The Effect of Dynamical Gauge Field on the Chiral Fermion on a Boundary

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Abstract

We study the effect of dynamical gauge field on the Kaplan’s chiral fermion on a boundary in the strong gauge coupling limit in the extra dimension. To all orders of the hopping parameter expansion, we prove exact parity invariance of the fermion propagator on the boundary. This means that the chiral property of the boundary fermion, which seems to survive even in the presence of the gauge field from a perturbative point of view, is completely destroyed by the dynamics of the gauge field.

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1 Introduction

The construction of a chiral gauge theory on a lattice is one of the long-lasting problems in field theory. The difficulty can be summarized in the famous no–go theorem by Nielsen and Ninomiya [1], which states that on a 2n–dimensional lattice the theory has to be vector–like under some fundamental assumptions such as locality, positivity, hermiticity and charge conservation.

Among various attempts to construct a chiral gauge theory, Kaplan’s proposal [2] seems one of the most promising ones at present and has been studied intensively for these two years [3]. The idea is to consider a (2n + 1)–dimensional Dirac fermion with a domain–wall type mass in the extra dimension, which yields a chiral zeromode on the domain wall. Equivalently one may consider a (2n + 1)–dimensional Dirac fermion with an ordinary constant mass but on the space with a boundary [4], which plays the same role as the domain wall in the Kaplan’s model. Our argument in this letter can be applied equally well to both the models, but to make the explanation as simple as possible, we state only for the case of the latter model.

To be concrete, let us consider a (2n + 1)–dimensional system with a boundary and denote the 2n–dimensional coordinate by \( x^\mu \) (\( \mu = 0, 1, \ldots, 2n - 1 \)) and the extra dimensional coordinate by \( s \). When one considers the system to have a finite extension in the extra dimension \( 0 \leq s \leq L \), one has two chiral zeromodes with opposite handedness on the boundaries \( s = 0 \) and \( s = L \). This fact can be regarded as a natural conclusion of Nielsen–Ninomiya’s theorem. But since the chiral fermions with opposite handedness live apart in the extra dimension, they are decoupled as long as the gauge field is treated as a background. A promising feature of the model is that the system allows a natural realization of fermion number anomaly through the flow of the fermion number current off the boundary [5], in contrast to other attempts such as the one using the Wilson–Yukawa system.

The problem is of course what happens if one couples the dynamical (2n + 1)–dimensional gauge field to the system. Let us denote the lattice gauge coupling in the extra dimension \( g_s \). When \( g_s = 0 \), adopting the axial gauge in the axis of the \( s \)–direction, one has copies of gauge configurations for each constant–s plane and can prove that the chiral zeromode exists on the boundary. Moreover the freedom of the gauge field in the extra dimension is completely fixed and the (2n + 1)-dimensional system of the gauge field is reduced to 2n-dimensional,
which sounds desirable for obtaining a $2n$-dimensional chiral gauge theory. There seems to be no problem if there is only one boundary at $s = 0$, that is, if the system is defined from the beginning to extend infinitely in the positive $s$-direction [6]. However, if one tries to define the system by first considering a finite system with the extension $0 \leq s \leq L$ and then take the $L \to \infty$ limit, one has another chiral zeromode with the opposite handedness on the other boundary at $s = L$, which can never be decoupled even in the $L \to \infty$ limit as long as $g_s = 0$. The system constructed here is nothing but a $2n$-dimensional vector gauge theory. Instead of setting $g_s = 0$, one might introduce some $L$-dependent parameters in the action and fine-tune them when one increases $L$ to infinity so that the opposite fermions can decouple and the relevant gauge field configurations for the chiral fermion may become effectively $2n$-dimensional. Although this is an interesting possibility, since we do not know a priori how many parameters we have to fine-tune, it might require many trials and errors in order to see if the scenario really works.

One may take another extreme case, the large $g_s$ case, where the correlation length of the gauge field is finite in the $s$-direction and therefore the dynamics of the gauge field can be regarded as $2n$-dimensional. This limit is against the continuum limit in the $s$ direction but it doesn’t lead us to throw away this possibility because, in the absence of the gauge field, the existence of the chiral zeromode is guaranteed even when the fermion has only finite correlation in the $s$ direction. Therefore, if the chiral zeromode survives after taking account of the dynamical gauge field, the whole system can be regarded as a desirable $2n$-dimensional chiral gauge theory. This possibility has been studied through the mean field analysis [7], but the conclusion is unfortunately a negative one which predicts that all the constant-$s$ planes become independent and that the fermion becomes vector-like in each plane. From a perturbative point of view, however, the presence of the chiral zeromode at the boundary is required from the induced Chern-Simons action in the bulk [8], which is non-renormalizable by the radiative corrections of the gauge field [9]. Thus the chiral zeromode seems to survive even when one considers the dynamics of the gauge field.

In this letter we study the above problem using the hopping parameter expansion in the strong coupling limit. Instead of calculating the fermion propagator explicitly, we define parity transformation in $2n$-dimensions and see if the fermion propagator is invariant under the parity transformation. If the fermion propagator is parity invariant, it means that the fermions are vector-like, actually much more than that, because it means the complete symmetry between the left-handed fermion and the right-handed fermion. Indeed, when
and neglecting the dynamical fermions, we prove that the fermion propagator in each layer is parity invariant.

The paper is organized as follows. In Section 2, we define parity transformation in $2n$-dimensions. In Section 3, we prove that to all orders of the hopping parameter expansion, the fermion propagator is parity invariant. Section 4 is devoted to the summary and the discussion.

## 2 Parity transformation in $2n$-dimensions

In order to obtain a chiral spectrum in $2n$-dimensions, it is necessary to violate the parity invariance. The parity transformation in $2n$-dimensions is defined as follows.

\begin{align}
\Psi & \rightarrow \gamma^0 \Psi \\
\bar{\Psi} & \rightarrow \bar{\Psi} \gamma^0 \\
x^i & \rightarrow -x^i \quad (i = 1, 2, \cdots, 2n - 1),
\end{align}

where $\Psi$ is the Dirac spinor in $2n$-dimensions.

We can see, for example, that the Dirac Lagrangian in $2n$-dimensions

\[ \mathcal{L} = i \bar{\Psi} \partial_{\mu} \gamma^\mu \Psi - m \bar{\Psi} \Psi \]

is invariant under the parity transformation.

In the present formulation, the chiral fermion in $2n$-dimensions is realized on a boundary of $(2n + 1)$-dimensional space. We, therefore, study the parity invariance (in the $2n$-dimensional sense) of the correlation functions

\[ \left\langle \Psi(x^0, x^i, s = 0) \bar{\Psi}(y^0, y^i, s = 0) \right\rangle = \left\langle \gamma^0 \Psi(x^0, -x^i, s = 0) \bar{\Psi}(y^0, -y^i, s = 0) \gamma^0 \right\rangle. \]

If this condition is satisfied, the spectrum is completely parity invariant which means that a chiral fermion cannot appear on the boundary.

It is instructive to see whether the condition can be satisfied due to the symmetry of the action. When there is no boundary (i.e., a system of the standard Dirac fermion in $(2n + 1)$-dimensions), the action has an invariance under the rotation about the $x^0$-axis by
angle $\pi$

\[ \Psi \rightarrow \gamma^0 \Psi \]
\[ \bar{\Psi} \rightarrow \bar{\Psi} \gamma^0 \]  \hspace{1cm} (2.4)

\[ x^i \rightarrow -x^i \quad (i = 1, 2, \cdots, 2n - 1) \]
\[ s \rightarrow -s. \]  \hspace{1cm} (2.5)

This is nothing but the parity transformation in $2n$-dimension (2.2) plus the inversion of the $s$-axis $(s \rightarrow -s)$. Since the inversion of the $s$-axis does not affect the boundary $(s = 0)$, the condition (2.3) is satisfied exactly and there is no chiral mode, as expected.

On the other hand, if there is a boundary, the action is not invariant under the inversion of the $s$-axis and we cannot expect that the condition (2.3) is satisfied, so that a chiral zero mode can appear on the boundary. In fact, the presence of the chiral zero mode can be explicitly shown when there is no gauge field.

## 3 Proof of parity invariance

In this section we prove that, to all orders of the hopping parameter expansion, the fermion propagator is parity invariant, although the action is not, provided that the gauge field coupling in the extra dimension is strong and the dynamical fermions are neglected.

To begin with, let us briefly review the hopping parameter expansion. The lattice action of the system with a $(2n + 1)$-dimensional Dirac fermion coupled to the dynamical gauge field can be written as

\[ S = \sum_{n,\mu} K [\bar{\Psi}_{n+\mu}(\lambda + \gamma^\mu)U_{n,\mu}\Psi_n + \bar{\Psi}_{n-\mu}(\lambda - \gamma^\mu)U_{n,\mu}^\dagger \Psi_n] + \sum_n \bar{\Psi}_n \Psi_n + \beta \sum_{(\mu,\overline{\nu})} U_{\mu} + \beta_s \sum_{(n,\overline{\mu})} U_{n,\overline{\mu}}, \]  \hspace{1cm} (3.1)

where

\[ K = \frac{1}{M + \lambda} \]  \hspace{1cm} (3.2)
is the hopping parameter and $M$, $\lambda$ are the mass and the coefficient of the Wilson term, respectively. $\beta = 1/g^2$ and $\beta_s = 1/g_s^2$ are the gauge coupling constants in the $(\mu, \nu)$ plane in $2n$-dimensions and in the $(s, \mu)$ plane in the extra dimension, respectively. The lattice spacing is set unity throughout this letter.

Let us define the gauge invariant two point fermion Green function by inserting link variables along the line connecting the two points as

$$\langle \Psi_n \prod U \bar{\Psi}_m \rangle. \quad (3.3)$$

Integrating the fermion field gives a summation of

$$K^i(\lambda \pm \gamma^\mu)(\lambda \pm \gamma^{\mu_2}) \cdots (\lambda \pm \gamma^\mu) \prod U \quad (3.4)$$

over the paths connecting the two points $n$ and $m$, where the $\pm$ signs in front of the $\gamma^\mu$'s are determined according to the direction of the arrows along the path (Fig.1). As for the gauge field, we have a product of $U$'s along the loops composed of the path considered in the above summation and the line originally introduce in order to make the fermion propagator gauge invariant. Integrating the link variables, then, gives a summation over the surfaces which have the loop as the boundary.

We now prove the parity invariance of the Green function

$$\langle \Psi_n \prod U \bar{\Psi}_m \rangle = \langle \gamma^0 \Psi_{n'} \prod U \bar{\Psi}_{m', \gamma^0} \rangle, \quad (3.5)$$

where $n'$ and $m'$ are the parity-transformed points of $n$ and $m$, respectively. This equation corresponds to eq.(2.3) in section 2. We prove it by classifying the paths of the fermion line as follows.

1. The case in which the path is restricted on the boundary

When the fermion Green function is parity-transformed, the corresponding fermion path in the transformed Green function can be obtained from the original fermion path by reversing all the $x^i$ coordinates (See Fig.2). Since the parity transformation reverses the direction of the arrows parallel to the $x^i$-axis, the contribution of the transformed path is given by reversing the signs in front of the $\gamma^i$'s in the expression for the contribution of the original path. However, the reversions of the signs in front of the $\gamma^i$'s are cancelled when $\gamma^0$ in the parity transformation passes through the product of $(\lambda \pm \gamma^i)$'s, due to the anti-commutation relation of $\gamma^0$ and $\gamma^i$ as

$$(\lambda - \gamma^0)(\lambda - \gamma^i) \cdots (\lambda - \gamma^0) = \gamma^0(\lambda - \gamma^0)(\lambda + \gamma^i) \cdots (\lambda - \gamma^0)\gamma^0. \quad (3.6)$$
This shows that there is a one-to-one correspondence of the paths contributing equally to the Green function and the parity-transformed one. Hence, as long as we consider paths on the boundary, parity invariance cannot be broken. This result can be naturally understood in the following way. When the fermion propagates on the boundary and not in the direction of the extra dimension, the left–right asymmetry cannot appear since the 2n-dimensional action is invariant under the parity transformation (see section 2).

2. The case in which the path goes around in the extra dimension (Fig. 3)

Since we are working in the strong coupling case, i.e. $\beta_s = 0$, the contribution of the above case vanishes after integrating over link variables. The cases to be considered in order to get a nonzero contribution are the followings.

3. Tube-like propagation in the extra dimension (Fig. 4)

We first consider the case in which the fermion propagates along the $s$-direction and then goes straight back along the same line. The product of $\gamma$-matrices corresponding to this tube-like propagation is

$$ (\lambda - \gamma^{2n})^l (\lambda + \gamma^{2n})^l = (\lambda^2 - 1)^l, $$

where $l$ is the number of times of the hopping in the extra dimension. This is a $c$-number and the $\gamma^0$ in the parity transformation can pass freely through the above expression, which means that the parity invariance is satisfied for this case as well.

4. Tube–like propagation with a closed loop on a constant-$s$ plane (Fig. 5)

We next consider the case in which the fermion propagates along the $s$-direction, moving around a closed loop on a constant-$s$ plane and then goes back along the same line it came along. Let us consider the sum of two such contributions with the following pair of loops, one of which is obtained by rotating the other on the constant-$s$ plane around the point on the tube. The expression to be considered is

$$ (\lambda - \gamma^{2n})^l (\lambda - \gamma^\mu) (\lambda - \gamma^{\mu_2}) \cdots (\lambda - \gamma^{\mu_n}) (\lambda + \gamma^{2n})^l $$

$$ + (\lambda - \gamma^{2n})^l (\lambda + \gamma^\mu) (\lambda + \gamma^{\mu_2}) \cdots (\lambda + \gamma^{\mu_n}) (\lambda + \gamma^{2n})^l, $$(3.8)

where $(\lambda \pm \gamma^{2n})^l$'s come from the tube–like propagation along the $s$-direction and $(\lambda \pm \gamma^\mu)$'s between them come from the propagation along the loops on the constant-$s$ plane. Note that the signs in front of $\gamma^\mu$'s are opposite due to the definition of the pair of loops.
We expand the product of the \((\lambda \pm \gamma^\mu)\)'s into the sum of \(\lambda^{m_1} \gamma^{r_1} \gamma^{r_2} \cdots \gamma^{r_{m_2}}\), where \(m_1 + m_2 = m\). When we consider a term where \(m_2\) is odd, the two contributions cancels. When \(m_2\) is even, the tube part \((\lambda \pm \gamma^{2n})^l\) can pass through this loop part \(\gamma^{r_1} \gamma^{r_2} \cdots \gamma^{r_{m_2}}\), and these tube parts become a \(c\)-number, commutable with \(\gamma^0\). Hence, these paths contribute equally to the Green function and the parity-transformed one as in the case \(I\), and parity invariance is satisfied as well.

5. Many tubes and loops (Fig. 6)

By induction, we can show parity invariance is satisfied as well. This completes the proof of the parity invariance of the fermion Green function.

4 Summary and Discussion

In this letter we have shown that the dynamics of the gauge field renders the fermion propagator vector–like in the strong gauge coupling limit in the extra dimension provided that dynamical fermions are neglected. The result is consistent with the mean field analysis.

When we include the effect of dynamical fermions, we should take account of such diagrams as Fig. 7, which may make the fermion propagator not exactly parity invariant. However, since adding dynamical fermions amounts to adding various types of Wilson loops to the gauge field action, we may say that our argument holds for such models that do not have Wilson loops including links in the \(s\)-direction after integrating dynamical fermions. We should also remind the readers that the parity invariance of the fermion propagator is much stronger than the fermion being vector–like. We might, therefore, expect that the disappearance of the chiral zeromode occurs generally in the strong coupling region of the gauge coupling in the extra dimension. Although our argument confirms that the strong coupling scenario of constructing chiral gauge theory seems unlikely to work, it does not exclude the other possibility in the weak coupling region explained in the Introduction.

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References


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