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Coherent Pulse Propagation in

a Discrete Non-Bragg

Resonant Medium

by

T.I. Lakoba

CLARKSON UNIVERSITY
Potsdam, New York 13699-5815, U.S.A.

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T.I.Lakoba*

February 1, 1994

Abstract

Equations of coherent pulse propagation in a resonance grating with non-Bragg-type modulation of its refractive index are derived for the first time and investigated. The main result of the work is that the velocity of a pulse in such a non-Bragg grating is determined by the structural parameter γ which measures how strong the non-Bragg harmonic in the spatial spectrum of the refractive index modulation of the grating is compared to the Bragg harmonic. For a certain range of values of γ it was found that stationary propagation of a localized pulse is impossible in the non-Bragg resonant grating.

*Department of Mathematics, Clarkson University, Potsdam, NY 13699-5817

1 Introduction

The propagation of electromagnetic pulses in nonlinear periodic structures was intensively studied in the recent years [1-9] in connection to its possible applications in all-optical switching devices. Most of the research in this direction was devoted to the case when the condition of strong Bragg reflection for the pulse's carrier was satisfied, i.e. when

$$d = \lambda/2 + \Delta, \quad |\Delta| \ll d, \quad (1)$$

where λ is the carrier's wavelength and d is the period of modulation of the medium's refractive index. In that case, the balance between the medium's nonlinearity and the spatial dispersion resulted from the coupling of the two counterpropagating modes leads to the possibility of stationary pulse propagation, with the slow envelopes of the two modes moving with the same group velocity in one direction. The existence of such two-mode pulses was predicted for periodically modulated fibers with the Kerr type of nonlinearity [1-3, 5-7], as well as for discrete gratings with the nonlinear refractive index of the resonant type [4]. Strong mode interaction due to the Bragg scattering can also take place for co-propagating modes if the period of modulation of the refractive index equals the length of spatial beats between these modes. This latter situation takes place for pulse propagation in twisted or periodically corrugated birefringent optical fibers [8,9].

It would be interesting to observe the effects caused by introducing a non-Bragg spatial harmonic in the spectrum of the refractive index modulation. In a linear medium in which the refractive index is modulated with spatial periods of both $\lambda/4$ and $\lambda/2$, with depth of the modulation being of order $\epsilon \ll 1$, the envelope of a pulse with the carrier's wavelength λ undergoes decay on the time scale ϵ^{-1} due to the Bragg scattering on the $\lambda/2$ - modulation harmonic, while the scattering on the non-Bragg $\lambda/4$ - harmonic has no effect on this time scale. In the present work we show that, in nonlinear media, by introducing even a weak non-Bragg modulation of the nonlinear refractive index one may completely change the dynamics of a propagating pulse as compared to the cases

with the purely Bragg modulation and without any modulation at all.

We consider propagation of an electromagnetic pulse in a discrete non-Bragg resonant grating (hereafter referred to as the NBRG). The NBRG is an array of equally distanced, parallel films, with the film's width being much smaller than the distance between each two of them. The NBRG interacts with an electromagnetic pulse whose carrier's wavelength is an integer multiple of the distance between the films and carrier's frequency is in resonance with one of the eigenfrequencies of atoms in the films. More will be said about this model in Section 2. The case of such a resonance grating, when the carrier's wavelength was twice the distance between two neighboring films and thus the Bragg condition *was* satisfied, was considered in [4]. There it was found that a quantity

$$\phi(x, t) = \int_{-\infty}^t (\mathcal{E}^+(x, t') + \mathcal{E}^-(x, t')) dt' ,$$

where $\mathcal{E}^{\pm}$ are slow envelopes of the two carrier modes with opposite group velocities and are supposed to be real, satisfies the Sine-Gordon equation (SGE):

$$\phi_{xx} - \phi_{tt} = \sin \phi .$$

In above, x and t are appropriately undimensionalized space and time coordinates. The forward and backward modes' amplitudes were related to the quantity ϕ by: $\mathcal{E}^{\pm} = \frac{1}{2}(\phi_t \mp \phi_x)$. Since the SGE is exactly integrable by means of the Inverse Scattering Transform (IST) method, then in a discrete resonant grating where the Bragg condition *is* satisfied, pulses can propagate with arbitrary velocities ranging from zero to the speed of light, depending on their width.

In the present work, we consider the situation when the wavelength of the pulse's carrier is *four* times the distance between two neighboring films, and thus the Bragg condition is strongly violated. We introduce a parameter γ defined as follows: the ratio of densities of resonance atoms in any three consecutive films is either $1 : \gamma : 1$ or $\gamma : 1 : \gamma$. The case with $\gamma = 0$ corresponds to the purely Bragg case considered in [4], and the one with $\gamma = 1$ corresponds to what we will refer to as the purely non-Bragg case. We will demonstrate that, even for small γ , the dynamics of a pulse in the NBRG is qualitatively different

from the dynamics in the purely Bragg case in that the pulse now can propagate only with discrete values of its velocity. In the opposite limiting case, i.e. when $\gamma \approx 1$, we show that no localized pulse can stationarily propagate in the NBRG. Finally, we point out that in the purely non-Bragg case the equations of the model in question may be, under certain conditions, reduced to the real-valued Self-Induced Transparency (SIT) equations for the continuous medium, with the difference from the continuous medium case arising if one considers collisions of counterpropagating pulses.

2 Derivation of the equations of the model and General discussion

Let us consider a periodic structure consisting of parallel, ultrathin films, with the film thickness being much less than the distance d between two neighboring films. Linear refractive indices of the films and the medium around them are equal to 1. A quasimonochromatic, linearly polarized optical pulse with the electric field

$$E(x, t) = \frac{1}{2}(\mathcal{E}^+(x, t)e^{ikx - i\omega t} + \mathcal{E}^-(x, t)e^{-ikx - i\omega t}) + c.c \quad (2)$$

propagates in the x -direction normal to the films; $\mathcal{E}^\pm$ are slowly varying envelopes of the two counterpropagating modes. The wavelength λ of the pulse's carrier satisfies the condition

$$\lambda/4 = d. \quad (3)$$

The carrier's frequency ω is exactly equal to a frequency of one of the radiational transitions between two atomic energy levels for atoms in the films; for this reason, we will term the atoms "two-level atoms". The dipole moment of the resonance transition is μ . Surface density of the two-level atoms is $\mathcal{N}$ for the films located at $x = \frac{\lambda}{2}m$, where m goes over all integers, and it is $\gamma\mathcal{N}$ for the films located at $x = \frac{\lambda}{2}(m + \frac{1}{2})$. In what follows we will use the letter m to denote only integer numbers. Without loss of generality one can take $\gamma \leq 1$. We assume that reflection due to the resonance interaction between the electromagnetic field and the two-level atoms is very weak; it is this assumption

that will allow us to perform spatial averaging of the initial discrete equation (4) below. In completing the description of our physical model, we note that relaxation and dissipation are neglected in what follows.

Following the paper [4], we can write down the equations for the slow amplitudes $\mathcal{E}^\pm$ and atomic variables:

$$(\mathcal{E}_t^+ + c\mathcal{E}_x^+ - \frac{ic}{2k}\mathcal{E}_{xx}^+)e^{ikx} + (\mathcal{E}_t^- - c\mathcal{E}_x^- - \frac{ic}{2k}\mathcal{E}_{xx}^-)e^{-ikx} = 4\pi in(x)\omega\mu\rho_{12}, \quad (4)$$

$$\frac{\partial\rho_{12}}{\partial t} = \frac{i\mu}{2\hbar}(\mathcal{E}^+e^{ikx} + \mathcal{E}^-e^{-ikx})(\rho_{22} - \rho_{11}); \quad (5a)$$

$$\frac{\partial(\rho_{22} - \rho_{11})}{\partial t} = \frac{i\mu}{\hbar}((\mathcal{E}^+e^{ikx} + \mathcal{E}^-e^{-ikx})\rho_{12} - c.c.). \quad (5b)$$

In (4), $n(x) = \sum_{m=-\infty}^{\infty}(\mathcal{N}\delta(x - 2md) + \gamma\mathcal{N}\delta(x - 2(m+1)d))$, where $\delta(\dots)$ is a Dirac delta-function. We omitted the terms $\mathcal{E}_{tt}^\pm$ on the grounds of the slowness of $\mathcal{E}^\pm$, but we had to retain the terms $\mathcal{E}_{xx}^\pm$ because, as it may be seen from the right-hand side of (4) and as it will be explicitly demonstrated below, $\mathcal{E}^\pm$ are discontinuous at $x = md$, and hence $\mathcal{E}_{xx}^\pm$ cannot be neglected *a priori*. Below, however, we will show that these terms may be omitted as well. In eqs (5), which are the standard Bloch equations for a two-level atom, $(\rho_{22} - \rho_{11})$ is the atomic inversion and ρ_{21} is a quantity proportional to polarization of the medium, with all the values being evaluated for $x = md$, i.e. on the films. Then, for the right-hand sides of eqs (5) to make sense, the quantity $(\mathcal{E}^+e^{ikx} + \mathcal{E}^-e^{-ikx})$ is to be continuous on the films. This is indeed the case; to see that, one needs to integrate eq (4) over x once in a small vicinity of a film and make use of slowness of $\mathcal{E}^\pm$ outside the films. As a by-product, we also obtain a useful expression for the values of the modes' discontinuities at a film:

$$\Delta\mathcal{E}^+ \equiv \mathcal{E}^+(x = 2md + 0) - \mathcal{E}^+(x = 2md - 0) = -\Delta\mathcal{E}^- = 2\pi i\mathcal{N}k\mu\rho_{21}(x = 2md), \quad (6)$$

and similar for $x = 2(m+1)d$. We will make use of this formula below to show that the discontinuities are weak compared to the amplitudes $\mathcal{E}^\pm$ themselves.

Below we will restrict our consideration to the particular case when $\mathcal{E}^\pm$ are real-valued throughout the grating. Then eqs (5) yield simple solutions for ρ_{21} :

$$\underline{x = 2md} \quad \rho_{21} = (-1)^{m+1} \frac{1}{2} \sin\left(\frac{\mu}{\hbar} \int_{-\infty}^t (\mathcal{E}^+(x, t') + \mathcal{E}^-(x, t')) dt'\right), \quad (7a)$$

$$x = (2m+1)d \quad \rho_{21} = (-1)^{m+1} \frac{1}{2} \sin\left(\frac{\mu}{\hbar} \int_{-\infty}^t (\mathcal{E}^+(x, t') - \mathcal{E}^-(x, t')) dt'\right), \quad (7b)$$

In order to obtain separate equations for $\mathcal{E}^+$ and $\mathcal{E}^-$, one proceeds as usual: to obtain the equation for $\mathcal{E}^+$, multiply both sides of eq (4) by e^{-ikx} and integrate them from $x - \lambda/2$ to $x + \lambda/2$, and do similarly for $\mathcal{E}^-$. Notice that we have gotten 4 films inside the region of averaging. The left-hand side of eq (4) becomes:

$$\langle \mathcal{E}_t^+ \rangle + c \langle \mathcal{E}_x^+ \rangle - \frac{ic}{2k} \langle \mathcal{E}_{xx}^+ \rangle - \langle \mathcal{E}_t^- e^{-2ikx} \rangle - c \langle (\mathcal{E}_x^- - \frac{1}{2ik} \mathcal{E}_{xx}^-) e^{-2ikx} \rangle, \quad (8)$$

where $\langle \dots \rangle \equiv \frac{1}{\lambda} \int_{x-\lambda/2}^{x+\lambda/2} dx' (\dots)$. Assuming that $\mathcal{E}^\pm$ are slowly varying functions outside the films and that the discontinuities $\Delta \mathcal{E}^\pm$ are small compared to $\mathcal{E}^\pm$ (we will show below that this always holds), we may introduce the averaged quantities $\tilde{\mathcal{E}}^\pm \equiv \langle \mathcal{E}^\pm \rangle$. Then the first two terms in eq (8) become: $\tilde{\mathcal{E}}_t^+ + c \tilde{\mathcal{E}}_x^+$, and it is clear that one can neglect the fourth term. The third and fifth terms give altogether:

$$-\frac{ic}{4\pi} (\mathcal{E}_x^+ + \mathcal{E}_x^- e^{-2ikx})|_{x+\lambda/2} + \frac{ic}{4\pi} (\mathcal{E}_x^+ + \mathcal{E}_x^- e^{-2ikx})|_{x-\lambda/2} = 0,$$

since, as we have already pointed out, the quantity $(\mathcal{E}_x^+ e^{ikx} + \mathcal{E}_x^- e^{-ikx})$ is continuous throughout the grating and $\mathcal{E}^\pm$ vary slowly away from the films. The rest of the averaging is carried out straightforwardly, and we obtain:

$$\frac{\mu}{\hbar} (\tilde{\mathcal{E}}_t^+ + c \tilde{\mathcal{E}}_x^+) = -\tau_c^{-2} \left(\sin\left(\int_{-\infty}^t \frac{\mu}{\hbar} (\mathcal{E}^+ + \mathcal{E}^-) dt'\right) + \gamma \sin\left(\int_{-\infty}^t \frac{\mu}{\hbar} (\mathcal{E}^+ - \mathcal{E}^-) dt'\right) \right), \quad (9)$$

where $\tau_c^{-2} = 2\mu^2 \omega^2 \mathcal{N} / (\hbar c)$. If we now replace $\mathcal{E}^\pm$ by the averaged quantities $\tilde{\mathcal{E}}^\pm$ in the right-hand side of eq (9) and after that introduce the notations: $\Omega^\pm = \frac{\mu}{\hbar} \tilde{\mathcal{E}}^\pm$, then we will obtain the equation for the (scaled) forward mode amplitude:

$$\Omega_t^+ + c \Omega_x^+ = -\frac{1}{2} \tau_c^{-2} \left(\sin \int_{-\infty}^t (\Omega^+ + \Omega^-) dt' + \gamma \sin \int_{-\infty}^t (\Omega^+ - \Omega^-) dt' \right),$$

and in the similar way — the equation for the scaled backward mode amplitude:

$$\Omega_t^- - c \Omega_x^- = -\frac{1}{2} \tau_c^{-2} \left(\sin \int_{-\infty}^t (\Omega^+ + \Omega^-) dt' - \gamma \sin \int_{-\infty}^t (\Omega^+ - \Omega^-) dt' \right).$$

Upon introducing the dimensionless coordinates $\tilde{x} = x/c\tau_c$, $\tilde{t} = t/\tau_c$ (below we will omit the tilde sign) and quantities $A = \int_{-\infty}^t \Omega^+(t')dt'$, $B = \int_{-\infty}^t \Omega^-(t')dt'$, we will rewrite the equations as follows:

$$A_{tt} + A_{xt} = -\frac{1}{2}(\sin(A+B) + \gamma \sin(A-B)) , \quad (10a)$$

$$B_{tt} - B_{xt} = -\frac{1}{2}(\sin(A+B) - \gamma \sin(A-B)) . \quad (10b)$$

Also, in the next section it will be convenient to us to deal with eqs (10) written in a different form:

$$\phi_{tt} + \psi_{xt} = -\sin \phi , \quad (11a)$$

$$\psi_{tt} + \psi_{xt} = -\gamma \sin \psi ; \quad (11b)$$

where $\phi = A+B$, $\psi = A-B$. Note that the quantities $-\cos \phi$ and $-\cos \psi$ are the inversions of the two-level atoms at $x = 2md$ and $x = (2m+1)d$, respectively.

Since in this paper we are interested in propagation of localized pulses in the NBRG, we first need to find out what asymptotic values of ϕ and ψ correspond to this situation. Clearly, these could be only the equilibrium points (i.e. such values of ϕ and ψ that all the derivatives vanish) of eqs (11): $\phi_0 = \pi n$, $\psi_0 = \pi n'$ (n and n' are intergers here and hereafter). A linear stability analysis around these points tells us that only $\phi_0 = 2\pi n$ and $\psi_0 = 2\pi n'$ may be stable asymptotics. Therefore, below we will consider propagation of 2π - $2\pi n$ kinks of eqs (11), i.e. when the ϕ -field changes by 2π and the ψ -field changes by $2\pi n$. The physical amplitudes are proportional to $\Omega^\pm = \frac{\partial}{\partial t}(\phi \pm \psi)$.

As it was pointed out in Section 1, the situation when $\gamma = 0$ corresponds to the *Bragg Resonant Grating* considered in [4]; indeed, for $\gamma = 0$, eqs (11) taken together with the initial condition $\phi(x, t = -\infty) = 0$ reduce, for the quantity ϕ , to the SGE in the form presented in Section 1. For $\gamma = 1$, when the Bragg harmonic is absent from the spatial spectrum of the grating, eqs (11) allow for a reduction $\phi = \psi$ and then become one equation:

$$\phi_{xt} + \phi_{tt} = -\sin \phi , \quad (12)$$

which is the other form of the SGE and is equivalent to the real-valued SIT equations in a continuous medium. The condition $\phi = \psi$ means that the backward mode in the NBRG is zero: $\Omega^- \equiv 0$. Hence, for $\gamma = 1$, a single-mode propagation regime is possible. A simple analysis of eqs (10) with $\gamma = 1$ and $|B| \ll 1$ shows that this regime is stable against small perturbations.

In the general case $0 < \gamma < 1$, eqs (11) are, to all appearance, non-integrable by the IST. This conclusion is grounded by the negative result of the Painleve test applied to the equations which were obtained from (11) by reduction to the single variable $\xi = x - Vt$. The results of computer simulation of two-kink collisions (see Section 5) also support this conclusion. Thus, in Sections 3 and 4 we present analytical results concerning the two near-integrable cases: $\gamma \ll 1$ and $(1 - \gamma) \ll 1$, respectively, and in Section 5 we present numerical results regarding the general non-integrable case $0 < \gamma < 1$.

To conclude the general discussion of our model, we will now demonstrate that our basic assumption

$$|\Delta\mathcal{E}^\pm|/|\mathcal{E}^\pm| \ll 1,$$

where $|\Delta\mathcal{E}^\pm|$ are defined by (6), indeed holds. Taking the film width as $100\text{\AA}$, the volume density of two-level atoms as $n_0 \sim 10^{16}\text{ cm}^{-3}$, $\mu \sim 10^{-18}$ SGSE, $\omega \sim 10^{15}\text{ sec}^{-1}$, $\rho_{21} \sim 1$, we obtain from (6) that $|\Delta\mathcal{E}^\pm| \sim 10^{-2}$ SGSE. In the Self-Induced Transparency experiments with subpicosecond pulses, in order to have considerable inversion of the resonance atoms, one needs to use $|\mathcal{E}^\pm|$ of order 10^2 SGSE or higher. Thus, our assumption that we utilized for averaging the discrete equations is very well satisfied.

3 Case $\gamma \ll 1$

When $\gamma = 0$, eqs (11) possess the kink solution:

$$\phi_0(x, t) = 4 \arctan(e^{-\xi}) \quad , \quad \psi_0(x, t) = \phi_0/V ; \quad (13)$$

where $\xi = (x - Vt - x_0)/\sqrt{1 - V^2}$, V is the kink's constant velocity, $x_0 = \text{const.}$ For $\gamma \ll 1$, we will seek an approximate solution of (11) in the form:

$$\phi(x, t) = \phi_0(x, t, \tau) + \gamma \phi_1(x, t) + \dots ,$$

$$\psi(x, t) = \psi_0(x, t, \tau) + \gamma \psi_1(x, t) + \dots ;$$

(see, for example, [10]), where the parameters V and x_0 are now functions of the slow time $\tau = \gamma t$. When $\gamma = 0$, the asymptotics of the ψ -field is completely determined by the kink's velocity, while for $\gamma \neq 0$ it must always be: $\psi(|x| \rightarrow \infty) \rightarrow 2\pi m$. Therefore, one can expect that at $\gamma \ll 1$ kinks will be able to move stationarily only with velocities $V_m = 1/m$. Below it is shown that this is indeed the case.

For $0 < \gamma \ll 1$, in the region of the kink's 'core', that is, where $|\xi| < 3$ or so, the right-hand side of equation (11b) is small compared to any term in the left-hand side, provided that the condition

$$\gamma \ll V \tag{14}$$

is satisfied. Then, to first order in γ , we obtain from (11b):

$$\psi_t = -\phi_x - \gamma \int_{-\infty}^t \sin\left(\frac{\phi_0(x, t')}{V}\right) dt' , \tag{15}$$

where $\phi_0(x, t)$ is given by (13). Substituting this expression into (11a), we arrive at the following perturbed SGE:

$$\phi_{xx} - \phi_{tt} = \sin \phi + \frac{\gamma}{V} \sin\left(\frac{\phi_0}{V}\right) . \tag{16}$$

Let us again stress that the above equation is accurate only around the kink's 'core', where one can make use of the approximations $|\phi_{xt}|, |\psi_{tt}| \gg \gamma$. Moreover, we can see that eqs (15) and (16) fail in the regions of the kink's 'tails': indeed, from (16) it follows that the asymptotics of ϕ is $\frac{\gamma}{V} \sin(\frac{2\pi}{V}) \pmod{2\pi}$, while the correct one is $0 \pmod{2\pi}$. Nevertheless, eq (16) is sufficient for determining the first-order adiabatic evolution of the kink's most important parameter, V . As it is well-known (see, for example, [10]), the adiabatic evolution of the parameter V of a kink in a perturbed SGE

$$\phi_{xx} - \phi_{tt} = \sin \phi + \gamma \mathcal{R}[\phi]$$

is given by the equation:

$$\frac{dV}{dt} = \gamma \frac{(1 - V^2)^{3/2}}{4} \int_{-\infty}^{\infty} d\xi \operatorname{sech} \xi \mathcal{R}[\phi(\xi)] . \quad (17)$$

For the perturbation $\mathcal{R}[\phi]$ given by (16) one obtains:

$$\frac{dV}{dt} = -\gamma \frac{(1 - V^2)^{3/2}}{4} \sin^2 \frac{\pi}{V} . \quad (18)$$

The fixed points (which are neutrally stable in the linear approximation) of eq (18) are: $V_m = 1/m$. These determine possible values of the kink's duration: $\tau_m = \sqrt{m^2 - 1}$. The physical velocity of the kink's center, which is $V + \frac{dx_0}{dt}$, differs from $V_m = 1/m$ by a quantity of order γ^1 . However, a correction of the same order to the velocity is also engendered by terms of order γ^2 in the equation for $\frac{dV}{dt}$. Computation of those terms, i.e. the second-order perturbations of the SGE, requires that a contribution from the kink's 'tails' be taken into account when deriving the higher-order approximations corresponding to eqs (15) and (16). This seems very difficult to do analytically. Therefore, we do not compute these corrections and hence the kink's physical velocity here.

Thus, the main result of this section is that kinks in the NBRG with a very weak non-Bragg component (i.e. such a grating is *almost* the Bragg grating) can propagate only with discrete values of velocity:

$$V = \frac{1}{m} + \mathcal{O}(\gamma) . \quad (19)$$

Such kinks were indeed observed in our numerical simulations of eqs (11) (see Section 5).

4 Case $(1 - \gamma) \ll 1$

As it was mentioned in Section 2, when in eqs (11) $\gamma = 1$, then these equations can support stationary propagation of a single-mode pulse, with the initial and boundary conditions selecting either of the two modes. Then for $(1 - \gamma) \ll 1$ one can expect that a weak second mode will be generated as the pulse is moving in such a NBRG. Below we show that this process leads to a decay of the initial

pulse, and hence a stationary pulse propagation in a NBRG with $(1 - \gamma) \ll 1$ is not possible.

In this section it will be more convenient to analyze the equations in form (10). Rescaling the coordinates as follows: $x' = \sqrt{\frac{1+\gamma}{2}} x$, $t' = \sqrt{\frac{1+\gamma}{2}} t$, we rewrite (10) in the form (the primes are omitted):

$$A_{tt} + A_{xt} = -\sin A \sin B - \frac{1-\gamma}{1+\gamma} \sin B \cos A, \quad (20a)$$

$$B_{tt} - B_{xt} = -\sin B \sin A - \frac{1-\gamma}{1+\gamma} \sin A \cos B. \quad (20b)$$

From (20) it follows that the amplitude of the backward mode, B , that arises because of scattering of the forward mode, A , has the order of magnitude $\mathcal{O}(1 - \gamma)$. Therefore, upon taking $\sin B \approx B$ and $\cos B \approx 1$ in (20), we find that to first order in the parameter $(1 - \gamma)$, equation (20a) decouples from (20b) and supports the kink solution:

$$A_0(x, t) = 4 \arctan(e^{-\frac{x-Vt}{\sqrt{V(1-V)}}}). \quad (21)$$

Here V is the velocity of the forward mode's kink, with this velocity being constant over the time scale $(1 - \gamma)^{-1}$, as we will show shortly. Introducing new coordinates:

$$y = \frac{x - Vt}{\sqrt{V(1-V)}}, \quad z = \frac{\sqrt{V(1-V)}}{2(1+V)} \left(\frac{1+2V}{V} x + t \right),$$

we find from (20b) a linear equation for the B -mode:

$$\left(\frac{\partial^2}{\partial y^2} - \frac{1}{4\sigma^2} \frac{\partial^2}{\partial z^2} \right) B = -\frac{1}{\sigma} B (1 - 2 \operatorname{sech}^2 y) - \frac{1-\gamma}{\sigma} \operatorname{sech} y \tanh y, \quad (22)$$

where $\sigma = \frac{1+V}{1-V} > 1$. The boundary conditions for the solution of (22) are set to be the following:

$$B(y \rightarrow +\infty) = 0; \quad \frac{\partial B}{\partial y}(y \rightarrow +\infty) = 0. \quad (23)$$

Notice that the asymptotic value of the B -field, $B(y \rightarrow -\infty)$, far behind the kink is uniquely determined by (22,23) and is not, in general, equal to zero. In

order to solve eq (22), we apply Fourier transform with respect to the z -variable to that equation and, upon taking into account (23), we obtain that

$$\hat{B}(\omega \neq 0, y) \equiv \int_{-\infty}^{\infty} B(z, y) e^{-i\omega z} dz = 0 .$$

This means that one can look for a z -independent solution of (22), so that (22) becomes an ODE for $B(y)$. Such a solution can be expressed through hypergeometric functions (see, for example, Chapter 2 in [11]). It will be clear from what follows that we do not need that exact solution but only the asymptotic value of $B(y)$ as $y \rightarrow -\infty$. In order to find this asymptotic value, it suffices to replace the hyperbolic profiles in (22) by box-like ones. Then one easily finds that

$$B(y \rightarrow -\infty) = K \frac{1-\gamma}{\sigma} \cos\left(\sqrt{\frac{1}{\sigma}} y + \varphi\right) , \quad (24)$$

where K and φ are unimportant, constant coefficient and phase, respectively. From (21,24) we conclude that, to first order in the small parameter $(1-\gamma)$, eqs (10) possess the solution which represents the kink (21) of the A -mode *plus* a weak B -mode which moves with the same group velocity as the A -kink, with this B -mode having a non-decaying, oscillating ‘tail’. Thus the backward mode’s amplitude, $\Omega^- = \frac{\partial B}{\partial t}$, is not localized in space. In order to support stationary propagation of such a non-localized mode, infinite energy is required, while the energy of the localized forward-mode pulse, which is the initial state in our problem, is finite. Therefore, an evolution of the solution of eqs (10) on the time scale larger than $(1-\gamma)^{-1}$ is the following. As the A -kink moves to the right, a weak B -mode is generated, with its front moving with the kink’s group velocity and its hind ‘tail’ extending to the left of the kink. Thus, in time course, more and more energy is being taken away from the forward mode and simultaneously stored in the non-localized backward mode. One can estimate the lifetime of the forward-mode kink as the inverse of its energy decay rate:

$$\text{lifetime} = \left(\frac{\text{energy stored in B-mode}}{(\text{time elapsed}) \times (\text{kink's energy})} \right)^{-1} \sim \frac{(1+V)^3}{(1-V)^{7/2} V^{1/2} (1-\gamma)^2} . \quad (25)$$

In deriving the above estimate, eqs (21) and (24) were taken into account.

Thus, the main result of this section is that no localized pulse can stationarily propagate inside an NBRG with $(1 - \gamma) \ll 1$. Our numerical simulations of eqs (11) showed that the kink's lifetime decreases with increasing the parameters $(1 - \gamma)$ and $(1 - V)$, in accordance with estimate (25).

5 Case $0 < \gamma < 1$

Below we will consider only stationary solutions of eqs (11), if such exist, for the non-integrable case $0 < \gamma < 1$. The quantities ϕ and ψ are then supposed to depend on the only variable $\xi = x - Vt$, and eqs (11) become the following ODE's:

$$\phi'' = (V \sin \phi + \gamma \sin \psi)/(V(1 - V^2)) , \quad (26a)$$

$$\psi'' = (\sin \phi + \gamma V \sin \psi)/(V(1 - V^2)) , \quad (26b)$$

where the prime denotes a derivative with respect to ξ . Since we are only looking for those solutions which correspond to the localized fields $\Omega^\pm$, then we need to impose the following boundary conditions:

$$\phi'(|\xi| \rightarrow \infty) = \psi'(|\xi| \rightarrow \infty) = 0 , \quad (27a)$$

$$\phi(\xi \rightarrow +\infty) = \psi(\xi \rightarrow +\infty) = 0 , \quad (27b)$$

$$\phi(\xi \rightarrow -\infty) = \psi(\xi \rightarrow -\infty) = 0 \pmod{2\pi} . \quad (27c)$$

We remark that eqs (26) are hamiltonian, with the (non-positive definite) Hamiltonian being:

$$H = \frac{P_\phi^2 + P_\psi^2}{2(1 - V^2)} + \frac{P_\phi P_\psi}{V(1 - V^2)} + (\cos \phi - 1) + \gamma(\cos \psi - 1) .$$

The existence or non-existence of solutions of (26,27) crucially depends on the behavior of the trajectories of the dynamical system (26) in the phase space $(\phi, \phi', \psi, \psi')$ in vicinities of the equilibrium points. By the latters we mean such values of ϕ and ψ that the right-hand sides of both eqs (26) vanish. Clearly, these are the points $(2\pi m, 0, 2\pi m', 0)$ (see also Section 2), and the heteroclinic trajectories going from the point $(0, 0, 0, 0)$ to any of the points $(2\pi, 0, 2\pi m, 0)$

correspond to the localized pulses of the physical quantities $\Omega^\pm$ (recall that in this paper m always stands for an integer number). Below we will briefly outline the method that allowed us to determine numerically the values of the parameters V and γ for which such heteroclinic trajectories of eqs (26) exist; a more detailed exposition of this technique may be found in [12].

Linearizing eqs (26) near the equilibrium points (EP), one finds that the characteristic exponents are:

$$\lambda_{1,2}^2 = \frac{K_1 + \sqrt{K_2}}{K_3} > 0, \quad \lambda_{3,4}^2 = \frac{K_1 - \sqrt{K_2}}{K_3} < 0; \quad (28)$$

$$K_1 = V^2(1 + \gamma), \quad K_3 = 2V^2(1 - V^2), \quad K_2 = K_1^2 + 2\gamma K_2.$$

Thus, for $\gamma \neq 0$, the first two of the characteristic exponents correspond to the two eigenvectors in the phase space along which phase trajectories can respectively enter and leave an EP. The other two characteristic exponents correspond to phase trajectories that wind around the EP. According to this, in the 4-dimensional phase space $(\phi, \phi', \psi, \psi')$, there is one 1-dimensional stable, one 1-dimensional unstable, and one 2-dimensional neutral (else called central) manifold of the EP. With both parameters V and γ in eqs (26) fixed, an orientation in the phase space of the unstable manifold and hence of the trajectory going out of the EP is also fixed, and there is no reason *a priori* why this trajectory should intersect with another EP, i.e. become a heteroclinic trajectory. So, in order to find a heteroclinic trajectory in the dynamical system (26), we must allow at least one parameter to vary, because varying, say, V , is equivalent to varying the direction of the unstable manifold of an EP, and then it becomes possible that the trajectory leaving the EP will end up exactly at another EP. Notice, however, that even then we might obtain no heteroclinic trajectory.

We numerically solved the ODE's (26) by the fourth-order Runge-Kutta method. For every value of γ in the interval $[0, 1]$ taken with the step of 0.01, we found a discrete set of values of the velocity V for which heteroclinic solutions of (26) existed (see Fig.1), and hence existed localized solutions for $\Omega^\pm$ in the original evolutionary equations. In order to make the computations as precise and non-time-consuming as possible, we made use of the following symmetry (i.e. the

transformation under which eqs (26) remain unchanged):

$$\phi \rightarrow 2\pi m_1 - \phi, \quad \psi \rightarrow 2\pi m_2 - \psi,$$

$$\xi \rightarrow -\xi;$$

where m_1, m_2 are integer numbers (see [12] for details). We restricted ourselves to finding only the trajectories going out from the EP ($\phi = 0 = \phi' = \psi = \psi'$) and into one of the EP's ($\phi' = \psi' = 0, \phi = 2\pi, \psi = 2\pi m$) with $m = 1, 2, 3, 4$. These trajectories are images in the phase space of the one-kink solutions of eqs (11). We term these solutions $2\pi-2\pi m$ kinks to reflect the fact that the ϕ -field changes by 2π and the ψ -field changes by $2\pi m$. Also, we considered only phase trajectories with monotonous dependence $\phi = \phi(\xi)$; the reason for that was that kinks with non-monotonous profiles are believed to be less robust objects in the original evolutionary PDE's.

The graphs of $V = V(\gamma, m)$ for $m = 2, 3, 4$ are plotted in Fig.1. Thus, for a given $\gamma < \frac{1}{2}$, a kink of eqs (10) can move only with a velocity whose value is determined from Fig.1. Notice that for $m = 1$ we found no heteroclinic solutions to eqs (26). It is remarkable that the graphs $V = V(\gamma, m)$ for $m = 2, 3, 4$ are just the straight lines

$$V = \frac{1}{m} - \gamma. \quad (29)$$

Unfortunately, we were unable to find an explanation to this fact. From Fig.1 one also sees that for $\frac{1}{2} < \gamma < 1$ there are no heteroclinic trajectories of eqs (26) and hence no localized solutions for the forward and backward modes in the NBRG. All these results, including formula (29), are in accordance with the results of Sections 3 and 4.

We conclude this section with the presentation of the results of numerical integration of PDE's (11). The numerical code we used was based on the second-order Method of Characteristics for hyperbolic systems. For $\gamma = 0.1$, we found a kink with $V = 0.4$ to be stable against perturbations of its initial profile, if those perturbations were within 25% in magnitude from the exact solution of eqs (26) (that exact solution was found from the numerical integration of the ODE's (26) and then substituted as an initial profile into eqs (11)). For the same value

of $\gamma = 0.1$, we found that a kink with $V = 0.23$ and whose parameters differed only slightly (by within 10%) from those of the exact solution, was able to "live" for a time of about seven or eight of its durations, before rapidly decaying. Thus, the solitary pulses corresponding to the branch with $m = 2$ in Fig.1 seem to be much more robust than those corresponding to the branches with larger values of m .

Finally, we consider the NBRG with $\gamma = 1$, in which case, as it was shown in Sections 2 and 4, eqs (11) allow the reduction $\phi = \psi$, which casts them into the integrable form (12) equivalent to the real-valued SIT equations in a continuous medium. Still, the process of collision of two counterpropagating pulses in the NBRG with $\gamma = 1$ has quantitative differences from that in a continuous resonant medium. In numerical simulations of collisions of two counterpropagating pulses in a continuous resonant medium, carried out in [13], it was shown that two pulses with durations larger than $\tau_{cr} \approx 1.4$ (in our dimensionless units) were completely destroyed after the collision. Narrower (and hence, faster) pulses survive the collision, with the single-mode propagation restoring after the pulses have moved apart. This means that no 'shadow pulses' (see, for example, [14]) are created as a result of a pulse collision in a continuous resonant medium. For two-pulse collisions in the NBRG we found the situation to be qualitatively the same, with the exception that now the value of $\tau_{cr} \approx 0.9$. Shadow formation does not occur in this case either. This shows that the purely non-Bragg Resonant Grating has properties very close to those of the continuous resonant medium.

6 Conclusion

In the present work, we derived and investigated the equations describing coherent propagation of an ultra-short electromagnetic pulse in the Non-Bragg Resonant Grating. The analysis of those equations demonstrated that the presence of even a weak non-Bragg modulation of the nonlinear refractive index can dramatically change the dynamics of a pulse within the grating, as compared to the case when only Bragg modulation is present. We showed that in a NBRG a

pulse can either move with certain discrete values of its velocity, or cannot remain localized at all, depending on the structural parameter γ , which determines how strong the non-Bragg harmonic in the spatial spectrum of the refractive index modulation is compared to the Bragg harmonic.

It is my pleasure to express gratitude to Professor V.M.Eleonsky and members of his research group for useful discussions and pointing out references [12] to me. This research was supported in part by the AFOSR and ONR.

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FIGURE CAPTION

Graphs $V = V(\gamma, m)$ determining the values of velocities with which pulses (i.e. $2\pi-2\pi m$ kinks of eqs (26)) can propagate in a non-Bragg Resonant Grating with a given value of γ . The solid line corresponds to $m = 2$, the dashed line — to $m = 3$, and the dashed-dotted line — to $m = 4$. Numerical simulations of eqs (26) were performed for $0.05 \leq V \leq 0.95$ in order to avoid small denominators.