On primordial magnetic fields

KARI ENQVIST
Nordita
Blegdamsvej 17, DK-2100 Copenhagen, Denmark

May 19, 1994

Abstract

A primordial magnetic field could be responsible for the observed magnetic fields of the galaxies. One possibility is that such a primordial field is generated at the electroweak phase transition because of the fluctuations in the Higgs field gradients. I describe a statistical averaging procedure which gives rise to a field of a correct magnitude. Another possibility, where the Yang-Mills vacuum itself is ferrromagnetic, is also discussed.

enqvist@nbivax.nbi.dk
1 Introduction

The very early universe is in notoriously short supply of observables that would have survived until today. It is likely that the baryon number of the universe and the density perturbations are one. Another, but a much more speculative possibility, might be the magnetic fields of spiral galaxies, the origin of which still largely remains a puzzle. The nearby galaxies have magnetic fields of the order of $B \simeq 10^{-6}$ G [1], which can be deduced from observations of the synchrotron radiation put out by electrons travelling through the fields, assuming equipartition of magnetic and particle energies. Recently, a field of a similar magnitude has been observed also in a object with $z=0.395$ [2].

The model for galactic magnetic fields most studied is the galactic dynamo [4], where differential rotation and turbulence of the ionized gas amplifies a weak seed field by several orders of magnitude. Not much is known about the seed field. As the dynamo growth time of the magnetic field cannot be smaller than the galactic rotation period $\tau \simeq 3 \times 10^8$ yrs, this gives a lower limit of $B_{\text{seed}} \gtrsim 10^{-19}$ G on a comoving scale of the protogalaxy, about 100 kpc. In the Milky Way and the Andromeda Nebula the dynamos appear to be rather weak and the growth time as long as $\tau \simeq 10^9$ yrs which would imply that $B_{\text{seed}} \gtrsim B \exp(-t/\tau) \simeq 10^{-10}$ G. Moreover, in the Milky Way the magnetic field changes its direction by about $180^\circ$ between the Sagittarius and the Orion spiral arms [5], and it has been argued [6] that such a reversal implies a stringent lower bound of $B_{\text{seed}} \gtrsim 10^{-7}$ G on the seed field. As such a reversal has only been observed in the Milky Way, it might not be a generic feature.

One interesting possibility is that the seed field is truly primordial, with an origin that predates nucleosynthesis. In that case the protogalaxy collapsed with a frozen-in magnetic field, which enhanced the primordial cosmological field by a factor of $10^4$ [7]. Thus at the scale of 100 kpc the dynamo mechanism requires a primordial field somewhere in the ballpark of $10^{-18}$ G, with an uncertainty of a few orders of magnitude.

From a theoretical point of view, however, the generation of a sufficiently large persistent magnetic field in the early universe is rather difficult. There are various attempts, relying on phase transitions such as the cosmic inflation or the QCD phase transition [8], but the field often comes out to be too small to be of cosmological interest. It has been suggested that a large field might actually be generated at the electroweak phase transition because of random fluctuations in the Higgs field [9]. If one assumes a stochastic, uncorrelated distribution of the Higgs field gradients, or of the magnetic field itself, one finds [10] today at 100 kpc a root-mean-square field of the order of $10^{-19}$ G, which could well serve as the origin of the seed field. This positive result is based on calculating the statistical averages along an arbitrary curve. This is not the only possibility, but averaging over areas or volumes would produce a field far too small to be of relevance for the dynamo effect.

\footnote{Equipartition may not to be valid for certain irregular galaxies [3].}
In Yang-Mills theories there is also the possibility [11] that the vacuum is an analog of the ferromagnet with a non-zero background magnetic field. This is a non-perturbative effect, and the resulting field is typically very small. If one is willing, however, to go up all the way to the GUT scale one finds that a typical GUT phase transition could have given rise to a background field large enough to serve as the seed field [12].

2 EW magnetic fields

Electromagnetism first occurs when the standard electroweak $SU(2) \otimes U(1)_Y$ theory is broken down to $U(1)_{em}$. It is therefore particularly attractive that Vachaspati [9] has explained the origin of a primordial field in terms of the cosmological boundary condition that all physical quantities should be uncorrelated over distances greater than the horizon distance. Since we start with the group $SU(2) \otimes U(1)_Y$ before the electroweak phase transition, the resulting electromagnetic field can be constructed in a way which is different from the usual $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$. The result is [9]

$$F_{ij} = -i(V_i^j V_j^i - V_i^i V_j^j),$$

$$V_i = \frac{2}{|\phi|} \sqrt{\frac{\sin \theta}{g}} \partial_i \phi,$$

where $\phi$ is the Higgs field. At the electroweak phase transition the correlation length in the broken phase is $\sim 1/m_W$ (assuming that the Higgs mass is comparable to $m_W$). The field $F_{ij}$ is thus constant over a distance $\sim 1/m_W$, but it varies in a random way over larger distances. Its variation is due to the fact that the field $\phi$ makes a random walk on the vacuum manifold of $\phi$. The problem then is to estimate the field $F_{ij}$ over a length scale $\sim N/m_W$. If $N = 1$, it then follows that on dimensional grounds $F_{ij} \sim m_W^2 \sim 10^{24}$ G, with probably an uncertainty of $\pm 1$ in the exponent. For $N$ large, one should use an appropriate statistical argument. The issue is, which is the appropriate way to average over the random fields.

Let us consider random fields walking around in space in a certain number of steps. Thus we replace the continuum by a lattice, where the points are denoted by greek letters $\alpha, \ldots$. I wish to estimate the magnetic field over a linear scale (which at most is equal to the horizon scale). Thus, let us consider a curve consisting of $N$ steps in the lattice and define the mean value

$$\overline{B} = \frac{1}{N} \sum_{i=1}^{N} B^{\alpha_i},$$

where $B$ is a component of the magnetic field, and where the lattice points $\alpha_i$ are on the curve.

Now this curve is arbitrary, and we could take any other curve. Let us therefore define the average $\langle \ldots \rangle$, which averages over curves spanning an $N^3$ lattice, i.e. over
all space. Then, for example,
\[ \langle \mathcal{B} \rangle = \frac{1}{N} \left( \sum_{i=1}^{N} B^{n_i} \right) \],

which means that for each curve with \( N \) steps the mean value \( \mathcal{B} \) is computed, and this is done for a set of curves which span an \( N^3 \)-lattice, and the average is then computed. Therefore \( \langle \mathcal{B} \rangle \) depends in general on \( N \), but for simplicity of notation we shall leave out the explicit reference to this dependence. It should be emphasized that the ensemble average (3) takes into account the field value at each lattice point, so that the average is really over the whole lattice volume [10].

Similarly, one can define higher moments such as
\[ \langle \mathcal{B}^2 \rangle = \frac{1}{N^2} \sum_{i,j=1}^{N} \langle B^{n_i} B^{n_j} \rangle \],

together with quantities like \( \langle (\mathcal{B} - \langle \mathcal{B} \rangle)^2 \rangle \). Note that in (4) the sum is over curves of length \( N \) steps of the non-local quantity \( \langle B^{n_i} B^{n_j} \rangle \).

In [9] the stochastic variable was taken to be the Higgs field itself which varies over the vacuum manifold. Vachaspati argued that the gradients are of order \( 1/\sqrt{N} \), since \( \phi \) makes a random walk on the vacuum manifold with \( \Delta \phi \sim \sqrt{N} \), and since \( \Delta z \sim N \). Thus \( V_i \) is, in a root mean square sense, of the order \( 1/\sqrt{N} \), and hence \( F_{ij} \) is of order \( 1/N \). Taking further into account that the flux in a co-moving circular contour is constant, the field must decrease like \( 1/a(t)^2 \), where \( a(t) \) is the scale factor. Using the fact that in the early universe \( a \) goes like the inverse temperature, the field was then estimated to behave like \( \langle F_{ij} \rangle_T \sim T^2/N \). For a scale of 100 kpc this leads to \( \langle F_{ij} \rangle_{\text{now}} \sim 10^{-30} \text{ G} \), which is far too small to explain the galactic fields (unless there exists some large scale amplification mechanism). One should also point out that this assumption presumes that the total magnetic flux through a given surface is a stochastic sum of the fluxes through the unit cells of that surface, or in other words, that the fluxes through two adjoining unit cells are uncorrelated. Whether this is true or not is an open question.

It is however natural to assume that also the gradient vectors \( V_i \) are stochastic, as was done in [10]. This is because they directly specify whether there is a magnetic field or not, whereas this is only true indirectly for the Higgs field itself. Also, the vectors \( V_i \) are relevant for questions of alignment between neighbouring domains.

### 3 Random Higgs gradients

Consider the expression (1) for the magnetic field in terms of the Higgs gradients \( V_i \). It is convenient to split these fields into real and imaginary parts,
\[ V_i(x) = R_i(x) + i I_i(x) \].

3
where \( R_i \) and \( I_i \) are real vectors. Let us consider the system at a fixed time. The cosmological boundary condition is then that \( R_i \) and \( I_i \) are random fields. Let us make the following assumptions:

(i) The random fields have a Gaussian distribution. Thus, the mean value of some quantity \( Q \) is given by

\[
\langle Q \rangle = \prod_{\alpha,\beta} \int \frac{d^3 R_\alpha}{D} \frac{d^3 I_\beta}{D} Q e^{-\lambda (R_\alpha - \langle R_\alpha \rangle)^2 - \lambda (I_\beta - \langle I_\beta \rangle)^2},
\]

where \( D \) is a normalization factor defined such that \( \langle 1 \rangle = 1 \), and \( \lambda \) is a measure of the inverse width. The quantities \( \overline{R}_i \) and \( \overline{I}_i \) are the mean values of \( R_i \) and \( I_i \) defined along a curve of length \( N \) steps.\(^2\) Thus, eq. (6) is relevant for a 3-dimensional world which is an \( N^3 \) lattice.

(ii) I assume that the mean values are isotropic, i.e. \( \langle \overline{R}_1 \rangle = \langle \overline{R}_2 \rangle = \langle \overline{R}_3 \rangle \) and \( \langle \overline{I}_1 \rangle = \langle \overline{I}_2 \rangle = \langle \overline{I}_3 \rangle \).

Assumption (i) is certainly the simplest way of implementing lack of correlation of the gradient vectors over distances compatible with the horizon scale, whereas assumption (ii) is natural as there is no reason to expect any preferred direction.

It should be noted that the distribution (6) factorizes into an \( I \)-part and an \( R \)-part. Thus, for any expectation value consisting of \( I \)'s and \( R \)'s one has factorization,

\[
\langle R_{i_1} \ldots R_{i_n} \, I_{j_1} \ldots I_{j_m} \rangle = \langle R_{i_1} \ldots R_{i_n} \rangle \langle I_{j_1} \ldots I_{j_m} \rangle.
\]

This property turns out to be very useful in computing the higher moments.

The expectation value of a component \( B_i \) of the magnetic field can now easily be found. From the expression (1) one finds that

\[
B_i = \frac{1}{2} \varepsilon_{ijk} F_{jk} = -i \varepsilon_{ijk} V_j V_k = 2 \varepsilon_{ijk} R_j I_k.
\]

Thus

\[
\overline{B}_j = \frac{1}{N} \sum_{i=1}^{N} B_{i,j} = 2 \varepsilon_{jk} \frac{1}{N} \sum_{i=1}^{N} R_{i}^2 I_{k}^i.
\]

Hence

\[
\langle \overline{B}_j \rangle = \frac{2}{N} \varepsilon_{jk} \langle \sum_{i=1}^{N} R_{i}^2 I_{k}^i \rangle
= \frac{2}{N} \varepsilon_{jk} \langle \sum_{i=1}^{N} (R_{i}^2 - \langle R_i \rangle)(I_{k}^i - \langle I_k \rangle) + N \langle \overline{R}_i \rangle \langle \overline{I}_k \rangle \rangle.
\]

Now, due to the factorization (7), the first term on the right-hand side of the last Eq. (10) vanishes\(^3\), and hence

\[
\langle \overline{B}_j \rangle = 2 \varepsilon_{jk} \langle \overline{R}_i \rangle \langle \overline{I}_k \rangle = \varepsilon_{jk} \left( \langle \overline{R}_i \rangle \langle \overline{I}_k \rangle - \langle \overline{R}_k \rangle \langle \overline{I}_i \rangle \right) = 0
\]

\(^2\)Note that this implies that the mean value in a point is assumed to be equal to the mean value computed along all curves of length \( N \). Thus the mean values can depend on \( N \).

\(^3\)Because \( \langle R_i^2 - \langle R_i \rangle \rangle = \langle I_k^i - \langle I_k \rangle \rangle = 0 \) for symmetry reasons.
because of the isotropy assumption (ii). Consequently the mean value of the magnetic field vanishes.

The second order moment is given by

\[
\langle B_i^2 \rangle = \frac{4}{N^2} \sum_{\alpha\beta} (R^\alpha R^\beta \cdot J^{\alpha} J^{\beta} - R^\alpha J^\alpha \cdot J^\beta)
\]

\[
= \frac{4}{N^2} \sum_{\alpha\beta} \left\{ \langle R^\alpha R^\beta \rangle \langle J^\alpha J^\beta \rangle - \langle R^\alpha J^\alpha \rangle \langle J^\beta \rangle \right\} ,
\]

(12)

using the factorization (7). Now

\[
\langle R^\alpha_i R^\beta_j \rangle = \prod \int \frac{d^3 R^\gamma_i}{D} R^\beta_j e^{-\lambda(R^\gamma_i-R_i)^2}
\]

\[
= \frac{1}{2\lambda} \delta_{ij} \delta^{\alpha\beta} + \prod \int \frac{d^3 R^\gamma_i}{D} \left[ \langle R_i \rangle R^\beta_j + \langle R_i \rangle R^\alpha_i - \langle R_i \rangle \langle R_i \rangle \right] e^{-\lambda(R^\gamma_i-R_i)^2} ,
\]

(13)

and similarly for \(\langle J^\alpha_i J^\beta_j \rangle\). Further we have e.g.

\[
\prod \int \frac{d^3 R^\gamma_i}{D} R^\beta_j e^{-\lambda(R^\gamma_i-R_i)^2} = \prod \int \frac{d^3 R^\gamma_i}{D} (R^\beta_j - \langle R_i \rangle) e^{-\lambda(R^\gamma_i-R_i)^2} + \langle R_i \rangle
\]

\[
= \langle R_i \rangle ,
\]

(14)

i.e., the mean value in a given arbitrary point \(\beta\) on the lattice is equal to the mean value computed over all curves. Using Eqs. (13) and (14) in Eq. (12) we get

\[
\langle B_i^2 \rangle = \frac{4}{N^2} \sum_{\alpha} \left( \frac{3}{2\lambda^2} + \frac{1}{\lambda} (\langle J \rangle^2 + \langle R \rangle^2) \right) + \frac{4}{N^2} \sum_{\alpha\beta} \left( \langle R \rangle^2 \langle J \rangle^2 - (\langle R \rangle \langle J \rangle)^2 \right) .
\]

(15)

The first term is \(O(N/N^2) = O(1/N)\). The last term, being the square of the mean value, actually vanishes because of isotropy. Thus we conclude that the \(rms\) value of the magnetic field scales like

\[
\sqrt{\langle B_i^2 \rangle} = \frac{2}{N} \sqrt{\sum_{\alpha} \left( \frac{3}{2\lambda^2} + \frac{1}{\lambda} (\langle J \rangle^2 + \langle R \rangle^2) \right)} \sim O\left( \frac{1}{\sqrt{N}} \right) .
\]

(16)

The reason for this slow decrease is the fact that isotropy prevents the mean value from entering in \(\langle B_i \rangle\) and \(\langle B^2 \rangle\), and that the correlations of the gradient vectors are of short range.

4 Consequences of the electroweak magnetic field

Let us now assume that at the time of the electroweak phase transition, a magnetic field with a coherence length \(\xi_0\) is generated, with a scaling as given by Eq. (16). Such a field evolves according to usual magnetohydrodynamics

\[
\frac{\partial B}{\partial t} = \nabla \times (v \times B) - \sigma^{-1} \nabla \times \nabla \times B ,
\]

(17)
where the conductivity $\sigma \sim \infty$ in the early universe. Accordingly, the field is then imprinted on the charged plasma\(^4\).

At later times the original coherence length is redshifted by the expansion according to

$$\xi(t) = \frac{a(t)}{a_0} \xi_0 .$$

(18)

The frozen-in magnetic field is also redshifted by the expansion of the universe. Thus at later times at the physical distance scale $L = N \xi$ one finds,

$$B_{\text{rms}}(t, L) = B_0 \left( \frac{a_0}{a(t)} \right)^2 \frac{1}{\sqrt{N}} = B_0 \left( \frac{t_0}{t} \right)^{\frac{2}{3}} \left( \frac{t_0}{t} \right) \left( \frac{\xi_0}{L} \right)^{\frac{1}{3}},$$

(19)

where $T_0^2 t_0 = 0.301 \frac{M_P}{\sqrt{g_*(T_0)}}$ with $g_*$ the effective number of degrees of freedom, and $t_* \simeq 1.4 \times 10^9 (\Omega_0 h^2)^{-2}$ yrs is the time when the universe becomes matter dominated; for definiteness, we shall adopt the value $\Omega_0 h^2 = 0.4$, which is the upper limit allowed by the age of the universe.

It is not obvious what the coherence length $\xi$ actually is. It is likely that it is macroscopic and much larger than the interparticle separation. Let me however assume for brevity that $B_0 \simeq 10^{24} \, \text{G}$ and $\xi \simeq 1/T$. It is then easy to find from Eq. (19) the size of the cosmological field today. Taking $t = 1.5 \times 10^{10}$ yrs and $L = 100 \, \text{kpc}$ (corresponding to $N = 1.0 \times 10^{24}$), we find that today the electroweak magnetic field at the scale of intergalactic distances is

$$B_{\text{rms}} = 4 \times 10^{-19} \, \text{G} .$$

(20)

This seems to be exactly what is required for the numerical dynamo simulations to produce the observed galactic magnetic fields of the order $10^{-6} \, \text{G}$. The inherent uncertainties in the estimate (20) are: the value of $\Omega_0 h^2$ used for computing $t_*$; the time at which the magnetic field froze, or $T_0$; the actual value of the field $B_0$. Therefore one should view (20) as an order-of-magnitude estimate only.

We should also check what other possible cosmological consequences the existence of the random magnetic field, Eq. (19), may have. Let us first note that the energy density $\rho_B$ in the $\text{rms}$ field is very small. In the radiation dominated era we find that the energy density within a horizon volume $V$ is

$$\rho_B = \frac{1}{2V} \int_0^{r_H} \delta^3 r B_{\text{rms}}^2 = \frac{3}{4} B_0^2 \left( \frac{T}{T_0} \right)^4 \frac{1}{r_H T} .$$

(21)

The horizon distance is $r_H = 2t$ so that $\rho_B \sim T^4/M_P \ll \rho_\gamma$, and the magnetic field contribution to the total energy density is negligible.

In principle, magnetic fields could modify primordial nucleosynthesis, as discussed in [13]. However, the electroweak magnetic field at the time of nucleosynthesis and at the horizon scale is only $B_{\text{rms}} \simeq 1500 \, \text{G}$ which is far too small to give rise to any modifications of the Big Bang nucleosynthesis.

\(^4\)There is a possible caveat here: in very large magnetic fields such as considered here the velocity of the plasma might depend on the background magnetic field. In the following I will neglect this.
5 Ferromagnetic universe

There is also another, more exotic possibility for producing magnetic fields, which is based on the observation that, due to quantum fluctuations, the Yang–Mills vacuum is unstable in a large enough background magnetic field [11]. There are indications from lattice calculations that this is a non-perturbative result [14]. Such magnetic field fluctuations in the early universe could be sufficient to trigger the phase transition to a new, ferromagnet–like ground state with a magnetic field made permanent by the charged plasma. In this scenario the primordial field is thus generated as a non-perturbative quantum effect.

The new vacuum results provided the β–function has a Landau singularity:

$$\left| \int_{g}^{\infty} \frac{dx}{\beta(x)} \right| < \infty. \quad (22)$$

Then the effective Lagrangian has a minimum away from the perturbative ground state \(Tr F^2 = 0\), given by

$$\frac{1}{2} g^2 \tau^2 \left| F_{\mu\nu} \right|_{\text{min}} = \Lambda^4, \quad (23)$$

where \(\Lambda\) is the renormalization group invariant scale

$$\Lambda = \mu \exp \left( -\int_{\infty}^{g} \frac{dx}{\beta(x)} \right), \quad (24)$$

where \(\mu\) is a subtraction point associated with the definition of \(g\).

The condition for the minimum can be realized in many ways. One of them is a constant non–abelian magnetic field \(B^a_i = \epsilon_{ijk} F^a_{jk}\) with a non–zero component only in one direction in the group space, and with a length given by

$$g \sqrt{B^a B^a} = \Lambda^2. \quad (25)$$

Consider now SU(N) at the one–loop level. We then have the one–loop, zero temperature effective energy for a constant background non–abelian magnetic field which in pure SU(N) theory reads [11]

$$V(B) = \frac{1}{2} B^2 + \frac{11 N}{96 \pi^2} g^2 B^2 \left( \ln \frac{g B}{\mu^2} - \frac{1}{2} \right) \quad (26)$$

with a minimum at

$$g B_{\text{min}} = \mu^2 \exp \left( -\frac{48 \pi^2}{11 N g^2} \right) \quad (27)$$

and \(V_{\text{min}} \equiv V(B_{\text{min}}) = -0.029 (g B_{\text{min}})^2\). Thus the ground state (the Savvidy vacuum) has a non–zero non–abelian magnetic field, the magnitude of which is exponentially suppressed relative to the renormalization scale, or the typical momentum scale of the system. Thus, for example, for SU(2)\(_L\) at the electroweak scale the vacuum magnetic field would be very small. In the early universe, however, where possibly a grand unified
symmetry is valid, the exponential suppression is less severe. It is also attenuated by
the running of the coupling constant. For a set of representative numbers, one might
consider a (susy) SU(5) model with $\alpha_{\text{GUT}} \simeq 1/25$ and $T_{\text{GUT}} \simeq 10^{15}$ GeV, as in the
supersymmetric Standard Model. This yields $B \simeq 5 \times 10^{-8} \mu^2$, which turns out to be
a magnitude which is relevant for the dynamo mechanism.

In the early universe the effective energy picks up thermal corrections from fermion-
ic, gauge boson, and Higgs boson loops. In SU(2) these are obtained by summing the
Boltzmann factors $\exp(-\beta E_n)$ for the oscillator modes

$$E_n^2 = p^2 + 2gB(n + \frac{1}{2}) + 2gBS_3 + m^2(T),$$

where $S_3 = \pm 1/2$ ($\pm 1$) for fermions (vectors bosons). In Eq. (28) I have included
the thermally induced mass $m(T) \sim gT$, corresponding to a ring summation of the
relevant diagrams. Numerically, the effect of the thermal mass turns out to be very
important.

The detailed form of the thermal correction depends on the actual model, but we
may take our cue from the SU(2) one-loop calculation, which for the fermionic and
scalar cases can be extracted from the real-time QED calculation in [15]. The result is

$$\delta V_T^f = \frac{(gB)^2}{4\pi^2} \sum_{l=1}^{\infty} (-1)^{l+1} \int_0^\infty \frac{dx}{x^2} e^{-K_l^f(x)} \left[ x\coth(x) - 1 \right],$$

$$\delta V_T^s = \frac{(gB)^2}{8\pi^2} \sum_{l=1}^{\infty} \int_0^\infty \frac{dx}{x^3} e^{-K_l^s(x)} \left[ x\frac{1}{\sinh(x)} - 1 \right],$$

where the normalization is such that the correction vanishes for zero field, and

$$K_l^a(x) = \frac{gBl^2}{4xT^2} + \frac{m^2_a x}{gB},$$

where $a = f$, $b$ stands for fermions or bosons.

For vector bosons there is the added complication that there exists a negative,
unstable mode, which gives rise to an imaginary part. At high temperatures the
instability is absent for fields such that $gB < m^2(T)$, which is the case we are interested
in here, so that no regulation of the unstable $n = 0$, $S_3 = -1$ mode is needed. Thus
we find [12]

$$\delta V_T^v = \frac{(gB)^2}{8\pi^2} \sum_{l=1}^{\infty} \int_0^\infty \frac{dx}{x^3} e^{-K_l^v(x)} \left[ x\frac{\cosh(2x)}{\sinh(x)} - 1 \right].$$

At high temperature, the bosonic contributions are more important than the
fermionic ones. When $B \ll T^2 \simeq m^2(T)$, we find numerically that $\delta V_T^v \simeq 0.016 \times
(gB)^2$. This gives rise to a small correction to the magnitude of the field at the min-
umum as obtained from Eq. (27). We may thus conclude that the Savvidy vacuum
exists for all $T$.

5Physically the imaginary part is an indicator that the vacuum also contains vector particles [16].
The transition to this new ferromagnet-like vacuum is triggered by local fluctuations. Charged particles in the primeval plasma generate current \( j = \nabla \times B \). The typical interparticle distance is \( L \sim 1/T \) and a typical curl goes like \( 1/L \) so that \( B \sim jL \) where \( j \) is like charge density with one charge in the volume \( L^3 \). Thus the Maxwell equations imply that \( B \sim 1/L^2 = T^2 \), indicating that the creation of the Savvidy vacuum can take place locally. A constant non-abelian magnetic field, given by Eq. (27), is then imprinted on the plasma of particles carrying the relevant charges. The Maxwell magnetic field \( B_{em} \) is a projection in the space of non-abelian magnetic fields, and we take it to be of the size comparable to \( B \) in Eq. (32).

The magnetic flux remains conserved (recall that the primordial plasma is an extremely good conductor), and we may write

\[
B(T) = g_{GUT}^{1/2} \mu^2 \exp \left( -\frac{48\pi^2}{11Ng^2} \right) \left( \frac{T^2}{\mu^2} \right) \approx 3 \times 10^{42} G \left( \frac{a(t_{GUT})}{a(t)} \right)^2 ,
\]

where \( \mu \approx T \) and \( a(t) \) is the scale factor of the universe, and the last figure is for susy SU(5). This expression is valid because the energy \( E \) of the vacuum is redshifted by \( 1/a(t) \). Now in the minimum \( E \) is proportional to \( VB^2 \), where \( V \) is the volume. Since \( V \) is proportional to \( a^3 \), we get \( B \sim 1/a^2 \). Hence the magnetic energy per horizon is much less than the radiation energy.

As time passes, the universe undergoes a number of phase transitions. Each of these correspond to new types of ferromagnetic vacua, which in general have decreasing field strengths. However, the original GUT vacuum has existed for a time that is long enough for the plasma to interact with the vacuum field \( B \) given by Eq. (32). This interaction does not allow for the GUT flux to decrease once it has been created since it has become a feature of the plasma, which conserves the flux in the sense that the magnetic lines of force are frozen into the fluid. Even if the original field is suddenly removed by the creation of a new vacuum, the magnetic field will survive in a perfect conductor (see e.g. pp. 186-189 in ref. [17]).

From Eq. (32) we find that the Maxwell magnetic field at \( t_{\text{now}} \approx 10^{10} \) yr is given by \( B_{\text{now}} \approx 3 \times 10^{42} G(t_{GUT}/t_\star)(t_\star/t_{\text{now}})^{4/3} \approx 10^{-14} G \). Such a magnetic field appears comparable to what is needed for the seed field in galactic dynamo models. Note also that at nucleosynthesis one obtains \( B \sim 10^4 G \), which is well below the nucleosynthesis bound on magnetic fields [13].

\[ \text{6 Discussion} \]

The GUT causal domain \( l_0 \) has today the size of only about 1 m. Obviously, during the course of the evolution of the universe, domains with magnetic fields pointing to different directions have come into contact with each other. One might think that this results in domain walls. However, here it is important that the magnetic flux lines follow the plasma particles and cease to be homogenous. If inside each GUT horizon
a magnetic field line at a certain time passes through two plasma particles, then this is true at any later time. The magnetic field thus "aligns" with the plasma. When two horizon bubbles collide, the two plasmas rearrange and become one plasma, and the same is true for the magnetic field lines, which are part of the new plasma [18]. The field "realigns" with the "new" plasma and the root mean square of the magnetic field remains of the same order as before. Because there are no domains, no random walk factor appears at large distances.

In this argument it is important that $B_{\text{rms}}$ is much smaller than the square of the rms momentum, since otherwise the electrical conductivity would depend on $B$. This condition implies that the radius of curvature of a typical plasma particle is very large compared to the mean free path, or that the magnetic energy is much less than the kinetic energy of the plasma. This certainly is the case in the ferromagnetic universe model.

The size of the field is determined by the scale at which the ferromagnet vacuum is created, and the earlier this happens, the bigger the field. If there is a period of cosmic inflation, then the relevant field would be created after reheating. If the reheating temperature is comparable to GUT scales, the strength of the magnetic field would be given as in (32), with interesting consequences for the formation of galactic magnetic fields. In this scenario the primordial seed field is thus a relic from the GUT era.

If the origin of the magnetic field is the electroweak phase transition, the situation might be different because at scales $1/T$ the magnetic energy equals to the energy in radiation. In such fields the plasma might be trapped by the field, rather than the field being imprinted on the moving plasma. Very likely this would result in a domain structure, but in the absence of any true dynamical calculation, the details must remain unclear. It nevertheless seems clear that a primordial magnetic field would have many intriguing consequences, some of which might actually be observable. It would of course be of great interest to detect this relic field directly in the intergalactic space.

**Acknowledgements**

I wish to thank Poul Olesen for many enjoyable discussions on primordial magnetic fields.
References


