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A TEST THEORY OF THE LOCAL STRUCTURE OF SPACE-TIME;

A FINSLERIAN APPROCH¹

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Abstract

For rival theories within the framework of the test theory of Mansouri-Sexl a geometrical setting in terms of Finsler geometry have been found. It is shown that the time dilation and length contraction parameters measure the deviation from the Riemannian geometry.

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1 Introduction

In the last few years, we have been witnessing a revival of the experimental tests of the Special Theory of Relativity (SR)[1]. SR is the fundamental theory of spacetime with geometrical features (Minkowski metric) upon which much of the modern physics is based. The symmetry of the theory, Lorents invariance, is unanimously considered to be the symmetry of nature. Now a question can be posed that is experimentally relevant and theoretically important: how accurately the background spacetime of physical phenomena is, at least locally, a Minkowski spacetime? A rather different but closely related question is: would a slight violation of Lorents-Invariance, still be compatible with a geometrical theory of spacetime?

In order to get a partial answer to these questions, one should work within a framework which is rather well-understood. The well-known test theory of special relativity, due to Mansouri and Sexl (MS) [2] may provide us with an appropriate framework. Many of the recent experiments to test SR, have been analysed on the basis of MS test theory [3]. As a matter of fact, synchrony of clocks and procedures of synchronisation are at the base of any theory of spacetime. Synchronisation of distant clocks by electromagnetic signals (E-synchronisation) and by slow transport of clocks (T-synchronisation) are the two most natural procedures discussed extensively in literature. It is known since Eddington that the two procedures are equivalent in SR [2]. In MS scheme a class of theories, rival to SR, are introduced for which the

above equivalence may not hold. Synchronisation procedure enter the theory by way of a vector parameter $\vec{\epsilon}$. Other parameters, which carry information about time dilation and length contraction, called α, β , and δ also enter the MS theory. Therefore the class of rival theories corresponds to a manifold of parameters. Each point in the manifold defines a theory; SR itself is characterised by the set of parameters $\alpha = -\frac{1}{2}, \beta = \frac{1}{2}$, and, $\delta = 0$. MS showed that classical tests of SR can be used to estimate how accurate the SR values are.

From the experimental point of view all one can say in MS framework, is that the spacetime of physical phenomena is represented by a point located somewhere within a small ball in the manifold of parameters, whose radius is determined by the accuracy of experiments.

Theoretically it is shown elsewhere that in any Riemannian spacetime, whenever the T- and E-synchronisations are both possible, these two methods lead to equivalent results [4]. Therefore any test of relativity based on a comparison of these methods is in fact a test of the Riemannian structure of spacetime. Within the scheme of MS, if one is somehow restricted to speculate only about Riemannian manifolds for spacetime, then only SR is capable of admitting a metric structure. Therefore if we are after a geometrical setting for rival theories, we should go beyond Riemannian geometry. We know of no attempt to formulate rival theories in the framework of some geometrical structure. Finsler geometry, as a generalization of Riemannian geometry, is a natural alternative [5,6]. Recent interests in Finsler geometry is motivated by other goals than ours and aims at different objectives [7,8]. In this

paper a class of rival theories in the framework of Finsler geometries is introduced and it is shown that the parameters of MS indeed test the geometrical structure of spacetime, In other words, a test theory for the geometry of spacetime is introduced.

Section.2 is devoted to a discussion of the MS test theory of SR, the terminology of which is used throughout the paper. In Section.3 we introduce the fundamentals of the Cartan approach to Finsler geometry. In Section.4 a test theory of the geometry of spacetime is developed using Finsler geometry and MS as a framework. Section.5 is devoted to the conclusions and remarks.

2 MS test theory of SR

According to the MS test theory of SR [2]the transformation between the spacetime coordinates (T, X) in Σ , the preferred (ether) frame, and (t, x) in S , a frame moving with the velocity v in the x -direction relative to Σ , has the following general form.

$$\begin{aligned} t &= a(v)T + \epsilon(v)x \\ x &= b(v)(X - vT). \end{aligned} \tag{1}$$

Here $1/a(v)$ is the coefficient of time dilation, $b(v)$ is the coefficient of length contraction and $\epsilon(v)$ is a function which only depends on the choice of synchronisation. The special relativistic values for a and b are $1/a(v) = b(v) = \frac{1}{(1-v^2)^{1/2}}$. In the terminology of MS, using (1) one

can show [2] that $\epsilon_E = -\frac{av}{b(1-v^2)}$ and $\epsilon_T = \frac{1}{b} \frac{da}{dv}$ for Einstein and slow clock transport synchronisation respectively. It is easily seen that the two procedures are equivalent in SR and lead to $\epsilon = -v$. In order to work with these coefficients more easily and to formulate a test theory we expand the parameters in power series of v :

$$\begin{aligned} a(v) &= 1 + \alpha_1 v^2 + \alpha_2 v^4 + \dots \\ b(v) &= 1 + \beta_1 v^2 + \beta_2 v^4 + \dots \end{aligned} \quad (2)$$

We call the coefficients of these expansions, the parameters of MS test theory. For example if one neglects the higher order terms, $\alpha_1 = -\frac{1}{2}$, $\beta_1 = \frac{1}{2}$ gives SR as a point in this parameter space. All first order experiments measure $(2\alpha_1 + 1)v$ and the so-called second order experiments measure $(\alpha_1 - \beta_1 + 1)v^2$ (the Kennedy-Thorndike experiment).

3 Fundamentals of Finsler Geometry

A Finsler space [5] is a real differentiable manifold M endowed with a metric function $F(x, u)$ which gives the length of the vector $u \in T_x M$ at the point $x \in M$. The metric function should be positively homogeneous of degree one with respect to u , i.e.

$$F(x, \alpha u) = \alpha F(x, u); \forall \alpha > 0. \quad (3)$$

In order to endow the Finsler space with the so-called "Euclidean connection" of E. Cartan, one should consider the manifold X_{2n-1} of the

line elements (x^i, u^i) , which is $(2n - 1)$ -dimensional, since only the ratios of the u^i are necessary to define a direction in the tangent space $T_x M$. The coordinates x^i refer to the center of the line element (x^i, u^i) . All quantities, such as tensors, are to be defined as functions of line elements. A metric tensor with symmetric components $g_{ij}(x, u)$ is given, such that the square of the distance between the centers x^i and $x^i + dx^i$ of the two neighboring line elements (x^i, u^i) and $(x^i + dx^i, u^i + du^i)$ is given by the expression.

$$ds^2 = g_{ij}(x, u) dx^i dx^j. \quad (4)$$

No relation is assumed between the tensor g_{ij} and the metric function $F(x, u)$ at the present stage and this relation is to be deduced from later postulates. For an arbitrary vector X we define the square of its length to be $g_{ij}(x, u) X^i X^j$. In all these expressions the element of support (x^i, u^i) has to be chosen in advance and are treated as fixed parameters. In order to link the metric function $F(x, u)$ to the metric tensor $g_{ij}(x, u) X^i X^j$, one assumes that if the direction of a vector X^i coincides with that of its element of support (x^i, u^i) , its length is to be equal to $F(x, X)$, in other words one should have

$$F^2(x, u) = g_{ij}(x, u) u^i u^j. \quad (5)$$

To go further one should put some additional assumptions [5] which are very reasonable and enable us to solve $g_{ij}(x, u)$ as a function of

$F(x, u)$ uniquely, i.e. one gets

$$g_{ij}(x, u) = \frac{1}{2} \frac{\partial^2 F^2(x, u)}{\partial u^i \partial u^j}. \quad (6)$$

This relation can be thought of as a consistency relation in the sense that if one starts from a $g_{ij}(x, u)$ and defines $F(x, u)$ by (3), then putting the metric function into (4), one should get the same result for the metric tensor.

Now let's fix, for the moment, a point x and consider some linear transformation $u'^i = K^i_j u^j$ on the directional variable of the line element (x^i, u^i) . The requirement that the Finslerian metric function remains invariant under the transformation will read $F(x, u') = F(x, Ku) = F(x, u)$. Using this relation and (4) one can get

$$g_{ij}(x, u') = g_{mn}(x, u) K^m_i K^n_j \quad (7)$$

in which K^{*i}_j is the inverse transformation, namely, $K^j_i K^{*i}_j = \delta^j_j$. If this condition is satisfied the transformation K is said to belong to the so-called K-group of invariance of the metric function [5].

4 A test theory of the spacetime geometry

Now, we demand that the spacetime, with the general MS transformations in $(1+1)$ dimensions(6), has a Finsler geometrical structure. This is a real constraint for the transformations(6), which we have to work it out. First we remember the basic assumption of MS test theory:

The metric in the privileged frame Σ is $\eta_{\mu\nu}$. Therefore the spacetime metric in the moving frame S is given by

$$g_{\mu\nu}(v) = \eta_{\alpha\beta} K^{\alpha}_{\mu} K^{\beta}_{\nu}, \quad (8)$$

where $v = \frac{u^1}{u^0}$ is the absolute velocity of S (relative to Σ). In each frame one can take an orthonormal tetrad with respect to its own metric tensor in which the seroth (timelike) vector can be taken to be the 4-velocity of the frame; it is easy to show [5] that K^{μ}_{ν} is indeed the transformation between these tetrads. In other words K is the space-time transformation which is the generalisation of the Lorents transformation.

Now, if we put K to be equal to the spacetime transformation of MS and use (8) we can find the metric tensor in any moving frame. For simplicity, we have considered a 1+1 dimensional spacetime. We then have,

$$K = \begin{pmatrix} a - \epsilon b v & \epsilon b \\ -b v & b \end{pmatrix} \quad (9)$$

$$K^* = K^{\dagger^{-1}} = \begin{pmatrix} \frac{1}{a} & \frac{v}{a} \\ -\frac{\epsilon}{a} & \frac{a - \epsilon b v}{ab} \end{pmatrix} \quad (10)$$

$$g(v) = K^* \eta K^{*\dagger} = \begin{pmatrix} \frac{v^2 - 1}{a^2} & \frac{av + b\epsilon(1 - v^2)}{a^2 b} \\ \frac{av + b\epsilon(1 - v^2)}{a^2 b} & \frac{a^2 - 2abv\epsilon - \epsilon^2 b^2(1 - v^2)}{a^2 b^2} \end{pmatrix}. \quad (11)$$

This metric, to represent a Finslerian metric, has to fullfill condition

(3) and (4). The metric function $F(u)$ gives the length of the 4-velocity u . We therefore assume $F^2(u) = -1$.

Now, to work out the consistency condition (3) and (4), we use the expansions (7) and

$$\epsilon(v) = \mu_1 v + \mu_2 v^3 + \dots$$

This expansion is obvious from the fact that simultaneous change of sign of v and x in terms like $\epsilon(v)x$ should not affect the transformations. Conditions (3) and (4) leads then to algebraic equations for the parameters of expansions of $a(v)$, $b(v)$ and $\epsilon(v)$. The solution is

$$\begin{aligned} \alpha_1 &= -\frac{1}{2} & , & & \alpha_2 &= -\frac{1}{8} + q \\ \beta_1 &= \frac{1}{2} - 2q \\ \mu_1 &= -1 & , & & \mu_2 &= -\frac{14}{3}q, \end{aligned}$$

where q is an arbitrary parameter. It is important to note, that to have a deviation from special velocity, one has to take at least up to 4th order of the expansion of the coefficients of the transformation. The parameter β_2 is not fixed at this order of expansion. Finally, we obtain

$$K = \begin{pmatrix} 1 + \frac{1}{2}v^2 + (\frac{3}{8} + \frac{11}{3}q)v^4 & -v + (-\frac{1}{2} - \frac{8}{3}q)v^3 \\ -v + (-\frac{1}{2} + 2q)v^3 & 1 + (\frac{1}{2} - 2q)v^2 + \beta_2 v^4 \end{pmatrix} \quad (12)$$

$$g = \begin{pmatrix} -1 + 2qv^4 & -\frac{8}{3}qv^3 \\ -\frac{8}{3}qv^3 & 1 + 4qv^2 \end{pmatrix}. \quad (13)$$

We have shown that the class of rival theories of special relativity formulated by MS can be brought into a geometrical setting. This ge-

ometrical setting, the Finsler geometry, correlates the test-functions a and b with the function $\epsilon(v)$, which in the MS test theory is determined by convention and not experiments. This correlation, via the parameter q , makes $\epsilon(v)$ a function which is determined by experiment. We will call this value Finsler synchronisation ϵ_F . A simple calculation leads to $\epsilon_T = -v + 2qv^3$ and $\epsilon_E = -v - 2qv^3$ for Transport- and Einstein-synchronisation respectively. Therefore in Finsler geometry $\epsilon_T \neq \epsilon_E$, and besides this the spacetime structure leads to another synchronisation different from Einstein or Transport. Only in the first approximation $\epsilon_T = \epsilon_F = \epsilon_E$ and we are faced with Riemannian geometry and special relativity.

5 Conclusions

It is a well-known fact, at least since MS, that starting from an ether frame Σ , whose clocks are Einstein-synchronised, one can write transformation laws to moving frames the kinematic consequences of which are the same as Lorents transformations. The two sets of transformations differ in so far as the synchronisation convention is concerned. In the "ether theory" while the synchronisation convention is fixed in Σ , that of the moving frame is quite arbitrary and definitely non-Einsteinian. Such a theory, though kinematically equivalent to SR, is not a geometric theory like SR a'la Minkowski. A geometric theory is characterised by a symmetry which exists between different frames (e.g. Lorents invariance in SR). Such a symmetry requirement removes the above mentioned arbitrariness and consequently certain types of syn-

chronisation will be held in all frames. One should note that this symmetry requirement does not do away with the conventionality of synchronisation method. It only implies that, by a proper choice of convention, we have made the symmetry of nature to be manifest in our mathematical formulation. A question that may be raised is that whether this choice is unique. In SR we know of two basically different synchronisation procedures manifesting the Lorents invariance of spacetime: synchronisation of clocks by electromagnetic signals, and synchronisation by slow clock transport.

Now let us assume that SR is not the exact theory of spacetime. Then preferred frames exist, and we should have the theories of spacetime formulated in terms of transformations from the preferred to moving frames as in the MS test theory. With no synchronisation convention adopted, ϵ is arbitrary. Such spacetime theories are not geometrical. No symmetry for spacetime is assumed in their framework. Any symmetry requirement implies some synchronisation convention.

Here we adopted the class of theories discussed in MS test theory and required the following geometrical condition: spacetime is Finslerian and the linear K-group of invariance of this Finslerian space defines the transformation of our frames of reference. To make these symmetry conditions mathematically manifest the synchronisation must be given by $\epsilon_F = -v - \frac{14}{3}qv^3$. E- and T-synchronisations are also obtained as $\epsilon_E = -v - 2qv^3$ and $\epsilon_T = -v + 2qv^3$. We have $\epsilon_E = \epsilon_T = \epsilon_F$ only for $q = 0$, which corresponds to SR and Minkowski geometry of spacetime.

Therefore q is a parameter measuring the deviation from the Minkowskian

(or Riemannian) structure of spacetime; and the theory discussed here can be considered as a test theory of spacetime structure. The synchronisation is not conventional any more. The value of q (or rather the bounds on it) can be estimated from the traditional tests of SR via the expansion parameters of a and b eqn.(2).

The so called first order experiments, measuring the difference of E- and T-synchronisations are no longer of the "first order" in the framework of this theory,

$$\epsilon_T - \epsilon_E = 4qv^3$$

noting that $2\alpha_1 + 1 = 0$. Taking the measurement of Kaivola[9], and Krisher[1] into account, one has

$$q \leq 10^{-3}$$

assuming $v = 600$ km/sec.

The second order tests, such as Kennedy-Thorndike and Michelson-Morely experiments are also of second order in this theory. The KT experiment measures $(\alpha_1 - \beta_1 + 1)$. In MS test theory this experiment measures a combination of time dilation and length contraction effects. Here since α_1 is independent of q ($\alpha_1 = -\frac{1}{2}$) the experiment measures β_1 , i.e. length contraction. One obtains the limit

$$q \leq 10^{-6}$$

for second order experiments [10].

No truly first order effect is allowed within the framework of this theory.

References:

1. R. W. Mc Gowan *etal*, *Phys. Rev. Lett.* 70 (1993) 251; S. Sroder *etal*, *Phys. Rev. Lett.* 64 (1990) 2901; W. Henning, *Nucl. Phys. News. I* (1990) 13; T. R. Krisher *etal*, *Phys. Rev. Lett.* 46 (1990) 1697; R. Klein *etal*, *Zeitschr. F. Phys. A* (1992) 342, No. 4 , 455.
2. R. Mansouri and R. Sexl, *Gen. Rel. Grav.* 8 (1977) 496, 515, 809.
3. C. M. Will, *Phys. Rev. D*, Vol 45, No 2 (1992) 403-411; I. Vethar-
anian and G. E. Stedman, *Phys. Lett. A*, 183 (1993) 349; M. Kret-
schmar, *Zeitschr. F. Phys. A* (1992) 342, No. 4, 463-9.
4. M. R. H. Khajehpour, R. Mansouri, On Synchronisation of Clocks
in General Spacetime, *Sharif University Preprint 1993*.
5. H. Rund, *The Differential Geometry of Finsler Spaces*, Berlin (1959).
6. G. S. Asanov, *Finsler Geometry, Relativity and Gauge Theories*,
D.Reidl. (1985).
7. I. W. Roxburgh and R. K. Tavakol, *Gen. Rel. Grav.* 10 (1979)
307; R. K. Tavakol and N. Van der Burgh, *Gen. Rel. Grav.* 18 (1986)
849.
8. J. D. Bekenstein, The Relation between Physical and Gravitational
Geometry, *University of California, Santa Barbara, Preprint UCSB-*
TH-92-41, Nov. 1992.
9. M. Kaivola, O. Poulsen, E. Riis, Siu Au Lee, *Phys. Rev. Lett.* 54
(1985) 255.
10. D. Hils and J. L. Hall, *Phys. Rev. Lett.* 64 (1990) 1697.