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Stochastic Techniques in Condensed Matter Physics¹

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1. Introduction

The purpose of these lectures is to describe a few applications of stochastic techniques to selected problems in condensed matter physics. These applications are described in the sections 3,4 and 6. However, to make the lectures reasonably self-contained, I have included in the sections 2 and 5 two short reviews of the sign problem in Quantum Monte Carlo and of the deviations from scaling in fully developed turbulence. These reviews should not in any way be considered as exhaustive or even well-balanced and fair to the many authors that have contributed to these fields. They

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are simply a kind of minimum knowledge required to understand the applications that are developed in the other sections.

Finally sections 7 and 8 are of a slightly different nature in the sense that, rather than applying standard stochastic techniques to condensed matter problems, I have attempted to show how the notions of quantum group and q-deformation, that first appeared in the context of exactly integrable systems in condensed matter physics, may now be carried over to the theory of stochastic processes. Section 7 is a naive introduction to the notions of q-deformations and q-algebras which, for the sake of brevity, is not based on the symmetries of exactly integrable models but, instead, on the calculus of finite differences.

2. Quantum Monte-Carlo and the sign problem

Numerical simulation is an important technique to explore many of the systems of physical importance in Condensed Matter, which cannot either be solved exactly or dealt with by approximation techniques. Among the numerical simulation methods, Quantum Monte-Carlo plays a predominant role. It is used both to compute expectation values of operators and to estimate the ground state wave function. For a recent extensive review see Ref.[1].

Expectation values. The worldline method

The aim is to compute

$$\langle A \rangle = \frac{1}{Z} \text{Tr} A e^{-\beta H} \quad (2.1)$$

where

$$Z = \text{Tr} e^{-\beta H} = \sum_{i_1} \langle i_1 | e^{-\beta H} | i_1 \rangle \quad \text{and} \quad \beta = \frac{1}{KT} \quad (2.2)$$

To compute the partition function Z we divide the interval $(0, \beta)$ into L subintervals of length $\Delta\tau = \frac{\beta}{L}$

$$Z = \sum_{i_1, \dots, i_L} \langle i_1 | e^{-\Delta\tau H} | i_L \rangle \langle i_L | e^{-\Delta\tau H} | i_{L-1} \rangle \dots \langle i_2 | e^{-\Delta\tau H} | i_1 \rangle \quad (2.3)$$

Now split the Hamiltonian into diagonalizable pieces

$$H = H_1 + H_2 \quad (2.4)$$

and use Trotter's formula^[2]

$$e^{-\Delta\tau H} = e^{-\Delta\tau H_2} e^{-\Delta\tau H_1} \{1 + O(\Delta\tau^2)\} \quad (2.5)$$

to obtain

$$Z \simeq \sum_{i_1, \dots, i_{2L}} \langle i_1 | U_1 | i_{2L} \rangle \langle i_{2L} | U_2 | i_{2L-1} \rangle \dots \langle i_2 | U_2 | i_1 \rangle \quad (2.6)$$

where $U_i = e^{-\Delta\tau H_i}$. The sum (2.6) is now computed by importance sampling, that is, one generates a sequence of distributions $\{i_1(k) \dots i_{2L}(k)\}$, $k = 1, 2, 3, \dots$, with probabilities

$$p(k) = \frac{\langle i_1(k) | U_1 | i_{2L}(k) \rangle \dots \langle i_2(k) | U_2 | i_1(k) \rangle}{\sum_{i_1, \dots, i_{2L}} \langle i_1 | U_1 | i_{2L} \rangle \dots \langle i_2 | U_2 | i_1 \rangle} \quad (2.7)$$

The distributions are, for example, selected by the heat bath algorithm. After an allowed move in the worldlines one computes the ratio R of the numerators in Eq.(2.7). The new configuration is accepted with probability $P = R/(1 + R)$.

We have described the computation of the partition function. To compute the expectation values of the operators we have to consider three different cases:

- (i) Operators that are diagonal in the occupation number representation: We simply average over the intermediate states after statistical equilibrium is attained.
- (ii) Non-diagonal operators A conserving particle number within a two-site block: Decompose $A = A_1 + A_2$ and compute $\langle i_k | A_i U_i | i_{k+1} \rangle$.
- (iii) Operators that do not conserve the particle number: The calculation now involves disconnected world lines and is more delicate^[3].

Consider as an example a one-dimensional chain of spin-polarised fermions with nearest-neighbour interactions

$$H = -t \sum_i \left(c_i^\dagger c_{i+1} + c_{i+1}^\dagger c_i \right) + V \sum_i \left(n_i - \frac{1}{2} \right) \left(n_{i+1} - \frac{1}{2} \right) = \sum_{i=1}^N H_{i,i+1} \quad (2.8)$$

with $n_i = c_i^\dagger c_i$. Choose

$$H_1 = \sum_{i \text{ odd}} H_{i,i+1} \quad H_2 = \sum_{i \text{ even}} H_{i,i+1} \quad (2.9)$$

The terms in H_1 and H_2 commute. Then

$$U_1 = e^{-\Delta\tau H_1} = \prod_{i \text{ odd}} e^{-\Delta\tau H_{i,i+1}} \quad (2.10)$$

and the computation of the matrix elements is a simple two-site problem. Using a basis $|n_i n_{i+1}\rangle$, where $n_i=0,1$, one obtains

$$\begin{aligned} e^{-\Delta\tau H_{i,i+1}} |00\rangle &= |00\rangle e^{-\Delta\tau \frac{V}{4}} \\ e^{-\Delta\tau H_{i,i+1}} |11\rangle &= |11\rangle e^{-\Delta\tau \frac{V}{4}} \\ e^{-\Delta\tau H_{i,i+1}} |10\rangle &= \left\{ \cosh(t\Delta\tau) |10\rangle + \sinh(t\Delta\tau) |01\rangle \right\} e^{\Delta\tau \frac{V}{4}} \\ e^{-\Delta\tau H_{i,i+1}} |01\rangle &= \left\{ \cosh(t\Delta\tau) |01\rangle + \sinh(t\Delta\tau) |10\rangle \right\} e^{\Delta\tau \frac{V}{4}} \end{aligned}$$

It follows that, with this choice for H_1 and H_2 , all terms in Eq.(2.6) are positive. The interpretation of the quantities $p(k)$ as probabilities is consistent. Unfortunately for higher dimensional cases the situation is not so favourable.

Projector Monte-Carlo

For the ground state ($T \rightarrow 0$, $\beta \rightarrow \infty$) the computation of the partition function is unreliable because $\Delta = \frac{\beta}{L}$ is large. Instead one may use the fact that the quantity

$$e^{-\tau H} |\psi\rangle = e^{-\tau E_0} \left\{ |0\rangle \langle 0|\psi\rangle + \sum_{n>0} e^{-\tau(E_n - E_0)} \right\} \quad (2.11)$$

tends to a constant times the ground state when $\tau \rightarrow \infty$, provided $\langle 0|\psi\rangle \neq 0$. Therefore using Trotter's formula to compute $e^{-\tau H} |\psi\rangle$, we obtain a numerical computation of the ground state. As initial state $|\psi\rangle$ it is convenient to start already from a reasonable approximation of the ground state, for example a variational wave function.

Hubbard-Stratonovich transformation

For the practical applications of the Quantum Monte-Carlo method, the Hubbard-Stratonovich transformation is an useful tool. It replaces a two-body propagator by a coherent superposition of one-body terms containing auxiliary classical fields. For the Hubbard model, for example

$$H = K + V \quad (2.12a)$$

$$K = -t \sum_{\langle ij \rangle, \sigma} \left(c_{i\sigma}^\dagger c_{j\sigma} + c_{j\sigma}^\dagger c_{i\sigma} \right) \quad (2.12b)$$

$$V = U \sum_i \left(n_{i\uparrow} - \frac{1}{2} \right) \left(n_{i\downarrow} - \frac{1}{2} \right) - \sum_i (n_{i\uparrow} + n_{i\downarrow}) \quad (2.12c)$$

the transformation is

$$e^{-\Delta\tau U n_{i\uparrow} n_{i\downarrow}} = \text{Tr}_\sigma e^{\left\{ \lambda \sigma (n_{i\uparrow} - n_{i\downarrow}) - \Delta\tau \frac{U}{2} (n_{i\uparrow} + n_{i\downarrow}) \right\}} \quad (2.13)$$

with $\lambda = 2 \arctan \sqrt{\tanh(\Delta\tau U/4)}$. Therefore a quartic form in the electron operators is traded for a quadratic form plus a summation over the auxiliary spin variables σ that take values ± 1 .

The (minus) sign problem

If not all the terms in Eq.(2.6) are positive, their interpretation as probabilities for the importance sampling calculation is not straightforward. We may still write

$$\langle A \rangle = \frac{1}{Z} \text{Tr}(AP) = \frac{\text{Tr}(A |P| \text{sign}P)}{\text{Tr}(|P| \text{sign}P)} = \frac{\langle A \text{sign}P \rangle |P|}{\langle A \text{sign}P \rangle |P|} \quad (2.14)$$

and use $|P|$ as a probability for the Monte-Carlo simulation. However at low temperature the weight of the positive and negative terms is such that both the numerator and the denominator are small quantities. Therefore the statistical errors make the evaluation of $\langle A \rangle$ quite unreliable.

The physical origin of the sign problem^[4]

In the projector Monte-Carlo method, for example, the quantity

$$\psi_\tau = e^{-\tau H} |\phi_{\tau=0}\rangle$$

tends to the ground state when $\tau \rightarrow \infty$. The weights that are used for the statistical sampling are proportional to the wave function ψ_τ . However we know that, for fermions, ψ_τ cannot be positive for all configurations. Decompose the wave function into positive and negative parts

$$\psi_\tau = \psi_\tau^+ - \psi_\tau^-$$

At low energies both ψ_τ^+ and ψ_τ^- have non-vanishing projections on the “hard-core” bosonic ground state. Hence at low temperatures the simulation spends most of the time below the actual energy of the fermionic ground state. These configurations have to be cancelled in the sum, hence the minus signs. A similar effect is unavoidable in frustrated spin systems.

One therefore sees that at low temperatures the Quantum Monte-Carlo method may be quite unreliable and a need is felt for alternative simulation methods.

3. Ground state processes and numerical simulations^[5]

The ground state and the Euclidian dynamics of a quantum system are uniquely associated to a stochastic process. Let ρ and G be the unique invariant measure density and the generator of stochastic process such that

$$-\hbar \rho^{\frac{1}{2}} G \rho^{-\frac{1}{2}} = H \quad (3.1)$$

is the Hamiltonian operator of a physical system. Then $\rho^{1/2}$ is the (zero energy) ground state, i. e. $H\rho^{1/2} = 0$ ^[6,7] ($\rho^{1/2} \doteq \phi$). From the representation $T = \exp\{-tG\}$ for the operator semigroup

$$(T_t f)(x) = \int P(t, x, dy) f(y) \quad (3.2)$$

associated to the transition functions of the Markov process, it follows that the Euclidian correlations of the theory (Schwinger functions) are obtained from the stochastic correlations of the process. The stochastic process may therefore be used to study the dynamical properties of the quantum system.

Existence of a process with unique invariant measure insures the existence of a Schrödinger picture for the quantum system and stochastic techniques like the theory of small random perturbations of Wentzell and Freidlin^[8,9], may be used to obtain rigorous non-perturbative results, in general not accessible through functional analytic methods^[10,11]. The stochastic process involved in this construction is a classical stochastic process which happens to have $\rho = |\phi|^2$ as its invariant measure and generates a set of Euclidian correlations. It should not however be confused with the *quantum probability*^[12,13] process associated to real-time quantum evolutions.

This stochastic formulation of quantum systems has, in the past, been used

mostly for boson systems. Here I will concentrate on methods which are also appropriate for fermion systems. In the occupation number representation, fermion systems appear associated to jump processes which, for boson systems, are characteristic of non-local interactions. Because of the formal similarities I will first review the construction of the stochastic process for non-local interactions^[7,14].

Let H be an Hamiltonian operator and $\phi(x)$ its (real) lowest energy eigenstate (which is adjusted to zero energy by the addition of a constant to H).

$$(H \phi)(x) = -\frac{\hbar^2}{2m} \Delta \phi(x) + U(x) \phi(x) + \int d^n y V(x,y) \phi(y) = 0 \quad (3.3)$$

with $V(x,y)$ real finite and $V(x,y) = V(y,x)$.

From (3.1) one obtains for the generator of the ground state process with invariant measure $\rho = \phi^2$,

$$(G f)(x) = \left(\frac{\hbar}{m} \frac{\nabla \phi}{\phi} \cdot \nabla \right) f + \frac{\hbar}{2m} \Delta f - \frac{1}{\hbar} \int d^n y \phi^{-1}(x) V(x,y) \phi(y) \{f(y) - f(x)\} \quad (3.4)$$

Now we compare with the general differential Chapman-Kolmogorov equation

$$\begin{aligned} \partial_t p(xt|zt') = & -\nabla \cdot \{b(x,t) p(xt|zt')\} + \frac{1}{2} \sum_{i,j} \frac{\partial^2}{\partial x_i \partial x_j} \{a_{ij}(x,t) p(xt|zt')\} \\ & + \int d^n y \{W(x|y) p(yt|zt') - W(y|x) p(xt|zt')\} \end{aligned} \quad (3.5)$$

for which the generator $Gf = \lim_{\Delta t \downarrow 0} \frac{(T_{\Delta t} f - f)}{\Delta t}$ is

$$(Gf)(x) = b(x,t) \cdot \nabla_x f(x) + \frac{1}{2} a_{ij}(x,t) \frac{\partial^2}{\partial x_i \partial x_j} f(x) + \int d^n y W(y|x) \{f(y) - f(x)\} \quad (3.6)$$

Comparing Eqs. (3.4) and (3.6) one concludes that the ground state process associated to the non-local Hamiltonian has drift, diffusion coefficient and jumping kernel, respectively

$$b = \frac{\hbar}{m} \frac{\nabla \phi}{\phi} \quad (3.7)$$

$$a_{ij} = \frac{\hbar}{m} \delta_{ij} \quad (3.8)$$

$$W(y|x) = -\frac{1}{\hbar} \phi^{-1}(x) V(x,y) \phi(y) \quad (3.9a)$$

Notice the association, in processes ruled by non-local interactions, of diffusion and jumping³. This will also be typical of processes involving bosons and fermions (see below).

Likewise, if the state space of an Hamiltonian is spanned by a discrete set $\{ |f\rangle \}$ of basis states one sees, by analogy with (3.9a), that the jumping process with kernel

$$W(f|f') = -\frac{1}{\hbar} \operatorname{Re} \left\{ \frac{1}{\langle f'|\phi \rangle} \langle f'| H |f\rangle \langle f|\phi \rangle \right\} \quad (3.9b)$$

has invariant density $\rho(f) = |\langle f|\phi \rangle|^2$ as may easily be checked from the stochastic equation

$$\partial_t \rho(f) = \sum_{f'} \{ W(f|f') \rho(f') - W(f'|f) \rho(f) \} \quad (3.10)$$

Notice that in this equation the kernel is defined only up to terms $\text{const.} \times \delta_{f,f'}$.

In the Eqs. (3.7) and (3.9) the drift and the jumping kernel are both functions of the ground state wave function ϕ which in general is not known for a given Hamiltonian. It is however possible to formulate the dynamical problem in such a way that the drift and the jumping kernel are obtained directly without having to solve the equation $H \phi(x) = \lambda_0 \phi(x)$ for the lowest eigenvalue λ_0 . Let ϕ_0 be some initial state non-orthogonal to the ground state. Then, in the $t \rightarrow \infty$ limit, the solution $\phi_t = \exp\{-\frac{1}{\hbar} H t\} \phi_0$ of the Euclidian equation

$$\hbar \partial_t \phi_t(x) = -H \phi_t(x) \quad (3.11)$$

converges to the ground state. Namely $\exp\{\frac{1}{\hbar} \lambda_0 t\} \phi_t \rightarrow \phi$. Defining

$$b_t = \frac{\hbar}{m} \frac{\nabla \phi_t}{\phi_t} \quad ; \quad W_t(y|x) = -\frac{1}{\hbar} \phi_t^{-1}(x) V(x,y) \phi_t(y) \quad (3.12)$$

³In the modelization of quantum systems, there are potentials that supply the simplest examples for each type of physical behaviour. For example the harmonic oscillator is typical of a pure point spectrum, the Coulomb potential typical of systems with both point and continuous spectrum and accumulation of levels at the ionisation threshold, etc. For roton excitations the non-local convolution potentials^[14], of which the simplest one is the symmetric translation potential $(V\psi)(x) = \lambda\psi(x+a) + \lambda^*\psi(x-a)$, provide perhaps the simplest model. In the stochastic interpretation of the Euclidian process the roton is the situation where slow forward diffusion and backward jumps exactly compensate each other. This may be thought of as the rigorous formulation of an old image of Feynman according to which one type of roton excitation may be viewed as a group of children going down a slide, then going around and coming back to the slide again.

where ϕ_t is a solution of the Euclidian equation (3.11), one obtains evolution equations for b_t and $W_t(y|x)$

$$\partial_t b_t(x) = \frac{1}{2} \nabla b_t^2(x) + \frac{\hbar}{2m} \nabla (\nabla \cdot b_t(x)) - \frac{1}{m} \nabla U(x) + \frac{\hbar}{m} \nabla_x \int d^n y W_t(y|x) \quad (3.13a)$$

$$\begin{aligned} \partial_t W_t(y|x) = & W_t(y|x) \int d^n z \left\{ W_t(z|y) - W_t(z|x) \right\} + \frac{1}{2} W_t(y|x) \left\{ 2 U(x) - \frac{m}{\hbar} b_t^2(x) \right. \\ & \left. - \nabla \cdot b_t(x) - 2 U(y) + \frac{m}{\hbar} b_t^2(y) + \nabla \cdot b_t(y) \right\} \end{aligned} \quad (3.13b)$$

As ϕ_t converges to the ground state, the solutions of Eqs.(3.13) converge to the drift and jumping kernels associated to the ground state process. Therefore these quantities may be obtained directly by iteration of Eqs.(3.13), without having to solve the Schrödinger equation. In the same way, for the pure jump case associated to the discrete basis $\{|f\rangle\}$ representation, one obtains the (Euclidian) evolution equation

$$\partial_t K_t(f|f') = \sum_g K_t(f|f') \left\{ K_t(g|f) - K_t(g|f') \right\} \quad (3.14a)$$

for the quantity

$$K_t(f|f') = -\frac{1}{\hbar} \frac{1}{\langle f' | \phi_t \rangle} \langle f' | H | f \rangle \langle f | \phi_t \rangle \quad (3.14b)$$

the jumping kernel being

$$W_t(f|f') = \text{Re } K_t(f|f')$$

Starting from an initial W_0 , consistent with (3.14b), this equation converges in the $t \rightarrow \infty$ limit to the jumping kernel of the ground state process. We may start for example from

$$W_0(f|f') = -\frac{1}{\hbar} \text{Re } \langle f | H | f' \rangle$$

which corresponds to consider the initial ϕ_t to have the same projection on all basis states. Although Eq.(3.14a) by itself contains no explicit dynamical information on the interaction potential, it is able to generate the ground state kernel because it preserves under time evolution all the information introduced by the initial condition, namely $\partial_t \{W_t(f|f) W_t(f|f')\} = 0$. That is, what Eq.(3.14a) does is simply to balance the transition probabilities to be consistent with an invariant measure while, at the same time, preserving the information about the Hamiltonian that is supplied by the initial condition.

For a many-body quantum system, because the number of possible states is very

large, the Euclidian equations (3.14a) become a very large system. If the interactions are short-range, a possible solution is to make the iteration on a small cluster simulation of the system to obtain an approximation of the ground state stochastic kernels. These kernels may then be used to obtain the correlation functions by simulating the process in a much larger lattice. The simulation of the process uses Eqs. (3.5) and (3.10) or the corresponding stochastic differential equations.

Expectation values of observables for the process X_t defined by Eqs. (3.7-3.9) are the quantum expectation values in the ground state

$$E\{A(X_t)\} = \int A(x) \rho(x) dx = \int \phi(x) A(x) \phi(x) dx = (\phi, A\phi) \quad (3.15)$$

The multi-time correlations of the stochastic process are the Euclidian correlations of the quantum system. For example

$$\begin{aligned} E\{A(X_t)A(X_0)\} &= \int (T_t A)(x) A(x) \rho(x) dx = \int \phi^2(x) A(x) \left(e^{-\frac{t}{\hbar}(\phi^{-1} H \phi)} A \right)(x) dx \\ &= (\phi, A e^{-\frac{t}{\hbar} H} A \phi) \end{aligned} \quad (3.16)$$

and similarly for the general n-point correlation function.

Assume that one chooses to compute the stochastic kernels in a basis $\{|f\rangle\}$ that diagonalizes the observable A for which one wishes to obtain the correlation functions. Then, the calculation above also shows that, in such an “adapted occupation number basis”, there is no essential difference in the procedures to be used for boson or for fermion systems. In particular the “sign problem” will be avoided.

Examples

Here I will consider two examples where the ground state may be explicitly represented as a functional of the operators of the theory. The study of the examples clarifies the nature of the stochastic processes associated to the ground state. The first example is the BCS model. The BCS ground state

$$|\phi\rangle = \prod_{\mathbf{k}, \sigma} \left(u_{\mathbf{k}} + v_{\mathbf{k}} b_{\mathbf{k}\sigma}^\dagger b_{-\mathbf{k}, -\sigma}^\dagger \right) |O\rangle \quad u_{\mathbf{k}}^2 + v_{\mathbf{k}}^2 = 1 \quad (3.17)$$

is a zero energy eigenstate of the Hamiltonian

$$H = \sum_{\mathbf{k}, \sigma} E_{\mathbf{k}} u_{\mathbf{k}}^2 \left(b_{\mathbf{k}\sigma}^\dagger - \alpha_{\mathbf{k}} b_{-\mathbf{k}, -\sigma} \right) \left(b_{\mathbf{k}\sigma} - \alpha_{\mathbf{k}} b_{-\mathbf{k}, -\sigma}^\dagger \right) \quad (3.18)$$

with

$$\alpha_k = -\alpha_{-k} = \frac{v_k}{u_k} \quad (3.19)$$

Let $\{ |c_{k\sigma}\rangle = b_{k\sigma}^\dagger b_{-k,-\sigma}^\dagger |O\rangle ; k_i > 0 \}$ be a basis of Cooper pairs. Then from (3.9b) the ground state process is a jumping process with kernel

$$W(\{c_{k\sigma}\}|\{c_{k'\sigma'}\}) = \frac{\prod_{q \in \{c_{k\sigma}\}} v_q}{\prod_{q \in \{c_{k'\sigma'}\}} v_q} \left\{ -\sum_{q \in \{c_{k\sigma}\}} 2\sqrt{E_q^2 - \Delta_q^2} \delta_{\{c_{k\sigma}\}, \{c_{k'\sigma'}\}} + \right. \\ \left. \Delta_p \delta_{\{c_{k'\sigma'}\}, \{c_{k\sigma}\} + c_p} + \Delta_p \delta_{\{c_{k'\sigma'}\}, \{c_{k\sigma}\} - c_p} \right\}^{\frac{1}{\hbar}} \quad (3.20)$$

with $\Delta_p = 2 E_p u_p v_p$. $\{c_{k\sigma}\}$ denotes a set of occupied Cooper pairs and $\{c_{k\sigma}\} \pm c_p$ is the same set with one more or one less occupied pair. Eq.(3.20) is obtained from (3.9b) neglecting a term of the type $\text{const.} \times \delta_{\{c_{k\sigma}\}, \{c_{k'\sigma'}\}}$

Another formulation of the ground state process which has, at least, formal interest is obtained using a representation of the creation and annihilation operators in Grassman variables η_k^* . The representation is

$$b_{k\sigma}^\dagger \rightarrow \eta_{k\sigma}^* \quad b_{k,\sigma} \rightarrow \left(\frac{d}{d\eta_{k\sigma}^*} \right)_L \quad (3.21)$$

where

$$\eta_{k\sigma} \eta_{q\sigma'}^* + \eta_{q\sigma'}^* \eta_{k\sigma} = \eta_{k\sigma}^* \eta_{q\sigma'}^* + \eta_{q\sigma'}^* \eta_{k\sigma}^* = 0$$

and the subscript L in the differentiation symbol means that the variable $\eta_{k\sigma}^*$ in a monomial must be pushed to the extreme left before the differentiation is applied.

The representation (3.21) acts on functions of $\{\eta_{k\sigma}^*\}$ alone. In this representation the BCS ground state is

$$\phi(\eta^*) = \prod_{\substack{k,\sigma \\ (k_i > 0)}} (u_k + v_k \eta_{k\sigma}^* \eta_{-k-\sigma}^*) \quad (3.22)$$

and the Hamiltonian

$$H = \sum_{k,\sigma} E_k u_k^2 \left(\eta_{k\sigma}^* - \alpha_k \frac{d}{d\eta_{-k-\sigma}^*} \right) \left(\frac{d}{d\eta_{k\sigma}^*} - \alpha_k \eta_{-k-\sigma}^* \right) \quad (3.23)$$

Performing on H the transformation induced by $\phi(\eta^*)$ one obtains

$$-\phi^{-1} H \phi = \sum_{\mathbf{k}, \sigma} E_{\mathbf{k}} u_{\mathbf{k}} \frac{d}{d\eta_{-\mathbf{k}-\sigma}^*} \frac{d}{d\eta_{\mathbf{k}\sigma}^*} - \sum_{\mathbf{k}, \sigma} E_{\mathbf{k}} \eta_{\mathbf{k}\sigma}^* \frac{d}{d\eta_{\mathbf{k}\sigma}^*} \quad (3.24)$$

Comparing with the standard form of the elliptic operator,

$$\frac{1}{2} \sum_{i,j} \alpha_{ij}(x,t) \frac{\partial^2}{\partial x_i \partial x_j} + \sum_i \beta_i \frac{\partial}{\partial x_i}$$

generator of a diffusion process, one concludes that formally the BCS ground state process may be interpreted as a diffusion process in Grassman variables, with drift

$$\beta_{\mathbf{k}\sigma} = - E_{\mathbf{k}} \eta_{\mathbf{k}\sigma}^* \quad (3.25a)$$

and diffusion coefficient

$$\alpha_{\mathbf{k}\sigma, \mathbf{p}\sigma'} = 2 E_{\mathbf{k}} u_{\mathbf{k}} \delta_{\mathbf{k}, -\mathbf{p}} \delta_{\sigma, \sigma'} \quad (3.25b)$$

For practical purposes however the formulation of the ground state as a jump process in an occupation number basis is probably more useful.

For the second example consider a supersymmetric hopping model defined on a 2-dimensional square lattice by

$$H = \frac{1}{2} \sum_{(ij)} \left(-\hbar^2 \frac{\partial^2}{\partial \phi_{(ij)} \partial \phi_{(ij)}} + \frac{\partial W}{\partial \phi_{(ij)}} \frac{\partial W}{\partial \phi_{(ij)}} \right) + \frac{1}{2} \hbar \sum_{(ij)(kl)} \frac{\partial^2 W}{\partial \phi_{(ij)} \partial \phi_{(kl)}} [c_{(ij)}^\dagger, c_{(kl)}] \quad (3.26)$$

where $\phi_{(ij)}$ and $c_{(ij)}$ are bosonic and fermionic fields at the lattice point (ij) satisfying

$$[\phi_{(ij)}, \frac{\hbar}{i} \frac{\partial}{\partial \phi_{(ij)}}] = i \hbar \delta_{(ij), (kl)} \quad \{c_{(kl)}^\dagger, c_{(ij)}\} = \delta_{(ij), (kl)} \quad (3.27)$$

For the superpotential $W(\phi)$

$$W(\phi) = -\frac{t}{\hbar} \sum_{[(ij), (kl)]} \phi_{(ij)} \phi_{(kl)} \quad (3.28)$$

with $[(ij), (kl)]$ denoting a sum over nearest neighbours, the Hamiltonian becomes

$$H = \frac{1}{2} \sum_{(ij)} \left(-\hbar^2 \frac{\partial^2}{\partial \phi_{(ij)} \partial \phi_{(ij)}} + \left(\sum_{[(kl), (ij)]} \phi_{(kl)} \right)^2 \right) - \frac{t}{2} \sum_{[(ij), (kl)]} [c_{(ij)}^\dagger, c_{(kl)}] \quad (3.29)$$

Denote by $|0\rangle$ and $|-\rangle$ the empty and the filled sectors in the Fermion Fock space. Then the state

$$|\psi\rangle = e^{\frac{W(\phi)}{\hbar}} \sum_{(ij)} (-1)^{i+j} c_{(ij)} |-\rangle \quad (3.30)$$

is the ground state in the one-hole sector and is a zero-energy eigenstate of the operator $\bar{H} = H + 4t$. Computing the drift and the jumping kernel from Eqs. (3.7) and (3.9) one obtains

$$b_{(ij)} = -\frac{t}{\hbar m} \sum_{[(kl), (ij)]} \phi_{(kl)} \quad (3.31a)$$

$$W(\phi \ i \ j \mid \phi' \ k \ l) = -\frac{t}{\hbar} \delta_{[(kl), (ij)]} \delta_{\phi, \phi'} \quad (3.31b)$$

The sum in $b_{(ij)}$ is over all lattice points (kl) that are nearest neighbours to (ij) and $\delta_{[(kl), (ij)]}$ vanishes unless (ij) and (kl) are nearest neighbours. The jumping kernel is computed in the basis

$$|\phi \ i \ j\rangle = |\phi\rangle c_{(ij)} |-\rangle$$

One sees that the boson-fermion nature of the problem implies that the corresponding process involves both diffusion and jumping.

Numerical simulation of the ground state process. The Hubbard model

As explained above the essential step is the calculation of the jumping kernels $W(f|f)$ from the asymptotic solution of Eq.(3.14). Notice that $|f\rangle$ runs over all states of the system and therefore, for a system with a large number of degrees of freedom, Eq.(3.14) becomes a differential equation of very high dimension. However in a system like the Hubbard model, where the elementary interactions are short-range, the kernels may, with good approximation, be obtained by solving the equation for a small cluster, and then storing the solution in a look-up table that is used in a much larger lattice to extract the global properties of the ground state. Notice that the small-cluster does not even have to be very small because, for each set of parameters, Eq.(3.14) has to be solved only once.

Consider the Hubbard model. The Hamiltonian is

$$H = -t \sum_{\{i,j\}} c^\dagger(i) c(j) + U \sum_i n_\uparrow(i) n_\downarrow(i)$$

With $W_0(f'|f) = -\text{Re} \langle f' | H | f \rangle$ the initial values for the Eq.(3.14) are:

$$\begin{aligned} W_0(f'|f) &= U N_p & f' &= f \\ W_0(f'|f) &= -t & \text{if } f' \text{ differ from } f \text{ by an electron jump to a} \\ & & \text{neighbouring site} \\ W_0(f'|f) &= 0 & \text{for all other pairs } f, f' \end{aligned}$$

and from the (Euclidian) evolution equation (3.14) the only non-zero W 's are those which are non-zero for the initial condition. Therefore, in this case, a 3×3 lattice with periodic boundary conditions already provides an unbiased environment for the elementary jumps. However a 3×3 lattice already carries 4^9 possible different configurations, considering at each site no-electron, one-electron($\uparrow$ or $\downarrow$) or a pair. Using the symmetry properties these configurations may be grouped into classes, a class being defined as the configurations that can be transformed into one another by translations, rotations or mirror reflections. This finally leads to 4480 different classes which are classified in the table below according to the number of up and down electrons.

		Number of electrons $\uparrow$									
		0	1	2	3	4	5	6	7	8	9
Number of electrons $\downarrow$	0	1	1	2	4	5	5	4	2	1	1
	1	1	3	8	16	23	23	16	8	3	1
	2	2	8	26	54	78	78	54	26	8	2
	3	4	16	54	118	170	170	118	54	16	4
	4	5	23	78	170	250	250	170	78	23	5
	5	5	23	78	170	250	250	170	78	23	5
	6	4	16	54	118	170	170	118	54	16	4
	7	2	8	26	54	78	78	54	26	8	2
	8	1	3	8	16	23	23	16	8	3	1
	9	1	1	2	4	5	5	4	2	1	1

We may therefore limit ourselves to the study of a representative element in each class and their neighbouring configurations. The system of equations corresponding to the possible, and topologically distinct, transitions between the different classes is now solved numerically and the $W(f'|f)$'s corresponding to its (rapidly) convergent large time solution are stored in a look-up table. Look-up tables were generated for several U/t values in the range 0 to 10.

Two remarks should be made concerning the calculation of the jumping kernels. The ground state stochastic equations were derived under the assumption that the energy of the ground state is normalised to zero. Because the ground state energy is in general not known it means that the diagonal kernels $W(f'|f)$ are defined only up to an additive constant. This has no effect in the evolution of Eq.(3.14) because the additive constant cancels in sum of the right-hand side, but it means that the diagonal element values stored in the look-up table are not the actual $W(f|f)$'s. The way to obtain its actual values for the stochastic evolution in the ground state process is through

$$W(f|f) = 1 - \sum_{g \neq f} W(g|f) \Delta t$$

provided Δt is sufficiently small compared with the $W(g|f)$'s.

To obtain now the properties of the ground state the simulation starts from a lattice with a certain number of up and down electrons and the evolution of the stochastic ground state process is carried out by each time picking up a site at random and having its 3×3 neighbourhood evolve according to the jumping kernels of the look-up table. In practice all compatible W 's are piled-up according to their sizes as intervals in the line and the decision taken by a random number uniformly distributed in the interval $[0, \sum W]$.

I now present some results of a study of pairing and short range antiferromagnetic correlations in the ground state of the Hubbard model. The simulations were carried out in a 20×20 lattice with 198 up electrons and 198 down electrons. The boundary conditions are periodic and the parameter values explored were $U/t = 0, 1, 4$ and 10. The initial electron distribution is random (constrained by the Pauli principle). For each case the results we present correspond to 10^6 productive evolutions (i.e. evolutions where there was an actual change in the lattice configuration). Measures of average values are taken only after the number of double occupancies is stabilised.

The following quantities were measured:

- number of double occupied sites
- number of hole pairs
- short range antiferromagnetic correlation
- medium range antiferromagnetic correlation

The antiferromagnetic correlation is measured by looking at a neighbourhood of each point and counting +1 for the antiferromagnetically correlated neighbours, -1 for the ferromagnetic ones and zero for those that are either empty or have double occupancy. For the “short range” one considers a 3×3 neighbourhood and for the “medium range” a 5×5 neighbourhood. The results are the following:

	U/t			
	0	1	4	10
double occupancy	99	93	74	50
hole pairs	54	46	28	14
short rang antiferr.	53	84	92	117
medium rang antif.	33	75	76	32

A critical issue in models of high-temperature superconductivity is whether or not the effect is due to a type of Bose condensation of pre-existing local pairs of carriers. In particular the observed short coherence lengths seem to favour such a local pair picture. It is therefore important to search for any symptoms of dynamical hole pairing in the Hubbard model. One sees from the table that the number of hole pairs decreases with U/t , but that is to expected from the statistical effect of the decreasing double occupancy which also decreases the number of holes. To disentangle the statistics from dynamical effects compare, in the following table, the average distribution of holes in the Hubbard model ground state (at $U/t = 4$) with a purely random distribution for the same total number of holes.

Number of holes in the cluster	Hubbard model	Random distribution
1	37.61	32.97
2	9.19	8.11
3	3.51	3.44
4	1.75	1.54
5	0.67	0.72
6	0.46	0.35
7	0.32	0.17
8	0.13	0.09
9	0.12	0.04
10	0.03	0.02
11	0.08	0.01
12	0.03	0.00

The first column defines the size of the hole clusters and the second and the third contain the average number of such clusters in the Hubbard model and in a random distribution. The conclusion is that there is no essential distinction between the random unbiased distribution of holes and the Hubbard model. Therefore we find no evidence for hole pair formation in the Hubbard model. Incidentally this agrees with the analytical estimates of Ref.[15] where it was concluded that the Hubbard model by itself does not lead to pair formation and that an additional pairing interaction seems to be needed in models of high- T_c superconductivity.

What the results show however is evidence for short range antiferromagnetic correlation, although already at a modest medium range the correlation seems to be lost.

4. Probabilistic diagonalization

For most non-trivial many-body Hamiltonians it is extremely difficult to obtain the exact eigenstates or even reliable analytical approximations. It may however be possible to make probabilistic estimates of the coefficients of the exact eigenstates in some Hilbert space basis. The difficulty of obtaining exact analytical or numerical results increases with the number of degrees of freedom. However the accuracy of

probabilistic estimates also increases with the number of degrees of freedom. This idea was developed in Ref.[15] for models of strongly correlated electron systems. Consider the Hamiltonian

$$H_0 = -t \sum_{\langle i,j \rangle_{\sigma}} (1 - n_{i-\sigma}) c_{i\sigma}^{\dagger} c_{j\sigma} (1 - n_{j-\sigma}) \quad (4.1)$$

with $n_{i\sigma} = c_{i\sigma}^{\dagger} c_{i\sigma}$. H_0 is the part of the Hubbard Hamiltonian that transfers electrons from single occupied sites to empty sites without changing the eigenvalue of $D = \sum_i n_{i\uparrow} n_{i\downarrow}$. In the dynamics of the ground state for a less than half-filled band, in the $D=0$ subspace, H_0 controls the hole dynamics.

Let t be > 0 and consider the diagonalization problem for H_0 in the $D=0$ sector. Take a less than half filled band with N sites and n_h holes. The situations we are mostly concerned with are situations where there is a macroscopic number of holes, i. e. n_h/N is of order at least a few per cent. States with n_h holes will be denoted by $|i, \beta\rangle$, i and β being the labels respectively for the hole configuration and the electron spin configuration. H_0 has matrix elements between $|i, \beta\rangle$ and $|j(i), \beta\rangle$, $j(i)$ being the hole configurations that are connected to i by the motion of an hole to a nearest neighbour site. Let $|\psi\rangle$ be an eigenstate of H_0

$$H_0 |\psi\rangle = \lambda |\psi\rangle = \lambda \sum_{i, \beta} c_{i\beta} |i, \beta\rangle = \sum_{i, \beta} \sum_{j(i)} \frac{c_{j(i)\beta}}{c_{i\beta}} c_{i\beta} |i, \beta\rangle$$

Then

$$\sum_{j(i)} \frac{c_{j(i)\beta}}{c_{i\beta}} = \text{constant} = \lambda \quad (4.2)$$

The energy of the ground state will be minimal if $\text{sign}(c_{i\beta}) = \text{sign}(c_{j(i)\beta})$, i. e. configurations differing by the motion of an hole should have the same sign.

Successive applications of H_0 will connect states with all hole configurations and all electron spin distributions (preserving only the number of holes and the number of up and down spins). The eigenstates of H_0 are thus coherent superpositions of all hole configurations. The number of states to which a given $|i, \beta\rangle$ is connected by H_0 is called its "connectivity number" $\nu(i)$ and varies from $4n_h$ for a state without hole clusters to about $4\sqrt{n_h}$ for a state with a global cluster.

The connectivity number $\nu(i)$ is the number of terms in the sum of Eq.(4.2). Because for an eigenstate the sum in (4.2) is a constant for all $|i, \beta\rangle$ and the

connectivity varies with the formation of clusters, this provides a way to estimate the coefficients, hence the probability of cluster formation. An exact analytic calculation, i. e. an exact diagonalization of H_0 is not practical for a large number N of sites. However the very fact that N is a large number may be turned to our profit to obtain "exact" results in the probabilistic sense, i. e. to obtain ratios for the coefficients $c_{i\beta}$ which hold in the average.

In the expansion of the ground state $|\psi_0\rangle$ of H_0 consider the coefficients of the states with p hole pairs, which we denote by c_{2p} . When, starting from a given hole configuration, one moves one hole, the probability that a hole pair is formed by this motion is $3(n_h-1)/(N-1) \simeq 3n_h/N$. Noticing that a hole pair contributes 6 to the connectivity as opposed to 8 for two isolated holes, one obtains from (4.2) the following, probabilistically rigorous, equation

$$\begin{aligned} 4(n_h-2p) \left\{ \left(1 - \frac{3n_h}{N}\right) + \frac{3n_h}{N} \frac{c_{2p+2}}{c_{2p}} \right\} + 6p \left\{ \left(1 - \frac{3n_h}{N}\right) \frac{c_{2p-2}}{c_{2p}} + \frac{3n_h}{N} \right\} = \\ = 4 n_h \left\{ \left(1 - \frac{3n_h}{N}\right) + \frac{3n_h}{N} \frac{c_2}{c_0} \right\} \end{aligned} \quad (4.3)$$

Considering the addition of a hole pair to have an uniform effect on the ratio of the coefficients

$$\frac{c_{2p+2}}{c_{2p}} = \frac{c_{2p-2}}{c_{2p}} = \frac{c_2}{c_0} = z \quad (4.4)$$

and defining $x = \frac{3n_h}{N}$, one obtains

$$-4(1-x) - 4xz + 3(1-x)\frac{1}{z} + 3x = 0 \quad (4.5)$$

The solution is $z = \frac{3}{4}$ (the other solution of (4.5) being negative for $x < 1$). This implies

$$c_{2p} = \left(\frac{3}{4}\right)^p c_0 \quad (4.6)$$

which means that the probability to find a macroscopic number of hole pairs ($\frac{2p}{n_h} \sim O(1)$) vanishes in the thermodynamic limit ($N \rightarrow \infty$). A similar reasoning would lead to a similar conclusion for the probability of occurrence of other hole clusters in macroscopic numbers. The ground state $|\psi_0\rangle$ of H_0 is therefore a coherent superposition dominated by sparse hole configurations with coefficients of similar

magnitude and the same sign. For each fixed number of up and down spins one finds in $|\psi_0\rangle$ all possible spin configurations β with coefficients of similar magnitude, determined only by the cluster structure of the hole distribution. One therefore concludes that the introduction of a macroscopic number of holes destroys any antiferromagnetic long range order that might exist in the half-filled band. The ground state of H_0 is degenerate in the sense that we will have a set of states of equal energy $|\psi_0^{n_+ n_-}\rangle$ labelled by the number of up and down spins.

The other eigenstates of H_0 are also degenerate coherent superpositions of all hole and spin configurations. The highest eigenvalue will correspond to coefficients of magnitude identical to the ones in $|\psi_0^{n_+ n_-}\rangle$ and such that configurations differing by the motion of a hole have opposite signs. The basis of H_0 -eigenstates is called the “resonant hole configuration”(RHC) basis and the diagonalization of the full effective Hamiltonian of the Hubbard model may be carried out in this basis. For more details and the application to other interaction terms I refer to [15].

A different probabilistic idea to carry out the diagonalization of the Hamiltonians in many-body problems has been proposed by De Raedt, von der Linden and Frick^[16,17] and called “Monte-Carlo diagonalization” by these authors.

5. Scaling laws and deviations from scaling in fully developed turbulence

The modern understanding of fully developed turbulence is very much related to the old idea of the Richardson cascade^[18]. In this cascade an hierarchy of eddies transfers the energy from the macroscopic scale to smaller and smaller scales until the fragmentation process is stopped by dissipation. From the Navier-Stokes equation

$$\partial_t \mathbf{v} + (\mathbf{v} \cdot \nabla) \mathbf{v} = - \frac{1}{\rho} \nabla p + \nu \Delta \mathbf{v} + \mathbf{f} \quad (5.1)$$

one sees that dissipation is only important when $\Delta \mathbf{v}$ is large, that is, at small scales. We may therefore distinguish the following regimens or ranges:

The macroscopic (external) range

The inertial range (Δv small)

The dissipation range (Δv large)

In the inertial range the viscosity term $\nu \Delta \mathbf{v}$ may be neglected and the simplified Navier-Stokes equation is scale invariant, that is, invariant for the transformations

$$\vec{r} \rightarrow \lambda \vec{r} \quad \vec{v} \rightarrow \lambda^h \vec{v} \quad t \rightarrow \lambda^{1-h} t \quad (5.2)$$

This means that, in the inertial range of fully developed turbulence, there is scale invariance in a statistical sense. Let

$$\delta v(r, \ell) = v(r + \ell) - v(r)$$

be the increment in the velocity field along ℓ . From homogeneity, isotropy and the assumption of finite mean rate of dissipation per unit mass (ϵ), Kolmogorov in 1941^[19] derived the following relation for the third momentum of the velocity increment parallel to ℓ (*third-order longitudinal structure function*)

$$S_3(\ell) = \langle (\delta v_{||}(r, \ell))^3 \rangle = -\frac{4}{5} \epsilon \ell \quad (5.3)$$

Comparing with (5.2) we conclude, from scale invariance, that $h = 1/3$. Using scale invariance once more we may now derive the form of the higher order structure functions, namely

$$S_p(\ell) = \langle (\delta v_{||}(r, \ell))^p \rangle = C_p \epsilon^{\frac{1}{3}p} \ell^{\frac{1}{3}p} \quad (5.4)$$

Notice that C_p depends on the geometry of the turbulence producing mechanism. That is, there is scaling but not universality^[20]. Defining

$$\langle |\delta v_{||}(r, \ell)|^p \rangle \sim \ell^{\zeta(p)} \quad (5.5)$$

Kolmogorov's (41) theory therefore predicts

$$\zeta(p) = \frac{1}{3} p \quad (5.6)$$

It was only in the eighties that this prediction could be experimentally checked in an accurate way. The experimental data has shown deviations from Kolmogorov scaling. Instead of (5.6), the data^[21] shows:

- (i) For small p : $\zeta(p) \simeq \frac{0.83}{3} p + 0.17$
- (ii) For large p : is not linear in p

Fractal hypothesis

The deviation at small p is explained by the fractal hypothesis^[22,23]. This states that the energy transfer does not occur in the whole volume but only in a set of dimension $d_f < 3$. Numerical simulations of the Navier-Stokes equation^[24] seem also to confirm this hypothesis. Let S be the set that contains the active eddies at scale ℓ

$$\delta v(r, \ell) \sim \begin{cases} \ell^h & r \in S \\ 0 & r \notin S \end{cases}$$

For the time, energy and rate of dissipation per unit mass, at scale ℓ , we have the following estimates

$$t(\ell) \sim \frac{\ell}{\delta v(\ell)} \quad E(\ell) \sim \delta v^2(\ell) \quad \epsilon(\ell) \sim \frac{E(\ell)}{t(\ell)} \sim \frac{\delta v^3(\ell)}{\ell} \quad (5.7)$$

Then

$$\int \frac{\delta v^3(\ell)}{\ell} d^3y \sim \ell^{3h+2}$$

implies $h = (d_f - 2)/3$ and

$$\zeta(p) = \frac{d_f - 2}{3} p + 3 - d_f \quad (5.8)$$

Eq.(5.8) fits the data at small p for $d_f \simeq 2.83$. The (single) fractal model however does not explain the non-linearity at large p .

Multifractal models^[25,26]

Instead of a single scaling exponent ℓ^h assume that there is a range of scaling exponents $I = (h_{\min}, h_{\max})$. For each $h \in I$ there is a set $S_h \in \mathbf{R}^3$ of Hausdorff dimension $D(h)$. Then

$$\langle |\delta v(\ell)|^p \rangle \sim \int_I d\mu(h) \ell^{ph+3-D(h)} \quad (5.9)$$

For $\ell \rightarrow 0$ a steepest descent argument implies that the smallest exponent dominates

$$\langle |\delta v(\ell)|^p \rangle \sim \ell^{\zeta(p)} \quad \zeta(p) = \min_h (ph + 3 - D(h)) \quad (5.10)$$

Therefore $\zeta(p)$ is the Legendre transform of the dimension function $D(h)$.

The random β -model

A concrete implementation of the multifractal idea is the random β -model. Let, for example, the length scales be related by powers of two.

$$\ell_n = 2^{-n} \ell_0$$

where ℓ_0 is the macroscopic injection scale. Let $\delta v_n = \delta v(\ell_n)$ and call β the ratio of the volume occupied by the active eddies at scale ℓ_{n+1} to the volume of the parent eddy at scale ℓ_n . Because the energy transfer rate between scales must be constant in the inertial range, but the total mass of the active eddies is multiplied by β , we have for the transfer rate per unit mass

$$\epsilon_n = \beta \epsilon_{n+1}$$

Now, to implement the multifractal idea, one assumes that there are eddies of different types with different β 's for each type of transfer. Then, recalling the estimate (5.7) for ϵ , one obtains

$$\frac{\delta v_n^3(k)}{\ell_n} = \beta_{n+1}(k) \frac{\delta v_{n+1}^3(k)}{\ell_{n+1}} \quad (5.11)$$

where each k denotes a particular type of energy transfer. In this way, at step n , the velocity fluctuation inside an eddy depends on the precise history of its fragmentation. It is determined by the sequence $[\beta_1, \dots, \beta_n]$. The volume that it occupies is $\prod_{i=1}^n \beta_i \ell_0^3$. Then

$$\delta v_n \sim \ell_n^{1/3} \left(\prod_{i=1}^n \beta_i \right)^{-1/3} \quad (5.12)$$

and

$$\langle |\delta v_n|^p \rangle \sim \ell_n^{p/3} \int \prod_{i=1}^n d\beta_i \beta_i^{1-p/3} P(\beta_1, \dots, \beta_n) \quad (5.13)$$

where $P(\beta_1, \dots, \beta_n)$ is the probability of the fragmentation $[\beta_1, \dots, \beta_n]$.

If one assumes independent fragmentations

$$P(\beta_1, \dots, \beta_n) = \prod_{i=1}^n P(\beta_i) \quad (5.14)$$

one obtains

$$\zeta(p) = \frac{p}{3} - \ln_2 \langle \beta^{1-p/3} \rangle \quad (5.15)$$

where $\langle \rangle$ denotes average in the probability $P(\beta_1, \dots, \beta_n)$. The simplest model for P is a binomial model

$$P(\beta) = \gamma \delta_1 + (1-\gamma) \delta_{1/2} \quad (5.16)$$

The two values of β , $\beta_0=1$ and $\beta_1=1/2$, correspond respectively to dimensions $d_f=3$ and $d_f=2$. That is, the binomial model considers two types of eddy energy transfers, namely volume - to - volume and volume - to - surface. With (5.16) one now obtains

$$\zeta(p) = \frac{p}{3} - \ln_2 \left\{ \gamma + \frac{1-\gamma}{2^{1-p/3}} \right\} \quad (5.17)$$

which actually fits the existing data, for $\gamma = 0.875$.

6. Asymmetric stochastic processes on a tree and the turbulent cascade^[27]

In Section 5 we have seen that, at least at the level of precision of the existing experiments, the statistics of the velocity fluctuations in the turbulent cascade are well described by multifractal models like the random β -model. However the models and the quantities that are computed do not involve time-correlations and therefore are not a detailed test of the dynamics of the turbulent cascade. Indeed assumptions like (5.14) and (5.16) only define the probability distribution ρ_i at each level of the cascade tree (Fig.1). They make no statement concerning the time evolution and the time scales of the eddies in the cascade. In fact it is known that only a few restrictions are imposed on the form of the velocity fields by the predictions of the statistical models described in Section 5 and many distinct velocity fields may be compatible with the spectra and scaling laws. One way to further the research on the dynamical properties of the velocity field is to study time-correlations for the observables. To make a connection of the models with the time dependence of each observable one should first consider the dynamical aspects of the energy cascade itself. To do this a time hierarchy in the development of the turbulent cascade must be defined. In this section one deals with this issue.

For a binary cascade tree (Fig.1) we may use a dyadic labelling for the possible states at each level. The *state space* V_n at level ℓ_n is the set of all products $\beta^{(1)} \dots \beta^{(n)}$ with $\beta^{(i)} \in \beta_0, \beta_1$ (In Eq.(5.16) $\beta_0 = 1$ and $\beta_1 = 0.5$). There are 2^n elements in V_n and the probability of the state i is

$$\rho_i = P(\beta^{(1)} \dots \beta^{(n)}) = \gamma^{n_0(i)} (1-\gamma)^{n_1(i)} \quad (6.1)$$

where $n_0(i)$ and $n_1(i)$ are the number of zeros and ones in the dyadic labelling of the state i .

From the random β -model all one obtains is a statement about these probabilities. This suffices to interpret most of the current experimental results which concern mostly velocity increments over a distance r and turbulent energy dissipation over a box of size r . These quantities do not involve time-correlations and therefore do not make a detailed test of the dynamics of the cascade, they only test its invariant probability measure. To identify the physical mechanisms behind the structure of fully developed turbulence more information is needed. If one wants, for example, the time

FIG. 1

FIG. 2

correlations at a point moving with the free-stream velocity of the fluid one should explicitly consider models for the dynamics in state space at each level n . To the same invariant measure ρ_i correspond many different processes. The most unstructured process corresponds to the statement that, if at time zero one finds the state i , then the transition probability to the state j at time t is proportional to ρ_j . For the turbulent cascade the unstructured process does not seem to be natural, because if the lifetime of the eddies in the inertial range scales like $\ell_n/\delta v_n$, then we expect larger eddies to live longer than small eddies. That is, if at time t the fluctuation $\delta v_n(x)$ at the point x is receiving its energy through a fragmentation history leading to the state i then, a short time thereafter, we expect to find a different state which is nearby in the sense of the natural ultrametric distance in the tree.

To characterise a stochastic process on a tree one has to solve the Chapman-Kolmogorov equation for the transition probabilities

$$\partial_t p(zt|y0) = \int dx \{W(z|xt) p(xt|y0) - W(x|zt) p(zt|y0)\} \quad (6.2)$$

with kernels $W(z|xt)$ that reflect the (natural) ultrametric distance in the tree. For kernels that depend only on the distance $W(z|xt) = W(|z-x|)$, Ogielski and Stein^[28] found the solution of Eq.(6.2). Albeverio and Karwowski^[29,30] have also constructed the stochastic processes on arbitrary p -adic fields $\mathbb{Q}_p$ for the case where the jumping kernels depend only on the distance between p -adic balls (see also Brekke and Olson^[31]). However it is easy to see from the equation for the probability densities

$$\partial_t \rho(zt) = \int dx \{W(z|xt) \rho(xt) - W(x|zt) \rho(zt)\} \quad (6.3)$$

that if $W(z|xt)=W(|z-x|)$ then the invariant density is $\rho(z)=\text{const.}$ For the stochastic process of the turbulent cascade we require a non-constant invariant density as in Eq.(6.1) and the results of the authors of Refs.[28-31] cannot be used. From (6.3) it follows that with

$$W(z|xt) = \rho(z) f(|z-x|) \quad (6.4)$$

the invariant density is $\rho(z)$ and, at the same time, full account is taken of the dependence of the transition probability on the distance between the points z and x in state space. First we characterise the solutions of the Chapman-Kolmogorov equation for kernels of the form (6.4) and then show how to use these solutions to compute (or parametrise) the time correlations of the turbulent cascade.

Random walk on a tree with asymmetric distance-dependent jumping kernels

Eq.(6.3) in matrix form is

$$\frac{\partial}{\partial t} \rho(t) = W \rho(t) \quad (6.5)$$

where W is the matrix

$$\begin{bmatrix} W_{11} & \rho_1^{\epsilon_1} & \rho_1^{\epsilon_2} & \rho_1^{\epsilon_2} & \rho_1^{\epsilon_3} & \rho_1^{\epsilon_3} & \rho_1^{\epsilon_3} & \rho_1^{\epsilon_3} & \cdot & \cdot \\ \rho_2^{\epsilon_1} & W_{22} & \rho_2^{\epsilon_2} & \rho_2^{\epsilon_2} & \rho_2^{\epsilon_3} & \rho_2^{\epsilon_3} & \rho_2^{\epsilon_3} & \rho_2^{\epsilon_3} & \cdot & \cdot \\ \rho_3^{\epsilon_2} & \rho_3^{\epsilon_2} & W_{33} & \rho_3^{\epsilon_1} & \rho_3^{\epsilon_3} & \rho_3^{\epsilon_3} & \rho_3^{\epsilon_3} & \rho_3^{\epsilon_3} & \cdot & \cdot \\ \rho_4^{\epsilon_2} & \rho_4^{\epsilon_2} & \rho_4^{\epsilon_1} & W_{44} & \rho_4^{\epsilon_3} & \rho_4^{\epsilon_3} & \rho_4^{\epsilon_3} & \rho_4^{\epsilon_3} & \cdot & \cdot \\ \rho_5^{\epsilon_3} & \rho_5^{\epsilon_3} & \rho_5^{\epsilon_3} & \rho_5^{\epsilon_3} & W_{55} & \rho_5^{\epsilon_1} & \rho_5^{\epsilon_2} & \rho_5^{\epsilon_2} & \cdot & \cdot \\ \rho_6^{\epsilon_3} & \rho_6^{\epsilon_3} & \rho_6^{\epsilon_3} & \rho_6^{\epsilon_3} & \rho_6^{\epsilon_1} & W_{66} & \rho_6^{\epsilon_2} & \rho_6^{\epsilon_2} & \cdot & \cdot \\ \rho_7^{\epsilon_3} & \rho_7^{\epsilon_3} & \rho_7^{\epsilon_3} & \rho_7^{\epsilon_3} & \rho_7^{\epsilon_2} & \rho_7^{\epsilon_2} & W_{77} & \rho_7^{\epsilon_1} & \cdot & \cdot \\ \rho_8^{\epsilon_3} & \rho_8^{\epsilon_3} & \rho_8^{\epsilon_3} & \rho_8^{\epsilon_3} & \rho_8^{\epsilon_2} & \rho_8^{\epsilon_2} & \rho_8^{\epsilon_1} & W_{88} & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \end{bmatrix}$$

Notice that the matrix has increasingly larger non-diagonal blocks of size $2^i \times 2^i$ which have the common factor ϵ_i . These blocks correspond to jumps to a ultrametric distance i . A matrix element in a block of size $2^i \times 2^i$ at line k equals $\rho_k^{\epsilon_i}$, ϵ_i being the value of $f(|z-x|)$ in Eq.(6.4) for a jump to a distance i . The elements W_{ii} in the diagonal are such that the columns add to zero.

As in the symmetric case studied by Ogielski and Stein^[28] one finds the complete set of eigenvectors of the matrix W . For a matrix of dimension 2^n , which describes the stochastic process at the n th level of the tree, the eigenvectors are:

(i) The eigenvector

$$\begin{bmatrix} \rho_1 \\ \rho_2 \\ \cdot \\ \cdot \\ \cdot \\ \cdot \\ \rho_{2^n} \end{bmatrix}$$

with eigenvalue $\lambda_0 = 0$;

(ii) n classes with 2^{n-k} ($k=1 \dots n$) eigenvectors each, where each eigenvector has only 2^k non-zero elements. The first $2^k - 1$ elements are positive and the others are negative. The 2^k non-zero elements of an eigenvector have a common ancestor, in the tree, at the level $n-k$. The non-zero elements of an (unnormalized) eigenvector are formed by multiplying the corresponding ρ_i by the sum of the ρ_j 's of the complementary group in the non-zero set of elements. The formation rule is easier to understand from an example. Let $n=3$ (Fig. 2). There are then three classes of eigenvectors in the group (ii), typical examples of which are:

a)

$$\begin{bmatrix} \rho_1 \rho_2 \\ -\rho_2 \rho_1 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\lambda_1 = -(\epsilon_1(\rho_1 + \rho_2) + \epsilon_2(\rho_3 + \rho_4) + \epsilon_3(\rho_5 + \rho_6 + \rho_7 + \rho_8))$$

Four eigenvectors of this type corresponding to the independent groups of two elements with a common ancestor at level 2.

b)

$$\begin{bmatrix} \rho_1(\rho_3 + \rho_4) \\ \rho_2(\rho_3 + \rho_4) \\ -\rho_3(\rho_1 + \rho_2) \\ -\rho_4(\rho_1 + \rho_2) \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\lambda_5 = -(\epsilon_2(\rho_1 + \rho_2 + \rho_3 + \rho_4) + \epsilon_3(\rho_5 + \rho_6 + \rho_7 + \rho_8))$$

Two eigenvectors of this type.

c)

$$\begin{bmatrix} \rho_1(\rho_5 + \rho_6 + \rho_7 + \rho_8) \\ \rho_2(\rho_5 + \rho_6 + \rho_7 + \rho_8) \\ \rho_3(\rho_5 + \rho_6 + \rho_7 + \rho_8) \\ \rho_4(\rho_5 + \rho_6 + \rho_7 + \rho_8) \\ -\rho_5(\rho_1 + \rho_2 + \rho_3 + \rho_4) \\ -\rho_6(\rho_1 + \rho_2 + \rho_3 + \rho_4) \\ -\rho_7(\rho_1 + \rho_2 + \rho_3 + \rho_4) \\ -\rho_8(\rho_1 + \rho_2 + \rho_3 + \rho_4) \end{bmatrix}$$

$$\lambda_7 = -\epsilon_3(\rho_1 + \rho_2 + \rho_3 + \rho_4 + \rho_5 + \rho_6 + \rho_7 + \rho_8) = -\epsilon_3$$

One eigenvector of this type.

The rule of formation for the eigenvalues is clear from the example above. Each eigenvalue is a sum of terms

$$\lambda_i = - \sum_{j=l(i)}^n \epsilon_j \sum_k \rho_k \quad (6.6)$$

where $\epsilon_{l(i)}$ is the first ϵ which covers all the non-zero elements of the vector and the sum $\sum_k \rho_k$ contains all the probability densities of the states reached by an ϵ_j -jump. A solution of Eq.(6.5) is an arbitrary superposition

$$\rho(t) = \sum c_i e^{-\lambda_i t} v_i \quad (6.7)$$

of the eigenvectors above. From (6.7) it is now easy to construct the general solution of the Chapman-Kolmogorov equation. To obtain the transition probability $p(jt|i0)$ from the state i at time zero to the state j at time t , one chooses the coefficients c_i in (6.7) in such a way that, at time zero, only the i -th component is non-zero and then read the value of the j -th component at time t . Before stating the general result we illustrate it by writing the transition probabilities for transitions between typical states in each group for the case $n=3$.

$$\begin{aligned} p(1t|10) &= \frac{\rho_1}{\rho_1 + \dots + \rho_8} + \frac{\rho_1 \rho_2}{\rho_1(\rho_1 + \rho_2)} e^{-t\lambda_1} + \frac{\rho_1(\rho_3 + \rho_4)}{(\rho_1 + \rho_2)(\rho_1 + \rho_2 + \rho_3 + \rho_4)} e^{-t\lambda_5} + \\ &\quad \frac{\rho_1(\rho_5 + \dots + \rho_8)}{(\rho_1 + \dots + \rho_4)(\rho_1 + \dots + \rho_8)} e^{-t\lambda_7} \\ p(2t|10) &= \frac{\rho_2}{\rho_1 + \dots + \rho_8} - \frac{\rho_2 \rho_1}{\rho_1(\rho_1 + \rho_2)} e^{-t\lambda_1} + \frac{\rho_2(\rho_3 + \rho_4)}{(\rho_1 + \rho_2)(\rho_1 + \rho_2 + \rho_3 + \rho_4)} e^{-t\lambda_5} + \end{aligned}$$

$$\begin{aligned}
& \frac{\rho_2(\rho_5+\dots+\rho_8)}{(\rho_1+\dots+\rho_4)(\rho_1+\dots+\rho_8)} e^{-t\lambda_7} \\
p(3t|10) &= \frac{\rho_3}{\rho_1+\dots+\rho_8} - \frac{\rho_3(\rho_1+\rho_2)}{(\rho_1+\rho_2)(\rho_1+\rho_2+\rho_3+\rho_4)} e^{-t\lambda_5} + \\
& \frac{\rho_3(\rho_5+\dots+\rho_8)}{(\rho_1+\dots+\rho_4)(\rho_1+\dots+\rho_8)} e^{-t\lambda_7} \\
p(5t|10) &= \frac{\rho_5}{\rho_1+\dots+\rho_8} - \frac{\rho_5(\rho_1+\dots+\rho_4)}{(\rho_1+\dots+\rho_4)(\rho_1+\dots+\rho_8)} e^{-t\lambda_7} \tag{6.8}
\end{aligned}$$

Of course in this case $\rho_1+\dots+\rho_8=1$ but we have kept this term to emphasise the rule of formation of the coefficients. The general rule for the transition probability $p(jt|i0)$ between two states i and j at the level n in the tree is the following:

(i) $p(jt|i0)$ is a sum of terms, the first of which is ρ_j (the target probability), and has as many terms as the number of eigenvectors that have non-zero elements in both the i and the j positions.

$$p(j \ t|i \ 0) = \rho_j + \sum_k c_k e^{-t \lambda_k} \tag{6.9}$$

(ii) The exponential factor in each term contains the eigenvalue of the associated eigenvector.

(iii) The coefficients all contain in the numerator the target probability ρ_j multiplied by the sum of the probabilities of non-zero elements of the corresponding eigenvectors in the half that does not contain j . The denominator is the sum of the half that contains i multiplied by the sum of all probabilities associated to the non-zero elements of the eigenvector.

(iv) The sign of the coefficient is the product of the signs of the i and j entries in the eigenvector.

Here I have dealt with an asymmetric stochastic process on a finite tree. If however the regular tree is infinite the appropriate labelling set for the states is a field $\mathbb{Q}_p$ of p -adic numbers. The stochastic processes on the infinite tree are stochastic

processes in $\mathbb{Q}_p$ [32]. Processes on finite trees may actually be obtained from the processes on $\mathbb{Q}_p$ by considering p-adic balls of appropriate radius.

Time correlations in the turbulent cascade

Once the transition probabilities $p(zt|y0)$, solutions of the Chapman-Kolmogorov equation are known, the time correlations of the process are obtained from

$$\langle x(t) x(0) \rangle = \int dy dx \ y \ p(yt|x0) \ x \ \rho(x) \quad (6.10)$$

or

$$\langle x(t)x(0) \rangle = \sum_{i,j} x_j \ p(jt|i0) \ x_i \ \rho_i \quad (6.11)$$

for a discrete state space. Using the results derived above (Eq.(6.9)) one sees that at large times the time correlation at level n will be dominated by the largest non-zero eigenvalue $\lambda_{2n-1} = -\epsilon_n$. Assuming that the dynamics of the turbulent cascade is controlled by the decay of the eddies, the largest non-zero eigenvalue will always be the same, associated to the mean lifetime of large eddies. However the asymptotic long-time correlation will be difficult to measure because of the small values of $\langle x(t)x(0) \rangle$ at large t . If, as proposed, the time correlations in the turbulent cascade are described by a stochastic process with kernels that depend on (ultrametric) tree distances, a first qualitative prediction is the occurrence of several exponential slopes as the time increases. Of special interest is the slope of the short-time correlation which is controlled by the smallest eigenvalue. Using the dyadic expansion to label the points x_i in state space

$$x_i = \beta_0^{n_0(i)} \beta_1^{n_1(i)} \quad (6.12)$$

$$\rho_i = \gamma^{n_0(i)} (1-\gamma)^{n_1(i)} \quad (6.13)$$

where $n_0(i)$ and $n_1(i)$ are the number of zeros and ones in the dyadic expansion of i . Assuming $\gamma > (1-\gamma)$, the smallest eigenvalue for the dynamics at level n is

$$\lambda_1^{(n)} = -\left\{ \epsilon_1^{(n)}(\rho_1^{(n)} + \rho_2^{(n)}) + \epsilon_2^{(n)}(\rho_3^{(n)} + \rho_4^{(n)}) + \epsilon_3^{(n)}(\rho_5^{(n)} + \dots + \rho_8^{(n)}) + \right. \\ \left. \epsilon_4^{(n)}(\rho_9^{(n)} + \dots + \rho_{16}^{(n)}) + \dots \right\} \quad (6.14)$$

If the dynamics of the turbulent cascade is associated to the decay of the eddies of different sizes, it is reasonable to assume that

$$\epsilon_i^{(n-1)} = \epsilon_{i+1}^{(n)} \quad (6.15)$$

Using this relation and the relations between the probability densities at the levels n and $n-1$ one obtains

$$\lambda_1^{(n)} - \lambda_1^{(n-1)} = -(\epsilon_1^{(n)} - \epsilon_1^{(n-1)}) \rho_1^{(n-1)} \quad (6.16)$$

One concludes that the ratio of short-time correlations measures the difference between the lifetimes of the structures at different length scales. If the dynamics of the cascade is controlled by the decay of the eddies and these have different lifetimes at different scales, the ultrametric stochastic model is an appropriate way to parametrise the dynamics and to characterise it in quantitative terms. Other models yield different correlation structures. Notice that here we are concerned with the time fluctuations of the turbulent cascade itself, not with the changes induced by the overall motion of the fluid. This means that for a fluid in motion with free-stream velocity $\vec{U}$ the correlations to measure, for an observable Δ , are $\langle \Delta(\mathbf{x} + \vec{U}t, t) \Delta(\mathbf{x}, 0) \rangle$.

7. A short introduction to q-bialgebras

The concept of symmetry plays an essential role in the description of physical phenomena. In most cases this symmetry is related to covariance under the transformations induced by a Lie algebra. A generalisation of this mathematical structure, the q -deformed (or quantum) algebras, has recently emerged^[33–39]. q -deformed algebras, first discovered in the context of integrable lattice models, were later identified as an underlying mathematical structure in topological field theories^[40] and rational conformal field theories^[41]. Other attempts to apply the notion of q -deformed algebras cover a wide range of different domains, from space–time symmetries^[42–44] to gauge fields^[45], to quantum chemistry^[46].

The usual way to introduce q -algebras is through the discussion of the symmetries of exactly integrable models. Here, for the sake of those less familiar with exactly integrable models, I have decided to use the q -calculus approach.

q -calculus $\equiv$ calculus of finite differences on an exponential net

Consider the following finite-difference operators

$$D_+ f(x) = \frac{f(x_{n+1}) - f(x_n)}{x_{n+1} - x_n} \quad (7.1)$$

$$D_- f(x) = \frac{f(x_n) - f(x_{n-1})}{x_n - x_{n-1}} \quad (7.2)$$

$$D_0 f(x) = \frac{f(x_{n+1/2}) - f(x_{n-1/2})}{x_{n+1/2} - x_{n-1/2}} \quad (7.3)$$

For numerical integration, for example, one in general considers a uniform net $x_n = a + nh$. However to explore very large or very small scales it is conceivable that an exponential net might be more convenient, that is, $x_n = a q^{2n}$. Then the finite-difference operators take the form:

$$D_+ f(x) = \frac{f(q^2 x) - f(x)}{(q^2 - 1)x} \quad (7.4)$$

$$D_- f(x) = \frac{f(x) - f(xq^{-2})}{(1 - q^{-2})x} \quad (7.5)$$

$$D_0 f(x) = \frac{f(qx) - f(xq^{-1})}{(q - q^{-1})x} \quad (7.6)$$

Applying the symmetric difference operator to a power of x

$$D_0 x^n = \frac{q^n - q^{-n}}{q - q^{-1}} x^{n-1} = [n]_q x^{n-1} \quad (7.7)$$

The rule of derivation of powers is similar to the usual one if one defines *q-numbers* $[n]_q = (q^n - q^{-n})/(q - q^{-1})$. Notice that $[n]_q \rightarrow 1$ when $q \rightarrow 1$.

Some properties of q-numbers

$$\begin{aligned} [-n]_q &= -[n]_q & [n]_{q^{-1}} &= [n]_q \\ [a+b]_q &= [a]_q q^b + q^{-a} [b]_q \\ [a+1]_q [b+1]_q &- [a]_q [b]_q = [a+b+1]_q \\ [a]_q [b+c]_q &= [a+c]_q [b]_q + [a-b]_q [c]_q \\ [a]_q^2 - [b]_q^2 &= [a-b]_q [a+b]_q \end{aligned}$$

From now on, whenever no confusion arises, I will drop the subscript q in $[n]_q$, writing only $[n]$. The Leibnitz rule for q -calculus is

$$D_0 \{f_1(x)f_2(x)\} = f_1(qx) D_0 f_2(x) + f_2(q^{-1}x) D_0 f_1(x)$$

$$= f_1(q^{-1}x) D_0 f_2(x) + f_2(qx) D_0 f_1(x)$$

We now use the q -calculus derivative operator to define the standard algebras. Define creation and annihilation operators in the usual way

$$a^\dagger = x \quad a = D_0$$

They act on the basis

$$|n\rangle = \frac{x^n}{\sqrt{[n]!}} \quad [n]! = [n][n-1] \dots [1] \quad (7.8)$$

as follows

$$a^\dagger |n\rangle = \sqrt{[n+1]} |n+1\rangle \quad a |n\rangle = \sqrt{[n]} |n-1\rangle \quad (7.9)$$

Together with the number operator, $N |n\rangle = n |n\rangle$, they form the q -deformed Heisenberg algebra

$$a a^\dagger - q^{-1} a^\dagger a = q^N \quad (7.10a)$$

$$[N, a^\dagger] = a^\dagger \quad (7.10b)$$

$$[N, a] = -a \quad (7.10c)$$

q -calculus is symmetric for the replacement $q \rightarrow q^{-1}$. Hence

$$a a^\dagger - q a^\dagger a = q^{-N} \quad (7.10d)$$

In the undeformed case, the algebra of $SU(2)$ may be obtained from a pair of mutually commuting boson operators (Jordan-Schwinger map)

$$J_+ = a_1 a_2^\dagger \quad J_- = a_2 a_1^\dagger \quad J_z = \frac{1}{2} (N_1 - N_2) \quad (7.11)$$

Using the same definition with q -deformed creation and annihilation operators one obtains the $SU_q(2)$ algebra

$$[J_3^{(q)}, J_\pm^{(q)}] = \pm J_\pm^{(q)} \quad (7.12a)$$

$$[J_+^{(q)}, J_-^{(q)}] = [2 J_3^{(q)}]_q = \frac{q^{2J_3^{(q)}} - q^{-2J_3^{(q)}}}{q - q^{-1}} \quad (7.12b)$$

The $SU_q(2)$ irreducible representations have a structure similar to the $SU(2)$ case

$$J_3^{(q)} |j, m\rangle = m |j, m\rangle \quad (7.13a)$$

$$J_+^{(q)} |j, m\rangle = \sqrt{[j-m][j+m+1]} |j, m+1\rangle \quad (7.13b)$$

$$J_-^{(q)} |j\ m\rangle = \sqrt{[j+m][j-m+1]} |j\ m-1\rangle \quad (7.13c)$$

with Casimir operator

$$C_2 = J_-^{(q)} J_+^{(q)} + [J_3^{(q)} + \frac{1}{2}]^2 q$$

$$C_2 |j\ m\rangle = [j + \frac{1}{2}]^2 q |j\ m\rangle$$

The construction leading to $SU_q(2)$ may be generalised to all classical simple algebras. Let $\{X_i^\pm, H_i\}$ be the generators in the Chevalley-Weyl basis and (a_{ij}) the Cartan matrix. Then the q -deformed algebra is

$$[H_i, H_j] = 0 \quad [H_j, X_i^\pm] = a_{ij} X_i^\pm$$

$$[X_i^+, X_j^-] = \delta_{ij} [H_i]_q \quad \text{or} \quad [X_i^+, X_i^-] = 0 \quad \text{if } a_{ij} = 0$$

The q -deformation of the Lie algebras is not, in a strict sense, a deformation of the algebra because the commutation relations involve elements that are not in the algebra but in the universal enveloping algebra. Besides the commutation relations what else changes in the algebraic structure, under q -deformation? Suppose that we want to couple two of the q -deformed angular momentum representation defined in (7.13). One finds that for the commutation relations to be maintained for the total angular momentum, they have to be coupled in the following way:

$$J_3^{(q)}(1,2) = J_3^{(q)}(1) \otimes 1 + 1 \otimes J_3^{(q)}(2)$$

$$J_\pm^{(q)}(1,2) = J_\pm^{(q)}(1) \otimes q^{J_3^{(q)}(2)} + q^{-J_3^{(q)}(1)} \otimes J_\pm^{(q)}(2)$$

Therefore, besides the modification in the algebra product $\mathcal{A} \otimes \mathcal{A} \rightarrow \mathcal{A}$, there is also a modification in the coupling operation $\mathcal{A} \rightarrow \mathcal{A} \otimes \mathcal{A}$. The coupling of angular momenta is an example of a general operation called *comultiplication*. One sees that q -deformation involves in fact an algebraic structure with two operations, that is, a *bialgebra*. More precisely a q -deformed algebra (or q -bialgebra) is a q -deformation of a Hopf algebra.

Hopf algebras

A Hopf algebra is a bialgebra $(\mathcal{A}, m, \eta, \Delta, \epsilon)$ with an antipode S . m denotes the multiplication, Δ the comultiplication, η the unit and ϵ the counit. These operations are defined by the following commutative diagrams

$$m : \mathcal{A} \otimes \mathcal{A} \rightarrow \mathcal{A}$$

$$\eta : \mathcal{C} \rightarrow \mathcal{A}$$

m and η are linear

$$\begin{array}{ccc}
 \mathcal{A} \otimes \mathcal{A} \otimes \mathcal{A} & \xrightarrow{m \otimes 1} & \mathcal{A} \otimes \mathcal{A} \\
 \downarrow 1 \otimes m & & \downarrow m \\
 \mathcal{A} \otimes \mathcal{A} & \xrightarrow{m} & \mathcal{A}
 \end{array}
 \qquad
 \begin{array}{ccccc}
 & & \eta \otimes 1 & & 1 \otimes \eta \\
 \mathcal{C} \otimes \mathcal{A} & \xrightarrow{\quad} & \mathcal{A} \otimes \mathcal{A} & \xleftarrow{\quad} & \mathcal{A} \otimes \mathcal{C} \\
 \searrow \sim & & \downarrow m & & \nearrow \sim \\
 & & \mathcal{A} & &
 \end{array}$$

$$\Delta : \mathcal{A} \rightarrow \mathcal{A} \otimes \mathcal{A}$$

$$\epsilon : \mathcal{A} \rightarrow \mathcal{C}$$

Δ and ϵ are linear and $\Delta(ab) = \Delta(a) \Delta(b)$

$$\begin{array}{ccc}
 \mathcal{A} \otimes \mathcal{A} \otimes \mathcal{A} & \xleftarrow{\Delta \otimes 1} & \mathcal{A} \otimes \mathcal{A} \\
 \uparrow 1 \otimes \Delta & & \uparrow \Delta \\
 \mathcal{A} \otimes \mathcal{A} & \xleftarrow{\Delta} & \mathcal{A}
 \end{array}
 \qquad
 \begin{array}{ccccc}
 & & \epsilon \otimes 1 & & 1 \otimes \epsilon \\
 \mathcal{C} \otimes \mathcal{A} & \xleftarrow{\quad} & \mathcal{A} \otimes \mathcal{A} & \xrightarrow{\quad} & \mathcal{A} \otimes \mathcal{C} \\
 \searrow \sim & & \uparrow \Delta & & \nearrow \sim \\
 & & \mathcal{A} & &
 \end{array}$$

$$S : \mathcal{A} \rightarrow \mathcal{A}$$

$$m(S \otimes 1)\Delta = m(1 \otimes S)\Delta = \eta \circ \epsilon$$

Examples of Hopf algebras

(i) Continuous functions on a topological group G

$$(f \cdot h)(g) = f(g) h(g)$$

$$(\Delta(f))(g_1 \otimes g_2) = f(g_1 g_2)$$

$$\eta(x) = x \cdot 1 \quad x \in \mathbb{C}$$

$$\epsilon(f) = f(e)$$

$$S(f)(g) = f(g^{-1})$$

is a commutative Hopf algebra.

(ii) Universal enveloping algebra $\mathfrak{U}(\mathfrak{L})$ of a Lie algebra $\mathfrak{L}$

multiplication in $\mathfrak{U}(\mathfrak{L})$

$$\Delta(x) = x \otimes 1 + 1 \otimes x$$

$$\eta(\alpha) = \alpha \cdot 1$$

$$\epsilon(1) = 1 \quad \text{and zero otherwise}$$

$$S(x) = -x$$

is a co-commutative Hopf algebra.

8. q -deformed Brownian motion^[47]

The q -deformation idea leads to interesting structures in the context of Lie algebras and superalgebras. It is therefore natural to ask whether the notion of q -deformation can also be extended to other (non-algebraic) mathematical structures. Here I will extend it to stochastic processes. The starting point is the well-known embedding of Brownian motion in the Fock space of bosonic free fields in one dimension^[48,49]. Extending this correspondence to a one-parameter family of creation and annihilation operators satisfying a q -deformed algebra one establishes a q -deformation of Brownian motion.

Because q -deformation of the boson operators is invariant under the replacement $q \rightarrow q^{-1}$, the algebra may be written in an explicitly symmetric form

$$aa^\dagger - \frac{1}{2}(q + q^{-1}) a^\dagger a = \frac{1}{2}(q^N + q^{-N}) \quad (8.1)$$

Consider now a family $\{a_\tau, a_\tau^\dagger\}$ of q -deformed operators labelled by a continuous time parameter and a scalar field

$$\phi_\tau = a_\tau + a_\tau^\dagger \quad (8.2)$$

For the $\{a_\tau, a_\tau^\dagger\}$ family generalise the relations (8.1) and (7.10b,c) to

$$a_{\tau_1} a_{\tau_2}^\dagger - \frac{1}{2}(q + q^{-1}) a_{\tau_2}^\dagger a_{\tau_1} = \frac{1}{2}(q^N + q^{-N}) \delta(\tau_1 - \tau_2) \quad (8.3)$$

$$[N, a_\tau^\dagger] = a_\tau^\dagger \quad [N, a_\tau] = -a_\tau \quad (8.4)$$

This is the simplest extension of the relations to a family of q -deformed operators labelled by a continuous parameter. Other generalisations are possible, involving for example braid relations at different times. Notice also that, for the purpose of construction of a stochastic process, no assumptions are needed concerning the commutation properties of $a_{\tau_1} a_{\tau_2}$ and $a_{\tau_1}^\dagger a_{\tau_2}^\dagger$ at different times.

Smearing the fields with characteristic functions $\chi_{[0,t]}$ of the interval $[0,t]$

$$a_q(t) = a_q(\chi_{[0,t]}) = \int_0^t d\tau a_\tau \quad (8.5a)$$

$$a_q^\dagger(t) = a_q^\dagger(\chi_{[0,t]}) = \int_0^t d\tau a_\tau^\dagger \quad (8.5b)$$

$$\phi_q(t) = \phi_q(\chi_{[0,t]}) = \int_0^t d\tau \phi_\tau \quad (8.6)$$

the algebraic relations become

$$a_q(t_1) a_q^\dagger(t_2) - \frac{1}{2}(q + q^{-1}) a_q^\dagger(t_2) a_q(t_1) = \frac{1}{2}(q^N + q^{-N}) \langle \tau_1 | \tau_2 \rangle \quad (8.7)$$

where $\langle \tau_1 | \tau_2 \rangle = \min(t_1, t_2)$.

One now uses (8.7) to construct a q -deformation of Brownian motion. Let (Ω, F_t, μ, B_t) be the usual Brownian motion. Ω is the set of continuous functions vanishing at $t=0$, μ is the Wiener measure and F_t is the σ -ring generated by $\{B_s: 0 \leq s \leq t\}$. On the other hand let $(\mathfrak{H}, A_t, \psi_0, \phi_1(t))$ be the free quantum field over $K = L^2([0, \infty), \mathbf{R})$. $\phi_1(t) = \phi_1(\chi_{[0,t]})$ (for $q=1$), $\mathfrak{H}$ is the symmetric Fock space over K , ψ_0 is the Fock vacuum and A_t is the W^* -algebra generated by $\{\phi_1(s): 0 \leq s \leq t\}$. Then^[49], interpreting B_t as a multiplication operator in $L^2(\Omega, F_t, \mu)$, there is a unitary operator $V : L^2(\Omega, F_t, \mu) \rightarrow \mathfrak{H}$ such that $VB_tV^{-1} = \phi(t)$. I.e. $\phi(t)$ as a stochastic process with expectation

$$E(f(\phi(t))) = \langle \psi_0, f(\phi(t)) \psi_0 \rangle \quad (8.8)$$

coincides with Brownian motion. For this identification of the free scalar field with Brownian motion it is useful to characterise the filtration A_t by the conditional

expectation of Wick products^[50]

$$E(: \phi_1(u_1) \dots \phi_1(u_n) : | A_t) = : \phi_1(x_{[0,t]}^{u_1}) \dots \phi_1(x_{[0,t]}^{u_n}) : \quad (8.9)$$

Recall that the Wick products span the algebra generated by $\phi_1(u)$. Hence, by linearity, definition on Wick products suffices to define conditional expectations on the complete algebra.

One now uses a minimal version of this correspondence to define q -deformed Brownian motion.

Definition : q -deformed Brownian motion is the process $(\Omega, F_t, \mu_{\psi_0}, \phi_q(t))$ where

- (i) $\phi_q(t)$ is the operator defined by (8.5 – 8.7)
- (ii) Expectations of field functionals $f(\phi_q)$ are obtained by

$$E(f(\phi_q)) = \langle \psi_0, f(\phi_q) \psi_0 \rangle, \quad \psi_0 \text{ being defined by } a_q \psi_0 = 0 \quad (8.10)$$

- (iii) The filtration F_t is characterised by the conditional expectations of Wick products

$$E(: \phi_q(t_1) \dots \phi_q(t_n) : | F_s) = : \phi_q(x_{[0,s]}^{t_1}) \dots \phi_q(x_{[0,s]}^{t_n}) : \quad (8.11)$$

Notice that the algebraic relations (8.7) allow all elements of the algebra generated by $\phi_q(t)$ to be reduced to Wick products, hence all conditional expectations may be computed. Notice also that in this minimal definition the family F_t of measurable events is fixed in advance and we avoid an explicit realisation of the probability space Ω .

Theorem 8.1 : q -deformed Brownian motion has :

- (i) zero mean, $E(\phi_q(t)) = 0$
- (ii) variance $E(\phi_q(t) \phi_q(s)) = \min(s, t)$
- (iii) independent increments in the sense

$$E(\{\phi_q(t_1) - \phi_q(t_2)\} \{\phi_q(t_3) - \phi_q(t_4)\}) = E(\phi_q(t_1) - \phi_q(t_2)) E(\phi_q(t_3) - \phi_q(t_4)) = 0$$

if there is no overlap between the intervals $[t_1, t_2]$ and $[t_3, t_4]$

- (iv) is a martingale

Properties (i) to (iii) follow by a simple computation using (8.6) (8.7) and (8.10).

The martingale property is a consequence of (8.11). If $s < t$

$$E(\phi_q(t) \mid F_s) = \phi_q(x_{[0,s]} x_{[0,t]}) = \phi_q(s)$$

Theorem 8.1 summarises the similarities of q -deformed Brownian motion to the usual Brownian motion. The next result displays their main differences as well as an explicit characterisation in terms of cumulants.

Theorem 8.2 :

- (i) q -deformed Brownian motion is not a Gaussian process
- (ii) The cumulants are

$$E_T(\phi_q(t_1) \dots \phi_q(t_n)) = \sum c_{(i_1 i_2) \dots (i_{n-1} i_n)} \langle t_{i_1} \mid t_{i_2} \rangle \dots \langle t_{i_{n-1}} \mid t_{i_n} \rangle \quad (8.12)$$

where $\langle t_i \mid t_j \rangle = \min(t_i, t_j)$, the sum is over all $\binom{n}{2}$ different partitions of the set $(t_1 \dots t_n)$ into pairs and the coefficients $c_{(i_1 i_2) \dots (i_{n-1} i_n)}$ are obtained by the following graphical rules

- (ii.a) For each term in (8.12) one draws the $\langle t_{i_k} \mid t_{i_{k+1}} \rangle$ contractions as follows

- (ii.b) For each crossing of lines there is a factor

$$\left\{ \frac{1}{2}(q + q^{-1}) - 1 \right\} \text{ in } c_{(i_1 i_2) \dots (i_{n-1} i_n)}$$

- (ii.c) For each contraction contained at depth α inside other contractions there is a factor

$$\left\{ \frac{1}{2}(q^\alpha + q^{-\alpha}) - 1 \right\} \text{ in } c_{(i_1 i_2) \dots (i_{n-1} i_n)}$$

- (ii.d) If there are no crossings nor inner contractions the coefficient $c_{(i_1 i_2) \dots (i_{n-1} i_n)}$ vanishes.

In the example (8.13) the contribution of the diagram to the coefficient is

$$\{\frac{1}{2}(q+q^{-1}) - 1\}^2 \{\frac{1}{2}(q^2+q^{-2}) - 1\} \{\frac{1}{2}(q+q^{-1}) - 1\}$$

the first factor coming from the crossings and the last two from the inner contractions at level 2 and 1.

A necessary and sufficient condition for a process to be Gaussian is that it possesses cumulants of all orders and that they vanish for order higher than 2. Computing the 4-time correlation one obtains using (8.7) and (8.10)

$$E(t_1 t_2 t_3 t_4) = E(\phi_q(t_1) \phi_q(t_2) \phi_q(t_3) \phi_q(t_4)) =$$

$$= \langle t_1 | t_2 \rangle \langle t_3 | t_4 \rangle + \frac{1}{2}(q+q^{-1}) \langle t_1 | t_3 \rangle \langle t_2 | t_4 \rangle + \frac{1}{2}(q+q^{-1}) \langle t_1 | t_4 \rangle \langle t_2 | t_3 \rangle$$

implying that the cumulant

$$E_T(t_1 t_2 t_3 t_4) = E(t_1 t_2 t_3 t_4) - E(t_1 t_2)E(t_3 t_4) - E(t_1 t_3)E(t_2 t_4) - E(t_1 t_4)E(t_2 t_3)$$

does not vanish. Hence the process is not Gaussian.

The explicit expression for the cumulants of arbitrary orders is obtained by systematic reduction of the expectation values using the algebraic relations (8.7). The “crossing lines” factor comes from the coefficient of the second term in the l.h.s. of (8.7) and the “inner contractions” factor from the r.h.s. together with (8.4).

Bozejko and Speicher^[51] have also proposed a generalisation of Brownian motion using a modified algebra of boson operators. The algebra they use, however, is not the q -deformed Heisenberg algebra but the quon commutation relations of Greenberg^[52] which interpolate between bosons and fermions.

As a final remark I point out that using q -fermions b and $b^\dagger$ with

$$bb^\dagger + q b^\dagger b = q^M ; \quad [M, b^\dagger] = b^\dagger ; \quad [M, b] = -b \quad (8.14)$$

and a generalisation along the lines of (8.2 – 8.7) one may construct a q -deformation of the non-commutative Clifford process^[50].

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