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Scattering Systems with Damping

by

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Integrable Stimulated Raman Scattering Systems with Damping

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ABSTRACT

We construct stimulated Raman scattering systems with damping that can be solved by the Inverse Scattering Transform with a variable spectral parameter, and present the corresponding linear systems. We also analyze the nonstandard properties of the damped soliton solution; amplitude, speed, and form change as the soliton propagates through a media.

1 Introduction

The soliton equations integrable by the Inverse Scattering Transform (IST) are conservative equations. Even small damping destroys the integrability. However, the IST with a variable spectral parameter[1], which is a generalization of the traditional IST, allows one to study integrable systems in the presence of special forms of damping, not necessarily small. We find that there are special forms of damping which allow an integrable system to remain integrable. We call these special forms of damping, “integrable damping”. They can be regularly found by the IST with a variable spectral parameter. What we do here is to apply the method of the variable spectral parameter to the general stimulated Raman scattering (SRS) and two-photon propagation(TPP) equations[2]. We find that there are several interesting forms of integrable damping which allow the resulting equations to remain integrable.

Let us first review the basic undamped SRS/TPP equations. They are[2]

$$\begin{aligned} \partial_\tau r &= i(grs_3 + r_3s) & \partial_\tau r_3 &= \frac{i}{2}(r\bar{s} - \bar{r}s) \\ \partial_\lambda s &= i(gsr_3 + \epsilon s_3r) & \partial_\lambda s_3 &= \frac{i}{2}(s\bar{r} - \bar{s}r) \end{aligned} \quad (1)$$

where $\epsilon = +1$ is used for SRS, and $\epsilon = -1$ for TPP. The system is written in characteristic coordinates τ and λ , τ is the retarded time:

$$\tau = t - x/v$$

$\chi = x$ is the spatial coordinate, (x, t) is a laboratory coordinate system, v is a group velocity of electromagnetic waves and g is a dynamical Stark shift coefficient. The Stokes vector $\vec{s} = (\text{Re}s, \text{Im}s, s_3)$ characterizes the pump and Stokes electromagnetic waves propagating through an optical media, represented by the Bloch vector $\vec{r} = (\text{Re}r, \text{Im}r, r_3)$, $s_3 = I_p - I_s$ is the difference between the intensities of pump- and Stokes-beams; r_3 is the inversion corresponding to the forbidden transition between ground and upper levels. Equations (1) can be integrated by Inverse Scattering Transform (IST) with a constant spectral parameter. The corresponding Lax pair for (1) is the following [3–5]:

$$\phi_\tau + U\phi = 0 \quad (2a)$$

$$\phi_\lambda + V\phi = 0 \quad (2b)$$

$$U = \lambda u_1 + u_0 \quad V = v_0 + \frac{v_1}{\lambda + g/2} \quad (3)$$

$$u_1 = \begin{pmatrix} is_3 & -s \\ \epsilon\bar{s} & -is_3 \end{pmatrix} \quad v_1 = -\frac{1}{2} \begin{pmatrix} -i\epsilon r_3/2 & (if - g/2)\epsilon r \\ (if + g/2)\bar{r} & i\epsilon r_3/2 \end{pmatrix}$$

$$u_0 = -if \begin{pmatrix} 0 & s \\ \epsilon \bar{s} & 0 \end{pmatrix} \quad v_0 = -\frac{1}{2} \begin{pmatrix} i g r_3 & \epsilon r \\ -\bar{r} & -i g r_3 \end{pmatrix}$$

where the bars refer to complex conjugation. Here $f^2 = (\epsilon - g^2)/4$ and λ is a constant spectral parameter.

The main goal of this work is to construct new integrable SRS models that include damping. Of these forms of integrable damping that we have found, two interesting ones are:

$$\begin{aligned} \partial_\tau r &= i(g r s_3 + r_3 s) - \bar{c} r & \partial_\tau r_3 &= \frac{i}{2}(r \bar{s} - \bar{r} s) - \bar{c} r_3 \\ \partial_\chi s &= i(g s r_3 + s_3 r) & \partial_\chi s_3 &= \frac{i}{2}(s \bar{r} - \bar{s} r) \end{aligned} \quad (4)$$

$$\begin{aligned} \partial_\tau r &= i(g r s_3 + r_3 s) & \partial_\tau r_3 &= \frac{i}{2}(r \bar{s} - \bar{r} s) \\ \partial_\chi s &= i(g s r_3 + s_3 r) - a s & \partial_\chi s_3 &= \frac{i}{2}(s \bar{r} - \bar{s} r) - a s_3 \end{aligned} \quad (5)$$

where $\bar{c}, a = \text{real constants}$. We remark that (4) and (5) can only be obtained when

$$\epsilon = 1, \quad g^2 = 1 \quad (6)$$

which means that $f = 0$. In other words, (4) and (5) only are integrable for the SRS-case and only when the Stark coefficient has a special value. This is not surprising and frequently occurs in other systems when one demands integrability.

Let us now make some remarks on the variable spectral parameter method[1]. Equations (2)–(3) actually are a set of overdetermined linear equations with a spectral parameter. The condition necessary for a solution to exist (integrability condition) is (1), provided λ is a constant. However, there is no reason for λ to be a constant except for simplicity. It was proposed in [1] to consider λ as a function of time, coordinate and an additional complex constant, called the "hidden" spectral parameter. The proposed method was coined as the IST with a variable spectral parameter. For each soliton equation, integrable by the traditional IST, one can generate the whole class of new equations, integrable by IST with a variable spectral parameter. The elements of this class are called "deformations" of the original soliton system. So, systems (4)-(5) are "deformations" of the SRS system (1). To construct special solutions, e.g. solitons, for "deformations" one can use, among other things, the "dressing" technique that was developed originally for constant λ [6]. The new solitons behave in new and interesting fashions. All of the soliton's characteristics such as amplitude, speed, and form change as the soliton propagates through a media.

The method proposed in [1] is a development of ideas of [8–10]. A series of "deformations" was also obtained in [11–14]; a symmetry approach was designed in [15]. The method to produce the finite-gap solutions was developed in [7].

In section 2 we construct a general “deformation” of the SRS-system (1) and also analyze special “deformations”. In section 3 we derive soliton solution to the damped SRS-system (4).

2 Inverse Scattering Transform with a Variable Spectral Parameter

2.1 General Deformation of SRS(TPP) System

Now we shall use a variable spectral parameter to deform the system (1). To do so, we only need to make one modification in a traditional IST. λ is a function of τ, χ and some “hidden” spectral parameter z . The dependence on z is important as it allows the functions λ and $\frac{1}{\lambda + g/2}$ to be linear-independent. As we shall later see, the function $\lambda = \lambda(\tau, \chi, z)$ is not fully arbitrary.

If (2) is to be integrable, then it follows that

$$\frac{\partial U}{\partial \chi} - \frac{\partial V}{\partial \tau} = [U, V] \quad (7)$$

Evaluation of (7) generates an expression with terms proportional to $(\lambda + g/2)^{-1}$. Consider the derivatives of λ . If they are to generate no additional functions of λ , then we must take

$$\frac{\partial \lambda}{\partial \chi} = a\lambda + b + \frac{c}{\lambda + g/2} \quad (8a)$$

$$\frac{\partial}{\partial \tau} \frac{1}{\lambda + g/2} = \tilde{a}\lambda + \tilde{b} + \frac{\tilde{c}}{\lambda + g/2} \quad (8b)$$

where $a, b, c, \tilde{a}, \tilde{b}$ and $\tilde{c}$ are arbitrary coefficients to be determined. This gives us a closed system for the function $\lambda = \lambda(\tau, \chi, z)$. Now using (8) in (7) gives three matrix nonlinear equations

$$\partial_\tau v_1 + [u_0 - \frac{g}{2}u_1, v_1] = -\tilde{c}v_1 + cu_1 \quad (9a)$$

$$\partial_\chi u_1 + [v_0, u_1] = -au_1 + \tilde{a}v_1 \quad (9b)$$

$$\partial_\chi u_0 - \partial_\tau v_0 - [u_0, v_0] - [u_1, v_1] = -bu_1 + \tilde{b}v_1 \quad (9c)$$

Rewriting (9a) by component gives equations for r and r_3

$$\begin{aligned}\partial_\tau r &= i(g r s_3 + r_3 s) - \bar{c} r - 8c(if + g/2)s \\ \partial_\tau r_3 &= \frac{i}{2}(r\bar{s} - \bar{r}s) - \bar{c} r_3 + 4c s_3\end{aligned}\tag{9a'}$$

Doing the same with (9b) give

$$\begin{aligned}\partial_\lambda s &= i(g s r_3 + \epsilon s_3 r) - a s + \bar{a}(if - g/2)\epsilon r/2 \\ \partial_\lambda s_3 &= \frac{i}{2}(s\bar{r} - \bar{s}r) - a s_3 + \epsilon \bar{a} r_3/4\end{aligned}\tag{9b'}$$

System (9a'),(9b') represents a general class of deformations of SRS (TPP) system. If λ is constant, then the third matrix equation (9c) doesn't generate any additional equations for r, r_3, s and s_3 provided (9a),(9b) are true. When λ isn't a constant, then (9c) generates, in general, additional equations and causes the nonlinear system to be overdetermined. To avoid the overdeterminacy and to avoid constraining the r and s fields, one may assume additional constraints on coefficients $a, b, c, \bar{a}, \bar{b}$ and $\bar{c}$ which have been so far arbitrary. From (9c), one finds that the first set of these constraints must be

$$b = -2gc; \quad \bar{b} = -2\epsilon g\bar{c}\tag{10}$$

For the other restrictions contained in (9c), there are two possibilities. If it happens that $f = 0$, then (9c) automatically holds true, provided (9a,9b) are satisfied. Then in this case there would be no further constraints other than (10) ($f = 0$ only if $\epsilon = +1$ (SRS) and $g^2 = 1$). But if $f \neq 0$, then there will be a further set of constraints. These are

$$c = \frac{ifa/4}{\epsilon(if + g/2) - g/2}; \quad \bar{c} = \epsilon \frac{\bar{a}}{4}\tag{11}$$

Now there is only one other thing to consider. At this point we have equations (2) integrable, provided (8) is also integrable. As noted above, there are two cases. First, when i) $f = 0$ and (10) must hold and second, when ii) $f \neq 0$ and (10),(11) must hold. So, we still must insure that (8) is also integrable, subject to the algebraic constraints for these two subcases. We shall consider these integrability conditions for (8) for each of the above subcases below.

2.2 Specific Equations

It is convenient to summarize all following results on specific deformations of SRS(TPP) system in a graphical tree form

We will take up each one of these branches individually. First, for the left branch of either the SRS or TPP tree we always have $f \neq 0$ with both (10) and (11) being necessary. Then integrability of (8) along with (10) and (11) gives two possible solutions depending on whether or not $g = 0$. If $g \neq 0$, then the integrability of (8) only requires

$$a\bar{a} = 0 \quad (12)$$

so that either a or $\bar{a}$ must be zero. The corresponding nonlinear systems are, for $\bar{a} = 0$

$$\begin{aligned} \partial_\tau r &= i(gsr_3 + r_3s) - 8c(if + g/2)s \\ \partial_\tau r_3 &= \frac{i}{2}(r\bar{s} - \bar{r}s) + 4cs_3 \end{aligned} \quad (13)$$

$$\begin{aligned} \partial_\chi s &= i(gsr_3 + \epsilon s_3r) - as \\ \partial_\chi s_3 &= \frac{i}{2}(s\bar{r} - \bar{s}r) - as_3 \end{aligned}$$

and for $a = 0$

$$\begin{aligned} \partial_\tau r &= i(gsr_3 + r_3s) - \epsilon\bar{a}r/4 \\ \partial_\tau r_3 &= \frac{i}{2}(r\bar{s} - \bar{r}s) - \epsilon\bar{a}r_3/4 \end{aligned} \quad (14)$$

$$\begin{aligned} \partial_\chi s &= i(gsr_3 + \epsilon s_3r) + \epsilon\bar{a}(if - g/2)r/2 \\ \partial_\chi s_3 &= \frac{i}{2}(s\bar{r} - \bar{s}r) + \epsilon\bar{a}r_3/4 \end{aligned}$$

Here a and $\bar{a}$ may be not only constants, but any functions of χ and τ correspondingly. Note that the equations (13),(14) involve a sort of “cross-damping” where $s(s_3)$ drives $r(r_3)$ in (13). The same is true in (14) except that r and s are interchanged.

If $g = 0$, then the integrability of (8) only requires that

$$a_\tau + \epsilon a \bar{a}/2 = 0, \quad \bar{a}_\chi + 2\bar{a}a = 0 \quad (15)$$

Of course, this could be the same as (12) if we took a and $\bar{a}$ to be constant. However (15) allows a more general class of solutions. Applying substitution

$$a = \frac{1}{2}\epsilon(\ell n G)_\chi, \quad \bar{a} = 2(\ell n G)_\tau$$

reduce (15) to a wave equation for G .

$$G_{\tau\chi} = 0 \quad (16)$$

The related nonlinear system is now

$$\begin{aligned} \partial_\tau r &= ir_3 s - \epsilon \bar{a} r/4 - i\epsilon^{1/2} a s \\ \partial_\tau r_3 &= \frac{i}{2}(r\bar{s} - \bar{r}s) - \epsilon \bar{a} r_3/4 + \epsilon a s_3 \end{aligned} \quad (17)$$

$$\begin{aligned} \partial_\chi s &= i\epsilon s_3 r - a s + i\epsilon^{1/2} \bar{a} r/4 \\ \partial_\chi s_3 &= \frac{i}{2}(s\bar{r} - \bar{s}r) - a s_3 + \epsilon \bar{a} r_3/4 \end{aligned}$$

Note that we now have a more general type of “cross-damping” whereby both $r(r_3)$ and $s(s_3)$ drives the other one. Thus we have a more general linear coupling between the fields.

Now, all that is left is the right branch of SRS tree. In this case, when $f = 0$, (6) and (10) are true, we obtain an even more general nonlinear system for the coefficients

$$\begin{aligned} \bar{a}_\chi + 2a\bar{a} &= 0, \quad \bar{c}_\chi + a\bar{c} + \frac{1}{2}a\bar{a} + 3\bar{a}c = 0 \\ c_\tau + 2\bar{c}c &= 0, \quad a_\tau + \bar{c}a + 8c\bar{c} + 3\bar{a}c = 0 \end{aligned} \quad (18)$$

At the moment, there is no general solution of (18), however there are some interesting cases contained in the seemingly trivial solution

$$\bar{a} = 0 = c \quad (19)$$

which reduces (18) to

$$\bar{c}_\chi + a\bar{c} = 0, \quad a_\tau + \bar{c}a = 0 \quad (20)$$

If one takes $\bar{c} = (\ell n G)_\tau$, $a = (\ell n G)_\chi$, then (20) is transformed into (16).

Particular simple examples follow from (20) upon taking $a = 0$ and $\bar{c} \neq 0$ in which case, the general system (9) reduces to (4) which is SRS system with a damping that affects only the optical media. If we choose $a \neq 0$ and $\bar{c} = 0$, then (9) reduces to (5). In this case, the electromagnetic wave loses its energy because of conductivity. In [15] a special deformations (13),(14)($f \neq 0, g \neq 0$) of SRS(TPP) were constructed.

2.3 The SRS equation with conductivity

There is a very interesting case of integrable damping for the SRS equations which was obtained in [18]. This integrable damping is simply an ordinary conductivity and it is interesting to follow how the result of [18] fits into our scheme.

Assume that $g = 0$ and consider a very special case of the general deformation of SRS (9a'),(9b'): $\bar{a} = 0 = \bar{b} = \bar{c}; 4c = a = \text{const}$. Let's make the following scaling of every dependent and independent variable in the deformation:

$$\begin{aligned} \chi &= \underline{\chi}/\varepsilon & s &= \varepsilon \underline{s} & s_3 &= \varepsilon \underline{s}_3 \\ \tau &= \underline{\tau} & r &= \varepsilon \underline{r} & r_3 &= -1 + O(\varepsilon^2) \\ a &= \varepsilon \underline{a} \end{aligned} \quad (21)$$

and take the limit: $\varepsilon \rightarrow 0$. The bar under variable denotes a new variable and should not be confused with the bar over variable which means complex conjugation. Such an asymptotic expansion means that almost all atoms in optical media are in a ground state, while low amplitude, long wave optical pulses are propagating through the media. As a result, we obtain the SRS system with an ordinary conductivity:

$$\partial_{\underline{\tau}} \underline{r} = -i \underline{s} \quad (22)$$

$$\partial_{\underline{\chi}} \underline{s} = i \underline{s}_3 \cdot \underline{r} - \underline{a} \cdot \underline{s}$$

$$\partial_{\underline{\chi}} \underline{s}_3 = \frac{i}{2} (\underline{s} \cdot \underline{\bar{r}} - \underline{\bar{s}} \cdot \underline{r}) - \underline{a} \cdot \underline{s}_3$$

which has been analyzed in [18]. The authors of [18] derived Lax pair for (22) using equivalence between special SRS system (see (24) below) and the Maxwell-Bloch system[2]. System (22) is, in a way, a trivial deformation. We remark that so far, every soliton system with integrable damping obtained by the IST with a variable spectral parameter is trivial, except system (4). The Maxwell-Bloch system with damping[1,14](which is equivalent to (22)), equations (5) are also trivial. What happens is that there is a certain transformation of dependent and independent variables which will reduce these systems with integrable damping to a system with no damping. To see this, let

$$\eta = \underline{\tau} \quad ; \quad \xi = \underline{a}^{-1}(1 - e^{-\underline{a}\underline{\chi}}) \quad (23)$$

$$(\underline{s}, \underline{s}_3, \underline{\tau}, \underline{r}_3)(\underline{\tau}, \underline{\chi}) = (S, S_3, R, R_3)(\eta, \xi) \cdot e^{-\underline{a}\underline{\chi}}$$

Then this transformation reduces the deformed system (22) to the original one:

$$\partial_{\eta} R = -iS \quad (24)$$

$$\begin{aligned}\partial_\xi S &= iS_3 R \\ \partial_\xi S_3 &= \frac{i}{2}(S\bar{R} - \bar{S}R)\end{aligned}$$

The transformation (23) is possible only because the spectral parameter λ :

$$\lambda = \lambda(z, \underline{\chi}) = ze^{-a\underline{\chi}} \quad (25)$$

corresponding to the system (22), has a very simple structure: the z -dependence is separated from $\underline{\chi}$ -dependence. To construct the transformation (23) it's better to use, instead of nonlinear system (22), the corresponding Lax pair that contains the spectral parameter $\lambda(z, \underline{\chi})$. First, one substitutes (25) into the Lax pair and multiplies the result by $e^{a\underline{\chi}}$. Then, it becomes clear that one can make a transformation of the spatial coordinate $\underline{\chi}$:

$$e^{a\underline{\chi}} \frac{\partial}{\partial \underline{\chi}} = \frac{\partial}{\partial \xi}$$

and that one can redefine the $\underline{s}$, $\underline{s}_3$, $\underline{r}$, and $\underline{r}_3$ fields according to (23). Finally, one obtains the Lax pair for the undamped system (24) that contains z as a spectral parameter. One cannot construct such a transformation for the damped system (4) because the z -dependence of the corresponding $\lambda = \lambda(z, \tau)$ (see (27) below) cannot be separated from τ -dependence.

So, one can take any known exact solution $(R, S, S_3)(\eta, \xi)$, process it with the transformation (23) and generate an exact solution for a damped system (22). Now note how this transformation has changed the evolution. The system with integrable damping evolves practically almost like the undamped system, except that the spatial coordinate has been distorted and we have to multiply fields by $\exp(-a\underline{\chi})$ in (23). Thus as $\underline{\chi}$ becomes much larger than $(\varepsilon a)^{-1}$, the intensity of the physical variables is vanishing, thereby cutting off the interaction. Meanwhile the variables R, S and S_3 continue to evolve “normally”, however at a decreasing rate (consistent with a vanishing electromagnetic field), slowly approaching the final asymptotic state at $\xi = 1/a$ as $\underline{\chi} \rightarrow \infty$.

3 Soliton Solution (Damped and Undamped)

If one considered the most general form of SRS with realistic damping terms, then the most appropriate model would be

$$\begin{aligned}\partial_\tau r &= i(grs_3 + r_3s) - \frac{r}{T}, & \partial_\tau r_3 &= \frac{i}{2}(r\bar{s} - \bar{r}s) - \frac{r_3 - r_3^0}{T_3} \\ \partial_\chi s &= i(gsr_3 + s_3r) - \sigma s, & \partial_\chi s_3 &= \frac{i}{2}(s\bar{r} - \bar{s}r) - \sigma s_3\end{aligned} \quad (26)$$

where T is the time of relaxation for polarization, T_3 is the time constant for decay of the inversion and σ is a phenomenological conductivity. Unfortunately, even with a variable spectral parameter, (26) is not integrable. However, we do have two special cases of (26) which can be of interest. The first case corresponds to Eq. (4) when there is no conductivity, $\sigma = 0$, and the relaxation constants T, T_3 and r_3^0 are taken to be $T = T_3$ and $r_3^0 = 0$. If it were not for r_3^0 being zero, then this would be an important physical situation. However $r_3^0 = 0$ means that the stable equilibrium would have to be 50% excited with equal populations in each level. That can only occur physically in an infinitely high temperature. The second case is when $T = \infty = T_3$ and only the conductivity, σ , is present. This case is given by (5) and would model SRS in a conducting medium and was discussed above.

Now let us turn our attention to (4) and look at the soliton solution. We derive the one-soliton formula for (4) using the “dressing” technique [6,16], taking into account that λ must satisfy (8). This gives

$$\lambda = \lambda(z, \tau) = \frac{ze^{\tilde{c}\tau}}{1 + 2gz(1 - e^{\tilde{c}\tau})} \quad (27)$$

We have also used $a = 0 = b = c = \tilde{a}, \tilde{b} = -2g\tilde{c}, \tilde{c} = \text{real const.}, g = \pm 1$. Note that z is the “hidden” spectral parameter which is, in fact, just an integration constant for the system (8). Omitting standard calculations, we present the final formula:

$$\begin{aligned} \frac{s_3}{s_0} &= 1 - 2 \frac{\lambda_{1i}^2}{|\lambda_1|^2} \frac{1}{\cosh^2 \theta} \\ \frac{s}{s_0} &= 2 \frac{\lambda_{1i}}{|\lambda_1|^2} \frac{e^{iq}}{\cosh \theta} [-\lambda_{1r} + i\lambda_{1i} \tanh \theta] \end{aligned} \quad (28)$$

In the above, we have taken

$$z = z_1 = \xi + \eta$$

where

$$\lambda_1 = \lambda(z = z_1, \tau) = \lambda_{1r} + i\lambda_{1i}$$

Note that $\lambda(\tau = 0) = z$ so z is also the initial value of variable spectral parameter. Also

$$\begin{aligned} \theta &= \frac{gs_0}{\tilde{c}} \arg \varphi - \frac{r_0 \Delta \eta / 2}{|z_1 + g/2|^2} + \theta_0 \\ q &= \frac{gs_{30}}{\tilde{c}} \ln |\varphi| + r_0 \Delta \left\{ g - \frac{\xi + g/2}{|z_1 + g/2|^2} \frac{1}{2} \right\} + q_0 \end{aligned} \quad (29)$$

where θ_0 and q_0 are real constants that define the real and complex phases of the soliton, respectively; the dependence on retarded time τ and the coordinate χ is defined in the functions $\varphi(\tau)$ and $\Delta(\chi)$:

$$\begin{aligned}\varphi(\tau) &= 1 + 2g\bar{z}_1(1 - e^{\tilde{c}\tau}) \\ \Delta(\chi) &= \frac{v}{\tilde{c}}(1 - e^{-\tilde{c}\chi/v})\end{aligned}\quad (30)$$

If η goes to 0, then formulas (27)-(30) degenerate to :

$$\begin{aligned}s_3 &= s_0 & r_3 &= r_0 \exp[-\tilde{c}(\tau + \chi/v)] \\ s &= 0 & r &= 0\end{aligned}\quad (31)$$

which define the “background” solution. Transforming (31) into the laboratory frame, we get for r_3 :

$$r_3 = r_0 e^{-\tilde{c}t}$$

Soliton formulas (27)-(30) are parameterized by four real constants: s_0, r_0, θ_0, q_0 and by one complex constant: z_1 . In the limit $\tilde{c} \rightarrow 0$, the “damped” soliton (27)-(30) degenerates into the known SRS soliton [3,5,17]. Indeed as $\tilde{c} \rightarrow 0$, then $\lambda \rightarrow z$; $\Delta \rightarrow \chi$; $\frac{g}{\tilde{c}} \ln \varphi \rightarrow -\bar{z}_1 \tau$, and

$$\begin{aligned}\theta &\rightarrow s_0 \eta \tau - \frac{r_0 \eta \chi / 2}{|z_1 + g/2|^2} + \theta_0 \\ q &\rightarrow -s_0 \xi \tau + r_0 \chi g - \frac{\xi + g/2}{|z_1 + g/2|^2} \frac{1}{2} + q_0\end{aligned}\quad (32)$$

To get the undamped Raman soliton, one simply has to substitute expressions (32) into (28).

Although the structure of formula (28) for 1-soliton is exactly the same as in the absence of damping; nevertheless, the damping strongly affects the way τ and χ appear in the expression (28). Without damping, $\ln \varphi / \tilde{c}$ and $\Delta(\chi)$ are just linear functions of τ and χ , respectively. With damping, $\ln \varphi / \tau$ and Δ are nonlinear functions. It is they (together with expression (27) for $\lambda(z, \tau)$) that define how the soliton’s parameters change as the soliton propagates through an optical media. In the “future”, $t \rightarrow +\infty$, expressions (28)-(30) simplify and describe a soliton of asymmetric shape that moves with the group velocity v of electromagnetic waves in a transparent media. To obtain this result, let us switch to the reference system that moves with the group velocity, $x - vt = y$ and take the limit $t \rightarrow +\infty$ in (28)-(30). Then

$$\lambda = \lambda(z, y) = \frac{ze^{-\tilde{c}y/v}}{1 + 2gz(1 - e^{-\tilde{c}y/v})}\quad (33)$$

$$\varphi = \varphi(y) = 1 + 2g\bar{z}_1(1 - e^{-\tilde{c}y/v})\quad (34)$$

Figure 1: Damped soliton(solid line) to SRS equation (4) vs. undamped soliton(dashed line)

$$\Delta \rightarrow \frac{v}{c}$$

in (28)-(29). The asymptotic shape of soliton is assymetric, because functions $\lambda(z, y)$ and $\varphi(y)$ are not even functions of y any more. Fig.1 represents a plot of $s_3(y)$ (solid line) as $t \rightarrow \infty$. Dashed line denotes the shape of the soliton [5,17] for SRS without any damping. In the above, we have chosen the following values for the parameters:

$$s_0 = r_0 = 1, g = v = 1; \tilde{c} = 0.3, \eta = 0.1 \quad (35)$$

From (28)—(30) one can show that the soliton is almost standing still (in laboratory reference system) when $\tau \leq 0$. Then, around $t = 1$, the soliton picks up speed, approaching the group velocity, v , of electromagnetic waves. Such a scenario may be qualitatively explained by the expression

$$v_s = \frac{v}{1 + \frac{r_0 v / 2 s_0}{|z_1 + g/2|^2}} \quad (36)$$

for the velocity v_s of an *undamped* Raman soliton. Here the constants r_0 and s_0 are the background values of fields r_3 and s_3 , respectively. Now, switch on the damping of the type studied in this

Figure 2: The dynamics of the damped soliton

section. The first change that occurs in (36) would be that the background inversion constant, r_0 would no longer be a constant and would start to grow exponentially as $t \rightarrow -\infty$; in other words the optical medium would become too dense for the soliton to propagate. So, in this limit, $v_s \rightarrow 0$. For time going forward, $t \rightarrow +\infty$, the inversion r_0 would decay and therefore the soliton accelerates: $v_s \rightarrow v$. Fig.2 illustrates these dynamics of the damped soliton in (x, t) -plane. (To get a better result we plotted function $-s_3(x, t) + 1$ instead of $s_3(x, t)$.)

To explain other features of Fig.2, we analyze expressions (27) - (29) in the limits $\tau \rightarrow \pm\infty$. If $\tau \rightarrow +\infty$; $\chi = \chi_0$, then

$$\lambda \rightarrow -g/2 \quad (37a)$$

$$\theta \rightarrow \frac{gs_0}{\tilde{c}} \tan^{-1}\left[-\frac{\eta}{\xi}\right] - \frac{r_0\eta\Delta/2}{|z_1 + g/2|^2} + \theta_0 \quad (37b)$$

is finite. As the result of (37a) and (28)

$$\lim_{\tau \rightarrow +\infty} s_3 = s_0 \quad (38)$$

If we return to the laboratory reference system, the result (38) means that both limits: a) $t \rightarrow$

$+\infty; x = x_0$ and b) $t = t_0; x \rightarrow -\infty$, results in s_3 going to the constant $s_0 = 1$. If $\tau \rightarrow -\infty; \chi = \chi_0$, then

$$\lambda_1 \simeq \frac{z_1 e^{\tilde{c}\tau}}{1 + 2gz_1}$$

$$\frac{\lambda_1^2}{|\lambda_1|^2} \rightarrow \frac{\eta^2}{|z_1|^2 |1 + 2gz_1|^2} \quad (39a)$$

$$\theta \rightarrow \frac{gs_0}{\tilde{c}} \tan^{-1} \left[-\frac{2g\eta}{1 + 2g\xi} \right] - \frac{r_0 \eta \Delta/2}{|z_1 + g/2|^2} + \theta_0 \quad (39b)$$

Because of (39a) and (39b), $\lim s_3 \neq s_0$. Thus, both limits a) $t \rightarrow -\infty; x = x_0$ and b) $t = t_0; x \rightarrow +\infty$ do not produce s_0 for s_3 . (To take the latter limit, we had to take into account that $\Delta(\chi) \rightarrow \frac{\tau}{\tilde{c}}$ as $\chi \rightarrow +\infty$.) The results (39) mean that damped soliton (28) - (30) is not really a solitary wave at all: the magnitude s_3 does not go to the same value of s_0 on right end of x -axis as it does on the left end. Rather it is like a combination of a solitary and a shock wave. For simplicity, we present the asymptotics for the special case $\xi = 0$ only

$$\lim_{x \rightarrow +\infty} \left[\frac{s_3}{s_0} - 1 \right] = -\frac{2}{1 + 4\eta^2} \frac{1}{\cosh^2 \left[\frac{gs_0}{\tilde{c}} \tan^{-1}(2g\eta) + \frac{2\eta r_0 v}{\tilde{c}(1 + 4\eta^2)} + \theta_0 \right]} \quad (40)$$

which has an inverse dependence on the damping $\tilde{c}$. The less damping $\tilde{c}$ or the more η , the less is the right-hand side of (40) and the more the damped soliton is like an undamped one. But even for small values of the damping the “tail” is present.

In the absence of damping, the Raman soliton may be created in the following way. First, we have a pure ($s_0 = 1$) pump laser beam incident on an optical medium that has all atoms in the upper state ($r_0 = +1$). Such a state is a stable one, because there are no Stokes photons that would cause the atoms to decay from the upper to the ground state. Then, we switch on a Stokes beam for a certain time. As a consequence we destabilize the total configuration (pump, Stokes and medium) and atoms start to decay into the ground state, coherently emitting photons at the frequency of pump wave. After shutting off Stokes beam and waiting for some time, there will be remaining a Raman soliton in the medium which is shown by the dashed line in Fig.1. This optical pulse has all the regular properties of a soliton.

To create a soliton under the presence of our integrable damping $\tilde{c}$, we probably would have to change the experimental setup, described earlier, in only one respect: at the beginning, we would have to use a combination of pump and Stokes incident on the optical medium (see Fig.2). The right

Figure 3: The soliton generating input pulse

ratio of intensities is probably determined by both the value of damping $\tilde{c}(40)$ and the amplitude of the future soliton. Then we increase the intensity of Stokes beam for a certain time and finally turn off Stokes beam completely. An exact profile of the damped SRS soliton is shown in Fig.3. The plotted function $s_3(x = 10, t)$ represents the normalized difference between pump- and Stokes-intensities. We obtained this function from expressions (28)-(30) and (35). For convenience, we took the left edge of the optical sample to be at $x = 10$.

We suspect that there are some other recipes to create a soliton under the presence of integrable damping. Maybe it's not even necessary to increase the intensity of the Stokes component at all as in the previous examples. Maybe it's sufficient just to shut off the Stokes component. To prove (or disapprove) such a scenario, it would be necessary to analyze the scattering problem (2a) with λ being defined by (27). This problem is very different from that studied before [3], [5] without any damping. The reader should note that if (27) is inserted into (3), then the analytical structure of (2a) will be dramatically changed. In (2a), U will have a simple pole on real z -axis, causing Φ to have a branch cut along the z -axis. For example, let's fix $s_3 = 1$ and $s = 0$ in (2a). Then the solution of (2a) is

$$\Phi(z, \tau) = \exp(i \frac{\sigma_3}{2g\bar{c}} \ln[1 + 2gz(1 - e^{\bar{c}\tau}]))$$

If $\bar{c}$ goes to zero, then $\Phi \rightarrow \exp[-iz\tau\sigma_3]$ which is a regular function on a real z -axis. But if $\bar{c} \neq 0$, then $\Phi(z, \tau)$, for any value of τ , has a branch point at

$$z = -\frac{g/2}{1 - e^{\bar{c}\tau}}$$

When τ changes from $-\infty$ to $+\infty$ the branch point sweeps out all values from $-\infty$ to $+\infty$. Thus, to define Jost functions correctly, one would have to make a cut along the real z -axis. This is in stark contrast to the undamped case where Φ only has an essential singularity at $\lambda = \infty$. Due to this branch cut along the real z -axis, we cannot use any criteria from the undamped case. Rather, one would have to analyze the eigenvalue problem (2a) for conditions necessary and/or sufficient for the bound states to exist. So with this in mind, one should not be too surprised if the solution for the damped case turned out to be significantly different from the undamped case. Turning our attention to b) we see that V has the very same simple pole: $\lambda = -g/2$ and $z = -g/2$ for both the damped and undamped cases respectively. Consequently the evolution of the scattering data should be exactly the same in both cases.

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