Pseudo Dirac Neutrinos in Seesaw model

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Abstract

Specific class of textures for the Dirac and Majorana mass matrices in the seesaw model leading to a pair of almost degenerate neutrinos is discussed. These textures can be obtained by imposing a horizontal $U(1)$ symmetry. A specific model is discussed in which: (1) All three neutrino masses are similar in magnitude and could lie around eV providing hot component of the dark matter in the universe. (2) Two of these are highly degenerate and their $(mass)^2$ difference could solve the solar neutrino problem through large angle MSW solution. (3) The electron neutrino mass may be observable through Kurie plot as well as through search of the neutrinoless double beta decay.

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1 Introduction

Variety of experimental findings strongly suggest that possibly [1] all the neutrinos are massive. But these masses have to be much smaller than the masses of other fermions. This smallness of neutrino mass is theoretically well-understood in the framework of seesaw mechanism [2]. Within this framework, neutrino masses are described by the following mass matrix in the basis \((\nu^c_L, \nu^c_R)\)

\[
\mathcal{M}_\nu = \begin{bmatrix} 0 & m_D \\ m_D^T & M_R \end{bmatrix}
\]

(1)

\(m_D\) and \(M_R\) are \(3 \times 3\) matrices in the generation space labeled by \(i = 1, 2, 3\). When the scale associated with \(M_R\) is much higher than that of \(m_D\), the masses of the light neutrinos are described by the effective matrix

\[
m_{\text{eff}} = -m_D M_R^{-1} m_D^T
\]

(2)

The overall magnitude of \(M_R\) is determined either by the grand unification scale or by some intermediate symmetry breaking scale in grand unified theories [3, 4]. The Dirac mass matrix \(m_D\) arise in the same way as other fermion masses. Its eigenvalues are therefore expected to be similar to other fermion masses. These two aspects together lead to very light neutrinos through \(m_{\text{eff}}\). The exact pattern of neutrino masses depends upon the structures of \(m_D\) and \(M_R\). The \(m_D\) can be related to other fermion mass matrix, typically up-quark, in \(SO(10)\) models but \(M_R\) remains unconstrained. In the absence of any knowledge on \(M_R\), it is usually chosen [3, 4] to be proportional to identity. In this case, eq.(2) immediately implies that the light neutrino masses display strong hierarchy similar to other fermion masses. Typically [3, 4] for \(M_R \sim \text{GUT scale} \sim 10^{16}\text{GeV}\) one has \(m_{\nu_s} \sim 10^{-11}\text{eV}\), \(m_{\nu_e} \sim 1.5 \times 10^{-7}\text{eV}\) and \(m_{\nu_\mu} \sim 10^{-3}\text{eV}\) while for \(M_R\) in the intermediate range \(\sim 10^{12}\text{GeV}\), \(m_{\nu_e} \sim 10^{-7}\text{eV}\), \(m_{\nu_\mu} \sim 1.5 \times 10^{-3}\text{eV}\) and \(m_{\nu_s} \sim 10\text{eV}\). Immediate consequence of these values is that at most \(\nu_e\) can have mass in the eV range if \(M_R \geq 10^{13}\text{GeV}\). Thus one cannot hope to see electron neutrino mass through direct laboratory experiments. It is important to realize that the above masses, many times taken [3, 4] as “predictions” of seesaw model, crucially depend upon \(M_R\) being proportional to identity. One may naively expect that even if \(M_R\) has some structure, as long as its non-zero elements (more precisely eigenvalues [5]) do not display any hierarchy, the predictions would remain true, at least qualitatively. These naive expectations need not always hold if some symmetry enforces a texture for \(m_D\) and/or \(M_R\). The purpose of this paper is to present examples of different structures for \(M_R\) which display completely different hierarchy in neutrino masses without making any unnatural assumptions on values of parameters entering \(m_D\) and \(M_R\). In the examples to be presented, two of the neutrinos turn out to have almost equal but opposite eigenvalues. These therefore form a pseudo Dirac neutrino [6]. These examples can be realized through explicit models and we discuss a specific model in which all three neutrinos have similar masses whose values could be \(O(\text{eV})\). Two of them are highly degenerate and have typical mass square difference of \(O(10^{-3}\text{eV}^2)\) needed for solution to the solar neutrino problem through Mikheyev Smirnov Wolfenstein (MSW) mechanism.

In the next section, we present an explicit model with pseudo Dirac neutrinos. The expected patterns of neutrino masses and mixing as well as their implications are discussed.
in section 3 and conclusions are presented in section 4. Appendix gives a class of textures for \( M_R \) which are similar to the one discussed in text and which lead to pseudo Dirac neutrinos.

2 Pseudo Dirac neutrinos

We present here a specific structure for \( m_D \) and \( M_R \) which lead to pseudo Dirac neutrinos. These structures can be easily obtained by imposition of a horizontal \( U(1) \) symmetry. Horizontal \( U(1) \) symmetries have been used long ago in order to constrain the quark [7] as well as neutrino [8] masses. These have also been studied recently [9] with a view of obtaining large mixing among neutrinos required in order to solve the atmospheric neutrino problem [1, 3]. The \( U(1) \) assignments that we employ here and the resulting structure for the neutrino masses is somewhat different from the one considered in refs. [8, 9]. We shall require \( U(1) \) to be vectorial and assume that the \( i^{th} \) generation of leptons carry the \( U(1) \) charge \( X_i \) and that no two \( X_i \) are identical. The ordinary Higgs doublet is assumed to be neutral under the \( U(1) \) symmetry. As an immediate consequence, both the charged leptons as well as Dirac mass matrix \( m_D \) in eq. (1) are forced to be diagonal. The Majorana mass term can still have texture. We are interested in the Fritzsch type [10] of structures for \( M_R \). This can be easily obtained by introducing an \( SU(2) \otimes U(1) \) singlet field \( \eta \) with non-trivial \( U(1) \) charge \( X \). The \( M_R \) receives contributions from the following terms:

\[ - \mathcal{L}_R = \frac{1}{2} \nu_R^T (M_{ij} + \Gamma_i \eta + \Gamma'_{ij} \eta^*) C \nu_R + h.c \]  

(3)

All the three terms are possible if total lepton number is not assumed to be conserved [11].

Now consider the specific assignment

\[
X_1 = -\frac{1}{2} X_3 = - \frac{1}{2} X \\
X_1 = -\frac{3}{2} X_3
\]

(4)

with \( X_3 \neq 0 \). Then \( M_{ij} \) in eq. (3) are zero for all \( i \) and \( j \). Moreover only \( \Gamma_{13} = \Gamma_{31}, \Gamma_{22} \) and \( \Gamma'_{23} = \Gamma'_{32} \) are allowed to be non-zero. This leads to the following texture for \( M_R \):

\[
M_R = \begin{bmatrix}
0 & 0 & M_1 \\
0 & M_3 & M_2 \\
M_1 & M_2 & 0
\end{bmatrix}
\]

(5)

\( M_1 = \Gamma_{13}(\eta), M_3 = \Gamma_{23}(\eta) \) and \( M_2 = \Gamma'_{23}(\eta') \). If we denote the elements of the diagonal matrix \( m_D \) by \( m_i (i = 1, 2, 3) \) then the effective mass matrix for the light neutrinos is given by:

\[
m_{\text{eff.}} = -m_D M_R^{-1} m_D^T = - \begin{bmatrix}
m_1^2 M_1^2 & -M_1 M_2 m_1 m_2 & M_1 M_3 m_1 m_3 \\
-M_1 M_2 m_1 m_2 & M_2^2 m_2^2 & 0 \\
M_1 M_3 m_1 m_3 & 0 & 0
\end{bmatrix} \frac{1}{M_1^2 M_3}
\]

(6)

We shall assume parameters \( M_{1,2,3} \) to be similar in magnitude (often we will take them identical for some of the numerical estimates). In addition we will also assume hierarchy in the Dirac masses \( m_1 << m_2 << m_3 \). Both the above assumptions are natural assumptions made in the usual seesaw picture [3, 4]. But since \( M_R \) is different from identity, the resulting
pattern of neutrino masses is completely different from the usual seesaw predictions. The eigenvalues of \( m_{\text{eff}} \) are given as follows:

\[
\begin{align*}
{m_{\nu_1}} &\approx -\frac{m_1 m_3}{M_1} \left( 1 + \frac{1}{2} \frac{\epsilon}{\delta - 1} \right) \\
{m_{\nu_2}} &\approx -\frac{m_2^2}{M_3} \left( 1 + \frac{\epsilon}{1 - \delta^2} \right) \\
{m_{\nu_3}} &\approx \frac{m_1 m_3}{M_1} \left( 1 + \frac{1}{2} \frac{\epsilon}{\delta + 1} \right)
\end{align*}
\]  

(7)

where

\[
\begin{align*}
\epsilon &\equiv \left( \frac{m_1}{m_2} \right)^2 \left( \frac{M_2}{M_1} \right)^2 \\
\delta &\equiv \frac{m_1 m_3}{m_2^2} \left( \frac{M_3}{M_1} \right)
\end{align*}
\]  

(8)

The parameter \( \epsilon << 1 \) with the above stated assumptions while \( \delta \) could be \( O(1) \). In the \( \epsilon \to 0 \) limit, two of the neutrinos are exactly degenerate while the presence of the \( \epsilon \) term introduces small splitting with the (mass)\(^2\) difference

\[
\Delta_{31} \approx 2 \left( \frac{m_1 m_3}{M_1} \right)^2 \frac{\epsilon \delta}{\delta^2 - 1}
\]  

(9)

Hence for \( \epsilon << 1 \), these neutrinos form a pair of pseudo Dirac particles [6]. The occurrence of a pseudo Dirac neutrino here is somewhat of a surprise. One would have expected it [12] if both \( m_D \) and \( M_R \) possessed some approximate global symmetry. For example, if \( M_2 \) is taken to be small, \( M_R \) is approximately invariant under \( L_e - L_\tau \) and one could have expected a pseudo Dirac neutrino. But \( M_R \) in eq.(5) does not possess any approximate global symmetry as long as \( M_1 \sim M_2 \sim M_3 \). Despite this, the hierarchy in \( m_i \) (combined with specific texture for \( M_R \)) makes \( m_{\text{eff}} \), approximately invariant under \( L_e - L_\tau \), symmetry resulting in a pseudo Dirac neutrino. In practice \( \epsilon \) could be quite small, e.g., if \( m_1(m_2) \) is identified with \( m_u(m_e) \) then \( \epsilon \sim 10^{-5} \) for \( M_2 \sim M_1 \). Thus degeneracy of two neutrinos is expected to be quite good.

The mixing among neutrinos implied by eq.(6) can be easily worked out for \( \epsilon << 1 \). In general, the eigen vectors of \( m_{\text{eff}} \) are given by

\[
\Psi_i = N_i \begin{pmatrix} x_i \\ y_i \\ z_i \end{pmatrix}
\]

with

\[
\begin{align*}
x_i &= \frac{m_2^0 - m_{\nu_i}}{B} y_i \\
z_i &= -\frac{m_1^0}{m_{\nu_i}} x_i
\end{align*}
\]  

(10)
where
\[ m_{\nu_1}^0 = -\frac{m_1 m_3}{M_1}, \quad m_{\nu_2}^0 = -\frac{m_2}{M_3}, \quad m_{\nu_3}^0 = \frac{m_3^2}{M_3} \quad \text{and} \quad B = \frac{m_1 m_2}{M_1 M_3}. \]

With \( m_{\nu_i} \) given in eq.(7), one could determine \( \Psi_i \) and hence the mixing angles. The three wavefunctions are approximately given by
\[
\Psi_1 \approx \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}, \quad \Psi_2 \approx \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \quad \text{and} \quad \Psi_3 \approx \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}
\]

\( \Psi_1 \) and \( \Psi_3 \) are maximally mixed to form a pseudo Dirac neutrino. Deviation of this mixing angle from \( 45^\circ \) is very small. Using eqs.(10) and (7), we see that
\[
\tan \theta_{13} \approx 1 - \frac{1}{2 \delta - 1}
\]

(11)

The angle \( \theta_{13} \) represents mixing among \( \nu_e \) and \( \nu_\tau \) states produced in association with \( e \) and \( \tau \) respectively if the charged lepton mass matrix is diagonal as is the case here. This has important implications for the solar neutrino problem as we will see in the next section.

\( M_R \) of eq.(5) possesses a generalized Fritzsch type [10] structure with one pair of off-diagonal and two diagonal elements being zero. One could write down other similar structures. Just as in the example studied here, most other similar structures admit almost degenerate neutrinos in the limit \( m_1 << m_2 << m_3 \). We list these structures separately in the Appendix.

### 3 Phenomenological Implications

We shall now investigate the implications of the specific structures for the neutrino masses and mixing derived in the last section. These clearly depend upon the unknown values of the Dirac masses \( m_i \). To be specific, we shall assume these masses to coincide with the up-quark masses. Moreover, we shall assume \( M_1 = M_2 = M_3 \) and denote them by a common scale \( M \). The parameters \( \epsilon, \delta \) and \( M \) then determine neutrino masses and mixing. It follows from eq.(8) that

\[
\delta \equiv \left( \frac{m_u m_t}{m_c^2} \right) \left( \frac{M_3}{M_1} \right) \approx \frac{m_{\nu_e}}{m_{\nu_\tau}} \approx 0.4-0.8
\]

\[
\epsilon \approx \left( \frac{m_u}{m_c} \right)^2 \approx 4 \times 10^{-5}
\]

where we have chosen \( m_u = 10 M eV \), \( m_t = 100-200 GeV \) and \( m_c = 1.5 GeV \). Also from eq.(7), we have \( |m_{\nu_e}| \sim |m_{\nu_\tau}| \). Hence independent of the numerical value of \( M \), all three
neutrino masses are expected to be of the same order. This has to be contrasted with the conventional seesaw prediction

\[ m_{\nu_e} : m_{\nu_\mu} : m_{\nu_\tau} = m_u^2 : m_c^2 : m_t^2 \]

obtained with similar assumptions on parameters but with \( M_R \) proportional to identity. The common mass of all three neutrinos would lie in the range \( 10^{-7} - 1 \text{eV} \) for the Majorana mass scale \( M \sim (10^{16} - 10^9) \text{GeV} \). Hence for values of \( M \) in the intermediate range \( \sim 10^6 \text{GeV} \), all three neutrinos would have masses in the eV range. These neutrinos can together provide the necessary hot component of the dark matter \( \sigma \) which requires \( \sum m_\nu = 7 \text{eV} \). Moreover, such neutrino spectrum could have observable consequences for laboratory experiments as well. Note that two of the neutrinos are highly degenerate. Their mass difference is given by eq.(9)

\[ \Delta_{31} \approx 2(m_{\nu_e})^2 \frac{\epsilon^2}{3^2 - 1} \]  

(12)

It follows that if \( m_{\nu_e} \sim m_{\nu_\mu} \sim O(\text{eV}) \) then their mass difference is naturally expected to be around \( \sim 10^{-5} \text{eV}^2 \). This value falls in the range required to solve the solar neutrino problem through Mikheyev-Smirnov-Wolfenstein [14] mechanism. Thus one can solve the solar neutrino problem and at the same time obtain an electron neutrino with mass in the observable range unlike in the seesaw models considered so far in the literature [3, 4].

The detailed analysis of the four solar neutrino experiments constrain the mixing angle as well [15]. The mixing angle between \( \nu_e - \nu_\tau \) is predicted to be large in our case. It turns out in fact to be too large to be consistent with observations if charged leptons do not mix among themselves. If the vacuum mixing angle is close to \( \pi/4 \) then the survival probability for \( \nu_e \) is independent of energy. Such an energy independent survival probability is not favored when the results of all four experiments are combined. They do allow large angle solution but \( \sin^2 2\theta \) (in our case \( \theta \equiv \theta_{13} \)) is required to be [15]

\[ \sin^2 2\theta_{13} < 0.85 \]

This constrain is not satisfied by the angle \( \theta_{13} \) of eq.(11). \( \theta_{13} \) represents the mixing between physical \( \nu_e - \nu_\tau \) states if the charged lepton mass matrix is diagonal as is the case in section 2. The correction to \( \theta_{13} \equiv \theta_{13} \) is proportional to \( \epsilon \sim 10^{-5} \) and is too small to cause significant deviation from \( \pi/4 \). Hence, one must have contribution from the charged lepton mixing in order to obtain MSW solution consistently. This needs enlargement of the model. For example, consider adding two more Higgs doublets \( \Phi'_{1,2} \) with \( U(1)_X \) charges 0 and \( -\frac{2}{3} X_3 \) respectively to the model presented in the last section. With a suitable discrete symmetry \( (\Phi'_{1,2} \rightarrow -\Phi'_{1,2}; \ e_R \rightarrow -e_R) \), \( \Phi'_{1,2} \) can be made to contribute only to the charged lepton masses. These are now described by a mass matrix

\[ M_l = \begin{pmatrix} m_{ee} & 0 & m_{e\tau} \\ 0 & m_\mu & 0 \\ 0 & 0 & m_{\tau\tau} \end{pmatrix} \]  

(13)

The neutrino mass matrix \( m_D \) and hence \( m_{\text{eff}} \) remains the same as before. Because of the
structure for $M_l$ in eq.(13), the effective mixing angle describing $\nu_e-\nu_\tau$ mixing is now given by

$$\theta_{e\tau} \approx \theta_{13} - \phi$$

with

$$\tan 2\phi = \frac{2 m_{e\tau}}{m_{e\tau}^2 + m_{e\tau}^2 - m_{e\tau}^2}$$

Due to contribution from $\phi$, $\theta_{e\tau}$ need not be very close to $\pi/4$. The large angle MSW solution typically needs $|\sin^2 2\theta_{e\tau}| \approx 0.65-0.85$. With $\theta_{13} \sim 45^0$, this translates to $\phi \sim 10^0-20^0$.

The present model makes a definite prediction for the neutrinoless $\beta\beta$ decay. The amplitude for this process is related to the $(11)$ element of the neutrino mass matrix in the basis in which charged lepton mass matrix is diagonal [3]. It follows therefore from eq.(6) that in the model of the earlier section, neutrinoless $\beta\beta$ decay amplitude is proportional to

$$\frac{m_1^2}{M_3} \left( \frac{M_2}{M_1} \right)^2 \approx \epsilon m_{\nu_2}$$

Hence unless $m_{\nu_2}$ is very large $\sim 10^6 eV$, the $\nu$-less $\beta\beta$ decay is not observable. If the charged lepton mass matrix is non-diagonal as is required here in order to obtain the right amount of mixing for the MSW solution to work then the $\nu$-less $\beta\beta$ decay amplitude also changes and is now proportional to

$$\cos^2 \phi \epsilon m_{\nu_2} - 2 \sin \phi \cos \phi m_{\nu_1}$$

with $m_{\nu_2}$ and $m_{\nu_1}$ given by eqs.(7). For $\phi \sim 10^0-20^0$ and $m_{\nu_1} \sim m_{\nu_2}$ this corresponds to $\sim (0.3-0.6) m_{\nu_1}$. Hence, the $\nu_e$ mass $\sim 2 eV$ could lead to an observable signal in the $\nu$-less $\beta\beta$ decay [17]. With $m_{\nu_1} \sim m_{\nu_2} \sim m_{\nu_3} \sim 2 eV$, one can also obtain the $\sim 7 eV$ needed for obtaining hot component in the dark matter [13]. We have concentrated here on a specific structure among various possibilities (Appendix) that lead to pseudo Dirac neutrinos. The quantitative consequences of other structures could be considerably different. Moreover, within the specific texture, we have assumed $m_{1,2,3}$ to coincide with $m_{\nu_e,\mu,\tau}$. Such an identification need not hold [16]. But the qualitative conclusions, namely the occurrence of pseudo Dirac neutrino due to approximate $L_\ell - L_\tau$ symmetry of $m_{\nu,eff}$, is more general and holds as long as $m_1 << m_2 << m_3$ and $M_1 \sim M_2 \sim M_3$. This is a naturally expected pattern even if $m_{1,2,3}$ do not exactly coincide with $m_{\nu_e,\mu,\tau}$. Interesting predictions discussed above still remain true if $m_{1,2,3}$ are chosen to coincide with the corresponding charged lepton masses instead of the up-quark masses. Now $\epsilon \sim (m_\tau/m_\mu)^2$ and $\delta \sim m_\epsilon m_\mu/m_\tau^2$. Hence $\Delta_{31}$ given in eq.(12) now becomes

$$\Delta_{31} \sim \frac{1}{8} (m_{\nu_3})^2 \times 10^{-6}$$

Hence for $m_{\nu_3} \sim 1 eV$, one can still solve the solar neutrino problem through MSW mechanism. $m_{\nu_3} \sim \frac{m_{\mu\tau}^2}{M}$ falls in the eV range if $M \sim 10^{12} GeV$. 

7
4 Conclusions

Seesaw model as conventionally analyzed generally lead to hierarchical neutrino masses. In particular, if the Majorana masses of the right handed neutrinos are large (> $10^9$GeV) then at most $\nu_e$ could have mass around eV range and $\nu_e$ is not expected to have mass near its laboratory limit. We have presented and discussed specific textures for neutrino masses which lead to very different predictions. A particular model analyzed in detail has all three neutrino masses in the eV range. Hence the model can naturally provide hot component in the dark matter of the universe. Two of the neutrinos are nearly degenerate and their $(mass)^2$ difference could naturally be in the range required to solve the solar neutrino problem. In addition, this model is capable of producing $\nu_e$ mass which could be observed directly in Kurie plot or as well as through $\nu$-less $\beta\beta$ decay. The atmospheric neutrino problem [3] cannot be solved in this model easily since the simultaneous solution of the solar $\nu$, atmospheric $\nu$ and dark matter problem needs nearly degenerate neutrinos [1]. Interesting aspect of the model worth reemphasizing here is the fact that (near) degeneracy of two of the neutrinos result here in spite of the fact that $M_R$ does not possess any global symmetry. The hierarchy in eigenvalues of $m_D$ and texture of $M_R$ conspire to make $m_{\nu e}$ approximately invariant under a global $U(1)$ symmetry resulting in almost degenerate pseudo Dirac neutrinos. This feature is shown to follow if $m_D$ is diagonal and $M_R$ has a generalized Fritzsch structure. Both these can be enforced by a global $U(1)$ symmetry. The study made here highlights the fact that the seesaw model can accommodate a completely different pattern of neutrino masses which is not thought to be among the conventional predictions of the scheme.
Appendix

In this appendix we give different structures of the Majorana mass matrix having generalized Fritzsch structure. These lead to a pair of almost degenerate light neutrinos when $m_D$ in eq.(1) is diagonal, i.e. $m_D = \text{diag}(m_1, m_2, m_3)$ as in the text. The following structures are possible for $M_R$ if $\det M_R$ is required to be non-zero [5].

\[
\begin{bmatrix}
0 & 0 & 0 \\
0 & 0 & 0 \\
0 & 0 & 0 \\
\end{bmatrix} (a1); \quad \begin{bmatrix}
0 & 0 & 0 \\
0 & 0 & 0 \\
0 & 0 & 0 \\
\end{bmatrix} (a2); \quad \begin{bmatrix}
0 & 0 & 0 \\
0 & 0 & 0 \\
0 & 0 & 0 \\
\end{bmatrix} (a3)
\]

\[
\begin{bmatrix}
0 & 0 & 0 \\
0 & 0 & 0 \\
0 & 0 & 0 \\
\end{bmatrix} (a4); \quad \begin{bmatrix}
0 & 0 & 0 \\
0 & 0 & 0 \\
0 & 0 & 0 \\
\end{bmatrix} (a5); \quad \begin{bmatrix}
0 & 0 & 0 \\
0 & 0 & 0 \\
0 & 0 & 0 \\
\end{bmatrix} (a6)
\]

Here $\times$ denotes a non-zero entry. Any of these structures can be obtained by imposing a $U(1)$ symmetry similar to the one considered in the text. If all the entries in a given $M_R$ are assumed to be identical (and denoted by $M$) then eigenvalues $\lambda_i(i=1,2,3)$ of $m_{\text{eff}}$ satisfy the following characteristic equations:

\[
\lambda_1 \lambda_2 \lambda_3 = \frac{m_1^2 m_2^2 m_3^2}{M^3} \quad (14)
\]

\[
\lambda_1 \lambda_2 + \lambda_1 \lambda_3 + \lambda_2 \lambda_3 = \frac{m_1^2 m_2^2}{M^2} \quad (15)
\]

\[
\lambda_1 + \lambda_2 + \lambda_3 = \frac{(-m_p^2 - m_i^2)}{M} \quad (16)
\]

with $i \neq j \neq k$ and $p$ either $j$ or $k$. In the absence of $\frac{m_p^2}{M}$ in eq.(16), the above eqs. are solved by the eigenvalues, $\frac{m_1 m_2}{M}$, $\frac{-m_1 m_2}{M}$ and $\frac{-m_3}{M}$. Hence as long as $m_p^2$ term represents a small correction, one would get a Pseudo Dirac neutrino. This naturally depends upon the exact value of the masses $m_{1,2,3}$. If $m_{1,2,3}$ are identified with $m_{\mu,\tau}$ (or $m_{e,\mu,\tau}$), the $m_p^2$ term amounts to a small corrections and one would obtain Pseudo Dirac neutrino in all cases except $(a4)$ and $(a5)$. 

9
References


[15] A recent review can be found in Smirnov, ref.[1]
[16] Due to vectorial nature of the $U(1)_H$ symmetry, this symmetry cannot commute with the usual $SO(10)$ group.

[17] There seems to be some indication of a non-zero mass in the $\nu$-less $\beta\beta$ decay, see for example, M. K. Moe, Talk at the Third Int. Workshop on Theory and Phenomenology in Astroparticle and Underground Physics, Gran Sasso, Italy (1993) and Smirnov in ref.[1]