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**ASPECTS OF LONG TIME BEHAVIOUR OF SOLUTIONS OF
NONLINEAR HAMILTONIAN EVOLUTION EQUATIONS**

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0. Introduction

In this paper we will be mainly concerned with the behaviour of solutions of (space periodic) nonlinear wave equations

$$u_{tt} = \Delta u + p(u; t, x) \quad (x \in \mathbb{T}^d) \quad (1)$$

and nonlinear Schrödinger equations

$$-i u_t = -\Delta u + V(x)u + \frac{\partial}{\partial \bar{u}} G(u, \bar{u}; t, x) \quad (x \in \mathbb{T}^d). \quad (2)$$

Most of the techniques used have a wider range of applicability however.

We are interested in nonlinear Hamiltonian PDE in a general (nonintegrable or close to integrable) context where one does not expect KAM results. Assuming local existence of solutions established, there are many issues to be addressed about their behaviour for $t \rightarrow \infty$ and the properties of the flow S_t in phase space, such as

$$\text{- global existence, blowup behaviour} \quad (3)$$

$$\text{- asymptotic stability} \quad (4)$$

$$\begin{aligned} &\text{- behaviour of higher Sobolev norms of smooth solutions when} \\ &\quad t \rightarrow \infty; \text{ spreading of energy to higher modes} \end{aligned} \quad (5)$$

$$\text{- recurrence properties in time} \quad (6)$$

Global wellposedness may be often derived from local wellposedness and conservation laws, without much further insight in the behaviour of the solutions.

On the other hand, the Hamiltonian property of the equation permits to exploit invariants of the flow in an appropriate phase space. For instance, Liouville's theorem leads to invariant Gibbs measures that permit to establish Poincaré recurrence (in appropriate topology). More recently, other invariants were discovered, such as symplectic capacities and applied to Hamiltonian mechanics by various authors. Symplectic capacities permit to prove certain nonsqueezing properties of S_t , which are of relevance for (4) and (5). In a recent work [K2], S. Kuksin adjusted the finite dimensional theory to an infinite dimensional phase space setting, provided the map S_t is of the form

$$S_t = \text{linear operator} + \text{compact smooth operator} \quad (7)$$

(essentially speaking). In this statement (7), the phase space is welldefined, due to the finite dimensional normalization. For instance, for equation (1), the "symplectic Hilbert space" is $H^{1/2}(\mathbb{T}^d) \times H^{1/2}(\mathbb{T}^d)$, while for equation (2) it is $L^2(\mathbb{T}^d)$. Examples of results obtained along these lines in [K2] is the nonsqueezing of balls in cylinders defined with respect to a Darboux basis, in the case of the nonlinear string equation

$$u_{tt} = u_{xx} + p(u; t, x) \quad (x \in \mathbb{T}) \quad (8)$$

where p is a smooth function which has at most polynomial growth as $|u| \rightarrow \infty$ and the quadratic nonlinear wave equation

$$u_{tt} = \Delta u + a(t, x)u + b(t, x)u^2 \quad (x \in \mathbb{T}^2). \quad (9)$$

These are special cases of (1).

Thus the squeezing theorem states that

$$S_t(B_R) \subset \mathbb{T}_r^{(k)} \Rightarrow R \leq r. \quad (10)$$

Here B_R denotes an R -ball in the symplectic Hilbert space Z (not necessarily centered at 0) and $\mathbb{T}_r^{(k)}$ stands for a translate of the cylinder $\{\sum p_j \varphi_j^+ + q_j \varphi_j^- \mid p_k^2 + q_k^2 < r^2\}$ where $\{\varphi_j^\pm\}$ is a Darboux basis of Z .

The squeezing theorem implies in particular that if B_ρ is a ball centered at some initial point, then the diameter of the set $S_t(B_\rho)$ can not tend to zero. This fact is referred to in [K2] as the failure of "uniform asymptotic stability" as $t \rightarrow \infty$ of bounded solutions of the equation.

Another consequence of the squeezing theorem relates to spreading of the energy to higher frequencies. It follows indeed from (10) that for any time t and mode k , one cannot have

$$p_k^2 + q_k^2 < (\rho - \varepsilon)^2 \quad (11)$$

for all numbers of $S_t(B_\rho)$.

In this paper, we will continue some of these investigations by extending Kuksin's results to other equations, requiring more PDE analysis or with flow map not of the form (7).

PROPOSITION 1 *The nonsqueezing theorem (10) in $H^{1/2} \times H^{1/2}$ holds for the wave equation (1) with at most degree 4 nonlinearity (resp. at most quadratic nonlinearity) in dimension 2 (resp. dimension 3, 4). In fact, the flow map is of the form (7) in these cases.*

The symplectic Hilbert space of the NLSE (2) is the space $L^2(\mathbb{T}^d)$ of complex functions. We restrict the discussion to $d = 1$. In the interesting cases, such as for instance the cubic NLSE $i u_t + u_{xx} \pm u|u|^2 = 0$, the map S_t is not of form (7). However, in [B4], we proved the nonsqueezing theorem for equations of the form

$$i u_t + u_{xx} + a(x, t)u + b(x, t) u|u|^2 = 0 \quad (12)$$

where a, b are sufficiently smooth real functions, both periodic in x . The property is derived from a direct approximation argument by finite dimensional models

$$-i U_t = -U_{xx} + P_N \frac{\partial}{\partial \bar{U}} G(U, \bar{U}; t, x) \quad U = P_N U \quad (13)$$

where P_N is the usual Dirichlet projection. An L^2 -analysis for the 1-dimensional NLSE with L^2 -local wellposedness theorem seems presently only available for cubic nonlinearity, i.e. $G(u, \bar{u}; t, x)$ is a polynomial of degree ≤ 4 in $u, \bar{u}$. The argument in [B5] depends moreover on the conservation of the L^2 -norm $\int_{\mathbb{T}} |u|^2 dx$ which holds for G of the form $G(|u|^2; t, x)$ as in (12). We consider here the more general case and prove the following.

PROPOSITION 2 *Consider the NLSE*

$$i u_t + u_{xx} + \frac{\partial}{\partial \bar{u}} G(u, \bar{u}; t, x) = 0 \quad (14)$$

where G is a real polynomial in $u, \bar{u}$ of degree ≤ 4 . Then bounded solutions of (14) are not uniformly asymptotically stable in L^2 for $t \rightarrow \infty$. The same statement holds without degree restriction on G in H^s , $s > \frac{1}{2}$.

One may compare solutions of (13), (14) for data $\varphi = P_N \varphi$ which “tail” Fourier coefficients are sufficiently small and invoke a result of Ekeland and Hofer [E-H] according to which in $2n$ -dimensional phase space the product $B^2(r) \times \cdots \times B^2(r)$ (n copies) can not be symplectically embedded in a ball $B^{2n}(\rho)$ with $\rho < \sqrt{n} r$.

In the last section, we exhibit smooth global solutions of Hamiltonian NLSE with smooth nonlinear term which develop large $\|u(t)\|_{H^s}$ -norm for $t \rightarrow \infty$. Here the Sobolev exponent s_0 is numerical. The construction used leads to equations which are close to the linear Schrödinger equation. Similar arguments apply for other evolution equations as well.

PROPOSITION 3 (1) *There is a Hamiltonian NLSE with smooth nonlinear term depending on u and projections Pu of u on the trigonometric system, such that for smooth data the solution u satisfies $\overline{\lim}_{t \rightarrow \infty} \|u(t)\|_{H^{s_0}} = \infty$ for some fixed exponent s_0 .*

(2) *There is a Hamiltonian NLSE with smooth and local nonlinearity such that $S_t(B^s(\delta))$, $t > 0$ is not a bounded subset of H^{s_0} , for any $s < \infty$ and $\delta > 0$. Here $B^s(\delta)$ denotes $\{\varphi \in H^s \mid \|\varphi\|_s < \delta\}$.*

In previous constructions, we use the fact that solutions of the linear equation $i u_t + u_{xx} = 0$ are periodic in time (for periodic boundary conditions). The last example discussed deals with perturbations of a linear Schrödinger operator $i u_t + u_{xx} + V(x)u$ where $V(x)$ is a real smooth potential with nearly resonant spectrum, in the sense that for some n_0 and infinite sequence $\{n_j\}$ one has $\text{dist}(\lambda_{n_j}, \mathbb{Z} \lambda_{n_0}) \rightarrow 0$ rapidly for $j \rightarrow \infty$. In this context we construct a Hamiltonian perturbation $\Gamma(u) = \frac{\partial}{\partial \bar{u}} G$ such that if $u_{\varepsilon, q}$ denotes the solution of the IVP

$$\begin{cases} -i u_t = -u_{xx} + V(x)u + \varepsilon \Gamma(u) \\ u(0) = q \end{cases} \quad (15)$$

then

$$\inf_{q \in \gamma} \sup_t \|u_{\varepsilon, q}(t)\|_{H^{s_0}} \rightarrow \infty \quad \text{for } \varepsilon \rightarrow 0 \quad (16)$$

where γ is some open subset of H^{s_0} .

This example is related to infinite dimensional versions of Lyapounov's theorem in KAM theory on persistency of invariant tori and the work of Kuksin on small Hamiltonian perturbations of integrable equations [K1].

1. Nonlinear wave equations

We consider the nonlinear wave equation

$$u_{tt} = \Delta u - u - f(u; t, x) \quad (1)$$

where $x \in \mathbb{T}^d$ and f is a polynomial in u with smooth coefficients in x, t .

Denoting B the operator $(-\Delta + 1)^{1/2}$ we may write (1) in Hamiltonian form as

$$\begin{cases} u_t = -Bv \\ v_t = Bu + B^{-1} f(u; t, x). \end{cases} \quad (2)$$

The symplectic Hilbert space here is $H^{1/2}(\mathbb{T}^d) \times H^{1/2}(\mathbb{T}^d)$. We study the Cauchy problem for (2) with data in $H^s(\mathbb{T}^d)$ (s close to $\frac{1}{2}$) local in time. From the integral equation we get

$$\begin{aligned} u(t) = & \sum_{\xi} \left[\frac{\widehat{u(0)}(\xi) + i\widehat{v(0)}(\xi)}{2} e^{i((x,\xi) + (1+|\xi|^2)^{1/2}t)} + \frac{\widehat{u(0)}(\xi) - i\widehat{v(0)}(\xi)}{2} e^{i((x,\xi) - (1+|\xi|^2)^{1/2}t)} \right] \\ & + \sum_{\xi} \int d\lambda \frac{\widehat{f}(\xi, \lambda)}{\lambda^2 - 1 - |\xi|^2} e^{i(x,\xi)} \\ & \left[e^{i\lambda t} - \frac{1}{2} \left(1 + \frac{\lambda}{(1+|\xi|^2)^{1/2}} \right) e^{i(1+|\xi|^2)^{1/2}t} - \frac{1}{2} \left(1 - \frac{\lambda}{(1+|\xi|^2)^{1/2}} \right) e^{-i(1+|\xi|^2)^{1/2}t} \right] \end{aligned} \quad (3)$$

where $\widehat{f}$ denotes the Fourier transform of $f(u; t, x)$ on $\mathbb{T}^d \times I$. The expression for $v(t)$ is obtained as $-B^{-1}u_t$.

The assumptions on dimension d and f are the following

$$d = 2 \quad f \text{ of the form } a_1(x, t)u + a_2(x, t)u^2 + a_3(x, t)u^3 + a_4(x, t)u^4 \quad (5)$$

$$d = 3, 4 \quad f \text{ of the form } a_1(x, t)u + a_2(x, t)u^2. \quad (6)$$

Consider following norm for functions $A(x, t) = \sum_{\xi} \int d\lambda \widehat{A}(\xi, \lambda) e^{i((x,\xi) + \lambda t)}$ on $\mathbb{T}^d \times I$

$$\|A\|_s = \left(\sum_{\xi} \int d\lambda (1 + |\xi|)^{2s} (1 + |\lambda| - |\xi|)^{2\rho} |\widehat{A}(\xi, \lambda)|^2 \right)^{1/2} \quad (7)$$

(to be understood as a restriction norm with respect to $\mathbb{T}^d \times I$).

Here ρ is choosen a bit larger than $\frac{1}{2}$. Observe that

$$\|A(t)\|_{H^\bullet} \leq c \|A\|_s \quad \text{for } t \in I. \quad (8)$$

The expression (3) defines a linear operator of $u(0), v(0)$; observe that

$$\|B(3)\|_s + \|B^{-1}(3)_t\|_s \leq c (\|u(0)\|_{H^\bullet} + \|v(0)\|_{H^\bullet}). \quad (9)$$

Our purpose is to show that for some $s_1 < \frac{1}{2} < s_2$

$$\|B(4)\|_{s_2} + \|B^{-1}(4)_t\|_{s_2} \leq C \sigma^c (1 + \|u\|_{s_1}^3) \|u\|_{s_1} \quad (10)$$

where $\sigma = |I|$, for some constants $0 < c, C < \infty$. Replacing s_2 by s_1 in the left member of (10), one gets as a first consequence, letting σ be sufficiently small,

$$\|u\|_{s_1} + \|v\|_{s_1} \leq c (\|u(0)\|_{H^{s_1}} + \|v(0)\|_{H^{s_1}}). \quad (11)$$

Hence, from (8), (10), (11), it follows that for $t \in I$

$$\|B(4)\|_{H^{s_2}} + \|B^{-1}(4)_t\|_{H^{s_2}} \leq \|u(0)\|_{H^{s_1}} + \|v(0)\|_{H^{s_1}} \quad (12)$$

and hence the nonlinear part of the flow map acts boundedly from $H^{s_1} \times H^{s_1}$ to $H^{s_2} \times H^{s_2}$ ($s_1 < \frac{1}{2}, s_2 > \frac{1}{2}$) which is the required condition to make the results from [K2] applicable.

The $a(x, t)$ -coefficients will play little role in the verification of (10) and we ignore them for simplicity sake. In fact the relevant calculation appear for $f(u; t, x) = u^4$ in $d = 2$ and $f(u; t, x) = u^2$ in $d = 3, d = 4$.

Consider the expression (4) with $||\lambda| - |\xi|| < 10$. One easily verifies because t is local (multiply the expression with a localizing function $\varphi(t)$) that the corresponding contribution to (7) is bounded by

$$\left[\sum_{\xi} (1 + |\xi|)^{2(s-1)} \int_{||\lambda| - |\xi|| < 10} |\hat{f}(\xi, \lambda)|^2 \right]^{1/2}. \quad (13)$$

Hence $\|B(4)\|_{s_2}$ may be estimated by the sum of

$$\left[\sum_{\xi} |\xi|^{2s_2} \int d\lambda \frac{|\hat{f}(\xi, \lambda)|^2}{(|\xi| + |\lambda| + 1)^2 (||\xi| - |\lambda|| + 1)^{2(1-\rho)}} \right]^{1/2} \quad (14)$$

$$\left[\sum_{\xi} |\xi|^{2(s_2-1)} \left(\int d\lambda \frac{|\hat{f}(\xi, \lambda)|}{||\xi| - |\lambda|| + 1} \right)^2 \right]^{1/2} \quad (15)$$

and hence, from Hölder's inequality ($\rho > \frac{1}{2}$)

$$\left[\sum_{\xi} \int d\lambda \frac{|\widehat{f}(\xi, \lambda)|^2}{(1 + |\xi|)^{2(1-s_2)} (1 + ||\xi| - |\lambda||)^{2(1-\rho)}} \right]^{1/2}. \quad (16)$$

To estimate $|||B^{-1}(4)_t|||_{s_2}$, we need to introduce an extra factor $\frac{\lambda^2}{1+|\xi|^2}$ in (14). Hence (16) is an estimate on both $|||(4)|||_{s_2}$ and $|||B^{-1}(4)_t|||_{s_2}$.

Consider first the case $d = 2$. Letting $f = u^4$, one has $\widehat{f} = \widehat{u} * \widehat{u} * \widehat{u} * \widehat{u}$. According to (7), define

$$c(\xi, \lambda) = (1 + |\xi|)^{s_1} (1 + ||\lambda| - |\xi||)^{\rho} |\widehat{u}(\xi, \lambda)| \quad (17)$$

hence

$$||c||_2 = |||u|||_{s_1}. \quad (18)$$

In the sequel, we will write $|\dots|$ instead of $1 + |\dots|$.

By duality (16) may be estimated by

$$\sum_{\substack{\xi=\xi_1+\xi_2+\xi_3+\xi_4 \\ \lambda=\lambda_1+\lambda_2+\lambda_3+\lambda_4}} \int \prod_{i=1}^4 \frac{c(\xi_i, \lambda_i)}{|\xi_i|^{s_1} ||\lambda_i| - |\xi_i||^{\rho}} \frac{d(\xi, \lambda)}{|\xi|^{1-s_2} ||\lambda| - |\xi||^{1-\rho}} \quad (19)$$

where $d(\xi, \lambda) \geq 0$, $||d||_2 \leq 1$.

Observe that in the problem of estimating (19), the discrete character of the summation plays clearly no role and we may as well replace $\sum_{\xi=\xi_1+\xi_2+\xi_3+\xi_4}$ by $\int_{\xi=\xi_1+\xi_2+\xi_3+\xi_4}$ (all denominators are taken > 1).

At this stage, we invoke Strichart's inequality on the Fourier transform of an L^2 -density carried by a cone in $\mathbf{R}^{d+1}$ (see [Str]).

Let $q = \frac{2(d+1)}{d-1}$. Then

$$\left\| \int_{|\xi| \sim R} a(\xi) e^{i(\langle x, \xi \rangle + t|\xi|)} d\xi \right\|_{L^q(dx dt)} \leq C R^{1/2} \left(\int |a(\xi)|^2 d\xi \right)^{1/2}. \quad (20)$$

In our case $d = 3$, $q = 6$. It follows from (20) and Hölder's inequality that

$$\begin{aligned} & \left\| \int_{|\xi| \sim R} d\xi \int d\lambda \frac{c(\xi, \lambda)}{||\lambda| - |\xi||^{\frac{1}{2} +}} e^{i(\langle x, \xi \rangle + \lambda t)} \right\|_{L^6(dx dt)} \\ & \leq C R^{1/2} \left(\int \int |c(\xi, \lambda)|^2 d\xi d\lambda \right)^{1/2} \end{aligned} \quad (21)$$

and hence, interpolating with the obvious (Parseval) L^2 -inequality

$$\left\| \int d\xi \int d\lambda c(\xi, \lambda) e^{i((x, \xi) + \lambda t)} \right\|_{L^2(dx dt)} \leq \left(\int \int |c(\xi, \lambda)|^2 d\xi d\lambda \right)^{1/2} \quad (22)$$

we get the inequality

$$\begin{aligned} & \left\| \int_{|\xi| \sim R} d\xi \int d\lambda \frac{c(\xi, \lambda)}{||\lambda| - |\xi||^{\frac{9}{20} +}} e^{i((x, \xi) + \lambda t)} \right\|_{L^5(dx dt)} \\ & \leq C R^{9/20} \left(\int \int |c(\xi, \lambda)|^2 d\xi d\lambda \right)^{1/2} \end{aligned} \quad (23)$$

which is used to bound (19). Restricting ξ_i, ξ to dyadic regions

$$\begin{cases} |\xi_i| \sim R_i & (i = 1, 2, 3, 4) \\ |\xi| \sim R \end{cases} \quad (24)$$

one gets

$$(R_1 R_2 R_3 R_4)^{-s_1} R^{-(1-s_2)} \sum_{\substack{\xi = \Sigma \xi_i, \lambda = \Sigma \lambda_i \\ |\xi_i| \sim R_i, |\xi| \sim R}} \int \prod_{i=1}^4 \frac{c(\xi_i, \lambda_i)}{||\xi_i| - |\lambda_i||^\rho} \frac{d(\xi, \lambda)}{||\xi| - |\lambda||^{q-\rho}}. \quad (25)$$

Define $F_i = F_i(x, t)$, $G = G(x, t)$ letting $\widehat{F}_i(\xi, \lambda) = \frac{c(\xi_i, \lambda_i)}{||\xi_i| - |\lambda_i||^\rho} \chi_{[|\xi_i| \sim R_i]}(\xi)$, $\widehat{G}(\xi, \lambda) = \frac{d(\xi, \lambda)}{||\xi| - |\lambda||^{q-\rho}} \chi_{[|\xi| \sim R]}(\xi)$. Thus (25) equals

$$\begin{aligned} & (R_1 R_2 R_3 R_4)^{-s_1} R^{-(1-s_2)} \int \prod_{i=1}^4 F_i \cdot G dx dt \\ & \leq (R_1 R_2 R_3 R_4)^{-s_1} R^{-(1-s_2)} \prod_{i=1}^4 \|F_i\|_5 \cdot \|G\|_5. \end{aligned} \quad (26)$$

Assume $s_1 < \frac{1}{2} < s_2$ choosen such that $s_1, 1 - s_2 > \frac{9}{20}$ and $\rho, 1 - \rho > \frac{9}{20}$. It follows from (23) that $\|F_i\|_5 \leq C R_i^{9/20} \|c\|_2$ ($1 \leq i \leq 4$) and $\|G\|_5 \leq C R^{9/20}$, so that by (18)

$$(26) \leq (R_1 R_2 R_3 R_4)^{\frac{9}{20} - s_1} R^{-\frac{11}{20} + s_2} \|u\|_{s_1}^4 \quad (27)$$

which is summable for dyadic values of R_i, R .

Considering a small time interval $I, |I| = \sigma$, there is an extra saving of σ^c , for some $c > 0$, which is inequality (10). Consider functions u which are supported

on a 2σ neighborhood of 0. It follows from the definition of the norm (7), in particular the $\|\lambda| - |\xi|\|^\rho$ -multiplier, that localizing (3), (4) to I will affect the $\|\cdot\|_{s_2}$ -norm by a factor $(\frac{1}{\sigma})^{\rho-\frac{1}{2}}$ ($\rho > \frac{1}{2}$). On the other hand, repeating the previous L^5 -estimate, one gets factors

$$\begin{aligned} & R^{9/20} \left\| \frac{c(\xi, \lambda)}{\|\lambda| - |\xi|\|^{\rho-\frac{9}{20}}} \chi_{|\xi| \sim R} \right\|_{L_{\xi, \lambda}^2} \\ &= R^{9/20+s_1} \left\| \|\lambda| - |\xi|\|^{9/20} |\widehat{u}(\xi, \lambda)| \chi_{|\xi| \sim R} \right\|_{L_{\xi, \lambda}^2} \end{aligned}$$

which by interpolation are bounded by

$$\begin{aligned} & R^{\frac{9}{20}+s_1} \left\| \widehat{u} \chi_{|\xi| \sim R} \right\|_2^{1-\frac{9}{20\rho}} \left\| \|\lambda| - |\xi|\|^\rho |\widehat{u}(\xi, \lambda)| \chi_{|\xi| \sim R} \right\|_2^{\frac{9}{20\rho}} \\ &\leq R^{\frac{9}{20}+(1-\frac{9}{20\rho})s_1} \cdot \left(\|\widehat{u}(\xi)\|_{L_{|\xi| \sim R}^2 L_t^2} \right)^{1-\frac{9}{20\rho}} \cdot \|c\|_2^{\frac{9}{20\rho}}. \end{aligned} \quad (28)$$

Since $\text{supp } u \subset \mathbb{T}^d \times I$, $\|\widehat{u}(\xi)\|_{L_t^2} \leq \sigma^{1/2} \|\widehat{u}(\xi)\|_{L_t^\infty} \leq \sigma^{1/2} \|\lambda| - |\xi|\|^\rho |\widehat{u}(\xi, \lambda)| \|_{L_\lambda^2}$ by Hölder's inequality and (28) $\leq R^{\frac{9}{20}} \sigma^{\frac{1}{2}(1-\frac{9}{20\rho})} \|c\|_2 \leq R^{\frac{9}{20}} \sigma^{\frac{1}{20}} \|u\|_{s_1}$. For $\rho > \frac{1}{2}$ close enough to $\frac{1}{2}$, this clearly implies inequality (10) with the σ^c -factor. From the earlier discussion, Proposition I follows in case (5).

The proof of (6) is completely analogous. For the argument to work, one needs the exponent $q = \frac{2(d+1)}{d-1}$ from Strichartz's inequality to fulfill the condition $q > k+1$, where k is the degree of $f(u; t, x)$ in u . Thus for $d \geq 3$, $\frac{d+3}{d-1} > k \geq 2$ only permits (6).

2. Behaviour of solutions of nonlinear Schrödinger equations with nonlinear term of degree at most 3

As pointed out in [K2], the symplectic Hilbert space for the NLSE

$$i u_t = -\Delta u + \frac{\partial}{\partial \bar{u}} G(u, \bar{u}; t, x) \quad (1)$$

where G is a real valued smooth function periodic in $x \in \mathbb{T}^d$ is the space $L^2(\mathbb{T}^d)$. Consider dimension $d = 1$, G a polynomial in $u, \bar{u}$. At this point, there is only a local wellposedness theory in $L^2(\mathbb{T})$ for degree ≤ 4 (see [B1]) and this will be our assumption here. Thus G is a sum of following terms

$$a u + \bar{a} \bar{u} \quad (\text{linear})$$

$$\begin{cases} a u^2 + \bar{a} \bar{u}^2 \\ a_r |u|^2 \end{cases} \quad (\text{degree 2})$$

$$\begin{cases} a u^3 + \bar{a} \bar{u}^3 \\ (a u + \bar{a} \bar{u}) |u|^2 \end{cases} \quad (\text{degree 3})$$

$$\begin{cases} a u^4 + \bar{a} \bar{u}^4 \\ (a u^2 + \bar{a} \bar{u}^2) |u|^2 \\ a_r |u|^4 \end{cases} \quad (\text{degree 4})$$

where a (resp. a_r) is a smooth (resp. real) function of x, t , periodic in x . In the case G has the form $f(|u|^2, x, t)$, the equation preserves the L^2 -norm and the local wellposedness property is global.

In [B4], we considered equations of the form

$$i u_t + u_{xx} + a(x, t)u + b(x, t) u|u|^2 = 0 \quad (2)$$

and proved nonsqueezing of balls in cylinders, i.e.

$$S_t(B_r) \subset \mathbb{T}_R^{(k)} \Rightarrow R \geq r. \quad (3)$$

Here B_r is a translate of the L^2 -ball of radius r

$$\{u \in L^2(\mathbb{T}) \mid \|u\|_2 < r\}$$

and $\mathbb{T}_R^{(k)}$ a translate of the cylinder defined with respect to the k^{th} -element of the (Darboux) basis $\varphi_k(x) = \sqrt{2} \cos kx$ or $\sqrt{2} \sin kx$

$$\{u \in L^2(\mathbb{T}) \mid |\langle u, \varphi_k \rangle| < R\}.$$

S^t denotes the flow map of (1), (2). Recall that canonical coordinates are $\text{Re } u$, $\text{Im } u$ here.

The proof uses a certain (uniform) approximation property of solutions of (1), (2) by solutions of a “truncated” equations

$$i U_t + U_{xx} + P_N \left[\frac{\partial}{\partial \bar{U}} G(U, \bar{U}; t, x) \right] = 0 \quad (4)$$

where $U = P_N U$ and P_N stands for the usual Dirichlet projection on the space $[e^{ikx} \mid -N \leq k \leq N]$. The phase space for equation (4) is finite dimensional and the squeezing property follows from the symplectic capacity theory. To obtain previous approximation result, the conservation of $\int_{\mathbf{T}} |u|^2 dx$ is used in an essential way (besides getting a priori L^2 -bounds) in the case of equation (2). This conservation fails if we allow a general degree ≤ 4 expression for G as described earlier. Our main purpose here is to study these more general equations. Essentially speaking, we show that there is an L^2 -approximation of the solutions of the IVP's

$$\begin{cases} i u_t + u_{xx} + \frac{\partial}{\partial \bar{u}} G(u, \bar{u}, t, x) = 0 \\ u(0) = \phi \end{cases} \quad (5)$$

$$\begin{cases} i U_t + U_{xx} + P_N \left[\frac{\partial}{\partial \bar{U}} G(U, \bar{U}; t, x) \right] = 0 \\ U(0) = \phi \end{cases} \quad (6)$$

($U = P_N U$, $\phi = P_N \phi$) on a time interval $[0, T]$, provided there is no blowup and the data ϕ satisfies a condition on Fourier coefficient size

$$|\hat{\phi}(k)| < \delta \quad (N_0 < |k| < N) \quad (7)$$

where N depends on N_0 , $\|u(t)\|_2$ for $t < T$ and the approximation ε

$$\|u(t) - U(t)\|_2 < \varepsilon \quad (0 < t < T) \quad (8)$$

and δ will be any power N^{-c} , $c > 0$.

This fact will imply the absence of “uniform asymptotical stability” for $t \rightarrow \infty$, in the sense that for any ρ -ball B_ρ the diameter of the set $S_t(B_\rho)$ can not tend to zero. We may derive this from the result of [E-H] about nonsqueezing of a translate of the set

$$\left(\frac{\rho}{\sqrt{N}} B^2 \right) \times \dots \times \left(\frac{\rho}{\sqrt{N}} B^2 \right) \quad (N \text{ copies})$$

in a ball of radius ρ , considering here the (Hamiltonian) flow $S_N(t)$ corresponding to (6)¹.

We first recall some facts on the analysis of (1) ($n = 1$) with cubic nonlinearity (see [B1] for details). Consider an equation of the form

¹This property becomes in fact already evident in the discussion of solutions of (5) assuming (7).

$$i u_t = u_{xx} + F(u, \bar{u}, x, t) \quad (9)$$

where F is smooth in x, t , periodic in x and a polynomial in $u, \bar{u}$ of degree ≤ 3 (not necessarily Hamiltonian). Consider a sufficiently small time interval $[0, \sigma]$, where σ depends on the L^2 -norm size $\|\phi\|_2$ of the data $u(0) = \phi$. A wellposedness result on $[0, \sigma]$ may then be proved using Picard's contraction principle and the equivalent integral equation

$$u(t) = S(t) \phi + i \int_0^t S(t-\tau) w(\tau) d\tau \quad ; \quad w(\tau) = F(u(\tau), \bar{u}(\tau), x, \tau) \quad (10)$$

where $S(t)$ is the unitary group solving the linear equation $i u_t = u_{xx}$. The fixpoint argument is applied in the space

$$|||u||| = \left(\sum_n \int d\lambda (|\lambda - n^2| + 1)^{2\rho} |\hat{u}(n, \lambda)|^2 \right)^{1/2} \quad (11)$$

assuming

$$u(x, t) = \sum_n \int d\lambda \hat{u}(n, \lambda) e^{i(nx + \lambda t)} \quad \text{on } \mathbb{T} \times [0, \sigma].$$

In (11), the exponent ρ is chosen slightly larger than $\frac{1}{2}$, so that

$$|||u||| \geq \|u(t)\|_2 \quad \text{for } t < \sigma \quad (12)$$

(by Hölder's inequality). The main estimate used in the analysis of the $||| \cdot |||$ -norm of the nonlinear term in (10) is following L^4 -inequality

$$\|u\|_{L^4(\mathbb{T} \times [0, 1])} \leq c \left(\sum_n (|\lambda - n^2| + 1)^{3/4} |\hat{u}(n, \lambda)|^2 \right)^{1/2} \quad (13)$$

which is general (and sharp).

Write F as a sum of monomials and consider say the cubic terms $a u_1 u_2 u_3$ where $u_i = u$ or $\bar{u}$. Writing using Fourier transform

$$\begin{aligned} & \int_0^t S(t-\tau) (a u_1 u_2 u_3)(\tau) d\tau \\ &= \sum_n e^{inx} \int d\lambda (a u_1 u_2 u_3)^\wedge(n, \lambda) \frac{e^{i\lambda t} - e^{in^2 t}}{\lambda - n^2} \end{aligned} \quad (14)$$

the $||| \cdot |||$ -norm of (14) may be estimated by

$$\left[\sum_n \int d\lambda \frac{|(a \ u_1 \ u_2 \ u_3)^\wedge(n, \lambda)|^2}{|\lambda - n^2|^{2(1-\rho)}} \right]^{1/2} \quad (15)$$

(denominators $|\lambda - n^2|$ will mean $|\lambda - n^2| + 1$ in the sequel). Define

$$c(n, \lambda) = (1 + |\lambda - n^2|)^\rho \cdot |\widehat{u}(n, \lambda)|. \quad (16)$$

Since $(a \ u_1 \ u_2 \ u_3)^\wedge = \widehat{a} * \widehat{u}_1 * \widehat{u}_2 * \widehat{u}_3$ (convolution), one may estimate by duality (15), taking $d(n, \lambda) \geq 0$, $\sum_n \int d\lambda \ d(n, \lambda)^2 \leq 1$

$$\sum_{\substack{n=n_0+\epsilon_1 n_1+\epsilon_2 n_2+\epsilon_3 n_3 \\ \lambda=\lambda_0+\epsilon_1 \lambda_1+\epsilon_2 \lambda_2+\epsilon_3 \lambda_3}} \int |\widehat{a}(n_0, \lambda_0)| \frac{c_1(n_1, \lambda_1)}{|\lambda_1 - n_1^2|^\rho} \frac{c(n_2, \lambda_2)}{|\lambda_2 - n_2^2|^\rho} \frac{c(n_3, \lambda_3)}{|\lambda_3 - n_3^2|^\rho} \frac{d(n, \lambda)}{|\lambda - n^2|^{1-\rho}}. \quad (17)$$

Because a is smooth

$$|\widehat{a}(n_0, \lambda_0)| < (|n_0| + |\lambda_0|)^{-c} \quad (18)$$

where c is an arbitrary exponent.

Consider for fixed n_0, λ_0 the expression

$$\sum_{\substack{n=n_0+\epsilon_1 n_1+\epsilon_2 n_2+\epsilon_3 n_3 \\ \lambda=\lambda_0+\epsilon_1 \lambda_1+\epsilon_2 \lambda_2+\epsilon_3 \lambda_3}} \int \frac{c(n_1, \lambda_1)}{|\lambda_1 - n_1^2|^{3/8}} \frac{c(n_2, \lambda_2)}{|\lambda_2 - n_2^2|^{3/8}} \frac{c(n_3, \lambda_3)}{|\lambda_3 - n_3^2|^{3/8}} \frac{d(n, \lambda)}{|\lambda - n^2|^{3/8}} \quad (19)$$

and define functions $F_i = F_i(x, t)$, $G = G(x, t)$ letting

$$\widehat{F}_i(n, \lambda) = \frac{c_i(\epsilon_i \ n, \epsilon_i \ \lambda)}{|\lambda - n^2|^{3/8}} \quad \text{and} \quad \widehat{G}(n, \lambda) = \frac{d(n, \lambda)}{|\lambda - n^2|^{3/8}}.$$

Then (19) = $\int (F_1 \ F_2 \ F_3 \ G) \ dx \ dt$, bounded by $\|F_1\|_4 \|F_2\|_4 \|F_3\|_4 \|G\|_4$. Hence, (13), (16) yield an estimate by $c\|u\|^3$.

Since the exponents $\rho, 1 - \rho > \frac{3}{8}$, it follows from (18) that (17) $\leq c\|u\|^3$. In fact, since $\rho > \frac{3}{8}$, a choice of a time interval of small size σ will give an extra saving of a factor σ^c , thus

$$(17) < c \ \sigma^{c_1} \ \|u\|^3 \quad (20)$$

for some fixed constant $c_1 > 0$. Indeed, consider functions u supported by a 2σ -neighborhood of 0. Observe from the multiplier in in (11) that multiplying a function (here given by (14)) with a localizing function in the t -variable to a σ -neighborhood will increase at most the $\| \cdot \|$ -norm by a $(\frac{1}{\sigma})^{\rho-\frac{1}{2}}$ factor.

On the other hand, repeating previous L^4 -estimate, one gets factors

$$\left\| \frac{c(n, \lambda)}{|\lambda - n^2|^{\rho - \frac{3}{8}}} \right\|_{\ell_n^2 L_\lambda^2} = \| |\lambda - n^2|^{3/8} |\widehat{u}(n, \lambda)| \|_{\ell_n^2 L_\lambda^2}$$

which by interpolation and (13) is bounded by

$$\|\widehat{u}\|_2^{1 - \frac{3}{8\rho}} \| |\lambda - n^2|^\rho |\widehat{u}(n, \lambda)| \|_2^{\frac{3}{8\rho}} \leq \sigma^{\frac{1}{4}(1 - \frac{3}{8\rho})} \|u\|_4^{1 - \frac{3}{8\rho}} \|u\|_{\frac{8\rho}{3}}^{\frac{3}{8\rho}} \leq \sigma^{\frac{1}{4}(1 - \frac{3}{8\rho})} \|u\|.$$

Similarly, considering difference expressions for functions u, v , previous reasoning leads to estimates of the form $c \sigma^{c_1} (\|u\| + \|v\| + 1)^2 \|u - v\|$, from where the contraction principle may be derived for sufficiently small σ .

These estimates appear in [B1] in the context of the cubic NLS $i u_t + u_{xx} \pm u|u|^2 = 0$.

The purpose of the next considerations is to verify which systems of frequencies (n_0, n_1, n_2, n_3) considering $\widehat{a}(n_0) \widehat{u}_1(n_1) \widehat{u}_2(n_2) \widehat{u}_3(n_3)$ will be significant in the $\| \int_0^t S(t - \tau) (a u_1 u_2 u_3)(\tau) d\tau \|$ -estimate.

Further Analysis

Observe that since in (17) the exponents $\rho, 1 - \rho > \frac{3}{8}$, there will be an extra saving of a factor B^{-c_2} , for some constant $c_2 > 0$, unless each of the factors $\lambda_1 - n_1^2, \lambda_2 - n_2^2, \lambda_3 - n_3^2, \lambda - n^2$ is bounded by B , hence

$$|(n_0 + \varepsilon_1 n_1 + \varepsilon_2 n_2 + \varepsilon_3 n_3)^2 - \varepsilon_1 n_1^2 - \varepsilon_2 n_2^2 - \varepsilon_3 n_3^2| \leq |\lambda_0| + n_0^2 + B. \quad (21)$$

Consider a partial summation

$$\widetilde{F} = \sum_{(n_0, n_1, n_2, n_3) \in E} e^{i(n_0 x + \lambda_0 t)} \widehat{u}_1(n_1) \widehat{u}_2(n_2) \widehat{u}_3(n_3) \quad (22)$$

where E is some index set with properties to be specified.

It follows from the preceding that $\left\| \int_0^t S(t - \tau) \widetilde{F}(\tau) d\tau \right\|$ may be bounded by an expression

$$B^4 \sum_{\substack{(n_0, n_1, n_2, n_3) \in E \\ (21) \text{ holds}}}$$

$$\overline{c}_1(n_1) \overline{c}_2(n_2) \overline{c}_3(n_3) \overline{d}(n_0 + \varepsilon_1 n_1 + \varepsilon_2 n_2 + \varepsilon_3 n_3) + C B^{-c_2} \|u\|^3 \quad (23)$$

letting $\overline{c}_i(n_i) = c(n_i, \lambda_i)$ and $\overline{d}(n) = d(n, \lambda)$ for some λ_i (resp. λ) satisfying $|\lambda_i - n_i^2| < B$ (resp. $|\lambda - n^2| < B$). Our purpose is to exploit a small size

property of the $\bar{c}_i(n_i)$, hence $|\hat{u}(n, \lambda)|$, in estimating the first term in (23). In fact we estimate

$$\sum_{\substack{(n_0, n_1, n_2, n_3) \in E \\ (21) \text{ holds}}} \bar{c}_1(n_1) \bar{c}_2(n_2) \bar{c}_3(n_3). \quad (24)$$

Assume as first condition on E the absence of indices (n_0, n_1, n_2, n_3) with

$$n_0 = 0 \quad (25)$$

and one of following cases

$$\varepsilon_1 = 1, \varepsilon_2 = 1, \varepsilon_3 = -1 ; n_3 = n_1 \text{ or } n_2 \quad (26)$$

$$\varepsilon_1 = 1, \varepsilon_2 = -1, \varepsilon_3 = 1 ; n_2 = n_1 \text{ or } n_3 \quad (27)$$

$$\varepsilon_1 = -1, \varepsilon_2 = 1, \varepsilon_3 = 1 ; n_1 = n_2 \text{ or } n_3 \quad (28)$$

(unless $n_1 = n_2 = n_3$).

Fix n_3 say and estimate the number of pairs (n_1, n_2) satisfying (21). Thus we consider following quadratic expression in n_1, n_2

$$(1 - \varepsilon_1)n_1^2 + (1 - \varepsilon_2)n_2^2 + 2\varepsilon_1\varepsilon_2 n_1n_2 + 2(\varepsilon_3n_3 + n_0)(\varepsilon_1n_1 + \varepsilon_2n_2) + (\varepsilon_3n_3 + n_0)^2 - \varepsilon_3n_3^2. \quad (29)$$

Assume all indices n_1, n_2, n_3 are bounded by N . We denote N^ε an arbitrary small power of N (appearing here as $\exp \frac{\log N}{\log \log N}$ from divisor numbers).

Case $\varepsilon_1 = -1, \varepsilon_2 = -1$. This case reduces to lattice point counting on an ellips $x^2 + 3y^2 = A$, $A \leq N^2 + |\lambda_0| + n_0^2 + B$, and hence is a bound by N^ε (we assume N much larger than n_0, λ_0, B).

Case $\varepsilon_1 = 1, \varepsilon_2 = 1$. (29) yields then

$$2(n_1 + \varepsilon_3n_3 + n_0)(n_2 + \varepsilon_3n_3 + n_0) - (\varepsilon_3n_3 + n_0)^2 - \varepsilon_3n_3^2. \quad (30)$$

Considering divisors, there is again a bound by N^ε unless $n_1 + \varepsilon_3n_3 + n_0 = 0$ or $n_2 + \varepsilon_3n_3 + n_0 = 0$ and $|(\varepsilon_3n_3 + n_0)^2 + \varepsilon_3n_3^2| \leq |\lambda_0| + n_0^2 + B$. Hence either

$$|n_3| \leq |\lambda_0| + n_0^2 + B \quad (31)$$

or

$$n_0 = 0, \varepsilon_3 = 1 \quad \text{hence} \quad n_1 = n_3 \text{ or } n_2 = n_3$$

which is the excluded case (25), (26).

Case $\varepsilon_1 = 1, \varepsilon_2 = -1$. ($\varepsilon_1 = -1, \varepsilon_2 = 1$ similar)
(29) becomes then

$$2(n_1 - n_2)(\varepsilon_3 n_3 + n_0 - n_2) + (\varepsilon_3 n_3 + n_0)^2 - \varepsilon_3 n_3^2. \quad (32)$$

Thus there is again a bound by N^ε unless $n_1 = n_2$ or $n_2 = n_0 + \varepsilon_3 n_3$ and either (31) or $\varepsilon_3 = 1, n_0 = 0$. Since (27) is excluded, (31) only is possible.

The conclusion is that if E excludes (25)+(26), (25)+(27), (25)+(28) and

$$|n_3| \geq |\lambda_0| + n_0^2 + B \quad (33)$$

is given, the number of pairs (n_1, n_2) satisfying (21) is at most N^ε .

Consider the terms in (24) with $|n_1|, |n_2|, |n_3| \geq |\lambda_0| + n_0^2 + B$. Estimate by Hölder's inequality as

$$\|c\|_2 \left[\sum_{n_3} \left(\sum_{(n_0, n_1, n_2, n_3) \in E; (21)} \bar{c}_1(n_1) \bar{c}_2(n_2) \right)^2 \right]^{1/2}. \quad (34)$$

Since for fixed n_3 the number of pairs (n_1, n_2) is at most N^ε

$$(34) \ll N^\varepsilon \|c\|_2 \left[\sum_{(n_0, n_1, n_2, n_3) \in E; (21)} \bar{c}_1(n_1)^2 \bar{c}_2(n_2)^2 \right]^{1/2}. \quad (35)$$

Similarly, for fixed n_2 there are at most N^ε pairs (n_1, n_3) in the summation, so that

$$(35) \ll N^\varepsilon \|c\|_2^2 \max_n \bar{c}_1(n). \quad (36)$$

Consequently

$$(24) \ll N^\varepsilon \left(\min_{i=1,2,3} \max_n \bar{c}_i(n) \right) \|c\|_2^2. \quad (37)$$

Suppose now

$$\min(|n_1|, |n_2|, |n_3|) \leq |\lambda_0| + n_0^2 + B \quad (38)$$

say $|n_3| < |\lambda_0| + n_0^2 + B$. Rewrite (21) as

$$\begin{aligned} & (1 - \varepsilon_1)n_1^2 + (1 - \varepsilon_2)n_2^2 + 2\varepsilon_1\varepsilon_2 n_1 n_2 + 2(n_0 + \varepsilon_3 n_3)(\varepsilon_1 n_1 + \varepsilon_2 n_2) \\ & \leq (|\lambda_0| + n_0^2 + B)^2. \end{aligned} \quad (39)$$

Case $\varepsilon_1 = -1, \varepsilon_2 = -1$. Clearly

$$|n_1|, |n_2| \leq |\lambda_0| + n_0^2 + B. \quad (40)$$

Case $\varepsilon_1 = 1, \varepsilon_2 = 1$. Then (39) gives

$$|n_0 + n_1 + \varepsilon_3 n_3| |n_0 + n_2 + \varepsilon_3 n_3| \leq (|\lambda_0| + n_0^2 + B)^2$$

hence one of the following cases

$$(40)$$

$$n_0 + \varepsilon_1 n_1 + \varepsilon_3 n_3 = 0 \quad (41)$$

$$n_0 + \varepsilon_2 n_2 + \varepsilon_3 n_3 = 0. \quad (42)$$

Case $\varepsilon_1 = 1, \varepsilon_2 = -1$. Then

$$|n_1 - n_2| |n_0 + \varepsilon_2 n_2 + \varepsilon_3 n_3| \leq (|\lambda_0| + n_0^2 + B)^2$$

and hence (40) or

$$n_1 = n_2 \quad (43)$$

or

$$(42).$$

Considering these remaining cases (40)-(43), inequality (37) will be valid provided E excludes moreover following cases

$$\begin{cases} |n_1|, |n_2| \leq |\lambda_0| + n_0^2 + B \\ n_0 + \varepsilon_1 n_1 + \varepsilon_2 n_2 = 0 \\ \varepsilon_3 = 1 \end{cases} \quad \text{and cyclic} \quad (44)$$

$$\begin{cases} |n_1| \leq |\lambda_0| + n_0^2 + B \\ n_2 = n_3. \end{cases} \quad \text{and cyclic} \quad (45)$$

Thus the cases to be avoided in E are

(25)+(26), (25)+(27), (25)+(28)

(44) etc.

(45) etc.

Considering an expression $F = a u_1 u_2 u_3$, a satisfying (18), we may as a consequence of the preceding write F as a sum of certain terms of the form

$$\left(\int_{\mathbf{T}} a \, dx \right) \left(\int_{\mathbf{T}} |u|^2 \, dx \right) u \quad (46)$$

$$a_0 \left(\int |u|^2 \right) P_K(u) \quad (47)$$

$$a \left(\int |u|^2 \right) P_K \bar{u} \quad (48)$$

$$\left(\int a_0 |P_K u|^2 \right) u \quad (49)$$

$$\left(\int a (P_K u)^2 \right) u \quad (50)$$

$$\left(\int a (P_K \bar{u})^2 \right) u \quad (51)$$

and an expression $\tilde{F}$ that will satisfy

$$\begin{aligned} & \left\| \int_0^t S(t-\tau) \tilde{F}(\tau) \, d\tau \right\| \\ & < B^4 N^\epsilon \left(\min_i \max_{n,\lambda} |\hat{u}_i(n, \lambda)| \right) \|u\|^2 + C B^{-c_2} \|u\|^3. \end{aligned} \quad (52)$$

Here $a_0 = a - \int a \, dx$ and K in the projection P_K depends on B and a . We assume $|n| < N$.

Approximation of solutions of Hamiltonian NLS with at most cubic nonlinearity

Consider

$$F(u, \bar{u}; t, x) = \frac{\partial}{\partial \bar{u}} G(u, \bar{u}; t, x) \quad (53)$$

where G is a sum of degree 1, 2, 3, 4 terms in $u, \bar{u}$ as described earlier in this section. The corresponding terms for F are as follows

$$a u + \bar{a} \bar{u} \longrightarrow \bar{a} \quad (54)$$

$$\begin{cases} a u^2 + \bar{a} \bar{u}^2 \\ a_r |u|^2 \end{cases} \quad \begin{cases} \bar{a} \bar{u} \\ a_r u \end{cases} \quad (55)$$

$$\begin{cases} a u^3 + \bar{a} \bar{u}^3 \\ (a u + \bar{a} \bar{u}) |u|^2 \end{cases} \quad \begin{cases} \bar{a} \bar{u}^2 \\ a u^2 + 2\bar{a} \bar{u} u \end{cases} \quad (56)$$

$$\left\{ \begin{array}{l} a u^4 + \bar{a} \bar{u}^4 \\ a u^3 \bar{u} + \bar{a} \bar{u}^3 u \\ a_r |u|^4 \end{array} \right\} \quad \left\{ \begin{array}{l} \bar{a} \bar{u}^3 \\ a u^3 + 3\bar{a} \bar{u}^2 u \\ a_r |u|^2 u. \end{array} \right. \quad (57)$$

Recall that a_r stands for a real function. Fix B and choose K depending on the smooth coefficients a and B as above. The discussion done above for cubic expressions $F = a u_1 u_2 u_3$ may be done for lower degree too, so that (51) will hold after removal of certain specific parts of the expression. Looking at the formula's (55)-(57), we isolate following contributions

$$\left(\int a_r \right) u \quad (58)$$

$$\left[\int (a P_K u + \bar{a} P_k \bar{u}) \right] u \quad (59)$$

$$\bar{a} \int |u|^2 \quad (60)$$

$$\left[\int a (P_K u)^2 + \bar{a} (P_k \bar{u})^2 \right] u \quad (61)$$

$$\left(\int a_r \right) \left(\int |u|^2 \right) u \quad (62)$$

$$\left(\int a_r \cdot P_k u \cdot P_k \bar{u} \right) u \quad (63)$$

$$\bar{a} \left(\int |u|^2 \right) \cdot P_k \bar{u} \quad (64)$$

$$a_r \left(\int |u|^2 \right) \cdot P_k u. \quad (65)$$

Let $M > K$ and add to the list (58)-(65) the additional contributions of $F(P_M u, P_M \bar{u}; t, x)$. This clearly yields an (algebraic) decomposition

$$F(u, \bar{u}; t, x) = \Omega_{M,u}(t) \cdot u + F_{M,u} + \tilde{F}_{M,u} \quad (66)$$

where $\Omega_{M,u}(t)$ is a real function of t (depending on u), $F_M = P_{3M} F_M$ and $\tilde{F}_M$ satisfies the estimate

$$\begin{aligned} & \left| \int_0^t S(t-\tau) \tilde{F}_M(\tau) d\tau \right| \\ & < B^4 N^\epsilon \left(\sup_{|n| > M} |n|^\epsilon |\hat{u}(n)| \right) (1 + |||u|||)^2 + C B^{-c_2} (1 + |||u|||)^3. \end{aligned} \quad (67)$$

Consider the Cauchy problem (5)

$$\begin{cases} i u_t + u_{xx} + F(u) = 0 \\ u(0) = \phi \end{cases} \quad (68)$$

where $\phi = P_N \phi$ and satisfies (7), thus $|\widehat{\phi}(k)| < \delta$ ($N_0 < |k| < N$).

Assume $\|u(t)\|_2$ bounded on $[0, T]$, T given. We compare the solutions u, v of (67) and

$$\begin{cases} i v_t + v_{xx} + \Omega_{10M,v}(t) v + F_{M,v} = 0 \\ v(0) = \phi \end{cases} \quad (69)$$

where

$$F(v, \bar{v}; t, x) = \Omega_{10M,v}(t) v + F_{10M,v} + \tilde{F}_{10M,v} \quad (70)$$

corresponds to the decomposition (66). Hence

$$\Omega_{10M,v}(t) v + F_{M,v}(v) = F(v) + F_{M,v} - F_{10M,v} - \tilde{F}_{10M,v}. \quad (71)$$

Write the integral equations

$$u(t) = S(t) \phi + i \int_0^t S(t-\tau) F(u(\tau), \bar{u}(\tau); \tau, x) d\tau \quad (72)$$

and

$$v(t) = S(t) \phi + i \int_0^t S(t-\tau) F(v(\tau), \bar{v}(\tau); \tau, x) d\tau \quad (73)$$

$$+ i \int_0^t S(t-\tau) (F_{M,v} - F_{10M,v})(\tau) d\tau \quad (74)$$

$$- i \int_0^t S(t-\tau) \tilde{F}_{10M,v}(\tau) d\tau. \quad (75)$$

Consider a time interval $[0, \sigma]$, σ depending on $\|u(t)\|_2$ ($0 < t < T$). Subtract (72) and (73) and estimate (74), (75). We use the $|||$ $|||$ -estimates given earlier leading to (20) and inequality (67). Thus clearly on $[0, \sigma]$

$$|||u - v||| \leq c \sigma^{c_1} |||u - v||| (1 + |||u||| + |||v|||)^2 \quad (76)$$

$$+ c \sigma^{c_1} |||P_{10M} v - P_M v||| (1 + |||v|||)^2 \quad (77)$$

$$+ B^4 \left(\sup_{|n| > 10M} |n|^\epsilon |\widehat{v}(n)| \right) (1 + |||v|||)^2$$

$$+ C B^{-c_2} (1 + |||v|||)^3. \quad (78)$$

Write

$$|||P_{10M} v - P_M v||| \leq |||P_{10M} u - P_M u||| + 2|||u - v|||$$

and observe that by considering the (finite) sequence $M_0, 10M_0, 10^2M_0, \dots, 10^s M_0$, $M_0 = \max(N_0, K)$, one may clearly get an M -value at most $10^s M_0$ such that

$$|||P_{10M} u - P_M u||| < c \frac{|||u|||}{s^{1/2}}. \quad (79)$$

Assume $|n| > 10M$. Since $F_{M,v} = P_{3M}(F_{M,v})$, the n -th-Fourier coefficient $\widehat{v}(n) = \widehat{v}(n)(t)$ satisfies the equation

$$\begin{cases} i \frac{d\widehat{v}(n)}{dt} - n^2 \widehat{v}(n) + \Omega_{10M,v}(t) \widehat{v}(n) = 0 \\ \widehat{v}(n)(0) = \widehat{\phi}(n). \end{cases} \quad (80)$$

Thus, because in particular $M > N_0$, $|\widehat{\phi}(n)| < \delta$ by hypothesis. The fact that $\Omega_{10M,v}(t)$ is real implies that $|\widehat{v}(n)|^2$ is constant in time, hence

$$|\widehat{v}(n)| < \delta \quad \text{for } |n| > 10M$$

$$\widehat{v}(n) = 0 \quad \text{for } |n| > N. \quad (81)$$

Substituting (79), (81) in inequality (76)-(78) gives

$$\begin{aligned} |||u - v||| &< c \sigma^{c_1} |||u - v||| (1 + |||u||| + |||v|||)^2 \\ &+ \left(\frac{1}{s^{1/2}} + \delta B^4 N^\epsilon + B^{-c_2} \right) (1 + |||u||| + |||v|||)^3. \end{aligned} \quad (82)$$

Recall that $\delta = N^{-c}$ for some $c > 0$. Choosing s and B suitably large, the coefficient of the second term in (82) may be made small. Thus (82) yields an estimate on $[0, \sigma]$

$$\|u(t) - v(t)\|_2 \leq c |||u - v||| < \kappa \quad (83)$$

(σ has to be taken small enough depending on $|||u|||$, hence $\|u(t)\|_2, t < T$). The number M will depend on N_0, κ and $N \gg M$.

Instead of (5), consider now the Cauchy problem (6), i.e.

$$\begin{cases} i U_t + U_{xx} + P_N F(U) = 0 \\ U(0) = \phi \end{cases} \quad (84)$$

where $U = P_N U$. The integral equation for U is now

$$U(t) = S(t) \phi + i \int_0^t S(t - \tau) P_N F(U(\tau), \overline{U}(\tau); \tau, x) d\tau. \quad (85)$$

Since $v = P_N v$, it also follows from (73)-(75) that

$$\begin{aligned} v(t) &= S(t) \phi + i \int_0^t S(t-\tau) P_N F(v(\tau), \bar{v}(\tau); \tau, x) d\tau \\ &+ i \int_0^t S(t-\tau) (F_{M,v} - F_{10M,v})(\tau) d\tau \\ &- i \int_0^t S(t-\tau) P_M \tilde{F}_{10M,v}(\tau) d\tau. \end{aligned} \quad (86)$$

The same argument as for u permits now by subtraction of (85), (86) to establish the approximation on $[0, \sigma]$

$$\|U(t) - v(t)\|_2 < \kappa. \quad (87)$$

Hence, from (83), (87)

$$\|u(t) - U(t)\|_2 < \kappa \quad \text{for } 0 < t < \sigma. \quad (88)$$

To cover the whole interval $[0, T]$, we partition in $T\sigma^{-1}$ intervals of length σ and repeat previous approximation. At the second step, we redefine N_0 as $N_1 = 10M$ and replace $u(t_1)$, $U(t_1)$ by $\phi_1 = v(t_1)$, v given by equation (69), which is now the new data. The perturbation is at most κ , from (87). By (81), one has again

$$|\hat{\phi}_1(n)| < \delta \quad \text{for } |n| > N_1. \quad (89)$$

The continuation of the process is clear. For sufficiently good approximations at each stage, one ensures (8) at the end. The final size condition on N will relate to N_0 , T , ε and $\|u(t)\|_2$ for $f < T$ (determining σ).

As a conclusion of the preceding, we get following

PROPOSITION 1 *Consider the Cauchy problems (with G as above) on time interval $[0, T]$*

$$\begin{cases} i u_t + u_{xx} + \frac{\partial}{\partial \bar{u}} G(u, \bar{u}; t, x) = 0 \\ u(0) = \phi \end{cases} \quad (90)$$

$$\begin{cases} i U_t + U_{xx} + P_N \left[\frac{\partial}{\partial \bar{U}} G(U, \bar{U}; t, x) \right] = 0 \\ U(0) = \phi \end{cases} \quad (91)$$

where $\phi = P_N \phi$ and satisfies

$$\|\phi\|_2 < C \quad \text{and} \quad |\hat{\phi}(n)| < N^{-c} \quad \text{for } |n| > N_0 \quad (92)$$

for some $C, c > 0$. Assume the solution u of (90) does not blowup on $[0, T]$ and

$$\|u(t)\|_2 < A \quad \text{for } 0 \leq t \leq T. \quad (93)$$

Let $\varepsilon > 0$ and assume

$$N > N(N_0, A, \varepsilon, T). \quad (94)$$

Then the solution U of (91) is an ε -approximation of u on $[0, T]$, i.e.

$$\|u(t) - U(t)\|_2 < \varepsilon \quad \text{for } 0 \leq t \leq T. \quad (95)$$

(In fact, (94) is a condition on the ratio $\frac{N}{N_0}$ to be sufficiently large depending on A, ε, T .)

COROLLARY 1 Assume (90) wellposed for ϕ in an L^2 -ball B_ρ of radius $\rho > 0$, for $0 \leq t \leq T$. Denote S_t the flowmap. Then

$$\text{diam } S_t(B_\rho) > \rho \quad \text{for } 0 \leq t \leq T. \quad (96)$$

Proof. Denote ϕ_0 the center of B_ρ and assume $\text{supp } \widehat{\phi}_0 \subset [-N_0, N_0]$ (by approximation). Define the sets of trigonometric polynomials

$$R_{N,\kappa} = \left\{ \phi \mid \text{supp } \widehat{\phi} \subset [-N, N] \quad \text{and} \quad |\widehat{\phi}(n)| < \frac{\kappa}{\sqrt{2N+1}} \right\}. \quad (97)$$

Thus $R_{N,\kappa} \subset B(0, \kappa)$ for each N . By the results of Ekeland and Hofer (see [E-H]), there is no symplectic embedding in $2(2N+1)$ -dimensional phase space of $R_{N,\kappa}$ in $B_{2(2N+1)}(0, \kappa')$ for $\kappa > \kappa'$ (balls are defined wrt ℓ^2 -norm). The same statement holds for translates of these sets.

Choose $\varepsilon > 0$ and N satisfying (94). Here (93) results from the wellposedness hypothesis on B_ρ . Observe that $\phi_0 + R_{N,\rho} \subset B_\rho$. Let $0 < t < T$ and denote S_t^N the flowmap corresponding to the truncated equation (91). Since $\phi \in \phi_0 + R_{N,\rho}$ satisfies (92), with $c = \frac{1}{2}$, and $S_t \phi$ satisfies (93), it follows that $\|S_t \phi - S_t^N \phi\|_2 < \varepsilon$ ($t < T$) uniformly for $\phi_0 + R_{N,\rho}$. The (finite dimensional) symplectic nonsqueezing property mentioned above implies that $S_t^N(\phi_0 + R_{N,\rho}) \not\subset B(S_t \phi_0, \rho - \varepsilon)$. Hence there is $\phi \in \phi_0 + R_{N,\rho}$ such that $\|S_t \phi - S_t \phi_0\|_2 > \|S_t^N \phi - S_t \phi_0\|_2 - \varepsilon > \rho - \varepsilon$, implying $\text{diam } S_t(B_\rho) > \rho - \varepsilon$. The claim follows letting $\varepsilon \rightarrow 0$.

COROLLARY 2 Assume the solution of (90) exists for all time. Then $\text{diam } S_t(B_\rho) \not\rightarrow 0$ for $t \rightarrow \infty$, for any neighborhood $B_\rho = B(\phi, \rho)$ of the initial data ϕ (failure of uniform asymptotic stability for $t \rightarrow \infty$).

Proof. In case the statement fails, there is T_0 such that $\text{diam } S_t(B_\rho) < 1$ for $t > T_0$. Hence, by the L^2 -wellposedness of NLS with cubic nonlinearity in L^2 , there is some $0 < \rho' < \rho$ such that (90) is globally wellposed on $B_{\rho'} = \phi + B(0, \rho')$. It follows thus from previous corollary 1 that $\text{diam } S_t(B_\rho) > \text{diam } S_t(B_{\rho'}) \geq \rho' > 0$ for all t .

Remarks.

(1) Assume $G(u, \bar{u}; t, x)$ does not contain monomials of the form $a \bar{u}$, $a|u|^2 \bar{u}$. Consider the IVP

$$\begin{cases} i u_t + u_{xx} + \frac{\partial}{\partial \bar{u}} G = 0 \\ u(0) = \phi \end{cases} \quad (98)$$

and assume $\phi = P_N \phi$, $\|\widehat{\phi}\|_\infty < N^{-c}$ for some $c > 0$. Then, for $N > N(T, \varepsilon)$, (98) is wellposed on $[0, T]$ and the solution $u(t)$ satisfies

$$\left(\sum \left| |\widehat{u(t)}(n)| - |\widehat{\phi}(n)| \right|^2 \right)^{1/2} < \varepsilon. \quad (99)$$

Let

$$\frac{\partial}{\partial \bar{u}} G = \Omega_{M,u}(t) u + F_{M,u} + \tilde{F}_{M,u} \quad (100)$$

be the decomposition considered earlier and compare u with the solution v of

$$\begin{cases} i v_t + v_{xx} + \Omega_{M,v}(t) v = 0 \\ v(0) = \phi. \end{cases} \quad (101)$$

Recall that since $\Omega_{M,v}(t)$ is real, $|\widehat{v}(n)| = |\widehat{\phi}(n)|$ for all n .

It follows from (98), (100), (101) that (locally)

$$\begin{aligned} |||u - v||| &\leq \left\| \int_0^t S(t-\tau) F_{M,v}(\tau) d\tau \right\| \\ &+ \left\| \int_0^t S(t-\tau) \tilde{F}_{M,v}(\tau) d\tau \right\| + c \sigma |||u - v||| \end{aligned} \quad (102)$$

and for the hypothesis on G and (67)

$$\begin{aligned} |||u - v||| &\leq |||P_M v||| + M^4 N^\varepsilon \sup_n |\widehat{v}(n)| + M^{-c_2} \\ &\leq |||P_M \phi|||_2 + M^4 N^{\varepsilon-c} + M^{-c_2} \\ &< M^{1/2} N^{-c} + M^4 N^{\varepsilon-c} + M^{-c} \\ &< N^{-c'} \end{aligned}$$

for appropriate choice of M . Hence also

$$\begin{aligned} \left(\sum \left(|\widehat{u(t)}(n)| - |\widehat{\phi}(n)| \right)^2 \right)^{1/2} &= \left(\sum \left(|\widehat{u(t)}(n)| - |\widehat{v(t)}(n)| \right)^2 \right)^{1/2} \\ &\leq \|u(t) - v(t)\|_2 < N^{-c'}. \end{aligned} \quad (103)$$

Since in particular $\|u(t)\|_2$ remains bounded, the statement follows.

(2) The limitation to $G(u, \bar{u}; t, x)$ of degree ≤ 4 in $u, \bar{u}$ is due to the fact that an L^2 -analysis is only developed for NLS with degree ≤ 3 nonlinearity (cf. [B1]). On the other hand, if L^2 is replaced by the space H^s , $s > \frac{1}{2}$, one has a local wellposedness theorem for any $F(u, \bar{u}; t, x)$ which is a polynomial in $u, \bar{u}$ say. One considers now the norm

$$\|u\|_s = \left[\sum_n (1 + |n|^2)^s \int d\lambda (1 + |\lambda - n^2|)^{2\rho} |\widehat{u}(n, \lambda)|^2 \right]^{1/2}. \quad (104)$$

Letting again

$$c(n, \lambda) = (1 + |n|)^s \cdot (1 + |\lambda - n^2|^\rho) |\widehat{u}(n, \lambda)| \quad (105)$$

the estimation of $\left\| \int_0^t S(t - \tau) F(\tau) d\tau \right\|$ reduces to bounding the expression

$$\sum_{\substack{n=n_0+\sum \epsilon_i n_i \\ \lambda=\lambda_0+\sum \epsilon_i \lambda_i}} \int \frac{|n|^s d(n, \lambda)}{|\lambda - n^2|^{1-\rho}} \prod_i \frac{c(n_i, \lambda_i)}{|n_i|^s |\lambda_i - n_i^2|^\rho} \quad (\|d\|_2 \leq 1). \quad (106)$$

Assume $|n_1| = \max |n_i|$, hence $|n_1| \geq c|n|$. Replacing (106) by

$$\sum_{\substack{n=n_0+\sum \epsilon_i n_i \\ \lambda=\lambda_0+\sum \epsilon_i \lambda_i}} d(n, \lambda) c(n_1, \lambda_1) \prod_{i \geq 2} \frac{c(n_i, \lambda_i)}{|n_i|^s |\lambda_i - n_i^2|^\rho} \quad (107)$$

one immediately gets an estimate by $\prod_i \|c_i\|_2$. In fact, there is clearly a saving if

$$\max_{i \geq 2} |n_i| \quad (108)$$

or

$$\max (|\lambda - n^2|, |\lambda_i - n_i^2|) \quad (109)$$

is large. Thus up to a small additional contribution B^{-c} , we may assume

$$|n| \approx |n_1|, \max_{i \geq 2} |n_i| < B \quad (110)$$

$$\left| \left(\varepsilon_1 n_1 + \sum_{i \geq 2} \varepsilon_i n_i + n_0 \right)^2 - \varepsilon_1 n_1^2 - \sum_{i \geq 2} \varepsilon_i n_i^2 \right| < B \quad (111)$$

and thus, from (110), (111)

$$|n_1| \leq B \quad (112)$$

or

$$\varepsilon_1 = 1, \sum_{i \geq 2} \varepsilon_i n_i + n_0 = 0. \quad (113)$$

This means that the contributing part of $F(u, \bar{u}; t, x) = \frac{\partial}{\partial \bar{u}} G(u, \bar{u}; t, x)$ in evaluating $\left\| \int_0^t S(t-\tau) F(\tau) d\tau \right\|$ comes from

$$\left[\int \frac{\partial^2}{\partial u \partial \bar{u}} G(P_B u, P_B \bar{u}; t, x) dx \right] u \quad (114)$$

$$F(P_B u, P_B \bar{u}; t, x). \quad (115)$$

Hence, there is a good approximation of u by the solution of

$$\begin{cases} i v_t + v_{xx} + \Omega_{M,v}(t) v + P_M F(P_M v, P_M \bar{v}; t, x) = 0 \\ v(0) = u(0) \end{cases} \quad (116)$$

on $[0, T]$, provided $\|u(t)\|_{H^s}$ remains controlled on $[0, T]$. Here $\Omega_{M,v}(t)$ is the real function appearing in (114) and M depends on $\|u(t)\|_s$, T and the approximation. (There is no smallness hypothesis on the Fourier coefficients of $u(0)$ here.)

From (116), the failure of uniform asymptotic stability in H^s , $s > \frac{1}{2}$ is obtained directly.

PROPOSITION 2 *Bounded solutions in H^s ($s > \frac{1}{2}$) of a Hamiltonian NLSE $i u_t + u_{xx} + \frac{\partial}{\partial \bar{u}} G(u, \bar{u}; t, x) = 0$, G a real polynomial in $u, \bar{u}$, are not uniformly asymptotically stable for $t \rightarrow \infty$.*

3. On the behaviour of higher Sobolev norms for large time in nonlinear Hamiltonian PDE's

The main purpose of this section is to exhibit smooth global solutions of smooth Hamiltonian PDE's with periodic boundary conditions (on $\mathbf{T}$) that develop large $\|u(t)\|_{H^{s_0}(\mathbf{T})}$ norm for $t \rightarrow \infty$. Here the exponent s_0 will be some fixed constant. Thus the Hamiltonian property does not imply necessarily bounds on higher order derivatives. The basic construction used is a general (small) perturbation argument of linear equations with resonant or almost resonant spectrum (under appropriate nonresonance conditions, the KAM results of S. Kuksin [K1] would make this impossible). Such argument was worked out in detail in [B5] for certain KdV-type equations

$$u_t + u_{xxx} + u_x f(u) = 0. \quad (1)$$

The examples considered here are nonlinear Schrödinger equations (NLSE), obtained as small Hamiltonian perturbation of a linear Schrödinger equation

$$i u_t = u_{xx} + V(x) u \quad (2)$$

with V a smooth real periodic potential. In fact the NLS case is easier and will be used as model to illustrate certain phenomena.

Consider a nonlinear equation

$$-i u_t = -u_{xx} + V(x) u + \varepsilon \Gamma(u). \quad (3)$$

Here V is a real smooth periodic potential. Denote (λ_n) the periodic spectrum of $-\frac{d^2}{dx^2} + V(x)$ and (φ_n) the corresponding normalized eigenfunctions basis (taking into account possible multiplicity) $\varepsilon > 0$ is a small parameter. We consider a (not necessarily local) nonlinear map $\Gamma : L^2(\mathbf{T}) \rightarrow L^2(\mathbf{T})$, such that Γ has bounded Lipschitz constant, thus

$$\|\Gamma(\phi)\|_2 \leq c (1 + \|\phi\|_2) \quad \text{and} \quad \|\Gamma(\phi) - \Gamma(\psi)\|_2 \leq c \|\phi - \psi\|_2. \quad (4)$$

More specifically Γ will be of the form

$$\Gamma(u) = \frac{\partial}{\partial \bar{u}} \left\{ \sum_j \int_{\mathbf{T}} F_j(P_j u, P_j \bar{u}) \right\} \quad (5)$$

where the P_j are projections on parts of the eigenfunction basis (φ_n) and the F_j real smooth functions.

Denoting $S(t) \phi = \sum \langle \phi, \varphi_n \rangle \varphi_n e^{i\lambda_n t}$ the solution operator to the linear problem

$$-i u_t = -u_{xx} + V(x) u \quad ; \quad u(0) = \phi. \quad (6)$$

We may rewrite (3) in integral equation form

$$u(t) = S(t) \phi + i \varepsilon \int_0^t S(t-\tau) \Gamma(u)(\tau) d\tau. \quad (7)$$

Since the group $S(t)$ are L^2 -isometries, it follows from (4), letting $\varepsilon|t| \ll 1$,

$$\|u(t) - S(t) \phi\|_2 \leq \varepsilon \int_0^t \|\Gamma(u(\tau))\|_2 d\tau \leq C \varepsilon t \left(1 + \max_{0 \leq \tau \leq t} \|u(\tau)\|_2\right) \quad (8)$$

$$\|u(t) - S(t) \phi\|_2 \leq C \varepsilon t (1 + \|\phi\|_2) \leq C \varepsilon t. \quad (9)$$

Writing

$$\begin{aligned} u(t) &= S(t) \phi + i \varepsilon \int_0^t S(t-\tau) \Gamma(S(\tau)\phi) d\tau \\ &+ i \varepsilon \int_0^t S(t-\tau) [\Gamma(u(\tau)) - \Gamma(S(\tau)\phi)] d\tau \end{aligned} \quad (10)$$

it follows thus from (4) and (9) that the last term in (10) is at most $C(\varepsilon t)^2$ in L^2 . Hence, in L^2

$$u(t) = S(t) \phi + i \varepsilon \int_0^t S(t-\tau) \Gamma(S(\tau)\phi) d\tau + 0(\varepsilon^2 t^2). \quad (11)$$

Projecting on a basis element φ_{n_0} gives

$$|\langle u(t) - \phi, \varphi_{n_0} \rangle| = \varepsilon \left| \int_{\mathbf{T}} \int_0^t \Gamma(S(\tau)\phi) \varphi_{n_0}(x) e^{-i\lambda_{n_0}\tau} dx d\tau \right| + 0(\varepsilon^2 t^2). \quad (12)$$

Assume $V = 0$. Then $\lambda_n = n^2$, $\varphi_n = \sqrt{2} \cos nx$, $\sqrt{2} \sin nx$ and $(S(t)\phi)(x)$ is periodic in x and t . Hence the integral term in (12) equals

$$t \int_{\mathbf{T}^2} \Gamma(S(\tau)\phi) e^{i(n_0 x - n_0^2 \tau)} dx d\tau + 0(1). \quad (13)$$

Assume

$$\frac{1}{t} \ll \int_{\mathbf{T}^2} \Gamma(S(\tau)\phi) e^{i(n_0 x - n_0^2 \tau)} dx d\tau \sim \varepsilon t. \quad (14)$$

From (12), (13) it follows now

$$\left| \widehat{u(t)}(n_0) - \widehat{\phi}(n_0) \right| \sim \left[\int_{\mathbf{T}^2} \Gamma(S(\tau)\phi) e^{i(n_0 x - n_0^2 \tau)} dx d\tau \right]^2. \quad (15)$$

Choose $\Gamma(u)$ of the form

$$\frac{\partial}{\partial \bar{u}} \left\{ \frac{1}{K^2} \int_{\mathbf{T}} \sin K F(Pu, P\bar{u}) \right\} = -\frac{1}{K} P \left\{ (\cos K F) \cdot \frac{\partial F}{\partial \bar{U}} (U, \bar{U}) \right\}, \quad (16)$$

$$U = Pu$$

where F is a real smooth function, P a projection on the eigenfunction basis and K a suitably chosen large number. Assume $\hat{P}(n_0) = 1$. Writing

$$\cos K F = \frac{1}{2} (e^{iKF} + e^{-iKF})$$

the integral in (15) will lead to oscillatory integrals

$$\int_{\mathbf{T}^2} e^{i(\pm KF(S(\tau)P\phi, \overline{S(\tau)P\phi}) + n_0 x - n_0^2 \tau)} \frac{\partial F}{\partial \bar{U}} (S(\tau)P\phi, \overline{S(\tau)P\phi}) dx d\tau. \quad (17)$$

The main idea now is to exploit critical points of the phase function. Assume

$$F(S(\tau)P\phi(x), \overline{S(\tau)P\phi(x)})$$

as a function of $(x, \tau) \in \mathbf{T}^2$ has a nondegenerate critical point. For K sufficiently large (mainly depending on n_0), there will be a solution $(\bar{x}, \bar{t})$ of the equation

$$\nabla F(S(\tau)P\phi, \overline{S(\tau)P\phi}) = \left(\mp \frac{n_0}{K}, \pm \frac{n_0^2}{K} \right) \quad (18)$$

and one expects a contribution $O(\frac{1}{K})$ of this stationary point $(\bar{x}, \bar{t})$ to the integral (17). If this contribution is not cancelled out by the contributions of other possible stationary points, the expression (15) will yield $O(K^{-4})$. This amounts to

$$\left| \widehat{u(t)}(n_0) - \widehat{\phi}(n_0) \right| \geq |n_0|^{-8} \quad \text{hence} \quad \left| \widehat{u(t)}(n_0) \right| \geq |n_0|^{-8} \quad (19)$$

assuming n_0 large.

In the preceding K is chosen depending on n_0 . One then chooses ε such that the perturbation $\varepsilon \Gamma$ in (3) has bounded (or small) derivatives up to any specified order (assuming F smooth). Finally, one picks a time t such that (14) is satisfied. Observe also that one expects the preceding to be true for data ϕ which are “generic”.

The conservation of the Hamiltonian

$$\int_{\mathbf{T}} |u_x|^2 + \frac{\varepsilon}{K^2} \int_{\mathbf{T}} \sin K F(Pu, \bar{P}u) \quad (20)$$

implies an a priori bound on $\|u(t)\|_{H^1}$. Analyzing the IVP, it follows that for data $\phi \in H^s(\mathbb{T})$, $s \geq 1$, there is global wellposedness and the solution u satisfies $u(t) \in H^s$ for all time, with bounds possibly depending on t .

Next, we make previous discussion more precise. Let f be a smooth real function and denote Φ its primitive, $\Phi'(0) = 0$. Since $\frac{\partial}{\partial u} \int_{\mathbb{T}} \Phi(\operatorname{Re} Pu) = Pf(\operatorname{Re} Pu)$, we make take $\Gamma(u) = Pf(\operatorname{Re} Pu)$ as Hamiltonian perturbation in (3), provided (4) holds. Let $f(t) = \frac{1}{K} \sin Kt$ and P a projection on an eigenspace containing $\{e^{\pm im_0 x}, e^{\pm in_0 x}\}$. Take for the initial data

$$\phi = (a + ib) \cos m_0 x \quad (21)$$

hence

$$S(\tau)P\phi = (a + ib) \cos m_0 x \cdot e^{im_0^2 \tau}$$

and

$$\operatorname{Re} S(\tau)P\phi = \cos m_0 x (a \cos m_0^2 \tau - b \sin m_0^2 \tau). \quad (22)$$

Substituting (22), the integral (17) becomes

$$\int_{\mathbb{T}^2} e^{i[\pm K \cos m_0 x (a \cos m_0^2 \tau - b \sin m_0^2 \tau) + n_0 x - n_0^2 \tau]} dx d\tau. \quad (23)$$

Assume $n_0 = n \cdot m_0$. By a change of variable $x' = m_0 x$, $\tau' = m_0^2 \tau$, (23) yields

$$\int_{\mathbb{T}^2} e^{i[\pm K \cos x (a \cos \tau - b \sin \tau) + nx - n^2 \tau]} dx d\tau. \quad (24)$$

Write $a \cos \tau - b \sin \tau = (a^2 + b^2)^{1/2} \cos(\tau + \varphi)$, $\varphi = \varphi(a, b)$. Another change of variable gives

$$e^{in^2 \varphi} \int_{\mathbb{T}^2} e^{i[\pm K \sqrt{a^2 + b^2} \cos x \cdot \cos \tau + nx - n^2 \tau]} dx d\tau. \quad (25)$$

The stationnary points are given by

$$\begin{cases} \sin x \cdot \cos \tau = \pm \frac{n}{K \sqrt{a^2 + b^2}} \\ \cos x \cdot \sin \tau = \mp \frac{n^2}{K \sqrt{a^2 + b^2}}. \end{cases} \quad (26)$$

Assuming K large with respect to n^2 (hence n_0^2), solutions of (26) will be close to $\frac{\pi}{2}$. Assuming n a multiple of 4, (26) will therefore essentially be

$$e^{in^2 \varphi} \int_{\mathbb{T}^2} e^{\pm i K \sqrt{a^2 + b^2} \cos x \cdot \cos \tau} dx d\tau \quad (27)$$

$$= e^{in^2 \varphi} \left\{ \int_{\mathbb{T}^2} e^{\pm i \frac{K}{2} \sqrt{a^2 + b^2} \cos \theta} d\theta \right\}^2 = 0(K^{-1}). \quad (28)$$

Thus the solution u of

$$\begin{cases} -i u_t = -u_{xx} + \varepsilon \Gamma(u) \\ u(0) = \phi = (a + ib) \cos m_0 x \end{cases} \quad (29)$$

letting

$$\Gamma(u) = \frac{1}{K} \sin K \operatorname{Re} u \text{ or } \Gamma(u) = \frac{1}{K} P[\sin K \operatorname{Re}(Pu)], \quad \widehat{P}(m_0) = 1 = \widehat{P}(n_0) \quad (30)$$

will satisfy for some time t

$$\left| \widehat{u(t)}(n_0) \right| > |n_0|^{-C} \quad (31)$$

where C is some numerical constant. The construction permits moreover to make the data ϕ and the perturbation $\varepsilon \Gamma(u)$ arbitrary small and smooth. This exhibits the behaviour of solutions of certain NLS mentioned in the beginning of this section. The Sobolev exponent s_0 depends on C in (31) and is thus fixed.

Take $\Gamma(u) = \frac{1}{K} P[\sin K \operatorname{Re}(Pu)]$ in (30).

Observe that then by construction $\operatorname{supp} \widehat{u(t)} \subset \operatorname{supp} p$ for the solution u of (29).

Then one may clearly piece together such examples considering a sequence of projections $\{q_j\}$ on disjoint subsets of the eigenfunction basis. By previous observation, the corresponding ivp's are decoupled and one gets a Hamiltonian perturbation $\Gamma(u)$ of the form (5) and a smooth data $\phi = \sum \delta_j \cos m_j x$ such that the solution u of

$$\begin{cases} -i u_t = -u_{xx} + \Gamma(u) \\ u(0) = \phi \end{cases} \quad (32)$$

satisfies

$$\overline{\lim}_{t \rightarrow \infty} \|u(t)\|_{H^{s_0}} = \infty. \quad (33)$$

This solution u is obtained as $u = \sum u^j$ where $u^j = P_j u^j$ solves

$$\begin{cases} -i u_t^j = -u_{xx}^j + \varepsilon_j \Gamma_j(u^j) \\ u^j(0) = \phi_j = \delta_j \cos m_j x \end{cases} \quad (34)$$

as in (29)-(31), thus

$$\left| \widehat{u^j(t_j)}(n_j) \right| > |n_j|^{-C}. \quad (35)$$

The nonlinear term $\Gamma(u) = \sum \varepsilon_j \Gamma_j(u)$ is smooth.

COROLLARY 1 *There is a Hamiltonian NLS $-i u_t = -u_{xx} + \Gamma(u)$ with $\Gamma(u)$ smooth of the form (5) such that for some smooth data ϕ , the (global and smooth) solution u of the IVP (32) satisfies (33), for some Sobolev exponent s_0 .*

In the next construction, we will exhibit a local nonlinearity $\Gamma(u)$, such that the flow map of $-i u_t = -u_{xx} + \Gamma(u)$, denoted $U(t)$, has following property

$$\overline{\lim}_{\delta \rightarrow 0} \sup_{t > 0} \|U(t)(\delta \cdot \cos x)\|_{H^s} = \infty. \quad (36)$$

Consider first (29)

$$\begin{cases} -i v_t^j = -v_{xx}^j + \varepsilon_j \Gamma_j(v^j) \\ v^j(0) = \delta_j \cos x \end{cases} \quad (37)$$

where $\Gamma_j(v)$ is now defined by

$$\Gamma_j(v) = \frac{1}{K_j^2} (\cos K_j \operatorname{Re} v) \chi_j(\operatorname{Re} v) \quad (38)$$

where $0 \leq \chi_j \leq 1$ is 0 on a neighborhood of 0 of size $\frac{1}{K_j^2}$, is 1 outside a neighborhood of 0 of size $\sim \frac{1}{K_j^2}$ and $|\chi_j'| < c K_j^2$. Thus Γ_j fulfils (4). For the integral (17) one gets instead of (23)

$$\int_{\mathbb{T}^2} e^{i[\pm K_j \delta_j \cos x \cos \tau + n_j x - n_j^2 \tau]} \chi_j(\delta_j \cos x \cos \tau) dx d\tau. \quad (39)$$

The evaluation $0(K_j^{-1})$ in (28) will be affected by an error bounded by

$$\operatorname{mes} \left[(x, \tau) \in \mathbb{T}^2 \mid |\cos x \cdot \cos \tau| < \frac{1}{K_j^2 \delta_j} \right] \ll \frac{1}{K_j} \quad (40)$$

for K_j choosen large enough with respect to δ_j . Hence we keep the lower estimate $0(K_j^{-1})$ for (39). The $\frac{1}{K_j^2}$ -factor in (38) yields now

$$\left| \widehat{v^j(t_j)}(n_j) \right| \geq K_j^{-3}$$

hence still

$$\left| \widehat{v^j(t_j)}(n_j) \right| > |n_j|^{-C} \quad (41)$$

for some fixed constant C . Define

$$\Gamma(u) = \sum \varepsilon_j \frac{1}{K_j^2} (\cos K_j \operatorname{Re} v) \chi_j(\operatorname{Re} v). \quad (42)$$

Fix j and analyze the effect of a replacement of Γ_j in (37) by the sum (42) on the solution v^j for $0 \leq t \leq t_j$. Observe that from the conservation of the Hamiltonian

$$\int_{\mathbf{T}} |v_x^j|^2 + \frac{\varepsilon_j}{K_j^2} \int_{\mathbf{T}} D^{-1}[\cos K_j t \cdot \chi_j(t)] (\operatorname{Re} v^j) \quad (43)$$

there is the a priori bound

$$\|v - \widehat{v}(0)\|_{\infty} \leq \|v - \widehat{v}(0)\|_{H^1} < \delta_j + \varepsilon_j^{1/2}. \quad (44)$$

Also, since $\Gamma_j(v)$ is real in (37), $\frac{d}{dt} (\int_{\mathbf{T}} \operatorname{Re} v) = 0$, hence $\operatorname{Re} \widehat{v(t)}(0) = 0 = \widehat{v(0)}(0)$. Consequently $\|\operatorname{Re} v^j\|_{\infty} < \delta_j + \varepsilon_j^{1/2} \ll \frac{1}{K_j^2}$ for $j' < j$ (by construction).

Hence, for $j' < j$, $\chi_{j'}(\operatorname{Re} v^j) = 0$ and the addition of the $j' < j$ terms in (42) does not affect the solution of (37). Now, for $j' > j$, the $\varepsilon_{j'}$ may clearly be taken small enough so that on $[0, t_j]$ the solution v^j is only slightly perturbed. The conclusion of previous considerations is that the inductive construction may be done such that

$$\begin{cases} -i v_t^j = -v_{xx}^j + \varepsilon_j \Gamma_j(v^j) \\ v^j(0) = \delta_j \cos x \end{cases} \quad \text{and} \quad \begin{cases} -i u_t^j = -u_{xx}^j + \Gamma(u^j) \\ u^j(0) = \delta_j \cos x \end{cases}$$

have approximately the same solutions on $[0, t_j]$ and in particular, by (41)

$$\left| (U(t_j)(\delta_j \cos x))^\wedge(n_j) \right| = \left| \widehat{u^j(t_j)}(n_j) \right| > |n_j|^{-C}. \quad (45)$$

Thus (36) follows. As corollary, one has

COROLLARY 2 *There is a Hamiltonian NLS $-i u_t = -u_{xx} + \Gamma(u)$ with $\Gamma(u)$ smooth and local, such that for all exponents $s > 0$ and all $\delta > 0$, the set*

$$\{U(t) \phi \mid \|\phi\|_{H^s} < \delta, t > 0\}$$

is unbounded in H^{s_0} , for some fixed exponent s_0 .

The next construction is more KAM related and shows in particular the failure of Liapounov's theorem on the persistency of 1-dimensional tori in ε -perturbed infinite dimensional Hamiltonian systems corresponding to an equation of the form (3), when the spectrum $\{\lambda_n\}$ of $-\frac{d^2}{dx^2} + V(x)$ satisfies an almost resonance assumption

$$|\lambda_{n_j} - m_j \lambda_{n_0}| < n_j^{-c_1} \quad (46)$$

and also some diophantine property

$$|\xi_1 \lambda_{n_0} + \xi_2 \lambda_{n_j}| \geq \kappa_j (|\xi_1| + |\xi_2|)^{-c_2} \quad \text{for all } \xi_1, \xi_2 \in \mathbb{Z} \quad (47)$$

for some fixed n_0 and infinite sequence $\{n_j\}$, where c_1, c_2 are constants and $\kappa_j > 0$.

Remarks.

(1) KAM theory on the perturbation of invariant tori may be developed in infinite dimensional systems (for finite dimensional tori). A general Melnikov-type theory with application to quasiperiodic solutions of PDE appears in [K1]. For an infinite dimensional version of Liapounov's theorem on the persistency of periodic solutions, see [C-W].

(2) There is an asymptotic formula (cf. [P-T])

$$\lambda_n = \pi^2 n^2 + \sum_{j=0}^{j_1} c_j(V) n^{-j} + o(n^{-j_1-1}) \quad (48)$$

for the periodic spectrum of a sufficiently smooth periodic potential V (depending on j_1). Thus an assumption such as (46), (47) is realistic, assuming say $\lambda_{n_0} = \pi^2$ and $c_j(V) = 0$ for j up to c_1 . This fact was pointed out by S. Kuksin to the author.

We will prove following fact.

PROPOSITION 3 *Assuming (46), (47), one may construct a smooth Hamiltonian perturbation $\Gamma(u)$ of the form (5), the smoothness depending on the exponent c_1 in (46), and a fixed Sobolev exponent s_0 such that*

$$\inf_{q \in \gamma} \sup_t \|u_{\epsilon, q}(t)\|_{H^{s_0}} \rightarrow \infty \quad \text{for } \epsilon \rightarrow 0. \quad (49)$$

Here $u_{\epsilon, q}$ denotes the solution of the IVP

$$\begin{cases} -i u_t = -u_{xx} + V(x)u + \epsilon \Gamma(u) \\ u(0) = q \end{cases} \quad (50)$$

and γ denotes a bounded set in H^{s_0} such that $|(q, \varphi_{n_0})| > c$ for $q \in \gamma$.

In particular, there is no invariant torus in an H^{s_0} -neighborhood of the periodic solution $u_{0, \varphi_{n_0}} = a \varphi_{n_0}(x) e^{i \lambda_{n_0} t}$ of the linear equation, no matter how small ϵ .

The equation in (50) appears as

$$-i u_t = \frac{\partial}{\partial u} H_\epsilon(u)$$

where $H_\varepsilon(u) = \int_{\mathbf{T}} |u_x|^2 + \int_{\mathbf{T}} V(x)|u|^2 + \Sigma \delta_j \int_{\mathbf{T}} D^{-1} F_j(P_j \operatorname{Re} u)$, thus

$$\Gamma(u) = \sum \delta_j \Gamma_j(u) \quad \text{with} \quad \Gamma_j(u) = P_j F_j(\operatorname{Re} P_j u) \quad (51)$$

and we will again take F_j of the form $F_j(t) = \frac{1}{K_j} \cos K_j t$.

In particular, Γ will fulfill the Lipschitz estimate (4).

Fix j_0 and consider $\varepsilon > 0$ and times t_1, t all depending on j_0 (to be specified). We may from (11) write

$$u(t_1) = S(t_1)\phi + i \varepsilon \int_0^{t_1} S(t_1 - \tau) \Gamma(S(\tau)\phi) d\tau + o((\varepsilon t_1)^2) \quad (52)$$

for the solution u of the IVP

$$\begin{cases} -i u_t = -u_{xx} + V(x)u + \varepsilon \Gamma(u) \\ u(0) = \phi. \end{cases} \quad (53)$$

Write

$$\begin{aligned} \int_0^{t_1} S(t_1 - \tau) \Gamma(S(\tau)\phi) d\tau &= \sum_{j < j_0} \delta_j \int_0^{t_1} S(t_1 - \tau) \Gamma_j(S(\tau)\phi) d\tau \\ &+ \sum_{j \geq j_0} \delta_j \int_0^{t_1} S(t_1 - \tau) \Gamma_j(S(\tau)\phi) d\tau. \end{aligned} \quad (54)$$

Let P_j be the projection on the eigenfunctions $\varphi_{n_0}, \varphi_{n_j}$. From the form (51) of Γ_j , it follows that (assuming ϕ real to simplify the formulas)

$$\begin{aligned} \int_0^{t_1} S(t_1 - \tau) \Gamma_j(S(\tau)\phi) d\tau &= \\ \frac{1}{K_j} \varphi_{n_0}(x) e^{i\lambda_{n_0} t_1} &\left\{ \int_{\mathbf{T}} \int_0^{t_1} \cos K_j [\langle \phi, \varphi_{n_0} \rangle \varphi_{n_0}(x) \cos \lambda_{n_0} \tau \right. \\ &+ \langle \phi, \varphi_{n_j} \rangle \varphi_{n_j}(x) \cos \lambda_{n_j} \tau] \cdot \varphi_{n_0}(x) e^{-i\lambda_{n_0} \tau} d\tau dx \Big\} \end{aligned} \quad (55)$$

$$\begin{aligned} &+ \frac{1}{K_j} \varphi_{n_j}(x) e^{i\lambda_{n_j} t_1} \left\{ \int_{\mathbf{T}} \int_0^{t_1} \cos K_j [\langle \phi, \varphi_{n_0} \rangle \varphi_{n_0}(x) \cos \lambda_{n_0} \tau \right. \\ &+ \langle \phi, \varphi_{n_j} \rangle \varphi_{n_j}(x) \cos \lambda_{n_j} \tau] \cdot \varphi_{n_j}(x) e^{-i\lambda_{n_j} \tau} d\tau dx \Big\}. \end{aligned} \quad (56)$$

We will show first that for $j < j_0$ these terms are small (compared with t_1), because of the assumption (47). We will use that for $t_1 = t_1(j_0)$ sufficiently

large, $\lambda_{n_0}\tau$ and $\lambda_{n_j}\tau$ behave almost like independent variables on $[0, t_1]$. We use following simple lemma proved by Fourier expansion.

LEMMA Consider a smooth function $f = f(\theta, \dots, \theta_k)$ on $\mathbb{T}^k$ and let $\lambda_1, \dots, \lambda_k \in \mathbb{R}$ be rationally independent numbers such that

$$|\sum \xi_j \lambda_j| > \kappa (\sum |\xi_j|)^{-M} \quad (57)$$

for all $\xi_j \in \mathbb{Z}$. Here M is some constant and $\kappa > 0$ some small number. Then

$$\left| \frac{1}{T} \int_0^T f(\lambda_1 \tau, \dots, \lambda_k \tau) d\tau - \int_{\mathbb{T}^k} f(\theta_1, \dots, \theta_k) d\theta \right| < \|f\|_{C^{M+k+1}} (T\kappa)^{-1}. \quad (58)$$

In the application to (55) say, we let $k = 2$, $\lambda_1 = \lambda_{n_0}$, $\lambda_2 = \lambda_{n_j}$. Hence $\kappa = \kappa_j$, $M = C_2$ by (47). The function $f = f(\theta_1, \theta_2)$ is given by

$$f(\theta_1, \theta_2) = \cos K_j [\langle \phi, \varphi_{n_0} \rangle \varphi_{n_0}(x) \cos \theta_1 + \langle \phi, \varphi_{n_j} \rangle \varphi_{n_j}(x) \cos \theta_2] e^{-i\theta_1} \quad (59)$$

(where x is fixed). Application of (58) for the τ -integration gives

$$\begin{aligned} & \int_0^{t_1} \cos K_j [\langle \phi, \varphi_{n_0} \rangle \varphi_{n_0}(x) \cos \lambda_{n_0} \tau + \langle \phi, \varphi_{n_j} \rangle \varphi_{n_j}(x) \cos \lambda_{n_j} \tau] e^{-i\lambda_{n_0} \tau} d\tau \\ &= t_1 \int_{\mathbb{T}^2} f(\theta_1, \theta_2) d\theta_1 d\theta_2 + O(K_j^{C_2+3} \kappa_j^{-1}) \\ &= O(K_j^{C_2+3} \kappa_j^{-1}) \end{aligned} \quad (60)$$

since the $\mathbb{T}^2$ -integral vanishes, as is clear from (59), making a variable change $(\theta_1, \theta_2) \mapsto (\theta_1 + \pi, \theta_2 + \pi)$. In case of a complex data ϕ , the conclusion would be the same (there would be additional terms $\sin \theta_1, \sin \theta_2$). Hence (55) is bounded by $K_j^{C_2+2} \kappa_j^{-1}$ and so is (56). Hence the summation $\sum_{j < j_0}$ in (54) is at most $\sum_{j < j_0} \delta_j K_j^{C_2+3} \kappa_j^{-1} < K_{j_0-1}^{C_2+2} \kappa_{j_0-1}^{-1}$ (assuming the κ_j decreasing). Thus

$$\begin{aligned} & \int_0^{t_1} S(t_1 - \tau) \Gamma(S(\tau)\phi) d\tau = \\ & \sum_{j \geq j_0} \delta_j \int_0^{t_1} S(t_1 - \tau) \Gamma_j(S(\tau)\phi) d\tau + O(K_{j_0-1}^{C_2+2} \kappa_{j_0-1}^{-1}) \end{aligned} \quad (61)$$

and the main idea is to have the error term small compared with t_1 . Denote

$$\Gamma^{j_0} = \sum_{j \geq j_0} \delta_j \Gamma_j. \quad (62)$$

We will compare the solution u of (53) with the solution v of the IVP

$$\begin{cases} -i v_t = -v_{xx} + V(x)v + \varepsilon \Gamma^{j_0}(v) \\ v(0) = \psi \end{cases} \quad (63)$$

on $[0, t_1]$.

From (52), (61)

$$u(t_1) =$$

$$S(t_1)\phi + i \varepsilon \int_0^{t_1} S(t_1 - \tau) \Gamma^{j_0}(S(\tau)\phi) d\tau + 0 \left(\varepsilon K_{j_0-1}^{C_2+2} \kappa_{j_0-1}^{-1} + (\varepsilon t_1)^2 \right) \quad (64)$$

and from (63)

$$v(t_1) = S(t_1)\psi + i \varepsilon \int_0^{t_1} S(t_1 - \tau) \Gamma^{j_0}(S(\tau)\psi) d\tau + 0((\varepsilon t_1)^2). \quad (65)$$

Hence, by (64), (65), (62) and the Lipschitz bounds on the Γ_j

$$\|\Gamma^{j_0}(\phi_1) - \Gamma^{j_0}(\phi_2)\|_2 \leq \delta_{j_0} \|\phi_1 - \phi_2\|_2 \quad (66)$$

$$\|u(t_1) - v(t_1)\|_2 \leq (1 + C \varepsilon t_1 \delta_{j_0}) \|\phi - \psi\|_2 + 0 \left(\varepsilon K_{j_0-1}^{C_2+2} \kappa_{j_0-1}^{-1} + (\varepsilon t_1)^2 \right). \quad (67)$$

Consider the interval $[0, t]$ now and subdivide in intervals of length t_1 . Iterating the estimate (67) clearly yields following comparison, if the data for $t = 0$ are both ϕ

$$\|u(t) - v(t)\|_2 \leq C \frac{t}{t_1} (1 + C \varepsilon \delta_{j_0} t_1)^{\frac{t}{t_1}} \left(\varepsilon K_{j_0-1}^{C_2+2} \kappa_{j_0-1}^{-1} + (\varepsilon t_1)^2 \right). \quad (68)$$

Assume $t = t(j_0)$ satisfies

$$\varepsilon \delta_{j_0} t < 1 \quad (69)$$

one gets from (68)

$$\|u(t) - v(t)\|_2 < \left\{ \frac{K_{j_0-1}^{C_2+2} \kappa_{j_0-1}^{-1}}{\delta_{j_0} t_1} + \frac{\varepsilon t_1}{\delta_{j_0}} \right\} \varepsilon \delta_{j_0} t \quad (70)$$

and we denote γ the expression $\{ \}$ in (70), which will be taken small. From (66), (11), the solution v of

$$\begin{cases} -i v_t = -v_{xx} + V(x)v + \varepsilon \Gamma^{j_0}(v) \\ v(0) = \phi \end{cases} \quad (71)$$

satisfies

$$v(t) = S(t)\phi + i \varepsilon \int_0^t S(t-\tau) \Gamma^{j_0}(S(\tau)\phi) d\tau + o((\delta_{j_0} \varepsilon t)^2). \quad (72)$$

It follows from (70), (72)

$$\begin{aligned} \langle u(t), \varphi_{n_{j_0}} \rangle - \langle S(t)\phi, \varphi_{n_0} \rangle = \\ i \varepsilon \delta_{j_0} \langle \int_0^t S(t-\tau) \Gamma^{j_0}(S(\tau)\phi) d\tau, \varphi_{n_{j_0}} \rangle + o((\gamma + \delta_{j_0} \varepsilon t) \delta_{j_0} \varepsilon t) \end{aligned} \quad (73)$$

where the integral is the expression (56), thus

$$\begin{aligned} \frac{1}{K_{j_0}} e^{i\lambda_{n_{j_0}} t} \int_{\mathbf{T}} \int_0^t \cos K_{j_0} [\langle \phi, \varphi_{n_0} \rangle \varphi_{n_0}(x) \cos \lambda_{n_0} \tau \\ + \langle \phi, \varphi_{n_{j_0}} \rangle \varphi_{n_{j_0}}(x) \cos \lambda_{n_{j_0}} \tau] \cdot \varphi_{n_{j_0}}(x) e^{-i\lambda_{n_{j_0}} \tau} d\tau dx. \end{aligned} \quad (74)$$

Recall that by (46)

$$|\lambda_{n_{j_0}} - m_{j_0} \lambda_{n_0}| < n_{j_0}^{-C_1} \quad (75)$$

which permits in (74) to replace $\lambda_{n_{j_0}}$ by $m_{j_0} \lambda_{n_0}$ provided say

$$n_{j_0}^{-C_1} t < K_{j_0}^{-10} \quad (76)$$

the error being $t K_{j_0}^{-10}$.

After this replacement, the $\frac{2\pi}{\lambda_{n_0}}$ -periodicity in τ yields the expression

$$\begin{aligned} \frac{1}{K_{j_0}} e^{i\lambda_{n_{j_0}} t} (\varepsilon \delta_{j_0} t) \int_{\mathbf{T}^2} \cos K_{j_0} [\langle \phi, \varphi_{n_0} \rangle \varphi_{n_0}(x) \cos \tau \\ + \langle \phi, \varphi_{n_{j_0}} \rangle \varphi_{n_{j_0}}(x) \cos m_{j_0} \tau] \cdot \varphi_{n_{j_0}}(x) e^{-im_{j_0} \tau} d\tau dx \end{aligned} \quad (77)$$

for the second term in (73), up to an error term $(\varepsilon \delta_{j_0} t) \cdot K_{j_0}^{-10}$.

Since ϕ is the function q in Proposition 3 controlled in H^{s_0} -norm, one has

$$|\langle \phi, \varphi_{n_{j_0}} \rangle| \leq n_{j_0}^{-s_0} \quad (78)$$

and the second term between $[\dots]$ in (77) may be deleted, up to an error term $(\varepsilon \delta_{j_0} t) n_{j_0}^{-s_0}$. Thus the second term in (73) becomes

$$\begin{aligned} & \frac{1}{K_{j_0}} e^{i\lambda_{n_{j_0}} t} (\varepsilon \delta_{j_0} t) \int_{\mathbf{T}_2} \cos(K_{j_0} \langle \phi, \varphi_{n_0} \rangle \varphi_{n_0}(x) \cos \tau) \cdot \\ & \varphi_{n_{j_0}}(x) e^{-im_{j_0} \tau} d\tau dx \\ & + \varepsilon \delta_{j_0} t O(K_{j_0}^{-10} + n_{j_0}^{-s_0}). \end{aligned} \quad (79)$$

Assuming K_{j_0} sufficiently large with respect to n_{j_0} , say

$$K_{j_0} > n_{j_0}^c \quad (80)$$

the integral in (79) will essentially be

$$\sum_{(x_c, \tau_c)} \varphi_{n_{j_0}}(x_c) e^{-im_{j_0} \tau_c} \int_{(x, \tau) \approx (x_c, \tau_c)} \cos(K_{j_0} \langle \phi, \varphi_{n_0} \rangle \varphi_{n_0}(x) \cos \tau) dx d\tau \quad (81)$$

where the summation extends to the critical points

$$\nabla [\varphi_{n_0}(x) \cos \tau] (x_c, \tau_c) = 0 \quad (82)$$

and hence one expects a typical size $O(K_{j_0}^{-1})$ again, since $|\langle \phi, \varphi_{n_0} \rangle| > c$. This gives for the first term in (79) size $\frac{1}{K_{j_0}^2} (\varepsilon \delta_{j_0} t)$ and hence the right side of (73) has size

$$(\varepsilon \delta_{j_0} t) \left(\frac{1}{K_{j_0}^2} + O(K_{j_0}^{-10} + n_{j_0}^{-s_0} + \gamma + \varepsilon \delta_{j_0} t) \right). \quad (83)$$

Choosing

$$\varepsilon \delta_{j_0} t \sim \frac{1}{K_{j_0}^2} \quad (84)$$

one gets a lower estimate $K_{j_0}^{-4} = n_{j_0}^{-C}$, provided

$$n_{j_0}^{-s_0} < K_{j_0}^{-10}, \quad \gamma < K_{j_0}^{-10}. \quad (85)$$

By (73), this leads to an estimate

$$|\langle u(t), \varphi_{n_{j_0}} \rangle| > n_{j_0}^{-C} \quad (86)$$

where $u(t) = u_{\varepsilon, \phi}(t)$, $\varepsilon = \varepsilon_{j_0}$, $t = t_{j_0}$. This last fact permits to obtain large H^{s_0} -norm for $u(t)$, leading to (49).

It remains to keep track of the conditions on the various parameters.

From (70), (76), (80), (84), (85), we get

$$\delta_{j_0} = K_{j_0}^{-C_3} \quad (87)$$

(c_3 determines the smoothness of Γ)

$$\gamma = \frac{K_{j_0-1}^{C_2+2}}{\kappa_{j_0-1} \delta_{j_0} t_1} + \frac{\varepsilon t_1}{\delta_{j_0}} \quad (88)$$

$$n_{j_0}^{-C_1} t < K_{j_0}^{-10} \quad (89)$$

$$K_{j_0} > n_{j_0}^C \quad (90)$$

$$\varepsilon \delta_{j_0} t \sim \frac{1}{K_{j_0}^2} \quad (91)$$

$$n_{j_0}^{-s_0} < K_{j_0}^{-10} \quad (92)$$

$$\gamma < K_{j_0}^{-10}. \quad (93)$$

Assume the construction performed up to stage $j_0 - 1$. Choosing n_{j_0} large, depending on K_{j_0-1} , κ_{j_0-1} , let $K_{j_0} = n_{j_0}^C$, $\delta_{j_0} = n_{j_0}^{-CC_3}$, $s_0 > 10C$. Condition (88), (93) leads to a choice of $t_1 > n_{j_0}^{CC_3+11C}$, $\varepsilon < n_{j_0}^{-2CC_3-21C}$ and the condition on t given by (89) is compatible with (91) for C_1 large enough (depending on C_3).

This concludes the proof of Proposition 1.

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