A Model Cylindrical Magnetron
Vlasov Distribution Function

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Abstract

We extend the analysis of the planar magnetron Vlasov distribution function [Phys. Fluids 31, 2362 (1988)] to the cylindrical case. In momentum space, the model distribution function is \( f(w,p\theta) = Ne^{-\beta_w w - \frac{m_e}{2e}(p\theta - p_0)^2} \) where \( w(p\theta) \) is the single particle energy (angular momentum), \( \beta_w(\beta_\theta) \) is the inverse of the thermal energy associated with variations in \( w(p\theta) \), \( p_0 \) is the angular momentum at the cathode and \( \Omega \) is the electron cyclotron frequency (\( = eB_0/mc \)). The problem is shown to be too "stiff" numerically to permit a pure numerical solution even using very high accuracy and state-of-the-art numerical schemes. We show that one may use a global singular perturbation expansion, similar to, but significantly more complex than, the one used in the planar case, to solve the resulting nonlinear ODE for the spatial dependence of the distribution function, the density, the electrostatic potential and the drift velocity.

1 Introduction

Models of magnetrons and crossed-field amplifiers (CFA's) in planar geometry have existed for some time\(^1\)-\(^2\). This geometry has quite convenient mathematically for studying such devices\(^3\)-\(^4\). Planar
models have also been used in numerical simulations\(^5,6\). Cylindrical devices have also been studied theoretically in the cold-fluid limit\(^7\) as well as numerically with particle simulations. However there has existed no adequate Vlasov model in cylindrical geometry up to the present time. The planar geometry is quite adequate for low aspect ratio devices such as most CFA's. (The aspect ratio is the ratio of the anode radius to the cathode radius). However magnetrons typically have high aspect ratios wherein cylindrical deviations from planar geometry can be expected to be quite important. In this paper, we provide a model Vlasov distribution function for a cylindrical magnetron. In many ways, it is quite similar to the planar model, however there are important geometrical deviations. Still, the basic governing function is exactly the same function \(J_0(x)\) defined in the planar model\(^2\).

In Section II, we describe the model and transform it to velocity space. Then in Section III, we show that we may use a singular perturbation expansion to solve the nonlinear ODE (Poisson's equation). In Section IV, we give results for typical medium and high aspect ratio devices.

2 The Cylindrical Model

We shall use the geometry detailed in Fig. 1 where \(a(b)\) is the cathode (anode) radius, the \(\vec{B}\)-field (assumed uniform) is directed into the paper and the anode is at some positive potential, creating an electric field from the anode to the cathode. We are here only interested in the stationary characteristics so we assume azimuthal symmetry, with all physical quantities independent of the azimuthal angle, \(\theta\).

The form of the electrostatic field and potential will follow from Poisson's equation,

\[
(\partial^2_x + \frac{1}{r} \partial_r)\phi = 4\pi ne
\]

where \(n\) is the azimuthally symmetric electron density. The anode-cathode voltage will pull electrons out of the cathode, creating a sheath next to the cathode. This sheath is magnetically insulated from the anode by the magnetic field, \(B_0\), and will grow until we have a space-charge limited current at the cathode, with a vanishing \(E\)-field at the cathode. Taking the potential to be zero at the cathode, then our boundary conditions on (1) at the cathode are

\[
\phi(r = a) = 0 = \partial_r\phi(r = a)
\]

This will uniquely determine \(\phi(r)\) from (1) once the electron density is determined.

The electron density will be determined by a model Vlasov distribution function. The model that we will use will be
\[ f(w, p_\theta) = Ne^{-\beta_w w}e^{-\frac{p_\theta^2}{\Omega_0^2}(p_\theta - p_0)^2} \]  

(3)

In the above, \( N \) is a normalization constant to be determined (it will be a function of the anode voltage). The next term is the standard Gibbs term for the distribution of single-particle energies, \( w \), where \( \beta_w \) is the inverse of thermal energy. The last term arises because all electrons leave the cathode at near zero velocity. The coefficient \( \beta_\theta \) determines the spread in these velocities about zero velocity. \( p_\theta \) is the single particle angular momentum (a constant of the motion) and \( p_0 \) is its value at the cathode. The quantity \( \Omega \), which is the electron cyclotron frequency \( (= eB_0/(mc)) \), is there simply to normalize \( \beta_\theta \) to an inverse energy, and the factor of 4 is simply a convenient scale factor. Since both \( w \) and \( p_\theta \) are constants of the single particle motion, this \( f(w, p_\theta) \) automatically satisfies the Vlasov equation

\[ \partial_t f + \vec{v} \cdot \vec{\nabla} f - \frac{e}{m}(\vec{E} + \frac{1}{c}\vec{v} \times \vec{B}_0) \cdot \vec{\nabla}_v f = 0 \]  

(4)

where

\[ \vec{E} = -\vec{\nabla} \phi \]  

(5)

In cylindrical coordinates, ignoring variations in the \( z \)-direction, the single particle equations in the presence of an electric field and a magnetic field are

\[ \ddot{r} - r \dot{\theta}^2 = \frac{e}{m} \partial_r \phi + \Omega r \dot{\theta} \]  

(6a)

\[ r \ddot{\theta} + 2r \dot{\theta} = -\Omega \dot{r} \]  

(6b)

when \( \partial_\phi = 0 \). These equations have the two constants of the motion

\[ w = \frac{m}{2}(v_r^2 + v_\theta^2) - e\phi \]  

(7)

\[ p_\theta = mr \nu_\theta + \frac{1}{2}mr^2 \Omega \]  

(8)

where

\[ v_r = \dot{r}, \quad v_\theta = r \dot{\theta} \]  

(9)
As stated above, we define $p_0$ to be that value of $p_\theta$ at the cathode for which $v_\theta = 0$. Thus

$$p_0 = \frac{1}{2}ma^2\Omega$$  \hspace{1cm} (10)

For a laminar flow, we have $\ddot{r} = 0$ and $\ddot{\theta} = 0$. Then by (6) there are two possible values for the drift velocity, $v_d = r\dot{\theta}$, both of which are negative.

$$v_d = -\frac{1}{2}r\Omega \pm \sqrt{\frac{1}{4}r^2\Omega^2 - \frac{e}{m}(r\partial_r \phi)}$$  \hspace{1cm} (11)

The upper sign corresponds to the drift velocity of the planar case. It also vanishes at the cathode and therefore corresponds to the typical values present in our model distribution function. The lower sign has a nonvanishing value at the cathode ($v_d = -a\Omega$) and for nominal values of $\beta_\theta$, can be ignored as being unphysical.

To calculate the electron density and the average particle velocity, we have to convert $f$ from a function of $w$ and $p_\theta$ to a function of $u_r$ and $v_\theta$. This is easily done with (7), (8) and (10). The result is

$$f = Ne^\chi e^{-\frac{m}{2}\beta_\omega w^2}e^{-\frac{e}{m}(\beta_\omega + \frac{r^2}{2}\beta_\theta)(v_\theta + u)^2}$$  \hspace{1cm} (12)

where

$$\chi = \beta_\omega e\phi - h(r)$$  \hspace{1cm} (13)

$$u = \frac{\beta_\theta r(r^2 - a^2)\Omega}{2(a^2\beta_\omega + r^2\beta_\theta)}$$  \hspace{1cm} (14)

$$h(r) = \frac{m\Omega^2\beta_\omega \beta_\theta (r^2 - a^2)^2}{8(a^2\beta_\omega + r^2\beta_\theta)}$$  \hspace{1cm} (15)

Taking the velocity averages of (12) now gives

$$n = \int \int f du_r du_\theta = \frac{2\pi Ne^\chi}{\sqrt{\beta_\omega(\beta_\omega + r^2\beta_\theta/a^2)}}$$  \hspace{1cm} (16)

$$\bar{v}_\theta = \frac{1}{n} \int \int fu_\theta du_r du_\theta = -u$$  \hspace{1cm} (17)

showing $u$ to be the negative of the average azimuthal velocity.
Near the cathode, we expect the electron plasma frequency to be just below the electron cyclotron frequency by some temperature factor. With this in mind, we rescale $N$ as

$$2\pi N = \frac{m}{4\pi e^2}\Omega^2\beta_\theta \left( \frac{\beta_w}{\beta_w + \beta_\theta} \right)^\frac{1}{2} (1 - \epsilon)$$  (18)

where $\epsilon$ is now the free parameter. With this, (16) gives

$$\omega_p^2 = \frac{4\pi e^2}{m} = \frac{\beta_0\Omega^2(1 - \epsilon)e^x}{\sqrt{\beta_w + \beta_\theta}\sqrt{\beta_w + \beta_\theta r^2/a^2}}$$  (19)

which at the cathode is

$$\omega_p^2(r = a) = \frac{\beta_0\Omega^2(1 - \epsilon)}{\beta_w + \beta_\theta}$$  (20)

Now the only remaining equation to solve is Poisson’s. From (1) we then have

$$(\partial_r^2 + \frac{1}{r}\partial_r)(\frac{e}{m}\phi) = \omega_p^2$$  (21)

where $\omega_p^2$ is given by (19). This is a nonlinear ODE since $\chi$ in (19) is proportional to $\phi$ [see (13)]. This is a particularly complex ODE because of the geometrical factors of $r/a$ in (19) and (15). Attempts to solve this equation numerically have failed, even with 40 place accuracy. However as we shall shortly see, these cylindrical factors actually only introduce some minor variations from the planar case. The key point is that the factor of $e^x$ in $\omega_p^2$ in (21) and (19) dominates all geometrical effects. This was the key for the asymptotic expansion given in the next section.

3 The Asymptotic Expansion

Due to secondary emission at the cathode, one expects values for $\beta_w$ and $\beta_\theta$ to be in the order of inverse electron volts. Since $ma^2\Omega^2$ is typically kev’s, we have here a small parameter for expansion. And as we shall see later, $\epsilon$ in (19) is typically $10^{-8}$ or even much smaller, giving us a second small parameter.

But before we expand in these parameters, let us rescale (21) and (19). We start by rescaling $r$ as

$$r = ae^\nu$$  (22)

so that
\[ \partial_r^2 + \frac{1}{r} \partial_r = \frac{e^{-2\nu}}{a^2} \partial_c^2 \]  

(23)

Thus \( \nu \) ranges from zero to \( \ell n(b/a) \). We introduce the ratio of the temperatures

\[ \rho = \beta_w / \beta_0 \]  

(24)

and define the large unitless parameter \( L \) by

\[ L^2 = \frac{a^2 \Omega^2 m \beta_w}{1 + \rho} \]  

(25)

Then (15) and (21) become

\[ h = \frac{L^2}{8} \frac{1 + \rho}{e^{2\nu} + \rho} (e^{2\nu} - 1)^2 \]  

(26)

\[ \omega_r^2 = \frac{L^2 (1 - \epsilon)}{a^2 \beta_w m} e^x \mu \]  

(27)

where

\[ \mu \equiv \left( \frac{1 + \rho}{e^{2\nu} + \rho} \right)^{\frac{1}{2}} \]  

(28)

Define the scaled Laplacian of \( h \) to be

\[ H \equiv \frac{a^2}{L^2} (\partial_r^2 + \frac{1}{r} \partial_r)h \]  

(29)

which is

\[ H = \frac{(\rho + 1)}{2(\rho + e^{2\nu})^3} [e^{6\nu} + 3\rho e^{4\nu} + e^{2\nu} + \rho(1 + 2\rho)(2e^{2\nu} - 1)] \]  

(30)

and is clearly positive definite.

With the above and (13), (21) becomes

\[ e^{-2\nu} L^{-2} \partial_c^2 \chi + H - \mu(1 - \epsilon)e^x = 0 \]  

(30)

For large \( L \) and small \( \epsilon \), the solution of this equation can be expanded in the following asymptotic series in \( \nu \) and \( z \).

\[ \chi = -J(z) + F_0(v) + \sum_{m,n} \frac{e^m}{L^n} J_{mn}(z) F_{mn}(v) \]
\[ + \sum_{m,n} \frac{\epsilon^n}{L^n} K_{mn}(z)G_{mn}(v)[\epsilon \sinh(L\nu)] \]  

where the sum is from \( m \) and \( n \) = 0 to infinity and

\[ z = \epsilon \cosh(L\nu) - \epsilon \]  

where \( \nu(v) \) is some function of \( v \), still to be determined.

Before we start the expansion, we should remark on certain features of it, so that one can more clearly see how the various terms will balance in the various orders. First we note that the \( e^x \) term in (30) will simply generate another series like (31). However, what will happen to the \( \cosh \) in (32) and its derivatives when one does the differentiation in (30), requires a detailed discussion. Let us look at each type of term, one by one. First, for \( J(z) \) we have

\[ \frac{1}{L^2} \delta_v^2 J(z) = \nu_v^2(z^2 J'' + z J') + \epsilon \nu_v^2 (2z J'' + J') + \frac{1}{L} \nu_v \epsilon \sinh(L\nu) J' \]  

where \( J' = dJ/dz \) and \( \nu_v = d\nu/dv \). Note how this is a sum of products of functions of \( z \) and functions of \( v \), with higher-order terms present \([\epsilon, L^{-1} \epsilon \sinh(L\nu)]\). It is the same form as (31). Similarly

\[ \frac{1}{L^2} \delta_v^2 F_0(v) = \frac{1}{L^2} F_0'' \]  

has only the higher-order term \([L^{-2}]\). Continuing

\[ \frac{1}{L^2} \delta_v^2 (J_{mn} F_{mn}) = \nu_v^2 F_{mn}(z^2 J''_{mn} + z J'_{mn}) + \epsilon \nu_v^2 F_{mn}(2z J''_{mn} + J'_{mn}) \]

\[ + \frac{1}{L} J'_{mn}(\nu_v F_{mn} + 2\nu_v F'_{mn})[\epsilon \sinh(L\nu)] + \frac{1}{L^2} J_{mn} F''_{mn} \]  

\[ \frac{1}{L^2} \delta_v^2 [K_{mn} G_{mn} \epsilon \sinh(L\nu)] = \nu_v^2 [(z^2 K''_{mn})' + (K_{mn} z)^' G_{mn}[\epsilon \sinh(L\nu)] \]

\[ + \epsilon \nu_v^2 [2(K'_{mn} z)' + K_{mn} G_{mn}[\epsilon \sinh(L\nu)] + \frac{1}{L} (K''_{mn} z^2 + K_{mn} z) (2\nu_v G'_{mn} + \nu_v G_{mn}) \]

\[ + \frac{\epsilon}{L} (2K'_{mn} z + K_{mn} (2\nu_v G'_{mn} + \nu_v G_{mn}) + \frac{1}{L^2} K_{mn} G''_{mn} \epsilon \sinh(L\nu) \]  

each of which has the same structure as (31). Thus when (31) is inserted into (30), we will generate a general sum of the form of (31) consisting of the sum of products of functions of \( z \) times functions of \( v \). The coefficients will be of various orders of \( 1/L, \epsilon \) and \( \epsilon \sinh(L\nu) \) as in (31). Thus it will be possible to collect like terms.
It is now simply a matter of inserting these results into (30), collecting the various powers and solving for the various functions of $v$ and $z$. The leading order gives

$$e^{-2v}v^2(z^2J'' + zJ') + H - \mu e^{-J}e^{F_0} = 0$$  \hspace{1cm} (34)

Separating the functions of $z$ and $v$ gives

$$F_0 = \ell n(H/\mu)$$  \hspace{1cm} (35)

$$\nu_v^2 = e^{2v}H$$  \hspace{1cm} (36)

$$z^2J'' + zJ' - 1 + e^{-J} = 0$$  \hspace{1cm} (37)

where we have chosen the separation ratios to be unity. Since $H$ and $\mu$ are known and are positive definite, solutions exist for $F_0$ and $\nu_v$. Eq. (37) is exactly the same as in the planar case and is Eq. (31) in Ref. 2. It satisfies

$$J(0) = 0, \quad J'(0) = 1$$  \hspace{1cm} (38)

Thus we have a solution in leading order. If we go one more order, we only need

$$\chi = -J(z) + F_0(z) + \epsilon J_{10}(z)F_{10}(v)$$  \hspace{1cm} (39)

which inserted into (30) gives

$$F_{10}(v) = 1$$  \hspace{1cm} (40)

$$z^2J_{10}'' + zJ_{10}' - e^{-J}J_{10} = -e^{-J} + 2zJ'' + J'$$  \hspace{1cm} (41)

The latter is a linear second-order nonhomogeneous ODE, whose solution is

$$J_{10} = 1 - (1 + z/3)J'$$  \hspace{1cm} (42)

for $J_{10}(0) = 0 = J_{10}'(0)$. The higher-order terms ($\epsilon^2, \epsilon/L, L^{-2})$ will not be given here since they will be at least two orders of magnitude smaller.

Finally, we will give expressions for $\phi$, $\delta_\phi$ and $\omega_p^2$ based on (39). From (13), (22), (25) and (27)
\[
\frac{e\phi}{ma^2\Omega^2} = \frac{(e^{2\nu} - 1)^2}{8(e^{2\nu} + \rho)} + \frac{1}{(1 + \rho)L^2}[-J + \ell_n(H/\mu) + \epsilon J_{01}] \tag{43}
\]

\[
\frac{e}{ma^2\Omega^2} a \partial_r \phi = e^\nu \frac{(e^{2\nu} - 1)(e^{2\nu} + 1 + 2\rho)}{4(e^{2\nu} + \rho)^2} - \frac{e^{-\nu} \nu}{L(1 + \rho)}(J' - \epsilon J_{01})[\epsilon \sinh(L\nu)]
+ \frac{e^{-\nu}}{L^2(1 + \rho)} \left( \frac{H'}{H} - \frac{\mu'}{\mu} \right) \tag{44}
\]

\[
\frac{\omega_p^2}{\Omega^2} = \frac{H}{1 + \rho} e^{-J}(1 - \epsilon + \epsilon J_{01}) \tag{45}
\]

Now, in (43) and (44), although the terms involving \( J \) and \( J' \) are of lower order, once \( z \) becomes large (outside the sheath), \( J \) does become of order \( L^2 \) and \( J' \) becomes of order \( L \). So these terms do have to be retained in general.

4 Results

For parameters, we use those for the Varian VMS-1873 magnetron\(^8\), where \( a = 1.65\,cm, b = 2.61\,cm \) and \( \Omega = 5.88 \times 10^9\,Hz \), corresponding to \( B_0 = 2.1\,kG \). We call this parameter set a medium aspect ratio case. For the thermal parameters, we can only guess as to their values. Although the values of \( \beta_0 = 1/(10\,eV) \) and \( \beta_w = \beta_0/8 \) might be on the small side, they are not unreasonable, based on other considerations\(^9\). As a second case, we also look at a large aspect ratio device where we keep \( \Omega \) and the \( \beta \)'s the same, but use \( a = 0.125'' \) and \( b = 2'' \) instead. Equilibrium distributions are found to exist in both cases with stable (albeit large) particle orbits.

A. Pure Numerical Solution

Poisson’s Equation (21) (with (13) and (19)) can have solutions for the potential \( \phi \) blowing up exponentially (due to the driving term on the right of the form \( e^\chi = e^{\beta_0 \epsilon \chi} \)). Clearly, to prevent such a blowup, \( \chi \) must remain close to zero, or go negative for \( r \in [a, b] \). It was therefore deemed to be more advantageous to consider the equivalent equation for the quantity \( \chi \), since this is the quantity critically affecting the nature of the solutions. Taking the second derivative of (13) and using (19) and (21) yields

\[
\frac{d^2\chi}{dr^2} = \beta_w \left[ \frac{m\beta_0 \Omega^2 (1 - \epsilon)e^\chi}{\sqrt{\beta_0 + \beta_\omega \sqrt{\beta_\omega + \beta_\omega r^2/a^2}}} - \frac{1}{r} \frac{d\phi}{dr} \right] - h''(r) \tag{46}
\]

At the cathode, \( r = a \), using (2), (13) and (15), the initial conditions on \( \chi \) are
\[ \chi(r = a) = 0 = \partial_r \chi(r = a) \]  

(47)

To ensure that \( \chi \) stays close to zero or assumes negative values for \( r \in [a, b] \), the best one may do is look for regimes where \( \chi \) (which starts at zero with a zero slope at \( r = a \)) is concave down in the vicinity of the cathode, i.e., choose values of \( \epsilon \) such that \( \frac{\partial^2 \chi}{\partial r^2}(r = a) < 0 \). Using (46) and (47),

\[ \frac{d^2 \chi}{dr^2}(r = a) = \frac{m_0^2 \beta \xi_0 (1-\epsilon)}{\beta_u + \beta_s} - \frac{k''(a)}{\epsilon} < 0 \]

yields simply

\[ \epsilon > 0 \]  

(48)

Numerical solutions of (46) and (47) were then performed for various positive values of \( \epsilon \).

However, it was found that in order to obtain anode voltages, \( \phi(b) \), of the order of hundreds of kilovolts typical of such devices, \( \epsilon \) would have to assume very small values indeed. For instance, \( \phi(b) > 100kV \) required \( \epsilon < O(10^{-20}) \), and reliable numerical solutions were found to be virtually impossible to obtain, even employing quad-precision and extremely stable composite backward difference schemes. However, the global asymptotic solution of Section III had no such limitation, and we turn next to results obtained from this solution.

B. Asymptotic Solutions

Notice that as the anode voltage increases, the electron plasma sheath extends further from the cathode, until it eventually fills the entire cathode-anode gap. This last configuration where the sheath hits the anode corresponds to a voltage "limit", or the highest operating voltage attainable for the device corresponding to a particular set of parameters. A good estimate of this limit may be obtained from the asymptotic solution. From Fig. 3 of Ref. 2, the function \( J \) of (37) - (38) is positive for all \( z \); Enforcing this in (43), yields to \( O(1) \):

\[ \phi < \phi_{\text{limit}} \equiv \frac{m_0^2 \Omega^2}{e} \left[ \frac{(e^{2v} - 1)^2}{8(e^{2v} + \rho)} + \frac{\ell n(H/\mu)}{L^2(\rho + 1)} \right] \]  

(49)

For our parameters corresponding to the typical medium and large aspect ratio cases, this voltage limit is 226.1 kV and 383.7 kV, respectively.

In order to obtain the full numerical solution, (36) (with (30)) was first solved for \( \nu(v) \) for each parameter set. From this, one has \( \nu(v_f) \equiv v_f \) where \( v_f = v_{\text{anode}} = \ell n(b/a) \). Hence \( z(\text{anode}) = z_f \) is obtained in terms of \( \epsilon \) via (32), so that \( J(z_f) \) is known from (37) - (38) (or Fig. 3 of Ref. 2). Using this, one may now use (43) to obtain the value of \( \epsilon \) required for each value of the anode voltage \( \phi(b) \).

Table 1 shows the required \( \epsilon \) values for various anode voltages. At this stage, the reason for the problem being "stiff" or intractable using a pure numerical solution becomes apparent. For instance

\[ \text{C. Particle Orbits} \]

To analyze the stability of the particle orbits, we perturb Eqs. (6). One finds that variations in the radial coordinate will evolve as
or negative. In terms of particle orbits, this means that wherever $\Delta^2/\Omega^2 \ll 1$, the orbits are on the verge of an instability$^{10}$, and that small perturbations can give large excursions.

ACKNOWLEDGMENTS

This research was supported in part by the ONR and AFOSR.

References


FIGURE CAPTIONS

Fig. 1 - The geometry for a smooth bore magnetron.

Fig. 2 - Stretched (z) versus physical (v) variables for the medium aspect ratio case.

Fig. 3 - The density profile in the anode-cathode gap for $\phi(\text{anode}) = 50kV$ in the medium aspect ratio case.

Fig. 4 - The same as Fig. 3 for $\phi(\text{anode}) = 200kV$.

Fig. 5 - The density profile for $\phi(\text{anode}) = 100kV$ in the high aspect ratio case.

Fig. 6 - The same as Fig. 5 for $\phi(\text{anode}) = 250kV$.

Fig. 7 - The same as Fig. 5 for $\phi(\text{anode}) = 350kV$. 
**TABLE CAPTION**

**TABLE 1.** The $\epsilon$ values required for various anode voltages in the medium and large aspect ratio cases.

<table>
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<tr>
<th>MEDIUM ASPECT RATIO</th>
<th></th>
<th>HIGH ASPECT RATIO</th>
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<tbody>
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<td>Anode Voltage (kV)</td>
<td>$\epsilon$</td>
<td>Anode Voltage (kV)</td>
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Stretched(z) and Physical(v) Variables, PHI = 25kV

Fig. 2
Fig. 3
Fig. 4
Large Aspect Ratio, PHI = 250kV

Fig. 6
Large Aspect Ratio, PHI = 350kV

Fig. 7