Reference = PEC vacuum chamber – same length as DUT
Lab measurements of beam impedance. Wire #3

Beam impedance:

\[ Z_{\text{Longitudinal}} = \frac{dZ}{dl} \cdot l = -2 \cdot Z_0 \cdot \ln \left( \frac{S21_{\text{DUT}}}{S21_{\text{REF}}} \right) \cdot \left( 1 + \frac{i \cdot \ln \left( \frac{S21_{\text{DUT}}}{S21_{\text{REF}}} \right)}{2 \cdot \theta} \right) \]

This is the improved log formula, which is used for wire measurements.
Lab measurements of beam impedance. Wire #4

\[
S_{21,DUT} = \frac{b_2}{a_1} = \frac{\frac{V_2 + Z_0 I}{2\sqrt{Z_0}}}{\frac{V_1 + Z_0 I}{2\sqrt{Z_0}}} = \frac{V_2 + Z_0 I}{V_1 + Z_0 I} = \frac{V_2 + Z_0 I}{V_2 + Z_L I + Z_0 I} = \frac{Z_0 I + Z_0 I}{Z_0 I + Z_L I + Z_0 I} = \frac{2Z_0}{2Z_0 + Z_L}
\]

\[
S_{21,REF} = \frac{\frac{V_2 + Z_0 I}{2\sqrt{Z_0}}}{\frac{V_1 + Z_0 I}{2\sqrt{Z_0}}} = 1
\]

\[
\frac{S_{21,DUT}}{S_{21,REF}} = \frac{2Z_0}{2Z_0 + Z_L}
\]

\[
\frac{Z_L}{Z_0} = 2\frac{S_{21,REF}}{S_{21,DUT}} - 2
\]

\[
V_1 - Z_L \cdot I = V_2
\]
Lab measurements of beam impedance. Wire #5

Vector Network Analyser

DUT

180° hybrid
Matching resistors
50 Ω
Adapted load

A

B

C

D

S = \frac{1}{\sqrt{2}}

\begin{pmatrix}
0 & 0 & -1 & 1 \\
0 & 0 & -1 & -1 \\
-1 & -1 & 0 & 0 \\
1 & -1 & 0 & 0
\end{pmatrix}
Lab measurements of beam impedance. Wire #6

Characteristic impedance \( Z_0 \) of two wires, each with diameter "\( d \)" and with distance between them "\( \Delta \)" is (See https://en.wikipedia.org/wiki/Twin-lead):

\[
Z_0 \approx \frac{120}{\sqrt{\varepsilon_r}} \ln \left[ 2 \frac{\Delta}{d} \right]
\]

Example:
\( \Delta = 10.0 \text{ mm} \quad \text{Z} = 120/1 \cdot \ln(40) \sim 450 \text{ Ohm} \)
\( d = 0.5 \text{ mm} \quad \text{i.e. 225 Ohm per wire} \)

Subtract 50 Ohm, as usual, this gives 175 Ohm per wire. So it is always 175 Ohm per wire – independent of the chamber diameter!
**Lab measurements of beam impedance. Wire #7**

**Voltage**

\[
\text{Voltage} = -((-I \cdot Z[-a,-a,0,0] + I \cdot Z[a,-a,0,0]) + (I \cdot Z[a,a,0,0] - I \cdot Z[-a,a,0,0]))
\[
= I \cdot (4a^2Z_{2D}) = \text{dipolar impedance}
\]

The distance between the wires is 2a:

\[\Delta = 2a\]
Another measure of transverse beam impedance!

An example of transverse impedance, that gives the beam a transverse kick! Here measured with the beam.
Lab measurements of beam impedance. Wire #8

Easy method to firmly straighten the wire. Make hole in connector and solder a thin wire to the resistor.

This method was invented by Muzhaffar Hazman
Lab measurements of beam impedance. Wire #8

Easy method to firmly straighten the wire. Make hole in connector and solder a thin wire to the resistor.

When soldering the resistor to the connector, keep the other soldering cold, otherwise it will dissolve. Use plier as heat sink.

This method was invented by Muzhaffar Hazman
Lab measurements of beam impedance. Wire #9
Lab measurements of beam impedance. Wire #10

MKE Kicker measurements

T. Kroyer, F. Caspers, E. Gaxiola
An example of serigraphy in the SPS Extraction Kicker Magnets (SPS-MKE) Lab measurements of beam impedance. Wire #11
Lab measurements of beam impedance. Wire #12

Kicker Transition piece, i.e. keep electrical connection with the vacuum chamber
Lab measurements of beam impedance. Wire #13

Kicker Transition piece, i.e. keep electrical connection with the vacuum chamber
Lab measurements of beam impedance. Wire #14

Tune shift vs half gap

Collimator measurement

Lab measurements of beam impedance. Probe #1

Two probe setup
- Short probe
- Copper adapter
- To VNA Port 1
- To VNA Port 2

One probe setup
- Long probe
- Copper adapter
- Close on adapted
- To VNA Port 1
Lab measurements of beam impedance. Probe #2
Lab measurements of beam impedance. Probe #3

Two probe setup

Short probe

DUT

Copper adapter

To VNA Port 1

To VNA Port 2

Amplitude in dB

\[ Q = \frac{f_c}{\Delta f} \]
Lab measurements of beam impedance. Probe #4

Smith chart

\[ \Gamma = \frac{Z_L - Z_0}{Z_L + Z_0} \]
A resonance is a circle in the Smith diagram.

Three different types of $Q$:
1) The loaded $Q$ ($Q_L$)
2) The unloaded $Q$ ($Q_0$)
3) The $Q$ of the external world ($Q_{ext}$).

We want $Q_0$, but we can only measure $Q_L$ and $\beta$:

$$Q_L = \frac{f_{res}}{\Delta f}$$

$$\beta = \frac{R_0}{R_{ext}}$$

$$Q_0 = Q_L (1 + \beta)$$
Lab measurements. Measure Q reflection. Probe #6

Ql ≈ 2018
1+β ≈ 2
Q0 ≈ 4036

Ql ≈ 2763
1+β ≈ 2
Q0 ≈ 5526

Courtesy of C. Vollinger and T. Kaltenbacher
感谢您的关注
1. **Measurement of transverse kick in CLIC accelerating structure in FACET** Hao Zha, Andrea Latina, Alexej Grudiev

2. Holomorphic decomposition. John Jowett
   \[\text{http://cern.ch/dfs/Projects/ILHC/MathematicaExamples/Accelerator/MultipoleFields.nb}\]

3. On single wire technique for transverse coupling impedance measurement. H. Tsutsui

4. Longitudinal instability of a coasting beam above transition, due to the action of lumped discontinuities V. Vaccaro

5. Wake Fields and Instabilities Mauro Migliorati
   \[\text{https://indico.cern.ch/event/683638/contributions/2801720/attachments/1589041/2513889/Migliorati-2018_wake_fields.pdf}\]

6. G.Rumolo, CAS Advanced Accelerator Physics Trondheim, Norway 18–29 August 2013

7. THE STRETCHED WIRE METHOD: A COMPARATIVE ANALYSIS PERFORMED BY MEANS OF THE MODE MATCHING TECHNIQUE M.R.Masullo, V.G.Vaccaro, M.Panniello

8. Two Wire Wakefield Measurements of the DARHT Accelerator Cell. Scott D. Nelson, Michael Vella
   \[\text{https://e-reports-ext.llnl.gov/pdf/236163.pdf}\]

9. Shunt impedance, RLC-circuit definition, Accelerator definition, Alexej Grudiev
   \[\text{https://impedance.web.cern.ch/lhc-impedance/Collimators/RLC_050211.ppt}\]

10. A.Mostacci
    \[\text{http://pcaen1.ing2.uniroma1.it/mostacci/wire_method/care_impedance.ppt}\]

11. **COUPLING IMPEDANCE MEASUREMENTS: AN IMPROVED WIRE METHOD** V.Vaccaro
    \[\text{http://cdsweb.cern.ch/record/276443/files/SCAN-9502087.tif}\]

12. Interpretation of coupling impedance bench measurements H. Hahn

Energy loss when beam passes through an equipment

**Shunt impedance**

**Accelerator definition:** $r$

$$V = \int E_z e^{\frac{oz}{z}} dz$$

$$r = \frac{V^2}{P}; \quad Q = \frac{\omega_0 U}{P}; \quad \frac{r}{Q} = \frac{V^2}{\omega_0 U}$$

$$V = 2qk_i; \quad k_i = \frac{U}{q}; \quad k_i = \frac{V^2}{4U}$$

$$k_i = \frac{\omega_0 r}{4Q}$$

**RLC-circuit definition:** $R$

$$Z_i(\omega) = \int W_i(t)e^{j\omega t} dt$$

$$k_i = \frac{1}{\pi} \int \Re\{Z_i(\omega)\} d\omega$$

$$\omega_0 = \frac{1}{\sqrt{LC}}; \quad Q = R \sqrt{\frac{L}{C}}$$

$$Z_i(\omega) = \frac{R}{1 + jQ(\omega/\omega_0 - \omega_0 / \omega)}$$

The energy lost, when the particle passes a resonance, is equal to: $E\downarrow\text{loss} = k \cdot q \cdot f^2$

Where $k$ is the loss factor, which is equal to:

$$k\downarrow\text{loss factor} = \omega_0 / 2 \cdot R / Q$$

And $q$ is the charge of the particle.

As you can see, the bigger $R$ over $Q$, the bigger the energy loss.

**Wake Loss Factor**

The wake loss factor ($k$) for the longitudinal component is calculated by:

$$k = - \int_{-\infty}^{\infty} \lambda(s) W_\parallel(s) ds$$

were lambda(s) describes the normed charge distribution function over s (to obtain it, you need to multiply this function by q1). It is given in [V / pC].

Use loss(kick) factor instead of impedance
Beam impedance modelled by lumped impedance

The energy lost, when the particle passes a resonance, is equal to: \( E_{\text{loss}} = k \cdot q \cdot f^2 \)

Where \( k \) is the loss factor, which is equal to: 
\[ k \cdot \text{loss factor} = \frac{\omega_0}{2} \cdot R/Q \]

And \( q \) is the charge of the particle.

As you can see, the bigger \( R \) over \( Q \), the bigger the energy lost.
The longitudinal beam impedance have 8 parameters:

The beam impedance is now decomposed into 13 parameters:

\[ Z_{1\parallel}[x_d, x_t, y_d, y_t] = Z_0 \]
\[ + Z_{1X \cdot D \cdot x_d} + Z_{1X \cdot T \cdot x_t} + Z_{1Y \cdot D \cdot y_d} + Z_{1Y \cdot T \cdot y_t} \]
\[ + Z_{2X \cdot Y \cdot D \cdot X \cdot Y \cdot D \cdot (x_d^2 - y_d^2)} + Z_{2X \cdot Y \cdot T \cdot X \cdot Y \cdot T \cdot (x_t^2 - y_t^2)} \]
\[ + Z_{2X \cdot D \cdot X \cdot T \cdot x_d \cdot x_t} + Z_{2X \cdot D \cdot Y \cdot D \cdot x_d \cdot y_d} + Z_{2X \cdot D \cdot Y \cdot T \cdot x_d \cdot y_t} \]
\[ + Z_{2X \cdot T \cdot Y \cdot D \cdot x_t \cdot y_d} + Z_{2X \cdot T \cdot Y \cdot T \cdot x_t \cdot y_t} + Z_{2Y \cdot D \cdot Y \cdot T \cdot y_d \cdot y_t} \]

The new formula is identical to the previous from Tsutsui:

\[ Z = Z_{0,0} + (x_1 - jy_1)Z_{1,0} + (x_1 + jy_1)Z_{-1,0} + (x_2 + jy_2)Z_{0,1} + (x_2 - jy_2)Z_{0,-1} \]
\[ + (x_1 - jy_1)^2 Z_{2,0} + (x_1 - jy_1)(x_2 - jy_2)Z_{1,-1} + (x_2 - jy_2)^2 Z_{0,-2} \]
\[ + (x_1 - jy_1)(x_2 + jy_2)Z_{1,1} + (x_1 + jy_1)(x_2 - jy_2)Z_{-1,1} \]
\[ + (x_1 + jy_1)^2 Z_{-2,0} + (x_1 + jy_1)(x_2 + jy_2)Z_{-1,1} + (x_2 + jy_2)^2 Z_{0,2} \]
\[ + O((x_1, y_1, x_2, y_2)^3). \]
The longitudinal beam impedance has 8 parameters. Interchanging the drive and test particles always give the same beam impedance.
The longitudinal beam impedance have 8 parameters

\[ Z_{\|}[xd, xt, yd, yt] = \]
\[ Z_0 + Z_{1X} (xd + xt) + Z_{1Y} (yd + yt) + Z_{2XYDXTYD} (xd^2 + xt^2 - yd^2 - yt^2) + Z_{2XDTYDT} (xd yd + xt yt) + Z_{2XDTYTD} (xd yt + xt yd) + Z_{2XDXT} xd xt + Z_{2YDYT} yd yt \]
What is beam impedance?

CST Wakefield example illustrating the 8 parameters

Prediction:

\[ Z_{2XYDTXYDT} (x_d^2 + x_t^2 - y_d^2 - y_t^2) \]

\[ Z_{2YTIm,Z2YDIm} \]

\[ Z_{2YTRe,Z2YDRe} \]

\[ Z_{2XTIm,Z2XDIm} \]

\[ Z_{2XTRe,Z2XDRe} \]
Prediction:
$$Z_{2XTYDT} \left( x_d y_d + x_t y_t \right)$$

$$Z_{2XTYTRe}, Z_{2XDYDRe}$$

$$Z_{2XTYTIm}, Z_{2XDYDIm}$$
CST Wakefield example illustrating the 8 parameters

Prediction:
Z2XDTYTD (xd yt +xt yd)
What is beam impedance?

CST Wakefield example illustrating the 8 parameters

Prediction versus simulation
3 examples

Example 1:
yd=2.5, yt=2.5

Example 2:
xd=1.0, yd=2.5

Example 3:
xd=-1.5, yd=2.0, xt=2.5

Prediction Real
Prediction Imaginary
Simulation Real
Simulation Imaginary
What is beam impedance?

The rotating wire method

In the additional slide, it is demonstrated how this measurement can derive all 8 parameters.

One wire represents the drive particle and the other wire represents the test particle.

In this measurement, we do not have a positive current in one wire and a negative current in the other.

Both wires are measured individually i.e. single-ended.

In the additional slide, it is demonstrated how this measurement can derive all 8 parameters.
The rotating wire method

Some implications of the new 8 parameter formula:

1) Transverse impedance

\[ Z_{\perp,x}(\omega) = Z_{1x} + 2Z_{2A} \cdot xt + Z_{2B} \cdot yt + Z_{2C} \cdot yd + Z_{2D} \cdot xd \]
\[ Z_{\perp,y}(\omega) = Z_{1y} - 2Z_{2A} \cdot yt + Z_{2B} \cdot xt + Z_{2C} \cdot xd + Z_{2E} \cdot yd \]

The offset term is not automatically zero, depends on the shape of the equipment

2) Transverse impedance

Is it possible to shape a collimator e.g. in three-fold symmetric form so that its transvers impedance is zero up to second order?

3) Transverse impedance

The beam oscillates during instability, is it possible to shape equipment in such a way that the drive position works against the instability?
Supporting material for slide

\[
\text{sol1} = \text{Solve} \left[ S_{rel} = \exp \left[ i \cdot v \cdot w \cdot \sqrt{\frac{CC}{L}} - i \cdot v \cdot w \cdot \sqrt{1 - \frac{i \cdot dZdl}{w \cdot L}} \cdot \frac{CC}{L} \right], dZdl \right]
\]

\[
\left\{ dZdl \rightarrow \frac{-2 \sqrt{CC \cdot L} \cdot w \cdot \log[S_{rel}] - 4 \log[S_{rel}]^2}{CC \cdot L^2 \cdot w} \right\}
\]

\[-2 \frac{L}{CC} \cdot \log[S_{rel}] \left(1 - \frac{4 \log[S_{rel}]}{2 \cdot 2 \sqrt{CC \cdot L} \cdot w} \right) = \frac{-2 \sqrt{CC \cdot L} \cdot w \cdot \log[S_{rel}] - 4 \log[S_{rel}]^2}{CC \cdot L \cdot w} \]

True
Longitudinal Beam Coupling Impedance of Device Under Test (DUT)

\[ Z_s = \text{Characteristic impedance of beam pipe} \]

Matching resistor = \( Z_s - 50 \, \Omega \) (removes reflections inside DUT)

50 \, \Omega \text{ Cable}

10 dB Attenuator
(to remove the reflected wave that goes back to the Network analyzer. If this wave hits the network analyzer, the measurement will be disturbed.)

NB! The calibration must be done without the attenuators. It is not yet understood exactly why, but it shown in test that including the attenuators give incorrect calibrations.

Vector Network Analyser

Vector network analyzer calibration to remove effects from cables and attenuators.
Measure transmission coefficient \( S_{21} \) (=forward transmission) and then calculate impedance.
Figure 3: **Normal** field patterns up to third order for respectively a dipole, quadrupole and sextupole magnet. The potentials are:

Dipole: $x$  
Quadrupole: $\frac{x^2}{2} - \frac{y^2}{2}$  
Sextupole: $\frac{x^3}{3} - xy^2$
Figure 4: **Skew** field patterns up to third order for respectively a dipole, quadrupole and sextupole magnet. The potentials are:

Dipole: $y$ \hspace{1cm} Quadrupole: $x \cdot y$ \hspace{1cm} Sextupole: $x^2y - \frac{y^3}{3}$