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Large-order estimates in perturbative QCD and non Borel summable series ^{*)}

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Abstract

Basic facts about the summation of divergent power series are reviewed, both for series with nonvanishing and for series with vanishing convergence radius. Particular attention is paid to the recent development that permits, in the former case, to define summation in the whole Mittag-Leffler star and, in the latter case, to define summation when the point of expansion lies at the tip of a horn-shaped analyticity domain with zero opening angle. Relevance of these results to perturbative QCD is stressed in relation to current discussions concerning large-order estimates of perturbative QCD expansion coefficients.

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1. INTRODUCTION

It is widely believed that perturbation series in quantum theory are mostly divergent. But the problem is not whether a power series is convergent or divergent, but rather whether it uniquely determines the expanded function or not. Perturbation theory yields - at least in principle - all coefficients of the power expansion. This can tell us whether the series is convergent or not, but what we want to know is whether it determines the physically relevant function. The series may determine the function uniquely even if it is divergent and may not do so even if it is convergent. This depends on additional conditions.

How to deal with divergent series and how to sum them, under what conditions a power series is able uniquely to determine the expanded function and how to give a series precise meaning are problems of paramount importance for quantum theory. Power expansions are badly needed in physics but to ensure that they have clear and unambiguous mathematical meaning additional conditions are required that are often difficult to fulfil.

The present lecture is intended to review some aspects of the recent progress in this field. But before going into details of this topic let me shortly mention, as a contrast to the generally difficult situation, a notable success that Halliday and Surányi reached [1] in calculating the energy eigenvalues of the anharmonic oscillator by means of an ingenious splitting of the perturbed Hamiltonian.

These authors construct an unconventional perturbation series to calculate the eigenvalues of the anharmonic oscillator Hamiltonian

$$H = H_h(m) + gx^4/4 , \quad (1.1)$$

where $H_h(m) = (p^2 + m^2 x^2)/2$ is the harmonic part and $g > 0$. They introduce an arbitrary frequency parameter Ω and write the Hamiltonian in the form

$$H = H_0(\Omega) + H_1(\Omega) ,$$

where $H_0(\Omega)$ is a squared Hamiltonian operator,

$$H_0(\Omega) = \frac{g}{\Omega^4} [H_h(\Omega)]^2$$

and

$$H_1(\Omega) = H_h(m) + x^4/4 - \frac{g}{\Omega^4} [H_h(\Omega)]^2.$$

Here, $H_0(\Omega)$ and $H_1(\Omega)$ are 4th order polynomials in the creation and the annihilation operators a^+ , a of frequency Ω . Halliday and Surányi prove that this perturbation expansion (which is not an expansion in powers of any particular parameter of the theory) converges.

But perturbation series can mostly have at best the meaning of asymptotic series, which can uniquely determine the expanded function only if special conditions are fulfilled. Indeed, according to an argument first raised by Khuri and 't Hooft [2], two-point Green's functions of four-dimensional renormalizable massless theories have a complicated singularity structure at the origin of the complex coupling constant plane, the analyticity domain being a wedge with a zero opening angle. The argument is based on the analytic structure of the Green functions and on the evidence of asymptotic freedom, which states that a measurable coupling parameter $g(\mu)$ is governed by

the equation

$$\frac{dg(\mu)}{d \ln \mu} = \beta(g(\mu)), \quad (1.2)$$

where $\beta(g)$ is determined by the theory considered. As a consequence, a two-point Green's function is not a function of the energy variable q^2 and the coupling g separately, but is a function of one single variable that is a special combination of these two. This implies that two-point Green's functions in QCD are analytic in g^2 in a such a domain that the origin $g^2=0$, around which the perturbation series is developed, lies on its border, on the tip of a horn-shaped region. (We shall expose details of this argument in Appendix A).

This urges us to study *power series under most difficult conditions of analyticity*, when the point $g^2=0$ is an accumulation point of singularities and the opening angle of the analyticity domain is zero, i.e., there is only one path along which the origin can be reached without leaving the analyticity domain.

We therefore devote the subsequent section 2 to basic facts about Borel summation and its generalization, placing emphasis on the recent development related to the zero opening angle of the analyticity domain.

2. BASIC FACTS ABOUT SUMMATION

To find a summation of the series

$$f(z) = \sum_{n=0}^{\infty} a_n z^n \quad (2.1)$$

means to give it precise meaning for some z for which the series is divergent.

Definition 1: Let $\mathcal{P}$ be a method of summation of the series (2.1). Then $\mathcal{P}$ is called regular if

$$\mathcal{P}\left(\sum_{n=0}^{\infty} a_n z^n\right) = \sum_{n=0}^{\infty} a_n z^n$$

whenever the right-hand side converges, and $\mathcal{P}$ is called analytic if

$$\mathcal{P}\left(\sum_{n=0}^{\infty} a_n z^n\right) = f(z)$$

whenever $\mathcal{P}\left(\sum_{n=0}^{\infty} a_n z^n\right)$ exists (while $f(z)$ denotes the analytic continuation of $\sum_{n=0}^{\infty} a_n z^n$).

Let us now state what is understood by Borel summability.

Definition 2: The series (2.1) is said to be Borel summable if

a) the series

$$B(t) = \sum_{n=0}^{\infty} a_n t^n / n! , \quad (2.2)$$

called the Borel transform of (2.1), converges inside some circle, $|t| < \delta$, $\delta > 0$;

b) $B(t)$ has an analytic continuation to a neighbourhood of the positive real semiaxis $\operatorname{Re} t \geq 0$, and

c) the integral

$$g(z) = \frac{1}{z} \int_0^{\infty} e^{-t/z} B(t) dt \quad (2.3)$$

converges (not necessarily absolutely) for some $z \neq 0$.

Then $B(t)$ and $g(z)$ is called the Borel transform and the Borel sum of the series (2.1) respectively.

Remark: If c) holds for some $z_0 \neq 0$, then it holds for every z in the disc $\operatorname{Re} \frac{1}{z} > \operatorname{Re} \frac{1}{z_0}$ (see [3] and [4]).

We stress that the Borel summation method is neither the only one nor always the most suitable one in practical applications. We therefore discuss also other summability methods, with particular emphasis on their varying validity and applicability domain.

The Borel summation method and its generalizations have a salutary effect on power series both in the regular case (when the convergence radius R is positive) and in the singular case (when it is zero). Let us first take the regular case.

2.1. Regular case: Beyond the convergence disc

2.1.1 The Borel polygon

Take first, as a simple example, the function

$$1/(1-z) = \sum_{n=0}^{\infty} z^n, \quad (2.4)$$

whose Taylor series converges in the open unit disc, in which the following operations are legal:

$$\sum_{n=0}^{\infty} z^n = \sum_{n=0}^{\infty} \frac{z^n}{n!} \int_0^{\infty} dt e^{-t} t^n = \int_0^{\infty} dt e^{-t} \sum_{n=0}^{\infty} \frac{(zt)^n}{n!} = \int_0^{\infty} dt e^{-(1-z)t}. \quad (2.5)$$

Outside the disc, we can use the integral on the right-hand side to define the sum (on the left-hand side) wherever the integral converges. This will cover the whole halfplane

$$\operatorname{Re} z < 1 . \quad (2.6)$$

Such a summation will be regular and analytic in the above sense because the integral is holomorphic in z .

We see on this example some remarkable features (both merits and shortcomings) that are typical for the Borel method in general. Its notable merit is that the integral in (2.5) defines the sum in a larger region (halfplane (2.6)) than the Taylor expansion in z does (unit disc). This halfplane is a special case of the so-called Borel polygon, which replaces the halfplane (2.6) when the expansion (2.4) is replaced with the general expansion (2.1). Details are seen from the following

Theorem 1 [5]: Let $f(z)$ be the principal branch of an analytic function regular at $z = 0$,

$$f(z) = \sum_{n=0}^{\infty} a_n z^n . \quad (2.7)$$

Then the integral of this series,

$$\int_0^{\infty} e^{-t} \sum_{n=0}^{\infty} a_n \frac{(zt)^n}{n!} dt = \int_0^{\infty} e^{-t} B(zt) dt , \quad (2.8)$$

exists in the Borel polygon and is there equal to the analytic function $f(z)$.

Definition 3: Let z_i be the position of the i -th singularity of $f(z)$, $i = 1, 2, 3, \dots, N$, where N is the number of

singularities of $f(z)$, and R_i is the ray connecting z_i with the origin and continuing beyond z_i to infinity. Draw the perpendicular P_i to R_i at $z=z_i$ for each $i=1,2,\dots,N$. The part of the complex plane that is closed up by the P_i and contains the origin is called the Borel polygon.

We can now see a major shortcoming of the Borel method in the regular case ($R > 0$): While the analytic continuation offered by the Borel summation method is restricted to the Borel polygon, being stopped at each singularity z_i by the perpendicular P_i , large areas may spread behind the P_i where $f(z)$ is still analytic but is no longer expressible in terms of the Borel integral (2.8), because these areas lie outside the Borel polygon. The Borel integral (2.8) is of no use outside the Borel polygon of $f(z)$. Note in this connection that for some functions $f(z)$ the Borel polygon may happen to be only slightly larger than the Taylor circle, which touches the nearest singularity of $f(z)$ ¹⁾.

We shall consider two methods of analytic continuation of the series (2.1) beyond the Borel polygon:

1/ Conformal mapping of the analyticity domain onto the unit disc, and

1)

There are special situations in which no improvement at all is possible. If $f(z)$ is such that the unit circle is the natural boundary of its analyticity domain, both the Borel polygon and the analyticity domain of $f(z)$ are identical with the unit disc. Example of such a function is (see [6])

$$\sum_{n=0}^{\infty} z^{n!} ,$$

whose singularities are dense on the unit circle.

2/ Extension of the Borel polygon to a larger domain, possibly to the whole analyticity domain.

2.1.2 Conformal mapping of the analyticity domain onto the unit disc

The series (2.1) is convergent on the convergence disc K_R , whose radius R is such that the disc reaches the nearest singularity of $f(z)$. R can be calculated from the formula

$$1/R = \lim_{n \rightarrow \infty} \sup (a_n)^{1/n} . \quad (2.9)$$

To extend the convergence domain outside the disc K_R and to improve the convergence properties of (2.1), we can use a conformal mapping [7] $w(z)$ of the complex z plane transforming a part D_w of the analyticity domain D onto the unit disc and satisfying the conditions

$$w(z=0) = 0 \quad \text{and} \quad w(z=-z_1) = 1, \quad (2.10)$$

where $-z_1$ is the position of the singularity nearest to the origin. Let us first consider the (fortunate) situation that there are only singularities on the negative ray of the real axis, $-\infty \leq z \leq -z_1$, with z_1 being a positive number. Let us denote by $w_m(z)$ the function

$$w_m(z) = \frac{1 - (1 + z/z_1)^{\frac{1}{2}}}{1 + (1 + z/z_1)^{\frac{1}{2}}} , \quad (2.11)$$

which transforms the whole complex z -plane cut along $[-\infty, -z_1]$

onto the unit disc centred at the origin, with (2.10) and the points of the cut mapped onto the boundary circle of the disc.

Let us then expand $f(z) = f(z(w_m))$ in powers of $w_m(z)$,

$$f(z) = \sum_{n=0}^{\infty} b_n w_m^n, \quad (2.12)$$

instead of (2.1). This expansion converges for all z that are mapped by means of $w_m(z)$ onto the unit disc of the w_m plane, i.e., in the whole cut z plane. In this way we ensure convergence at points where (2.1) is divergent. Moreover, the convergence of (2.12) at points where (2.1) is convergent is asymptotically faster than in the expansion (2.1).

The same procedure can be used in the case of a general location of singularities of $f(z)$, where w again maps a part of the analyticity domain D of $f(z)$ onto the unit disc. The fastest asymptotic convergence is reached by the conformal mapping $w_m(z)$ (where the index m means the maximal speed of convergence) that maps the whole analyticity domain D inside the unit circle. This can be shown [7] by comparing w_m with a general conformal mapping $w_1(z)$ transforming an arbitrary smaller domain $D_1 \subset D$ onto the unit disc. We observe that $w_m(z)$ transforms D_1 onto a domain D'_1 contained in the unit disc. The function $w_m = F(w_1)$ that transforms the w_1 -unit disc into D'_1 is analytic and has the following properties: $|F| \leq 1$ for each point of the w_1 -unit disc, and $F(0) = 0$. Now the well-known Schwarz lemma states that

If $f(z)$ is analytic for all $|z| < R$, $f(0) = 0$ and $|f(z)| \leq M$, then $|f(z)| \leq M|z|/R$.

This implies that $|w_m| \leq |w_1|$ for all points of the unit disc. Thus the convergence rate (which is determined by $|z|/R$ and, for $R=1$, simply by $|z|$) of the expansion in the powers of w_m is, because of $|w_m| \leq |w_1|$, asymptotically faster than that in

the powers of any other w_1 .

The improvement of the convergence properties by means of a conformal mapping can be illustrated numerically. Let us recall the example discussed in the original paper [7] in order to compare the convergence rates of different expansions: The function

$$A(z) = 1/(\tau+z) + 1/(\tau-z) , \quad (2.13)$$

where $z = \cos\theta$, θ being the scattering angle, is expanded in powers of (z^2-1) (note that A is an even function of z) and then in powers of $w(z)$, where

$$w(z) = (1-z^2)/(2\tau^2-1-z^2) \quad (2.14)$$

represents a conformal mapping (for which, for simplicity of calculations, not the optimal mapping w_m was chosen) of the complex z plane. The exact partial waves of $A(z)$ calculated from (2.13) are compared [7] with the first, second, etc. approximation calculated from the power expansion in (z^2-1) and from that in $w(z)$. The results are discussed for two values of τ ; the higher value, $\tau = 1.67$, exhibits convergence of the power series in both variables, (z^2-1) and w , but the expansion in the powers of w evidently converges faster. For the lower value of τ , $\tau = 1.28$, the expansion in powers of (z^2-1) is divergent, while the expansion in powers of w is convergent. Details are seen in Table 1.

TABLE 1

Exact and approximate S- and D-wave projections of $\lambda(z)$, eq. (2.13), without and with the conformal mapping w , eq. (2.14), of the $z=\cos\theta$ plane (from Ciulli and Fischer, [7]). The table shows how much faster the power expansion in w tends to the exact value than that in z^2-1 .

$$\tau = 5/3$$

		S-wave		D-wave	
Exact:		1.386		0.0831	
Expansion	in (z^2-1)	in w	in (z^2-1)	in w	
one term	1.87	1.87	0	0	
two terms	1.17	1.30	0.14	0.105	
three terms	1.49	1.403	0.05	0.078	
four terms	1.33	1.383	0.10	0.0836	
convergence	<i>slow</i>	<i>fast</i>	<i>slow</i>	<i>fast</i>	

$$\tau = 1.28$$

		S-wave		D-wave	
Exact:		2.10		0.265	
Expansion	in (z^2-1)	in w	in (z^2-1)	in w	
one term	4	4	0	0	
two terms	-0.16	1.4	0.8	0.4	
three terms	5	2.4	-0.7	0.19	
four terms	-2	1.99	1.7	0.298	
convergence?	<i>no</i>	<i>yes</i>	<i>no</i>	<i>yes</i>	

2.1.3 Extension of the Borel polygon

Let us now discuss an alternative method that uses a modification of the Borel summation procedure. If we were to say in one phrase how the method works in the simple example (2.4), we should say that it consists in replacing the straight line $\text{Re } z = 1$, i.e.,

$$r = (\cos \varphi)^{-1}, \quad z = r e^{i\varphi}, \quad |\varphi| < \frac{1}{2}\pi, \quad (2.15)$$

with the curve

$$r = [\cos(\varphi/\alpha)]^{-\alpha}, \quad |\varphi| < \frac{1}{2}\pi\alpha, \quad (2.16)$$

where $z = r e^{i\varphi}$ and $\alpha > 0$. The region situated on the left from the curve (2.16), i.e., the region

$$r^{1/\alpha} [\cos(\varphi/\alpha)] < 1 \quad (2.17)$$

will be denoted, as in ref. [8], by $\mathcal{B}_\alpha(f)$, where $f = (1-z)^{-1}$ in this example. The region (2.17) encroaches on the right half plane and reaches very far right beyond the line $\text{Re } z = 1$, especially when α is chosen small enough.

The generalization of the notion of $\mathcal{B}_\alpha(f)$ to the case of a general function $f(z)$ (defined by the expansion (2.1)) is straightforward. Note that the Borel polygon is obtained by the choice $\alpha=1$, i.e., $\mathcal{B}_1(f)$. The region $\mathcal{B}_\alpha(f)$ is larger than (and contains) the Borel polygon if $\alpha < 1$, see Fig. 1. The corresponding generalization of the Theorem 1 reads as follows:

Theorem 2: Let $f(z)$ be defined as in Theorem 1. Then the

integral $I(z_0)$,

$$I(z_0) = \int_0^\infty t^{\beta/\alpha-1} e^{-t^{1/\alpha}} \sum_{n=0}^\infty a_n (tz_0)^n / \Gamma(\alpha n + \beta) dt, \quad (2.18)$$

converges if and only if $z_0 \in \mathcal{B}_\alpha(f)$. Then

$$f(z_0) = I(z_0).$$

The convergence is absolute and uniform on any compact subset of $\mathcal{B}_\alpha(f)$ with nonzero distance from the boundary of $\mathcal{B}_\alpha(f)$ and one can differentiate inside the integration sign.

The Theorem shows what is the price to pay for the enlargement of the Borel polygon: since $\alpha < 1$, the suppression of the coefficients a_n in (2.18) is less pronounced, $a_n / \Gamma(\alpha n + \beta)$, instead of $a_n / n!$ for $\alpha=1$, $\beta=1$. As a consequence, the summation for a smaller value of α applies to a smaller class of sequences $\{a_n\}$ than the summation for α greater. If α approaches zero, the region $\mathcal{B}_\alpha(f)$ approaches the Mittag-Loeffler star $\text{MLS}(f)$ of the function $f(z)$, which is defined as follows:

Definition 4: Let z_i , R_i and N have the same meaning as in the Definition 3. Cut the complex z plane along that part of each ray R_i that lies behind the singularity z_i , and remove these cuts from the complex z plane. The part of the z plane remaining after these cuts are removed is called the Mittag-Loeffler star $\text{MLS}(f)$ of the function $f(z)$.

So, for instance, $\text{MLS}(1/(1-z))$ is the complex plane from which the cut along $[1, \infty]$ is removed.

2.1.4 Reaching the Mittag-Loeffler star

Theorem 2 represents, in comparison with Theorem 1, a considerable improvement because, for $\alpha < 1$, the region $\mathcal{B}_\alpha(f)$, in which the summation (2.18) represents the function (2.1), is larger than the Borel polygon $\mathcal{B}_1(f)$ of the same function f . It is also clear that we can choose $\alpha > 0$ arbitrarily small, thereby approaching arbitrarily close the Mittag-Loeffler star. On the other hand, we cannot choose $\alpha = 0$ and, consequently, the function $I(z)$ in (2.18) can never represent the sum in the whole holomorphy region of $f(z)$.

The summation method representing the expanded function $f(z)$ in the whole Mittag-Loeffler star nevertheless exists and is based on the following theorem.

Theorem 3: [8] Let $f(z)$ be the principal branch of an analytic function regular at the origin,

$$f(z) = \sum_{n=0}^{\infty} a_n z^n . \quad (2.19)$$

Then the integral $J(z_0)$,

$$J(z_0) = \int_0^{\infty} \exp(-e^t) \sum_{n=0}^{\infty} a_n (tz_0)^n / \mu(n) dt , \quad (2.20)$$

converges if and only if $z_0 \in \text{MLS}(f)$, where the $\mu(n)$ are defined as

$$\mu(n) = \int_0^{\infty} \exp(-e^t) t^n dt . \quad (2.21)$$

Then

$$f(z_0) = J(z_0) . \quad (2.22)$$

The convergence is absolute and uniform on any compact subset of $\text{MLS}(f)$ with nonzero distance from the boundary of $\text{MLS}(f)$ and one can differentiate inside the integration sign.

We can again ask what price is paid for this powerful result, which yields the analytic continuation of the power series (2.19) in the whole Mittag-Leoffler star. This question resembles the analogous question we posed after Theorem 2, and the answer is similar: the price we pay for this result is a relatively weak suppression of the large-order a_n by the moments $\mu(n)$, which asymptotically grow with $n \rightarrow \infty$ slower than the gamma function $\Gamma(\alpha n + \beta)$ for any positive α . Note the double exponential function in the integral (2.21).

By this we shall close up the discussion of the role of the Borel summation and its generalizations in the regular case. One may ask what is effectively the difference between the optimal conformal mapping method (section 2.1.2) and the summation method based on the Theorem 3. Indeed, both these methods represent $f(z)$ in compact form in the whole Mittag-Leoffler star. The former represents $f(z)$ in the form of the power series (2.12) with $w=w_m$, which converges in the whole Mittag-Leoffler star $\text{MLS}(f)$ of $f(z)$, where $w_m(z)$ conformally maps $\text{MLS}(f)$ onto the unit disc in the w_m plane. The summation method, on the other hand, yields for $f(z)$ the integral representation (2.20)-(2.22), which again applies to the whole $\text{MLS}(f)$. Evidently, both these methods make full use of the analyticity of $f(z)$. One might think that there had been no progress between 1961, when the conformal mapping method was introduced into elementary particle physics [7], and 1990, when Theorem 3 was published [8]. But the progress can be seen in the fact that, to make use of the optimal conformal mapping w_m , one must know in advance the exact location of all singularities of $f(z)$, in order explicitly to construct $w_m(z)$,

which places the boundary of $MLS(f)$ on the unit circle. Nothing of that kind is needed in the summation method based on Theorem 3. This method yields a convergent expression for $f(z)$ automatically in the whole $MLS(f)$, without the need in advance to know the location of the singularities of $f(z)$. One can see from eqs. (2.20) - (2.22) that they contain no reference to any singularity of $f(z)$.

2.2 Singular case: Strong asymptotic conditions

2.2.1 Theorems on Borel summability

Applications in quantum mechanics and in field theories lead to situations when the convergence radius R is zero (we refer the reader to the comprehensive reviews [9] containing both a mathematical introduction and physical applications). If R equals zero, then $f(z)$ has a singularity at $z = 0$ and the series (2.1) does not converge to $f(z)$ for any $z \neq 0$. The series (2.1) may have sense as an asymptotic series. Let us recall the definition of the asymptotic series. We shall restrict ourselves to series in powers of z .

Definition 5: Let S be a set of points with the origin $z=0$ as an accumulation point, and $f(z)$ a function defined on S . We shall say that $f(z)$ has on S the asymptotic series (2.1) and denote it by

$$f(z) \approx \sum_{n=0}^{\infty} a_n z^n \quad (2.23)$$

if

$$f(z) - \sum_{n=0}^N a_n z^n = o(z^N), \quad z \rightarrow 0, \quad z \in S.$$

The notion of asymptoticity of a series is not correlated with its possible convergence. If a series is asymptotic it means that the difference of $f(z)$ and its partial sum is small if compared, at small z , with the last term of this sum. This does not contradict the fact that many concrete examples of asymptotic series do yield convergent expansions.

The statement that a series is asymptotic to a function refers to its limit at a point, and says nothing about its values away from this point.

This statement must be accompanied with a statement about the set S on which the limit is reached, i.e., along which the limiting point is approached.

Power asymptotic series can be added and multiplied.

If the function, the set of "expanding functions" (i.e., the powers of z in our case) and the set S are determined, the asymptotic series (i.e., its coefficients) is uniquely determined.

This does not hold conversely. Generally speaking, there is an infinite set of functions having the same asymptotic series. If, for instance, $g(z) = ce^{-a/z}$, (with c arbitrary and a positive), then $g(z)/z^{n+1} \rightarrow 0$ for any n uniformly in any angle $-\frac{1}{2}\pi + \delta \leq \theta \leq \frac{1}{2}\pi - \delta$ for a positive δ ; thus, if this opening angle plays the role of the set S in (2.23), then any function of the form $f(z) + g(z)$ has the same asymptotic series as $f(z)$.

A more general form of this statement amounts to the interesting result that to any asymptotic series there corresponds at least one analytic function. More precisely, for any set $\{a_n\}$ and any $\alpha > 0$ there exist functions $f(z) = f(re^{i\theta})$ such that (2.23) holds for $r \rightarrow 0$ uniformly in the angle $|\theta| < \alpha\pi$.

If $a > 0$ and $2k\alpha < 1$, then $r^n \exp(-a/z^k) \rightarrow 0$ uniformly in this angle for any n . Consequently, as previously with $k = 1$, all functions of the form $f(z) + c \exp(-a/z^k)$ have the same asymptotic expansion in this angle.

In perturbation theory, we in principle know the expansion coefficients but do not know the resulting Green function. So it is crucial for us to be aware that the asymptotic expansion does not determine the function uniquely. But it is also crucial to know that one can supplement the asymptotic expansion with such additional conditions that the function is uniquely determined. A notable class of such additional conditions are the so called "strong asymptotic conditions". We shall mention some of them in the following.

As a first example, we present a theorem by Watson that gives a sufficient condition for the function $f(z)$ to equal the Borel sum $g(z)$ of its asymptotic power series (Watson, 1912, [10], see also [4]).

Theorem 4. Let $f(z)$ be analytic in a sector $|\arg z| < \frac{1}{2}\pi + \epsilon$, $|z| < \eta$, for some $\epsilon > 0$ and $\eta > 0$, and let $f(z)$ have there the asymptotic expansion

$$f(z) = \sum_{n=0}^{N-1} a_n z^n + R_N(z) \quad (2.24)$$

with

$$|R_N(z)| \leq A \sigma^N N! |z|^N \quad (2.25)$$

uniformly in N and z in the sector. Then

a') the Borel transform $B(t) = \sum_{n=0}^{\infty} a_n t^n / n!$ converges on the disc $|t| < 1/\sigma$;

b') $B(t)$ has an analytic continuation to the sector

$|\arg t| < \varepsilon$; and

c') the integral (2.3)

$$g(z) = \frac{1}{z} \int_0^{\infty} e^{-t/z} B(t) dt$$

converges absolutely in the open disc $\operatorname{Re} z^{-1} > \eta^{-1}$ and there equals $f(z)$.

Remark. We can easily see that $\operatorname{Re} z^{-1} > \eta^{-1}$ represents a disk of radius $\frac{1}{2}\eta$ touching the imaginary axis. One can see it if one writes separately the real and the imaginary part of the inequality to arrive at $(\operatorname{Re} z - \frac{1}{2}\eta)^2 + (\operatorname{Im} z)^2 \leq \frac{1}{4}\eta^2$. Simpler is to notice that $\operatorname{Re} z^{-1} = r^{-1}\cos\phi > \eta^{-1}$, i.e., $r < \eta\cos\phi$, what is the locus of all points of the disc (see Fig. 2).

The Watson theorem has been widely used in physical applications, in particular in perturbation expansions. Its disadvantage is, however, that one of its assumptions refers to the sector $|\arg z| < \frac{1}{2}\pi + \varepsilon$, which is strictly larger than that in which $f(z)$ is recovered by the Borel summation, i. e., the disc $\operatorname{Re} z^{-1} > \eta^{-1}$. Watson's result was superseded by Nevanlinna already in 1916 (see [10]), but this fact remained unnoticed by physicists, who had till 1980 universally relied on Watson's version, until Sokal [4] publicized Nevanlinna's version and presented a simple proof of it as well as a physical application.

Nevanlinna's improvement of Watson's theorem has the following form (see [10] and [4]):

Theorem 5. Let f be analytic in the disc

$$D_{\eta} = \left\{ z: \operatorname{Re} z^{-1} > \eta^{-1} \right\}$$

and satisfy there the estimates (2.24) and (2.25) uniformly in

N and in $z \in D_\eta$. Then $B(t) = \sum_{n=0}^{\infty} a_n t^n / n!$ converges for $|t| < 1/\sigma$ and has an analytic continuation to the strip-like region

$$S_\sigma = \{t: \text{dist}(t, \mathbb{R}_+) < 1/\sigma\},$$

satisfying the bound

$$|B(t)| \leq K \exp(|t|/\eta) \quad (2.26)$$

uniformly in every $S_{\sigma'}$, with $\sigma' > \sigma$ (where $\text{dist}(t, \mathbb{R}_+)$ is the distance of t from the positive real semiaxis $\mathbb{R}_+$). Furthermore, f can be represented by the absolutely convergent integral (2.3) for any $z \in D_\eta$.

Let us check, as an example, that the function $f(z) = e^{-1/z}$ does not satisfy the conditions of the Theorem 5. Certainly, $f(z)$ is analytic within the circle $\text{Re } z^{-1} > \eta^{-1}$. But the estimate (2.24), (2.25) does not hold uniformly. Indeed, since all a_n are identically zero, $R_N(z) = e^{-1/z}$ and, consequently,

$$|R_N(z)| = e^{-\text{Re}(1/z)} \leq e^{-1/\eta}.$$

Since the bound must be uniform, let us approach the origin along the boundary of D_η , with $\eta' < \eta$, i.e., $|R_N(z)| = e^{-1/\eta'}$. But $e^{-1/\eta'}$ can never be uniformly majorized by $A \sigma^N N! |z|^N$, the right hand side of (2.25), which tends to zero with z approaching the origin. Then, there is no surprise that $f(z)$ cannot be represented in the form of the integral (2.3).

2.2.2 Non Borel case: small opening angle

The Nevanlinna theorem can be easily generalized to the case that the opening angle of the analyticity domain is less than π , in which case the disc D_η transforms into a drop-like

domain with the thin end at the origin (see Fig. 3).

Indeed, let $f(z)$ be analytic in the "droplet" placed inside the sector $|\arg z| \leq \frac{1}{2}\pi\gamma$, $|z| < \eta'$, where $\gamma < 1$, the boundary of the droplet being defined by

$$|E_\alpha(1/z)| = \text{const}, \quad E_\alpha(z) = \sum_{n=0}^{\infty} z^n / \Gamma(\alpha n + 1).$$

We then require, in analogy with (2.24) and (2.25), the strong asymptotic condition in the form

$$|f(z) - \sum_{n=0}^{N-1} a_n z^n| < A \sigma^N (N!)^\rho |z|^N \quad (2.27)$$

with $\rho \leq \gamma$. The Borel transform will have the form

$$B_\rho(t) = \sum_{n=0}^{\infty} a_n z^n / \Gamma(n\rho + 1). \quad (2.28)$$

We see that the requirement (2.27) is more restrictive than (2.25) but, in compensation for it, its validity is required in a smaller angle. Correspondingly, the series $B_\rho(t)$ defined by (2.28) represents a weaker suppression of the coefficients a_n than the series $B(t)$ defined by (2.2) (note that $B(t)$ is $B_\rho(t)$ for $\rho = 1$).

Comparing this generalized form for different values of γ with the original Theorem 5 we see that there is a balance between the size of the opening angle of the analyticity domain of $f(z)$ and the restrictiveness of the corresponding strong asymptotic conditions required for the series to uniquely determine the expanded function. Just one value, $\rho = 1$, $\gamma = 1$, corresponds to the case of the Borel summation. This means that Borel summability plays no privileged role among the great variety of possible summation methods, being just a special

case in which the moment $\mu(n)$ is equal to $n!$. [To illustrate this, we have discussed in detail also the cases when $\mu(n) = \Gamma(\alpha n + \beta)$ and $\mu(n) = \int_0^\infty \exp(e^{-t}) t^n dt$].

In spite of this, many authors restrict themselves to the Borel method and focus on the problem of convergence of the Borel transform $B(t)$, but do not care about unique determination of the function expanded.

One may think of various methods ensuring uniqueness. One of them definitely is to look after the analyticity properties of $f(z)$ in the complex z plane and to check whether the opening angle of the analyticity domain equals π or more (as required by Theorem 5) or equals a positive number (as required by its generalization given in the section 2.2.2). If, however, the opening angle is equal to zero, it remains to use the subsequent Theorem 6 (see ref. [11]).

This theorem has the following motivation. Neither the Theorems 4 or 5 nor their generalization given by the formulae (2.27) and (2.28) are applicable to situations in which the opening angle is zero. Indeed, one would be forced to choose $\gamma=0$, but this is impossible because for $\rho=0$ (notice that $\rho \leq \gamma$) the series (2.28) would coincide with the original series (2.1) and its convergence would contradict our original assumption that the convergence radius of (2.1) is zero.

On the other hand, as was already mentioned in the Introduction, there are strong arguments that it is exactly this location of singularities that two-point Green's functions of QCD and of various other field theories have as functions of the renormalized coupling parameter g^2 : the origin is an accumulation point of singularities that gather around the origin in all directions except the positive real semiaxis, and the analyticity domain is a horn-shaped region, a wedge bisected by the positive real semiaxis and bounded above and

below by circles which are tangent to it (see [2]).

The argument of 't Hooft leading to this conclusion is described in detail in the Appendix A.

In the following section, we quote a theorem that tells the form of the strong asymptotic conditions for the case that the opening angle is zero (see Fig. 4).

2.2.3 Non Borel case: zero opening angle

Theorem 6 [11]: Let $f(z)$

- 1) be regular in the region M whose boundary is defined by the condition $|F(1/z)| = \text{const}$, $F(z) = \sum_{n=0}^{\infty} z^n / \mu(n)$ [with $\mu(n)$ given by (2.21)], and continuous in the closure $\bar{M}$,
- 2) have in M the asymptotic expansion (2.24),

$$f(z) = \sum_{n=0}^{N-1} a_n z^n + R_N(z) .$$

If $R_N(z)$ satisfies the bound

$$|R_N(z)| \leq A \rho^N |z|^N \exp[e^{-w(N)} + N \ln w(N)] \quad (2.29)$$

with $\rho \leq \eta$, uniformly in $z \in \bar{M}$ and N , $w(s)$ being the solution of the equation $w(s)\exp[w(s)] = s$, then the asymptotic series determines $f(z)$ uniquely. The condition (2.29) is strong in the following sense: if (2.29) can be fulfilled uniformly in $z \in \bar{M}$ and N only with some $\rho > \eta$, then there is a nonzero function with the trivial asymptotic expansion and the asymptotic series (2.24) does not determine $f(z)$ uniquely.

To get an idea how restrictive the bound (2.29) is, we can solve the determining equation $w(s)\exp[w(s)] = s$ iteratively, neglecting w with respect to e^w at each step. We obtain the following iterations steps:

$$(1) \, w(s) = \ln s$$

$$(2) \, w(s) = \ln(s/\ln s)$$

$$(3) \, w(s) = \ln[s/\ln(s/\ln s)]$$

etc. Thus, the growth of $w(N)$ with increasing N is asymptotically bounded from above and from below:

$$\ln(s/\ln s) < w(s) < \ln s . \quad (2.30)$$

These bounds can be refined in an evident manner, but even from the present form it is seen that the right hand side of (2.29) grows much slower with increasing N than that of (2.25) or (2.27). In other words, the uniform bound (2.29) on $R_N(z)$ in the Theorem 6 is very restrictive, i.e., the corresponding "strong asymptotic condition" is very strong: we have $(\ln N)^N$ instead of $N!$. This is the price paid for the extremely small analyticity region, characterized by the zero opening angle at the origin.

Summarizing, we can conclude that the knowledge of the coefficients of a divergent power series without the knowledge of the analyticity domain of the expanded function represents a very insufficient piece of information. Furthermore, Borel non-summability is not a reliable criterion for detecting possible non-perturbative effects; important is whether certain additional conditions (for, instance, the corresponding strong asymptotic conditions) are fulfilled, to make the problem unambiguous. I refer to the paper [12], where instructive non-trivial examples are discussed.

A final remark is in order. The proof of the Theorem 6 is closely related to the problem of finding an entire function that tends to zero in all radial directions $|z| \rightarrow \infty$ except one. Its asymptotic properties completely determine the region to which the summability method applies [13].

3. LARGE-ORDER ESTIMATES IN PERTURBATIVE QCD

3.1 Introduction

Asymptotic freedom dictates that perturbation theory governs the high energy behaviour of hadronic processes. An important quantity in this context is R , the ratio of the total hadronic e^+e^- annihilation cross section to that of the $\mu^+\mu^-$ pair production:

$$R = \frac{\sigma(e^+e^- \rightarrow \text{hadrons})}{\sigma(e^+e^- \rightarrow \mu^+\mu^-)} . \quad (3.1)$$

While in the limit $q^2 \rightarrow \infty$ (with q^2 being the c.m. energy squared in the reaction $e^+e^- \rightarrow X$) the naive parton model suggests

$$3 \sum_{i=1}^{N_f} Q_i^2$$

for R , QCD corrections have to be taken into account for finite values of q^2 . These can in principle be systematically calculated using QCD perturbation theory, which yields the following expansion:

$$R(q^2/\mu^2, \alpha_s(\mu^2)) = \left(3 \sum_{i=1}^{N_f} Q_i^2 \right) \sum_{n=0}^{\infty} r_n(q^2/\mu^2) \left(\alpha_s(\mu^2)/\pi \right)^n. \quad (3.2)$$

Here r_n and α_s denote the expansion coefficients and the strong coupling constant respectively. (Instead of the coupling constant, we prefer the term "the coupling parameter".) Whereas the first two coefficients, r_0 and r_1 , are equal to 1 independently of energy and of the renormalization scheme (RS),

the coefficients r_k , $k = 2, 3, \dots$, depend, along with the coupling parameter α_s , on the RS chosen. Note that just one part of the RS dependence is labelled by the scale parameter μ , while the full RS ambiguity is much wider. The additional ambiguity originates in the renormalization group differential equation (where the perturbation expansion parameter $g^2(\mu^2)$ denotes $4\pi \alpha_s(\mu^2)$)

$$\frac{dg(\mu^2)}{d \ln \mu} \equiv \beta(g) = -g^3(b_1 + b_2 g^2 + b_3 g^4 + \dots) , \quad (3.3)$$

in which only the first two coefficients, b_1 and b_2 , are uniquely determined by the number of flavours (we keep restricting ourselves to massless quarks), while the higher-order coefficients b_n , $n = 3, 4, \dots$, can be used to define, along with the renormalization scale μ , the renormalization scheme chosen; see ref. [14] for details.

The first nontrivial correction r_2 was calculated independently by three groups [15] with the result $r_2(1) = 1.41$ in the $\overline{\text{MS}}$ scheme with 5 flavours. The next coefficient (next-to-next-to-leading order, NNLO) r_3 was calculated with a tremendous effort later [16] and we seem currently to be near the border of what can be done with the available analytical and numerical tools. Since, on the other hand, higher perturbative terms play a central role in our understanding of QCD, it is important at least to look for a reasonable estimate of them, of course, without any *a priori* claim to its validity or applicability at a given *finite* order.

Recently two independent attempts, [17] and [18], have been made to obtain some information about the large-order behaviour of the perturbation expansion coefficients of certain physical quantities. Both of them arrive at the conclusion that

QCD perturbation series are factorially divergent. Whereas the former approach [17] considers the special problem of particle fractions in the e^+e^- annihilation, the procedure presented in [18] does explicitly use the information following from the general analytic properties of the corresponding Green functions and from the renormalization group equation. It is claimed in ref. [18] that this approach allows one to give answers to the following two important questions:

1. *how to reliably estimate higher-order terms in perturbation expansions of physical quantities, and*
2. *how to formulate a physically well-motivated summation method when these estimates prevent the use of the universally adopted Borel summation technique.*

In a subsequent paper [19], West formulates a summation procedure with the aim to obtain a reliable approximation of the series by using only a few lowest terms in (3.1) even in the low q^2 region.

But his approach aims still higher: the aim is to use the asymptotic estimates r_n^{est} to calculate approximate values of r_n for small n , $n = 3, 4, \dots$. So he compares the value $r_3^{\text{est}}(1) = -12.0$, which he obtains from his asymptotic, $n \rightarrow \infty$ formula [18]

$$r_n^{\text{est}}(1) \approx \dots - \frac{e^{1+b'}}{\pi} (4\pi^2 e b_1)^{n-1} \frac{\Gamma(n+b')}{(n+b')^2}, \quad (3.4)$$

with

$$b' = b_2/b_1^2$$

and $r_3^{\text{ex}}(1; \overline{\text{MS}}) = -12.8$ (as was obtained by explicit calculation for $n = 3$ carried out by two independent groups [16]), and he interprets the excellent numerical agreement as a sign of

legitimacy of the whole approach. Since, however, as is pointed out in ref. [14], the formula (3.4) is manifestly RS independent, the question arises why not to choose some other renormalization scheme for comparison. For instance, $r_3^{\text{ex}}(1; \text{MS}) = 16.3$, but formula (3.4), being renormalization scheme independent, yields the same value for the MS scheme as for the $\overline{\text{MS}}$ scheme.

West's derivation was examined by several authors (see, for example, [14], [17] and [20]). We shall occasionally mention some points of these analyses. We shall basically follow the reasoning presented in ref. [14].

3.2 Analyticity and the renormalization group

Let us first look at the physical background of the quantity R . The total cross section of the process $e^+e^- \rightarrow$ hadrons is given by [21]

$$\sigma(e^+e^- \rightarrow \text{hadrons}) = \frac{e^4}{2(q^2)^3} \ell^{\mu\nu} w_{\mu\nu}, \quad (3.5)$$

where $\ell^{\mu\nu}$ and $w_{\mu\nu}$ is the leptonic and the hadronic tensor respectively. They can be expressed as

$$\ell^{\mu\nu} = p_1^\mu p_2^\nu + p_1^\nu p_2^\mu - \frac{1}{2} q^2 g^{\mu\nu} \quad (3.6)$$

$$w_{\mu\nu} = \int d^4x e^{iq \cdot x} \langle 0 | [j_\mu(x), j_\nu(0)] | 0 \rangle, \quad (3.7)$$

where p_1 and p_2 is the momentum of the electron and the positron respectively and j_μ denotes the electromagnetic quark current. Using the Lorentz covariance and current conservation we obtain the general hadronic tensor structure

$$w_{\mu\nu} = -\frac{1}{6\pi} (q_\mu q_\nu - q^2 g_{\mu\nu}) w(q^2) . \quad (3.8)$$

Inserting (3.6) and (3.8) into (3.5) and dividing the result by the leptonic cross section $\sigma(e^+e^- \rightarrow \mu^+\mu^-)$ we arrive at the conclusion that $R = w(q^2)$. Contraction of (3.7) with $g^{\mu\nu}$ gives

$$R = -\frac{2\pi}{q^2} \int d^4x e^{iq \cdot x} \langle 0 | [j_\mu(x), j^\mu(0)] | 0 \rangle . \quad (3.9)$$

In massless theory, R is a function of q^2 , the renormalization scale μ and the renormalized coupling parameter $g^2 = 4\pi\alpha_s$ only. Being an observable quantity, it must be independent of the renormalization parameter μ . Consequently, it has to satisfy the homogeneous renormalization group equation (RGE)

$$\frac{d}{d\mu} R(q^2/\mu^2, g^2) = \left(\frac{\partial}{\partial \mu} + \frac{dg}{d\mu} \frac{\partial}{\partial g} \right) R(q^2/\mu^2, g^2) = 0 , \quad (3.10)$$

or, multiplied by μ ,

$$\left(\mu \frac{\partial}{\partial \mu} + \mu \frac{dg}{d\mu} \frac{\partial}{\partial g} \right) R(q^2/\mu^2, g^2) = \left(\mu \frac{\partial}{\partial \mu} + \beta(g) \frac{\partial}{\partial g} \right) R(q^2/\mu^2, g^2) = 0, \quad (3.11)$$

where the function β satisfies the equation (3.3). As was already discussed in Section 2, R is not a function of q^2 and g^2 separately but, rather, is a function of a single variable $z = q^2/\mu^2 e^{2K(g)}$. Inserting this into the equation (3.11) we arrive at the relation

$$\frac{dK(g)}{dg} = 1/\beta(g) . \quad (3.12)$$

Consequently, $K(g)$ has the following form:

$$K(g) = \int_0^g \frac{dg'}{\beta(g')} \approx \frac{1}{2b_1 g^2} - \frac{b_2}{2b_1^2} \ln(b_1 g^{-2} + b_2) + \dots \quad (3.13)$$

We now introduce the vacuum polarization tensor:

$$\Pi_{\mu\nu} = (q_\mu q_\nu - q^2 g_{\mu\nu}) \quad \Pi(q^2, g^2) = i \int d^4x \, e^{iq \cdot x} \langle 0 | T(j_\mu(x) j_\nu(0)) | 0 \rangle. \quad (3.14)$$

The vacuum polarization function Π is assumed to be analytic in the complex q^2 plane with a discontinuity along the positive real semiaxis. The optical theorem relates the ratio R and the imaginary part $\text{Im } \Pi$ in the following way:

$$\text{Im } \Pi(q^2/\mu^2, g^2) = \frac{1}{12\pi} R(q^2/\mu^2, g^2) \quad (3.15)$$

It is convenient to define also D ,

$$D(q^2/\mu^2, g^2) = \frac{\partial}{\partial t} \Pi(q^2/\mu^2, g^2), \quad (3.16)$$

where $t = \ln(q^2/\mu^2)$. D itself can be formally written as an asymptotic expansion

$$D(q^2/\mu^2, g^2) = \sum_{n=0}^{\infty} (-1)^n d_n(q^2/\mu^2) g^{2n}. \quad (3.17)$$

Besides the renormalization group equation (3.11), we shall use the analyticity property of $D(z)$, where $z = (q^2/\mu^2) e^{2K(g)}$. The corresponding dispersion relation has the form

$$D(z) = \frac{z}{12\pi^2} \int_0^\infty \frac{R(z')}{(z' - z)^2} dz' \quad (3.18)$$

3.3 Comments on West's asymptotic estimate

Combining these equations and definitions one obtains the following relation between the expansion coefficients of R and D for $n \geq 2$:

$$\left(\sum_i Q_i^2 \right) r_n = -(-4\pi^2)^{n+1} \frac{1}{\pi n b_1} \left(\text{Im } d_{n+1} + \frac{b_2}{b_1} \text{Im } d_n + \dots \right). \quad (3.19)$$

West's claim that one may deduce a unique large-order behaviour of $\{r_n\}$ from the renormalization group and analyticity alone is criticized by Brown and Yaffe [20]. These authors examine the constraints imposed by asymptotic freedom and analyticity on the large-order behaviour of perturbation theory for the current-current correlation function Π and its imaginary part (3.15). Denoting by c_n and r_n their perturbation expansion coefficients respectively and by $C(z)$ and $A(z)$ the corresponding Borel transforms,

$$C(z) = c_0 + \sum_{m=1}^{\infty} c_m z^m / (m-1)!$$

and

$$A(z) = \sum_{m=1}^{\infty} r_m z^m / (m-1)! ,$$

they find that the relation between the two sets of coefficients results in the remarkable relation between $A(z)$ and $C(z)$,

$$C(z) = A(z) / \sin(\pi b_0 z) . \quad (3.20)$$

This result shows to what extent the singularities of $C(z)$ are

different from those of $A(z)$: the existence of zeros in the $\sin(\pi b_0 z)$ denominator produces additional singularities of $C(z)$ at all integral values of $b_0 z$ unless $A(z)$ has compensating zeros (notice it has one at the origin). Since the nature and location of singularities of a function affect the large-order behaviour of its power-expansion coefficients, relation (3.20) amounts to a relation between the coefficients $\{c_n\}$ and $\{r_n\}$, provided that the zeros of $\sin(\pi b_0 z)$ are not too close to the origin.

As Brown and Yaffe show, equation (3.20) has implications for the domain of convergence of $C(z)$ and hence on the large-order behaviour of the dispersive coefficients, but puts no generally valid restriction on $A(z)$. Consequently, the renormalization group and analyticity alone do not determine the large-order behaviour of $\{r_n\}$.

Important information about d_n is obtained by inverting the relation (3.17) between D and d_n . Using the Cauchy theorem we can write

$$D(q^2/\mu^2, g_0^2) = \frac{1}{2\pi i} \oint D(q^2/\mu^2, g^2) (g^2 - g_0^2)^{-1} dg^2, \quad (3.21)$$

which can be used to express d_n in terms of D :

$$d_n = - \lim_{g_0^2 \rightarrow 0^+} \oint_{C(g_0)} \frac{dg^2}{2\pi i} (g_0^2 - g^2)^{-1-n} D(q^2/\mu^2, g^2). \quad (3.22)$$

$C(g_0)$ is a closed contour in the complex g^2 plane around g_0^2 . Its size has to be chosen sufficiently small to avoid cutting across the complicated singularity structure of $D(q^2/\mu^2, g_0^2)$ around the origin: $C(g_0)$ must not leave, with g_0 approaching zero, the wedge bisected by the real axis and bounded above and below by circles that are tangent to the real axis at the

origin (see Fig. 4 and Appendix A). Now it is evidently crucial that the order of the limit $g_0 \rightarrow 0$ and the integration in (3.22) cannot be interchanged. West uses a dispersion relation for D , as is given by formula (3.18).

Inserting (3.18) into (3.22) yields

$$d_n = - \frac{1}{12\pi^2} \lim_{g_0 \rightarrow 0} \oint_{c(g_0)} \frac{dg^2}{2\pi i} \frac{1}{(g_0^2 - g^2)^{1+n}} z \int_0^\infty \frac{R(z')}{(z' - z)^2} dz'. \quad (3.23)$$

This equation incorporates two essential features of the theory, renormalizability and analyticity. To interchange the order of integrations, as is done in ref. [18], is unjustified, as is shown in detail in ref [14]. In particular, the result would then be contour dependent. Finally, as is shown in ref. [14], the use of the saddle point technique [18] to estimate the contour integrals is not justified because the positions of the leading saddle points depend on the integration variable.

Thus the large order behaviour of the QCD perturbation expansion remains an open question.

Barclay and Maxwell [17] direct their attention at the remarkable agreement of the estimate $r_3^{\text{est}} = -12.0$ (see (3.4)) with the explicitly calculated value $r_3^{\text{ex}} = -12.8$, as obtained by the authors of [16]. They examine the N_f dependence of r_3^{est} and r_3^{ex} and conclude that the agreement between the value of r_3^{est} and that of r_n^{ex} is an accidental coincidence for $N_f = 5$, while $r_3^{\text{est}} = -10$ and $r_3^{\text{ex}} = -23$ for $N_f = 3$ and $r_3^{\text{est}} = -19.5$ and $r_3^{\text{ex}} = -0.7$ for $N_f = 10$.

In a subsequent paper [19] the author claims that, to determine the sum of a perturbation QCD series, it is sufficient to take into account only a few lowest order terms and that the relevant number of terms is inversely proportional to the value of the coupling parameter, i.e., it becomes

particularly small in the IR kinematic domain of the theory. To show this, the author proposes to estimate the remainder $R_N(z)$ of the asymptotic expansion,

$$R_N(z) = \left| F(z) - \sum_{n=0}^{N-1} a_n z^n \right| \quad (3.24)$$

by means of the first omitted term of the series, viz.,

$$R_N(z) \approx a_N z^N. \quad (3.25)$$

Taking $a_n = n!$ we find the critical $n = n_c$ (above which the terms start increasing at a given z) by setting $(n-1)!z^{n-1}/n!z^n = 1$, i.e., $n_c \sim 1/z$. Defined in this way, n_c does decrease when z moves away from the origin.

This argument is misleading for several reasons, for instance,

1. The idea (3.25) that $a_N z^N$ is an estimate of $R_N(z)$ is not correct and comes from a confusion of the asymptotic series with the Taylor series. $\sum_{n=0}^{\infty} a_n z^n$ is an asymptotic expansion of several, in general infinitely many, functions, which at a given z may differ from each other by an arbitrary number. Consider as an example the function $F(z) + A \exp(-z^{-\alpha})$, which for an appropriate α has the same asymptotic expansion as $F(z)$, but for a given $z \neq 0$ differs from $F(z)$ by an arbitrary number, provided that A is appropriately chosen. The point is that all perturbation theory can give is the coefficients a_n , but this is insufficient unless strong asymptotic conditions are known, which permit to determine the expanded function uniquely, as was explained in Section 2.

2. If $a_n = n!$ asymptotically, the choice $n = n_c \sim 1/z$ gives the smallest of the terms $a_n z^n$, $n = 1, 2, \dots$, at fixed z , but says nothing about the actual value of $a_n z^n$ at finite n .

For fixed n , the error radically increases with increasing z even if we restrict ourselves to the (unrealistic) estimate (3.25).

3.4 Discussion

We can conclude that analyticity and the renormalization group equation may serve as a general framework for obtaining useful information in perturbative QCD, but, as is seen from ref. [20], they alone are not sufficient to determine the large-order behaviour of the QCD perturbation coefficients. It seems that such information can hardly be obtained unless the scheme is supplemented with an additional assumption of non-perturbative nature. This conclusion is almost self-evident because the analyticity principle itself does have non-perturbative background. It seems that to give perturbation expansions good meaning we have to go beyond the purely perturbative framework. A number of papers (see, for example, [22] - [25]) choose this way.

There is no reason to expect that the use of asymptotic relations can reliably predict the values of low- n perturbative terms.

Another lesson to take is that the mathematical texture of the subject is so delicate that one cannot venture steps that seem normal elsewhere in physics (like interchanging integrations or limits in calculations), not even in order to obtain crude or qualitative estimates. The large-order behaviour of QCD series remains an open question.

4. CONCLUSION

A power series does not uniquely determine the function expanded. The degree of ambiguity is essentially determined by the location of singularities of the function; more specifically, by the circumstance how "easy" it is to approach the point of expansion from the analyticity region.

In QCD, the access to the point of expansion (i. e., to the origin) is particularly difficult; the opening angle of the analyticity domain at the origin appears to be zero (see [2], see also Fig. 4 and Appendix A).

One can remove the ambiguity in determining the expanded function from its power expansion by supplementing the scheme with strong asymptotic conditions (SAC, see section 2.2, Theorems 4 - 6). In the case of a zero opening angle (relevant to QCD), the strong asymptotic conditions are particularly restrictive; see Theorem 6.

So the problem of a unique determination of a function from its perturbation series in QCD reduces to that of finding appropriate strong asymptotic conditions. As is seen from the Theorem 6, they are very restrictive and are of mathematical form. It is a challenge to find a nonperturbative input able to ensure their fulfillment.

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APPENDIX A Analyticity region of $D(z)$

Starting from general renormalization group ideas, 't Hooft argues [2] that the analyticity domain of the two-point Green's functions of QCD in g^2 is a horn-shaped region, a wedge bisected by the positive real semiaxis and bounded above and below by circles that are tangent to it.

Since the theory is invariant under the renormalization group, $g(\mu)$ obeys the differential equation

$$\mu \frac{dg}{d\mu}(\mu) = \beta(g(\mu)) \quad (\text{A.1})$$

where μ is the renormalization parameter and β is the Callan-Symanzik function. Its solution with $\beta(g) = -a g^2$ is

$$1/g(\mu_0) - 1/g(\mu) = a \ln(\mu/\mu_0) \quad (\text{A.2})$$

where μ_0 is a normalization mass. The Green function depends on the energy variable q^2 and on μ_0 through the dimensionless combination q^2/μ_0^2 ; 't Hooft prefers to introduce its logarithm multiplied by $\frac{1}{2}a$; thus, G is a function of $\frac{1}{2}a \ln(-q^2/\mu_0^2)$. This quantity can be expressed, using (A.2), as a combination of a , q^2/μ^2 and g in the following form:

$$\frac{1}{2}a \ln(-q^2/\mu_0^2) = \frac{1}{2}a \ln(-q^2/\mu) + 1/g(\mu) - 1/g(\mu_0) \quad (\text{A.3})$$

with $g(\mu_0)$ real. Choosing now a fixed, positive (i.e., physical) value of $g(\mu)$ we know that G is analytic in q^2 except the positive real axis above some threshold value. The points of this cut in the q^2 plane correspond to the right hand side of (A.3) having the form $\frac{1}{2}a [\text{real} + (2n+1)\pi i]$. Thus, (A.3)

becomes

$$\frac{1}{2}a \ln(-q^2/\mu_0) = \frac{1}{2}a [\text{real} + (2n+1)\pi i]. \quad (\text{A.4})$$

If we fix, on the other hand, the value of q^2 and vary $g(\mu)$ the same form of the r.h.s. of (A.4) can be obtained by choosing g complex and satisfying the relation

$$1/g(\mu) = 1/g(\mu_0) + \frac{1}{2}a [\text{real} + (2n+1)\pi i] . \quad (\text{A.5})$$

This is, in the complex $1/g(\mu)$ plane, a set of equidistant straight lines parallel to the real axis. In the complex $g(\mu)$ plane, these parallel straight lines transform into a bunch of circles displaced on either side of the real axis and tangent to it, with radii varying from $1/(a\pi)$ down to zero (see Fig. 4). The singularities of $G[\frac{1}{2}a\ln(-q^2/\mu) + 1/g(\mu) - 1/g(\mu_0)]$ in the complex $g(\mu)$ plane are displaced along these circles. Consequently, the only way how to reach the origin of the complex g plane from the analyticity domain without crossing any singularity is to follow the path along the real axis from the positive side; the domain of analyticity is a wedge bisected by the real axis and bound from above and below by circles which are tangent to the real axis (see Fig. 4). In other words, the opening angle is zero. Corrections in $K(g^2)$ described by the equation (3.13) should result in slight deformations of the circles.

As is pointed out in ref. [12], this conclusion is essentially based on experimental information. First, we use the fact that the singularities of G in the q^2 plane are located along the positive real semiaxis. This *non-perturbative* information follows from the experimentally established

spectrum of elementary particles. Further, the experimental fact of asymptotic freedom implies the sign of the dominant term of $\beta(g)$ near $g = 0$. Finally, it is plausible to assume that, at sufficiently high momenta, we can ignore all rest masses, the only mass scale left being the integration constant μ_0 . It is remarkable that all this, essentially nonperturbative and experimental, information is used here to determine the analyticity properties of G in the g plane, which in turn determine the properties of the perturbation expansion. In this way we admix a nonperturbative ingredient into this perturbative result. I share the view that this fact does not diminish the usefulness of the result.

APPENDIX B Derivation of the dispersion relation

As a starting point we use Cauchy's theorem to write

$$\Pi(z) = -\frac{1}{2\pi i} \oint \frac{\Pi(z')}{z' - z} dz' \quad (\text{B.1})$$

where $[z = q^2/\mu^2 \exp[2K(g)]]$ and the closed integration contour encircles the point z . For D we thus obtain

$$\begin{aligned} D(z) &= \frac{\partial}{\partial t} \Pi(z) = -\frac{1}{2\pi i} \frac{\partial}{\partial t} \oint \frac{\Pi(z')}{z' - z} dz' \\ &= -\frac{1}{2\pi i} \left(\frac{\partial}{\partial z} \oint \frac{\Pi(z')}{z' - z} dz' \right) \frac{\partial z}{\partial t} \\ &= -\frac{1}{2\pi i} \left(\frac{\partial z}{\partial t} \right) \oint \frac{\Pi(z')}{(z' - z)^2} dz' , \end{aligned}$$

where

$$\frac{\partial z}{\partial t} = \frac{\partial}{\partial [\ln(q^2/\mu^2)]} \exp[\ln(q^2/\mu^2) + 2K(g)] = z .$$

So we easily see that

$$D(z) = -\frac{z}{2\pi i} \oint \frac{\Pi(z')}{(z' - z)^2} dz' . \quad (\text{B.2})$$

Let us suppose that $\Pi(z)$ is real analytic in the complex plane, $\Pi(z^*) = [\Pi(z)]^*$, has a discontinuity along the positive real axis, and has the behaviour $\Pi = o(|z|)$ for $z \rightarrow \infty$. We choose the integration wrapping the cut from either side starting from the branch point to $+\infty$ and encircling the cut plane with a

radius tending to ∞ . Performing the integration we obtain

$$\begin{aligned}
 \oint \frac{\Pi(z')}{(z'-z)^2} dz' &= \int_{-\infty}^0 \frac{\Pi(z'-i\varepsilon)}{(z'-z)^2} dz' + \int_0^{\infty} \frac{\Pi(z'+i\varepsilon)}{(z'-z)^2} dz' . \\
 &= \int_0^{\infty} \frac{\Pi(z'+i\varepsilon) + [\Pi(z'+i\varepsilon)]^*}{(z'-z)^2} dz' . \\
 &= 2i \int_0^{\infty} \frac{\text{Im } \Pi(z')}{(z'-z)^2} dz' .
 \end{aligned}$$

This result yields, together (B.2), the desired relation (3.18):

$$D(z) = \frac{z}{\pi} \int_0^{\infty} \frac{\text{Im } \Pi(z')}{(z'-z)^2} dz' = \frac{z}{12\pi^2} \int_0^{\infty} \frac{R(z')}{(z'-z)^2} dz' .$$

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FIGURE CAPTIONS

Fig. 1 Explanation of symbols:

C . . . convergence circle of the series (2.1)

P_1, P_2, P_3, P_4 . . . the Borel polygon

P'_1, P'_2, P'_3, P'_4 . . . extension of the Borel polygon
(Theorem 2)

R_1, R_2, R_3, R_4 . . . boundary of $MLS(f)$

(Definition 4, Theorem 3)

Fig. 2 Analyticity domains in the Theorems 4 and 5

Fig. 3 Analyticity domain of $f(z)$, formulae (2.27) and (2.28)

Fig. 4 The coupling constant singularities of the two point function given by equation (A.5) lie on circles tangent to the real axis. Assumed is $\beta(g) = ag^2$; modifications due to other ansätze (e.g., formula (3.3)) are straightforward

Fig. 5 The integration contour in formula (B.2)

Fig. 1

Fig. 4

Fig. 5