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On some integrable cases in surface theory

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Abstract. We show how to reformulate Gauss-Codazzi system for surfaces with arbitrary fixed Gaussian curvature $\mathcal{K}$ as one second-order differential equation. Similar reformulation is obtained for surfaces with arbitrary fixed mean curvature $\mathcal{H}$.

In the cases of 'Bianchi-two' surfaces ($\mathcal{K} = -(f(x) + g(y))^{-2}$, $f \neq \text{const}$, $g \neq \text{const}$ in asymptotic coordinates) and Bianchi-two surfaces with positive curvature ($\mathcal{K} = (h(z) + \bar{h}(\bar{z}))^{-2}$, $h \neq \text{const}$ in conformal coordinate of second fundamental form) these equations turn out to be "unitary" reductions $\mathcal{E}_1 = (\bar{\mathcal{E}}_2)^{-1}$ of coupled Ernst system in general relativity. For all classes of Bianchi surfaces: B_0 ($f = \text{const}$, $g = \text{const}$; this class is described by the sine-Gordon equation), B_1 ($f = \text{const}$, $g \neq \text{const}$), B_2 ($f \neq \text{const}$, $g \neq \text{const}$), B_0^+ ($h = \text{const}$, this class is described by the sinh-Gordon equation) and B_2^+ ($h \neq \text{const}$) we construct Bäcklund-Darboux transformation, multi-soliton and finite-gap (theta-functional) solutions. We show that the Gauss map of Bianchi-two surfaces with positive curvature defines axisymmetric harmonic map $\mathbf{R}^3 \rightarrow S^2$.

The main focus is on the finite-gap solutions. In particular, in the case of sine-Gordon equation we show how to provide reality conditions in pure constructive way in terms of analytical properties of Baker-Akhiezer function. In contrast to B_0 (i.e. sine-Gordon) case, where dynamics in finite-gap solutions is defined in standard way by straight winding of Jacobi variety $\mathcal{J}(\mathcal{L})$ of fixed algebraic curve, in the finite-gap solutions of B_2 -equation the dynamics is defined by deformation of basic curve with (x, y) -dependent branch points. The B_1 - case is intermediate: dependence on x is defined by the straight winding of Jacobi variety of the curve depending on y .

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1 Introduction

It is well-known that some of equations which were extensively studied during the last twenty years by inverse scattering method were first obtained in the framework of surface theory about one hundred years ago. The most famous equation of this kind is sine-Gordon (SG) equation

$$\phi_{xt} = \sin \phi \quad (1.1)$$

which coincides with the Gauss-Codazzi system for surfaces of constant negative Gaussian curvature $\mathcal{K}$. The Gauss-Codazzi system in other two cases: constant positive $\mathcal{K}$ and constant mean curvature $\mathcal{H}$ is equivalent to hyperbolic version of (1.1)

$$\phi_{z\bar{z}} + \sinh \phi = 0 \quad (1.2)$$

Integrability of these equations i.e. existence of the zero curvature representations with the spectral parameter (in this sense the integrability is equivalent to the possibility to insert an additional complex parameter into related Gauss-Weingarten system) allowed to develop the theory of exact solutions for (1.1), (1.2) (see [4] for references) which was then successfully applied to surface theory (for review see [9]).

One of the parts of the classical differential geometry in $\mathbf{R}^3$ is a theory of congruences (two-parametric families of lines). This theory turns out to be closely related to the theory of integrable non-linear differential equations. This link goes back to Bianchi [2] who considered a congruence as a way to transform one of its focal surfaces to another. So if the Gaussian curvature of these surfaces lying on the same straight line of the congruence coincide (therefore, related Gauss-Codazzi systems also coincide), we obtain, in modern terminology, the one-soliton Bäcklund-Darboux (or "dressing") transformation of the Gauss-Codazzi system (only a one-soliton dressing transformation shifts the surface along some tangent vector; the multi-soliton transformations include in generic case a shift in normal direction).

If the Gaussian curvature of two focal surfaces at the points lying on the same straight line of the congruence coincide and negative (in this case the congruence is called the Bianchi congruence and the focal surfaces - the Bianchi surfaces) then it has the following form:

$$\mathcal{K} = \frac{-1}{(f(x) + g(y))^2} \quad (1.3)$$

where (x, y) are asymptotical coordinates; f and g are arbitrary functions.

The transformation of one Bianchi surface to another along related congruence of straight lines (which are tangent to both these surfaces since the surfaces are focal surfaces of the congruence) is called Bianchi transformation (as we mentioned above, on the level of related Gauss-Codazzi system this is in fact one-soliton Bäcklund-Darboux transformation). Moreover, it turned out that such a transformations may be iterated: we can start from some Bianchi congruence C_1 that transforms one focal Bianchi surface (S_1) to another (S_2) and then consider another Bianchi congruence C_2 that transforms surface S_2 to new Bianchi surface S_3 . In modern terms, on the level of Gauss-Codazzi system this looks like a two-soliton Bäcklund-Darboux (or "dressing") transformation; investigation of permutation laws of such a transformations is a part of the theory of pairs of congruences [7].

Taking into account these classical results, it was reasonable to expect that the Gauss-Codazzi system for surfaces with the curvature (1.3) should be integrable in the sense of the soliton theory i.e. it should exist associate linear system with additional complex ("spectral") parameter which gives the Gauss-Codazzi system as its compatibility condition. Indeed, such a system was found in [15]. (Notice the similarity with the sine-Gordon equation, which is a partial case of Gauss-Codazzi system for (1.3): the Backlund -Darboux transformations and even some classes of finite-gap solutions were known for more than 100 years, but related $U - V$ pair was found quite recently)

The Bianchi surfaces may locally be divided into three classes:

a. B_0 -surfaces: $f = \text{const}, g = \text{const}$; then $\mathcal{K} = \text{const} < 0$ and Gauss-Codazzi equations reduce to (1.1).

b. B_1 -surfaces: $f = \text{const}, g \neq \text{const}$; if $g' \neq 0$, we can locally choose $f = 0, g = y$.

c. B_2 -surfaces: $f \neq \text{const}, g \neq \text{const}$; outside of the points where $f' = 0$ or $g' = 0$ we can locally choose $f = x, g = y$

So, in modern language, the classical result about the Gaussian curvature (1.3) of Bianchi surfaces means that all surfaces admitting one-soliton BDT which doesn't change the Gaussian curvature, obey condition (1.3), and, therefore, are integrable. (Notice, however, that existence of one-soliton BDT is not a necessary feature of integrable system: for example, for sinh-Gordon equation only $2n$ - soliton BDT exist.)

In [5] it was established integrability of Gauss-Codazzi equations in two new cases:

1. Surfaces 'Bianchi-plus' (B^+) i.e. surfaces having positive $\mathcal{K}$ of the following form:

$$\mathcal{K} = \frac{1}{(h(z) + \bar{h}(\bar{z}))^2} \quad (1.4)$$

where h is an arbitrary holomorphic function, z is a conformal coordinate of the second fundamental form. Here we can locally distinguish two essentially different situations:

a. B_0^+ -surfaces: $h = \text{const}$; then Gauss-Codazzi system coincides with sinh-Gordon equation (1.2).

b. B_2^+ - surfaces: $h \neq \text{const}$; if $h' \neq 0$ we can choose locally $h(z) = iz$. Here we establish the following fact: surface S is a B_2^+ -surface if and only if its Gauss map (i.e. unit normal vector) defines axisymmetric harmonic map $\mathbf{R}^3 \rightarrow S^2$.

2. Surfaces with harmonic inverse mean curvature

$$\mathcal{H} = \frac{1}{h(z) + \bar{h}(\bar{z})} \quad (1.5)$$

where z is a conformal coordinate of first fundamental form. Here we also have two essentially different cases: $h = \text{const}$, when Gauss-Codazzi system coincides with sinh-Gordon equation, and $h \neq \text{const}$, which seems to be new.

In this paper we obtain the following results:

We make the general observation that the Gauss-Codazzi system may be written as one complex differential equation of the second order in the following three cases:

1. Fixed $\mathcal{K} = -\rho^{-2}(x, y) < 0$, where ρ is an arbitrary (sufficiently smooth) real-valued function; (x, y) - asymptotic coordinates. Then Gauss-Codazzi system is equivalent to the following equation:

$$(\mathcal{E} + \bar{\mathcal{E}}^{-1})(\mathcal{E}_{xy} + \frac{\rho_x}{2\rho}\mathcal{E}_y + \frac{\rho_y}{2\rho}\mathcal{E}_x) = 2\mathcal{E}_x\mathcal{E}_y \quad (1.6)$$

where $\mathcal{E}(x, y)$ is a complex-valued function equal to $\tilde{\Phi}_{11}(\tilde{\Phi}_{12})^{-1}$ and $\tilde{\Phi} \in SU(2)$ is a solution of Gauss-Weingarten system (2.6) written in terms of 2×2 matrices.

2. Fixed $\mathcal{K} = \sigma^{-2}(z, \bar{z}) > 0$ where σ is an arbitrary (sufficiently smooth) real-valued function of the conformal coordinates of the second fundamental form $(z, \bar{z})$. Then Gauss-Codazzi system is equivalent to the following equation:

$$(\mathcal{E} + \bar{\mathcal{E}}^{-1})(\mathcal{E}_{z\bar{z}} + \frac{\sigma_z}{2\sigma}\mathcal{E}_{\bar{z}} + \frac{\sigma_{\bar{z}}}{2\sigma}\mathcal{E}_z) = \mathcal{E}_z\mathcal{E}_{\bar{z}} \quad (1.7)$$

where function $\mathcal{E}(z, \bar{z})$ is constructed in the same way as in (1.6) from solution $\tilde{\Phi} \in SU(2)$ of the Gauss-Weingarten system (5.3).

3. Fixed $\mathcal{H}(z, \bar{z})$, where $(z, \bar{z})$ are conformal coordinates of the first fundamental form. Then Gauss-Codazzi system is equivalent to

$$(\mathcal{E} + \bar{\mathcal{E}}^{-1})(\mathcal{E}_{z\bar{z}} - \frac{\mathcal{H}_z}{\mathcal{H}}\mathcal{E}_{\bar{z}}) = 2\mathcal{E}_z\mathcal{E}_{\bar{z}} \quad (1.8)$$

We hope that this alternative parametrization of the surface in terms of function $\mathcal{E}$ which is always equal to the ratio of two elements of the first (with almost the same result we could take the second) row of $SU(2)$ -valued solution of the Gauss-Weingarten system may sometimes be more convenient than the traditional ones. For the cases 1 and 2 function $\mathcal{E}$ is closely related to Beltrami coefficients of the surface which have the following form:

$$\mu_y^x = -\frac{\mathcal{E}_y}{\mathcal{E}_x} \quad \text{and} \quad \mu_z^{\bar{z}} = -\frac{\mathcal{E}_{\bar{z}}}{\mathcal{E}_z}$$

respectively.

In particular, our parametrization allows to establish a close link of Gauss-Codazzi system for B_2 - and B_2^+ -surfaces and the Ernst equation (Einstein equation with two commuting Killing vectors). Namely, Ernst equation (if both Killing vectors are space-like) is a reduction $\mathcal{E}_1 = \bar{\mathcal{E}}_2$ of the system

$$\begin{aligned} (\mathcal{E}_1 + \mathcal{E}_2)(\mathcal{E}_{1xt} + \frac{\mathcal{E}_{1x} + \mathcal{E}_{1y}}{2(x+y)}) &= 2\mathcal{E}_{1x}\mathcal{E}_{1y} \\ (\mathcal{E}_1 + \mathcal{E}_2)(\mathcal{E}_{2xt} + \frac{\mathcal{E}_{2x} + \mathcal{E}_{2y}}{2(x+y)}) &= 2\mathcal{E}_{2x}\mathcal{E}_{2y} \end{aligned} \quad (1.9)$$

Equation (1.6) in the case of B_2 -surfaces ($\rho = x + t$) is another ("unitary") reduction $\mathcal{E}_1 = \bar{\mathcal{E}}_2^{-1}$ of the same system.

The same link takes place between equation (1.7) for the B_2^+ -surfaces ($h = iz$) and the Ernst equation in the case of one time-like and one space-like Killing vectors (stationary axisymmetric Einstein equation).

It is known [19] that Ernst equation describes axisymmetric harmonic maps from $\mathbf{R}^3$ into upper half-plane with the standard metric

$$ds^2 = \frac{d\mathcal{E}d\bar{\mathcal{E}}}{(\mathcal{E} + \bar{\mathcal{E}})^2}$$

The B_2^+ equation ((1.6) with $h(z) = iz$) describes axisymmetric harmonic maps from $\mathbf{R}^3$ into sphere S^2 with the standard metric

$$ds^2 = \frac{d\mathcal{E}d\bar{\mathcal{E}}}{(1 + \mathcal{E}\bar{\mathcal{E}})^2}$$

Zero curvature representations for B -surfaces and B^+ -surfaces found in [15] and [5] respectively turn out to be equivalent to reduction $\mathcal{E}_1 = \bar{\mathcal{E}}_2^{-1}$ of zero curvature representation for the system (1.9) (and its reformulation for $\mathcal{K} > 0$ respectively) known since 1978 [16], [1]. (Notice, however, that existence of reduction $\mathcal{E}_1 = \bar{\mathcal{E}}_2^{-1}$ of the system (1.9) is rather non-obvious itself.) We shall call equations (1.6) and (1.7), where $\mathcal{K}$ is defined by (1.3) and (1.4), the *Bianchi* equation (B-equation) and *Bianchi-plus* equation (B^+ -equation) respectively.

Integrability of (1.8) for $\mathcal{H}$ defined by (1.5) and $h(z) \neq \text{const}$ established in [5] seems to be essentially new; however, as we shall see in Sect.5, related zero curvature representation is more difficult to use.

In the present paper we construct multisoliton Bäcklund-Darboux transformations (BDT) for B -surfaces and B^+ -surfaces (which are the same for all cases 0,1,2). Applying the BDT to simplest 'seed' solutions we get the one-soliton (or two-soliton, if one soliton is absent or trivial) solutions for all these cases.

The central analytical result of the paper is a construction of finite-gap (theta-functional) solutions in all the cases $B_{0,1,2}$ and $B_{0,2}^+$. Certainly the finite-gap solutions of B_0 - and B_0^+ - equations equivalent to (1.1) and (1.2) respectively are well-known, but it is instructive to consider them simultaneously with other cases of $B_{1,2}$ - and B_2^+ - equations to point out similar and different features of the construction. Besides that, our construction includes one new element even for equations (1.1) and (1.2) concerning the reality condition $\phi \in \mathbf{R}$. Traditionally this condition had to be satisfied in final formulae for the solutions by an appropriate choice of parameters; it was not clear how to satisfy this condition in constructive way i.e. in terms of the analytical properties of related Baker-Akhiezer function on basic algebraic curve. We fill this flaw and show how to satisfy the reality condition in pure deductive way.

The construction is presented in more details for B -surfaces. Basic algebraic curve $\mathcal{L}$ is defined in this case by the following equation

$$\omega^2 = (\lambda + f(x))(\lambda - g(y)) \prod_{j=1}^g (\lambda - E_j)(\lambda - F_j)$$

where $(\omega, \lambda) \in \mathbf{C}^2$; E_j and F_j are (x, y) -independent branch points.

In the case of B_0 -equation, which is equivalent to sine-Gordon, the curve $\mathcal{L}$ is (x, y) -independent and dynamics is defined by straight winding of Jacobi variety of $\mathcal{L}$. In intermediate case of B_1 -surfaces curve $\mathcal{L}$ depends on y ; so dynamics in y -direction is defined by the deformation of $\mathcal{L}$ while dynamics in x -direction is defined as before by the straight winding of $J(\mathcal{L})$. Finally, in B_2 -case, dependence on x and y is defined only by the deformation of curve $\mathcal{L}$ (in B_2 and B_2^+ cases the situation is analogous to Ernst equation [11],[12],[13]).

The paper is organized as follows. Section 1 contains description of B -surfaces on the level of equations. We derive Gauss-Weingarten and Gauss-Codazzi systems for arbitrary $\mathcal{K} < 0$ in a suitable form and show how to rewrite the Gauss-Codazzi system as (1.6). In Sect.2 we derive Bäcklund-Darboux transformations (one-soliton and multi-soliton) for B -surfaces and construct the simplest soliton solutions in all cases. In Sect.3 we construct finite-gap solutions in B_0, B_1 and B_2 cases. Especial attention is paid on the comparison of analytical properties of Baker-Akhiezer function. In Sect.4 we analyse B^+ -surfaces. Here our treatment contains less details since many facts may be obtained from the B -case by simple formal replacement $x \rightarrow z, y \rightarrow \bar{z}$. For general $\mathcal{K} > 0$ we show the equivalence of equation (1.7) equivalent to the Gauss-Codazzi system. We show that B_2^+ -surfaces may be alternatively characterized by the fact that their Gauss map is an axisymmetric harmonic map $\mathbf{R}^3 \rightarrow S^2$. Then we discuss how Bäcklund-Darboux transformations and finite-gap solutions differ from the B -case. The main focus will be on finite-gap solutions where the difference with B -case is most significant. In Sect.5 we prove equivalence of Gauss-Codazzi system and equation (1.8) for general $\mathcal{H}(z, \bar{z})$, discuss the zero curvature formulation for (1.8) and the difficulties which arise if we try to construct explicit solutions of this equation.

2 Bianchi surfaces

Consider some congruence C (two-parametric family) of straight lines in R^3 . In general case it has two focal surfaces: S_1 and S_2 .

Definition 1 *Congruence C is called Bianchi congruence (briefly B -congruence) if the Gaussian curvatures of S_1 and S_2 lying on the same straight line of the congruence are coinciding and negative. Surfaces S_1 and S_2 are called Bianchi surfaces.*

It seems that the Bianchi congruences and Bianchi surfaces had been first introduced by Bianchi [3]; proof of the following theorem may be found in [2]. For more recent presentation and references see [7].

Theorem 2.1 *Gaussian curvature $\mathcal{K}$ of Bianchi surface S obeys the following equation:*

$$\left[\frac{1}{\sqrt{-\mathcal{K}}}\right]_{xy} = 0 \quad (2.1)$$

where x, y are asymptotic coordinates on S i.e. coordinates in which the second fundamental form is off-diagonal.

Equation (2.1) implies

$$\mathcal{K} = \frac{-1}{\rho^2(x, y)} \quad \rho(x, y) = f(x) + g(y) \quad (2.2)$$

where $f(x)$ and $g(y)$ are arbitrary real-valued functions.

Consider embedding of arbitrary surface S in $\mathbf{R}^3$ defined by vector-function $F(x, y) : \mathbf{R}^2 \rightarrow \mathbf{R}^3$. Let $N(x, y)$ ($|N| = 1$) be normal vector to S . If S has negative Gaussian curvature

$$\mathcal{K} = \frac{-1}{\rho^2(x, y)} \quad \rho(x, y) > 0$$

we can always locally choose coordinates (x, y) along asymptotic lines (the curvature of asymptotic lines $F(x = \text{const}, y)$ and $F(x, y = \text{const})$ is equal to zero; not only vectors F_x and F_y , but also F_{xx} and F_{yy} are orthogonal to normal vector N).

Then the Ist and IInd fundamental forms may be parametrized as follows (taking into account that $\mathcal{K} \equiv \det I / \det II$):

$$I = \langle dF, dF \rangle = \rho^2[a^2 dx^2 + 2ab \cos \phi dx dy + b^2 dy^2] \quad (2.3)$$

$$II = - \langle dF, dN \rangle = 2\rho ab \sin \phi dx dy \quad (2.4)$$

where $\rho a = |F_x|$, $\rho b = |F_y|$; ϕ is an angle between F_x and F_y i.e. between the asymptotic lines. (We consider weakly regular surfaces $|F_x| \neq 0$, $|F_y| \neq 0$)

Derive Gauss-Weingarten system in quaternionic form (our exposition at this point mostly follows [5], where reader can find more details).

The quaternionic description of surfaces is based on the following standard identification of $\mathbf{R}^3$ with algebra $su(2)$:

$$X = (x_1, x_2, x_3) \in \mathbf{R}^3 \leftrightarrow X = \sum_{k=1}^3 x_k \frac{\sigma_k}{2i}$$

where

$$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \quad \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \quad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

are Pauli matrices. In the sequel all vectors of $\mathbf{R}^3$ will be represented by the elements of $su(2)$. The scalar product may be rewritten in matrix notations as follows:

$$\langle X, Y \rangle = -2\text{Tr}(XY)$$

Consider in R^3 the following basis (in quaternionic notations):

$$\rho a \left(\frac{\sigma_1}{2i} \cos \phi_1 + \frac{\sigma_2}{2i} \sin \phi_1 \right) \quad \rho b \left(\frac{\sigma_1}{2i} \cos \phi_2 - \frac{\sigma_2}{2i} \sin \phi_2 \right) \quad \frac{\sigma_3}{2i}$$

The angle between first two vectors is equal to $\phi_1 + \phi_2$. If $\phi_1 + \phi_2 = \phi$ we can find $SU(2)$ transformation which maps this basis into basis (F_x, F_y, N) :

$$F_x = \rho a \tilde{\Phi}^{-1} \left(\frac{\sigma_1}{2i} \cos \phi_1 + \frac{\sigma_2}{2i} \sin \phi_1 \right) \tilde{\Phi} \quad F_y = \rho b \tilde{\Phi}^{-1} \left(\frac{\sigma_1}{2i} \cos \phi_2 - \frac{\sigma_2}{2i} \sin \phi_2 \right) \tilde{\Phi} \quad (2.5)$$

$$N = \tilde{\Phi}^{-1} \frac{\sigma_3}{2i} \tilde{\Phi} \quad \phi_1 + \phi_2 = \phi$$

where $\tilde{\Phi} \in SU(2)$

Theorem 2.2 If $\phi_1 + \phi_2 = \phi$ then matrix $\tilde{\Phi}(x, y) \in SU(2)$ defined by (2.5) satisfies the following Gauss-Weingarten (GW) linear system:

$$\tilde{\Phi}_x = U \tilde{\Phi} \quad \tilde{\Phi}_y = V \tilde{\Phi} \quad (2.6)$$

where

$$U = \frac{i}{2} \left[-a \cos \phi_1 \sigma_1 - a \sin \phi_1 \sigma_2 + \left(\phi_{2x} + \frac{\rho_y a}{2\rho b} \sin(\phi_1 + \phi_2) \right) \sigma_3 \right] \quad (2.7)$$

$$V = \frac{i}{2} \left[b \cos \phi_2 \sigma_1 - b \sin \phi_2 \sigma_2 + \left(-\phi_{1y} - \frac{\rho_x b}{2\rho a} \sin(\phi_1 + \phi_2) \right) \sigma_3 \right] \quad (2.8)$$

Proof. Let $U \equiv \tilde{\Phi}_x \tilde{\Phi}^{-1}$, $V \equiv \tilde{\Phi}_y \tilde{\Phi}^{-1}$. Relations $\langle F_{xx}, N \rangle = \langle F_{yy}, N \rangle = 0$ imply:

$$0 = - \langle F_{xx}, N \rangle = \langle F_x, N_x \rangle = \frac{\rho a}{2} \text{Tr} \{ (\cos \phi_1 \sigma_1 + \sin \phi_1 \sigma_2) [\sigma_3, U] \}$$

$$0 = - \langle F_{yy}, N \rangle = \langle F_y, N_y \rangle = \frac{\rho b}{2} \text{Tr} \{ (\cos \phi_2 \sigma_1 - \sin \phi_2 \sigma_2) [\sigma_3, V] \}$$

Hence the off-diagonal parts of U and V are proportional to $\cos \phi_1 \sigma_1 + \sin \phi_1 \sigma_2$ and $\cos \phi_2 \sigma_1 - \sin \phi_2 \sigma_2$ respectively. Coefficients of proportionality $-ia/2$ and $-ib/2$ respectively may be calculated from the relation

$$\langle F_x, N_y \rangle = \langle F_y, N_x \rangle = -\rho ab \sin(\phi_1 + \phi_2)$$

As a result we obtain

$$U = -\frac{ia}{2}(\sigma_1 \cos \phi_1 + \sigma_2 \sin \phi_1) + \frac{iQ}{2}\sigma_3 \quad V = \frac{ib}{2}(\sigma_1 \cos \phi_2 - \sigma_2 \sin \phi_2) + \frac{iR}{2}\sigma_3 \quad (2.9)$$

To find Q and R we can use integrability condition $F_{xy} = F_{yx}$ of (2.5) which gives

$$Q = \phi_{2x} + q \quad R = -\phi_{1y} - r$$

$$q = \frac{(\rho a)_y - (\rho b)_x \cos \phi}{\rho b \sin \phi} \quad r = \frac{(\rho b)_x - (\rho a)_y \cos \phi}{\rho b \sin \phi} \quad \phi = \phi_1 + \phi_2 \quad (2.10)$$

Compartibility condition of (2.9) gives the Gauss equation

$$\phi_{xy} + 2r_x - 2q_y - ab \sin \phi = 0 \quad (2.11)$$

and the following alternative representations for q and r :

$$q = -\frac{a_y + b_x \cos \phi}{b \sin \phi} \quad r = -\frac{b_x + a_y \cos \phi}{a \sin \phi} \quad (2.12)$$

Comparing (2.10) and (2.12) we obtain the Codazzi equations

$$a_y + \frac{\rho_y}{2\rho}a - \frac{\rho_x}{2\rho}b \cos \phi = 0 \quad b_x + \frac{\rho_x}{2\rho}b - \frac{\rho_y}{2\rho}a \cos \phi = 0 \quad (2.13)$$

and the required form of q and r :

$$q = \frac{\rho_y a}{2\rho b} \sin \phi \quad r = \frac{\rho_x b}{2\rho a} \sin \phi$$

□

Simultaneously we derived the following

Theorem 2.3 *Compartibility condition of Gauss-Weingarten system (2.6), (2.7), (2.8) is the following Gauss-Codazzi system of non-linear equations for three real-valued functions $a(x, y) > 0$, $b(x, y) > 0$, $\phi(x, y) = \phi_1 + \phi_2$:*

$$\phi_{xy} + \frac{1}{2}\left(\frac{\rho_x b}{\rho a} \sin \phi\right)_x + \frac{1}{2}\left(\frac{\rho_y a}{\rho b} \sin \phi\right)_y - ab \sin \phi = 0$$

$$a_y + \frac{\rho_y}{2\rho}a - \frac{\rho_x}{2\rho}b \cos \phi = 0 \quad (2.14)$$

$$b_x + \frac{\rho_x}{2\rho}b - \frac{\rho_y}{2\rho}a \cos \phi = 0$$

It turns out that using quaternionic form (2.6), (2.7), (2.8) of Gauss-Weingarten system we can introduce instead of three real-valued functions (a, b, ϕ) one complex-valued function $\mathcal{E}(x, y)$. In terms of $\mathcal{E}$ the Gauss-Codazzi system may be equivalently rewritten as one second order differential equation:

Theorem 2.4 Let $\mathcal{E}(x, y) \equiv \tilde{\Phi}_{11}(\tilde{\Phi}_{12})^{-1}$, where matrix $\tilde{\Phi}(x, y) \in SU(2)$ is a solution of Gauss-Weingarten linear system (2.6), (2.7), (2.8) for surface S with arbitrary negative Gaussian curvature $\mathcal{K} = -\rho^{-2}$. Then $\mathcal{E}(x, y)$ satisfies the following equation:

$$(\mathcal{E} + \bar{\mathcal{E}}^{-1})(\mathcal{E}_{xy} + \frac{\rho_x \mathcal{E}_y}{2\rho} + \frac{\rho_y \mathcal{E}_x}{2\rho}) = 2\mathcal{E}_x \mathcal{E}_y \quad (2.15)$$

Functions a, b and ϕ solving Gauss-Codazzi system (2.14) may be expressed in terms of $\mathcal{E}$ as follows:

$$a = 2 \frac{|\mathcal{E}_x|}{1 + |\mathcal{E}|^2} \quad b = 2 \frac{|\mathcal{E}_y|}{1 + |\mathcal{E}|^2} \quad \phi = \pi + \arg \frac{\mathcal{E}_x}{\mathcal{E}_y} \quad (2.16)$$

Proof. Since $\tilde{\Phi} \in SU(2)$, it can always be parametrized as follows

$$\tilde{\Phi} = \frac{1}{\sqrt{|A|^2 + |B|^2}} \begin{pmatrix} A & B \\ -\bar{B} & \bar{A} \end{pmatrix}$$

Calculating $\tilde{\Phi}_x \tilde{\Phi}^{-1}$ and $\tilde{\Phi}_y \tilde{\Phi}^{-1}$ in terms of A and B and identifying with (2.7) and (2.8) respectively, we can easily check that a, b and ϕ depend only on the ratio $\mathcal{E} = A/B$ according to (2.16). $\square$

Remark 1. Defining our matrix $\tilde{\Phi}$ by (2.5) we obtained one-parametric family of $\tilde{\Phi}$ describing the same moving frame (F_x, F_y, N) (the only link between ϕ_1 and ϕ_2 was $\phi_1 + \phi_2 = \phi$). The usual way to pick one representative from this family [5, 15] is to choose $\phi_1 = \phi_2 = \phi/2$. We can follow another way: choose $\tilde{\Phi}$ in the coset space $SU(2)/U(1)$:

$$\tilde{\Phi} = \frac{1}{\sqrt{1 + |\mathcal{E}|^2}} \begin{pmatrix} \mathcal{E} & 1 \\ -1 & \bar{\mathcal{E}} \end{pmatrix} \quad (2.17)$$

Then in (2.5)

$$\phi_1 = \frac{\pi}{2} - \arg(\mathcal{E}_x) \quad \phi_2 = \frac{\pi}{2} + \arg(\mathcal{E}_y) \quad (\phi_1 + \phi_2 = \pi + \arg \frac{\mathcal{E}_y}{\mathcal{E}_x} = \phi)$$

This shows how to derive statement of Theorem 3 in deductive way. Notice also that $\tilde{\Phi}_{11}(\tilde{\Phi}_{12})^{-1} = \mathcal{E}$ for an arbitrary representative of the family of matrices $\tilde{\Phi}$ describing the same (F_x, F_y, N) : arbitrary two of them are related by

$$\tilde{\Phi}_1 = \begin{pmatrix} e^{i\beta(x, y)} & 0 \\ 0 & e^{-i\beta(x, y)} \end{pmatrix} \tilde{\Phi}, \quad \beta(x, y) \in \mathbf{R}$$

Remark 2. Calculating logarithmic derivatives of (2.17) we easily obtain a, b, ϕ according to (2.16). Other objects may be expressed in terms of $\mathcal{E}(x, y)$ as follows
- fundamental forms:

$$I = \frac{4\rho^2}{(1 + |\mathcal{E}|^2)^2} (\mathcal{E}_x dx - \mathcal{E}_y dy)(\bar{\mathcal{E}}_x dx - \bar{\mathcal{E}}_y dy) = \frac{4\rho^2}{(1 + |\mathcal{E}|^2)^2} |\mathcal{E}_x dx - \mathcal{E}_y dy|^2 \quad (2.18)$$

$$II = \frac{4i\rho}{(1 + |\mathcal{E}|^2)^2} (\bar{\mathcal{E}}_x \mathcal{E}_y - \mathcal{E}_x \bar{\mathcal{E}}_y) dx dy \quad (2.19)$$

- mean curvature:

$$\mathcal{H} \equiv \frac{1}{2} \text{Tr}(I^{-1}II) = \frac{\cot \phi}{\rho} = \frac{1}{\rho} \frac{\text{Re}(\mathcal{E}_y \bar{\mathcal{E}}_x)}{\text{Im}(\mathcal{E}_y \bar{\mathcal{E}}_x)}$$

- Beltrami coefficient:

$$\mu_y^x \equiv \frac{1}{g_{xx}}(g_{xy} - \sqrt{g_{xy}^2 - g_{xx}g_{yy}}) = -\frac{\mathcal{E}_y}{\mathcal{E}_x}$$

where g_{ij} is the matrix of our 1st fundamental form.

- moving frame:

$$\begin{aligned} F_x &= \frac{\rho}{(\mathcal{E} + \bar{\mathcal{E}}^{-1})^2} \begin{pmatrix} 1 & 0 \\ 0 & (\mathcal{E} \bar{\mathcal{E}}^{-1})^2 \end{pmatrix} \begin{pmatrix} (\mathcal{E} \bar{\mathcal{E}}^{-1})_x & (\bar{\mathcal{E}}^{-1} - \mathcal{E})_x \\ (\bar{\mathcal{E}} - \mathcal{E}^{-1})_x & (\bar{\mathcal{E}} \mathcal{E}^{-1})_x \end{pmatrix} \\ F_y &= \frac{\rho}{(\mathcal{E} + \bar{\mathcal{E}}^{-1})^2} \begin{pmatrix} 1 & 0 \\ 0 & (\mathcal{E} \bar{\mathcal{E}}^{-1})^2 \end{pmatrix} \begin{pmatrix} (\bar{\mathcal{E}} \mathcal{E}^{-1})_y & (\mathcal{E} - \bar{\mathcal{E}}^{-1})_y \\ (\mathcal{E}^{-1} - \bar{\mathcal{E}})_y & (\mathcal{E} \bar{\mathcal{E}}^{-1})_y \end{pmatrix} \\ N &= \frac{1}{2i} \frac{1}{1 + |\mathcal{E}|^2} \begin{pmatrix} |\mathcal{E}|^2 - 1 & 2\bar{\mathcal{E}} \\ 2\mathcal{E} & 1 - |\mathcal{E}|^2 \end{pmatrix} \end{aligned} \quad (2.20)$$

Remark 3. Equation (2.15) looks very similar to the Ernst equation in general relativity. This link really takes place in the case of B_2 -surfaces ($\rho = x + y$), when (2.15) and the Ernst equations are simply different real reductions of the coupled Ernst system (see below).

Remark 4. In terms of the Gauss map $N(x, y)$ (2.20) we can rewrite equation (2.15) as follows:

$$(\rho N_x N^{-1})_y + (\rho N_y N^{-1})_x = 0$$

or, in coordinates $u = (x + y)/2$, $v = (x - y)/2$:

$$(\rho N_u N^{-1})_u - (\rho N_v N^{-1})_v = 0$$

Remark 5. Equation (2.15) admits the following symmetry transformation:

$$\mathcal{E} \rightarrow \frac{a\mathcal{E} + b}{c\mathcal{E} + d} \quad \begin{pmatrix} a & b \\ c & d \end{pmatrix} \equiv C \in SU(2) \quad (2.21)$$

where matrix C is independent of (x, y) . This transformation doesn't change a, b, ϕ and corresponds to transformation $\tilde{\Phi} \rightarrow \tilde{\Phi}C$.

From the point of view of inverse scattering method linear system (2.6) may be interpreted as a zero curvature representation for equation (2.15). However, it has an essential shortage: the absence in general case of additional "spectral" parameter. In the case of Bianchi surfaces this parameter can be inserted in (2.6). Namely, for

$$\rho(x, y) = f(x) + g(y)$$

equation (2.15) takes the following form:

$$(\mathcal{E} + \bar{\mathcal{E}}^{-1})(\mathcal{E}_{xy} + \frac{f_x \mathcal{E}_y + g_y \mathcal{E}_x}{2(f+g)}) = 2\mathcal{E}_x \mathcal{E}_y \quad (2.22)$$

We shall call (2.22) the Bianchi equation (or B-equation).

Theorem 2.5 *Bianchi equation (2.22) is a compatibility condition of the following linear system:*

$$\Psi_x = U\Psi \quad \Psi_y = V\Psi \quad (2.23)$$

$$U = \begin{pmatrix} A & 0 \\ 0 & B \end{pmatrix} + \sqrt{\frac{\lambda - g(y)}{\lambda + f(x)}} \begin{pmatrix} 0 & A \\ B & 0 \end{pmatrix} \quad V = \begin{pmatrix} C & 0 \\ 0 & D \end{pmatrix} + \sqrt{\frac{\lambda + f(x)}{\lambda - g(y)}} \begin{pmatrix} 0 & C \\ D & 0 \end{pmatrix}$$

where λ is a spectral parameter; $\Psi(x, y, \lambda)$ is a 2×2 matrix-valued function. Coefficients $A, \dots, D$ may be expressed from the condition

$$\Psi(x, y, \lambda = \infty) = \begin{pmatrix} \mathcal{E} & 1 \\ \bar{\mathcal{E}}^{-1} & -1 \end{pmatrix} \quad (2.24)$$

as follows

$$A = \frac{\mathcal{E}_x}{\mathcal{E} + \bar{\mathcal{E}}^{-1}} \quad B = \frac{(\bar{\mathcal{E}}^{-1})_x}{\mathcal{E} + \bar{\mathcal{E}}^{-1}} \quad C = \frac{\mathcal{E}_y}{\mathcal{E} + \bar{\mathcal{E}}^{-1}} \quad D = \frac{(\bar{\mathcal{E}}^{-1})_y}{\mathcal{E} + \bar{\mathcal{E}}^{-1}} \quad (2.25)$$

The *Proof* may be obtained by simple calculation. $\square$

Representation (2.23), (2.24) may be equivalently rewritten in terms of a, b, ϕ inserting the spectral parameter into original Gauss-Weingarten system (2.6), (2.7), (2.8) (for $\rho = f(x) + g(y)$):

$$\Phi_x = \frac{i}{2} \left[-\sqrt{\frac{\lambda - g(y)}{\lambda + f(x)}} (a \cos \phi_1 \sigma_1 + a \sin \phi_1 \sigma_2) + Q \sigma_3 \right] \Phi \quad (2.26)$$

$$\Phi_y = \frac{i}{2} \left[\sqrt{\frac{\lambda + f(x)}{\lambda - g(y)}} (b \cos \phi_2 \sigma_1 - b \sin \phi_2 \sigma_2) + R \sigma_3 \right] \Phi$$

where

$$\tilde{\Phi} \equiv \Phi(\lambda = \infty) \in SU(2)$$

Notice that it is not necessary to set *a priori* coefficients $Q(x, y)$ and $R(x, y)$ according to (2.10); they are uniquely defined from the compatibility condition of (2.26).

It turns out that the knowledge of λ -dependent solution of (2.26) obeying condition $\Phi(x, y, \lambda \in \mathbf{R}) \in SU(2)$ (the latter condition may always be satisfied by the gauge transformation of Φ) allows to describe embedding of the Bianchi surface S in $\mathbf{R}^3$ in the following way [10]:

Theorem 2.6 *Let $\Phi(x, y, \lambda)$ be some solution of (2.26) satisfying condition $\Phi(\lambda \in \mathbf{R}) \in SU(2)$. Then formulae*

$$F = -2\Phi^{-1}\Phi_{1/\lambda}|_{\lambda=\infty} \quad (2.27)$$

$$N = \Phi^{-1}\frac{\sigma_3}{2i}\Phi|_{\lambda=\infty} \quad , \quad (2.28)$$

where subindex $1/\lambda$ means derivative of Φ with respect to the local parameter $1/\lambda$ at $\lambda = \infty$, define the embedding and normal vector of Bianchi surface S having the Ist and IIInd fundamental forms (2.3) and (2.4) respectively.

Proof. We have to check that F and N defined by (2.27) and (2.28) obey (2.5). For N this is obvious; for our F we have:

$$\begin{aligned} F_x &= (-2\Phi^{-1}\Phi_{1/\lambda})_x|_{\lambda=\infty} = 2\Phi^{-1}\Phi_x\Phi^{-1}\Phi_{1/\lambda}|_{\lambda=\infty} - 2\Phi^{-1}(U_{1/\lambda}\Phi + U\Phi_{1/\lambda})|_{\lambda=\infty} \\ &= -2\Phi^{-1}U_{1/\lambda}\Phi|_{\lambda=\infty} \end{aligned} \quad (2.29)$$

Taking into account that matrices U and V defined by (2.23) coincide at $\lambda = \infty$ with (2.7) and (2.8) respectively, we see that (2.29) coincides with the first equation in (2.5); analogously for F_y . $\square$

It is natural to distinguish (locally) the following three different types of Bianchi surfaces:

0. B_0 -surfaces: $f = \text{const}, g = \text{const}$. In this case without loss of generality we can put $f(x) = 0$, $g(y) = 1$; then $\mathcal{K} = -1$. Equation (2.22) reduces to

$$(\mathcal{E} + \bar{\mathcal{E}}^{-1})\mathcal{E}_{xy} = 2\mathcal{E}_x\mathcal{E}_y \quad (2.30)$$

Gauss-Codazzi system in terms of a, b, ϕ takes the following form:

$$\phi_{xy} = ab \sin \phi \quad a_y = b_x = 0 \quad (2.31)$$

If surface S is weakly regular i.e. $a \neq 0$, $b \neq 0$ then the change of variables $a(x)dx = dx'$, $b(y)dy = dy'$ reduces this system to sine-Gordon equation:

$$\phi_{x'y'} = \sin \phi \quad (2.32)$$

Correspondence (2.16) between (2.32) and (2.30) is one-to-one up to an arbitrary change of variables in (2.30) ((2.30) is invariant with respect to an arbitrary change of variables $x \rightarrow x'(x)$, $y \rightarrow y'(y)$ while (2.32) not). To fix the choice of coordinates in (2.30) we can impose on $\mathcal{E}$ the 'normalization' restriction $a = b = 1$ i.e.

$$2\frac{|\mathcal{E}_x|}{1+|\mathcal{E}|^2} = 2\frac{|\mathcal{E}_y|}{1+|\mathcal{E}|^2} = 1$$

1 B_1 -surfaces: $f(x) = \text{const}$, $g(y) \neq \text{const}$ and $g'(y) \neq 0$. Now without loss of generality we can put locally $f(x) = 0$, $g(y) = y$; then

$$\mathcal{K} = -\frac{1}{y^2}$$

Equation (2.22) reduces to

$$(\mathcal{E} + \bar{\mathcal{E}}^{-1})(\mathcal{E}_{xy} + \frac{\mathcal{E}_x}{2y}) = 2\mathcal{E}_x\mathcal{E}_y \quad (2.33)$$

2 B_2 -surfaces: $f(x) \neq \text{const}$, $g(y) \neq \text{const}$ (and $f' \neq 0$, $g' \neq 0$). Now without loss of generality we can locally choose $f(x) = x$, $g(y) = y$ and (2.22) reduces to

$$(\mathcal{E} + \bar{\mathcal{E}}^{-1})(\mathcal{E}_{xy} + \frac{\mathcal{E}_x + \mathcal{E}_y}{2(x+y)}) = 2\mathcal{E}_x\mathcal{E}_y \quad (2.34)$$

This equation is a 'unitary' reduction $\mathcal{E}_1 = \bar{\mathcal{E}}_2^{-1}$ of the following coupled system for two complex-valued functions $\mathcal{E}_{1,2}(x, y)$:

$$(\mathcal{E}_1 + \mathcal{E}_2)(\mathcal{E}_{1xy} + \frac{\mathcal{E}_{1x} + \mathcal{E}_{1y}}{2(x+y)}) = 2\mathcal{E}_{1x}\mathcal{E}_{1y} \quad (2.35)$$

$$(\mathcal{E}_1 + \mathcal{E}_2)(\mathcal{E}_{2xy} + \frac{\mathcal{E}_{2x} + \mathcal{E}_{2y}}{2(x+y)}) = 2\mathcal{E}_{2x}\mathcal{E}_{2y}$$

The 'real' reduction of this system $\mathcal{E} \equiv \mathcal{E}_1 = \bar{\mathcal{E}}_2$:

$$(\mathcal{E} + \bar{\mathcal{E}})(\mathcal{E}_{xy} + \frac{\mathcal{E}_x + \mathcal{E}_y}{2(x+y)}) = 2\mathcal{E}_x\mathcal{E}_y$$

is the Ernst equation which is essential part of the Einstein equations in the presence of two commuting space-like Killing vectors. System (2.35) admits zero curvature representation which coincides with (2.23) if we take $g(y) = y$, $f(x) = x$ and instead of condition (2.24) impose

$$\Psi(\lambda = \infty) = \begin{pmatrix} \mathcal{E}_1 & 1 \\ \mathcal{E}_2 & -1 \end{pmatrix} \quad (2.36)$$

Reduction $\mathcal{E}_1 = \bar{\mathcal{E}}_2$ gives $U - V$ -pair for Ernst equation; reduction $\mathcal{E}_1 = \bar{\mathcal{E}}_2^{-1}$ gives $U - V$ -pair for B_2 - equation (2.34). Representation (2.23), (2.36) (with $\mathcal{E}_1 = \bar{\mathcal{E}}_2$) was found in [18] and coincides in essence with $U - V$ pairs found earlier in [1] and [16].

3 Bäcklund-Darboux transformations of Bianchi surfaces

In this section we shall work with original Gauss-Codazzi system (2.14) for $\rho = f(x) + g(y)$:

$$\begin{aligned} \phi_{xy} + (\frac{f_x}{2(f+g)}\frac{b}{a}\sin\phi)_x + (\frac{g_y}{2(f+g)}\frac{a}{b}\sin\phi)_y - ab\sin\phi &= 0 \\ a_y + \frac{g_y}{2(f+g)}a - \frac{f_x}{2(f+g)}b\cos\phi &= 0 \\ b_x + \frac{f_x}{2(f+g)}b - \frac{g_y}{2(f+g)}a\cos\phi &= 0 \end{aligned} \quad (3.1)$$

The $U - V$ pair for (3.1) is given by (2.26).

Denote by $\mathcal{L}_0$ the Riemann surface of genus zero defined by the equation

$$\omega^2 = (\lambda + f(x))(\lambda - g(y))$$

Let τ be involution on $\mathcal{L}_0$ interchanging the sheets.

Theorem 3.1 *Let 2×2 matrix-valued function $\Phi(x, y, P)$ ($(x, y) \in \mathbb{R}^2, P \in \mathcal{L}_0$) satisfy the following conditions:*

1. *Logarithmic derivatives $\Phi_x \Phi^{-1}$ and $\Phi_y \Phi^{-1}$ have simple poles on $\mathcal{L}_0$ at $\lambda = -f(x)$ and $\lambda = g(y)$ (in local parameters $\sqrt{\lambda + f(x)}$ and $\sqrt{\lambda - g(y)}$) respectively.*
2. *Functions $\Phi_x \Phi^{-1}$ and $\Phi_y \Phi^{-1}$ are holomorphic on $\mathcal{L}_0$ everywhere outside of the points $\lambda = -f(x)$ and $\lambda = g(y)$ respectively.*
3. *Reduction restriction:*

$$\Phi(\lambda^\tau) = \sigma_3 \Phi(\lambda) M \quad (3.2)$$

where matrix M is independent of (x, y) .

4. *'Unitarity' conditions:*

$$\Phi(\lambda) \Phi^\dagger(\lambda^\sigma) = I \quad \det \Phi(\lambda) = 1 \quad (3.3)$$

where I is a unit matrix; $\sigma : \mathcal{L}_0 \rightarrow \mathcal{L}_0$ is an anti-holomorphic involution on $\mathcal{L}_0$ acting in the neighbourhood of $\lambda = \infty$ as $\lambda \rightarrow \bar{\lambda}$ on both sheets (this condition uniquely defines σ if we assume that the branch cut on $\mathcal{L}_0$ between $-f(x)$ and $g(y)$ doesn't pass through $\lambda = \infty$).

Then function $\Phi(x, y, P)$ satisfies the linear system (2.26) for some $(a, b, \phi_1, \phi_2, P, Q)$, and, therefore, functions $a, b, \phi = \phi_1 + \phi_2$ solve the Gauss-Codazzi system (3.1).

The *Proof* is straightforward. Conditions 1. and 2. provide the necessary analytical structure of the logarithmic derivatives $\Phi_x \Phi^{-1}$ and $\Phi_y \Phi^{-1}$ on $\mathcal{L}_0$:

$$\Phi_x \Phi^{-1} = X_1 \sqrt{\frac{\lambda - g(y)}{\lambda + f(x)}} + Y_1 \quad \Phi_y \Phi^{-1} = X_2 \sqrt{\frac{\lambda + f(x)}{\lambda - g(y)}} + Y_2$$

Reduction (3.2) implies that $Y_{1,2}$ are diagonal and $X_{1,2}$ are anti-diagonal matrices in accordance with (2.26). Finally, the unitarity condition (3.3) implies that $X_j \in su(2)$ and $Y_j \in su(2)$ and, therefore, X_j and Y_j may be parametrized by some (a, b, ϕ_1, ϕ_2) according to (2.26). $\square$

We shall derive Bäcklund-Darboux transformation for (3.1) in the framework of general 'dressing' method in the theory of integrable systems [8, 21].

Let (a_0, b_0, ϕ_0) be some 'seed' solution of (3.1) and Φ_0 be related solution of (2.26). We would like to find such a 'dressing' function $T(x, y, P)$ meromorphic on $\mathcal{L}_0$ that function $\Phi \equiv T \Phi_0$ is again a solution of (2.26) with some new coefficients (a, b, ϕ) .

Theorem 3.2 *Let function Φ_0 be a solution of (2.26) satisfying conditions 1.-4. of Th.3.1. Let matrix-valued function $T(x, y, P)$ obey the following conditions:*

1. *Function $T(P)$ is holomorphic and invertible at $\lambda = -f(x)$ and $\lambda = g(y)$.*
 2. *Functions $(T \Phi_0)_x (T \Phi_0)^{-1}$ and $(T \Phi_0)_y (T \Phi_0)^{-1}$ are holomorphic (on $\mathcal{L}_0$) at the poles of $T(P)$ and the zeros of $\det T(P)$.*
 3. *Coincides with item 3. of Th.3.1 with replacement $\Phi \rightarrow T$ and $M = \sigma_3$.*
 4. *Coincides with item 4. of Th.3.1 with replacement $\Phi \rightarrow T$.*
- Then function

$$\Phi = T \Phi_0 \quad (3.4)$$

satisfies all items 1.-4. of Th.3.1 and, therefore, solves (2.26) with some new coefficients (a, b, ϕ) which give some new solution of (3.1).

The *Proof* immediately follows from Th.3.1.

3.1 One-soliton and two-soliton BDT

It is convenient to start with two-soliton BDT and then obtain one-soliton one as some special case. Choose the following ansatz for $T(P)$:

$$T(P) = C(P, x, y) \tilde{T}(P) \quad \tilde{T}(P) \equiv \begin{pmatrix} q_0 + q_1 \lambda & \omega \\ -\omega & \bar{q}_0 + \bar{q}_1 \lambda \end{pmatrix} \quad (3.5)$$

where $\omega = \sqrt{(\lambda + f(x))(\lambda - g(y))}$; $q_{0,1}$ are two scalar functions on (x, y) which will be chosen later;

$$C = [\det \tilde{T}]^{-1/2} \quad (3.6)$$

is a normalization factor.

It is easy to verify that function $T(P)$ satisfies items 1., 3. and 4. of Th.3.2.

To satisfy item 2. use the following canonical procedure [8, 21].

Consider $\tilde{T}\Phi_0$, where $\tilde{T}$ is set by (3.5). It is easy to see that logarithmic derivatives $(\tilde{T}\Phi_0)_x(\tilde{T}\Phi_0)^{-1}$ and $(\tilde{T}\Phi_0)_y(\tilde{T}\Phi_0)^{-1}$ are holomorphic at the poles of $\tilde{T}$ on $\mathcal{L}_0$ i.e. at $\Phi = \infty^+$ and $\Phi = \infty^-$. It remains to provide regularity of these logarithmic derivatives at the zeros of $\det \tilde{T}$.

Lemma 3.1 *Let the determinant of 2×2 matrix-valued function $\Phi(\lambda, x, y)$ has simple pole at the point $\lambda = \lambda_0$ independent of (x, y) . Denote by $\vec{a}$ vector annihilated by $\Phi(\lambda_0)$: $\Phi(\lambda_0)\vec{a} = 0$. If $\vec{a}$ is independent of (x, y) then functions $\Phi_x\Phi^{-1}$ and $\Phi_y\Phi^{-1}$ are holomorphic at $\lambda = \lambda_0$ (i.e. λ_0 is a regular singularity of $\Phi(\lambda)$).*

The *Proof* is simple (see [8]).

Define the zeros of $\det \tilde{T}$ by γ_1 and γ_2 and assign to these points vectors $(1, \alpha_1)$ and $(1, \alpha_2)$ (where α_j are (x, y) -independent parameters) respectively. According to Lemma 3.1 impose on $\tilde{T}$ the following conditions:

$$\tilde{T}(\gamma_j)\Phi_0(\gamma_j) \begin{pmatrix} 1 \\ \alpha_j \end{pmatrix} = 0 \quad j = 1, 2 \quad (3.7)$$

Conditions (3.7) provide regularity of $(\tilde{T}\Phi_0)_x(\tilde{T}\Phi_0)^{-1}$ and $(\tilde{T}\Phi_0)_y(\tilde{T}\Phi_0)^{-1}$ at $P = \gamma_j, j = 1, 2$. In turn, this entails regularity of $(T\Phi_0)_x(T\Phi_0)^{-1}$ and $(T\Phi_0)_y(T\Phi_0)^{-1}$ at these points. Relations (3.7) give four linear equations for two variables: q_0 and q_1 .

Assume that for the seed function Φ_0 matrix M from (3.2) is equal to σ_3 . Then it is easy to verify using relations (3.2) and (3.3) for Φ_0 that system (3.7) is compatible in the following two cases:

$$1. \gamma_j \in [-f(x), g(y)] \subset \mathbf{R} \quad \text{i.e.} \quad \gamma_j^{\tau\sigma} = \gamma_j \quad (\text{Fig.1a}), \quad |\alpha_j| = 1 \quad j = 1, 2 \quad (3.8)$$

$$2. \gamma_1 = \gamma_2^{\tau\sigma} \quad (\text{Fig.1b}), \quad \alpha_1 = \frac{1}{\bar{\alpha}_2} \quad (3.9)$$

(in both cases we choose the branch cut $[-f(x), g(y)]$ such that the anti-involution σ acts in the neighbourhood of γ_1, γ_2 as $P = (\lambda, \omega) \rightarrow \bar{P}^\tau = (\bar{\lambda}, \bar{\omega})$)

If conditions (3.8) or (3.9) are fulfilled, solution of (3.7) is given by

$$q_0 = \frac{\gamma_1\beta_2\omega_2 - \gamma_2\beta_1\omega_1}{\gamma_2 - \gamma_1} \quad q_1 = \frac{\beta_2\omega_2 - \beta_1\omega_1}{\gamma_1 - \gamma_2} \quad (3.10)$$

where

$$\omega_j \equiv \omega(\gamma_j) = \sqrt{(\gamma_j + f(x))(\gamma_j - g(y))} \quad \beta_j = \frac{\Phi_{021} + \alpha_j \Phi_{022}}{\Phi_{011} + \alpha_j \Phi_{012}} \quad j = 1, 2 \quad (3.11)$$

In the case (3.8) we have $|\beta_1| = |\beta_2| = 1$; in the case (3.9) $\beta_1 = \bar{\beta}_2^{-1}$.

Factor $C = (\det \tilde{T})^{-1/2}$ in (3.5) provides condition (3.3). Besides that, this factor doesn't introduce new singularities of $(T\Phi_0)_x(T\Phi_0)^{-1}$ and $(T\Phi_0)_y(T\Phi_0)^{-1}$ in comparison with $(\tilde{T}\Phi_0)_x(\tilde{T}\Phi_0)^{-1}$ and $(\tilde{T}\Phi_0)_y(\tilde{T}\Phi_0)^{-1}$ respectively. This follows from the identity $(\log \det \Psi)_x = \text{Tr}(\Psi_x \Psi^{-1})$ which entails

$$(T\Phi_0)_x(T\Phi_0)^{-1} = (\tilde{T}\Phi_0)_x(\tilde{T}\Phi_0)^{-1} - \frac{1}{2} \text{Tr}[(\tilde{T}\Phi_0)_x(\tilde{T}\Phi_0)^{-1}]$$

Formulae (3.4), (3.5), (3.10) and (3.11) give the Bäcklund-Darboux transformation for Bianchi surfaces in terms of function Φ . Function $\mathcal{E} = \Phi_{11}\Phi_{12}^{-1}$ transforms as follows:

$$\mathcal{E} = \frac{q_1 \mathcal{E}_0 - 1}{q_1 + \bar{\mathcal{E}}_0} \quad \mathcal{E}_0 \equiv (\Phi_0)_{11}(\Phi_0)_{12}^{-1} \quad (3.12)$$

According to (2.16) we see that to know the transformation of fundamental forms we need only coefficient q_1 . However, to find the transformation of embedding F (i.e. surface S itself) we need both q_0 and q_1 .

Using (3.4), (3.5), (3.10) and (3.11) we find according to (2.27), (2.28) the following link between embedding F_0 of the 'seed' surface S_0 and embedding of 'new' surface S :

$$\begin{aligned} F &= -2\Phi^{-1}\Phi_{1/\lambda}|_{\lambda=\infty} = -2[\Phi_0^{-1}\Phi_{01/\lambda} + \Phi_0^{-1}T^{-1}T_{1/\lambda}\Phi_0]|_{\lambda=\infty} \\ &= F_0 - \Phi_0^{-1} \frac{2}{|q_1|^2 + 1} \begin{pmatrix} q_0 \bar{q}_1 + \frac{f-g}{2} & -\bar{q}_0 + \bar{q}_1 \frac{f-g}{2} \\ q_0 - q_1 \frac{f-g}{2} & q_1 \bar{q}_0 + \frac{f-g}{2} \end{pmatrix} \Phi_0 + \tilde{C}I \end{aligned}$$

where scalar function $\tilde{C}$ is determined by the condition $\text{Tr} F = 0$. As a result we obtain the two-soliton BDT in terms of embedding:

$$F - F_0 = \frac{4}{1 + |q_1|^2} \{ \text{Im}(q_0 \bar{q}_1) N_0 + \text{Im}(q_0 - \frac{f-g}{2} q_1) A_0 - \text{Re}(q_0 - \frac{f-g}{2} q_1) B_0 \} \quad (3.13)$$

where

$$N_0 = \Phi_0^{-1} \frac{\sigma_3}{2i} \Phi_0|_{\lambda=\infty} \quad (3.14)$$

is a normal vector of surface S_0 ; vectors

$$A_0 = \Phi_0^{-1} \frac{\sigma_1}{2i} \Phi_0|_{\lambda=\infty}, \quad B_0 = \Phi_0^{-1} \frac{\sigma_2}{2i} \Phi_0|_{\lambda=\infty}, \quad (3.15)$$

are two orthonormal vectors in the tangent plane of S_0 .

To obtain the most "elementary" one-soliton BDT let's put $\gamma_2 = -f(x)$ (formally this is not a partial case of two-soliton solution if $f(x) \neq \text{const}$ since we assumed previously that γ_1 and γ_2 are (x, y) -independent, but it is easy to verify that the logarithmic derivatives of Φ at $\lambda = \gamma_2 = -f(x)$ have the proper analytical structure). Then we obtain from (3.10):

$$q_0 = -f(x)\beta_1\mu_1 \quad q_1 = -\beta_1\mu_1 \quad \mu_1 = \sqrt{\frac{\gamma_1 - g(y)}{\gamma_1 + f(x)}}$$

and

$$F - F_0 = \omega_1 \Phi_0^{-1} \begin{pmatrix} 0 & \bar{\beta}_1 \\ \beta_1 & 0 \end{pmatrix} \Phi_0$$

or

$$F = F_0 + 2i\omega_1 \text{Re}\beta_1 A_0 + 2i\omega_1 \text{Im}\beta_1 B_0 \quad (3.16)$$

(note that ω_1 is imaginary since we assumed $\gamma_j \in [-f(x), g(y)]$).

Here we obtain a new surface by some shift of every point of the old one in the tangent plane. Transformation (3.16) includes two arbitrary parameters: $\gamma_1 \in [-f(x), g(y)]$ and α_1 ($|\alpha_1| = 1$). Two-soliton transformation (3.13) may be obtained as a superposition of two transformations (3.16) with different parameters (γ_1, α_1) and (γ_2, α_2) (in the case $\gamma_j \in \mathbf{R}$, $|\alpha_j| = 1$ (3.8)). For the case of conjugated pair (3.9) embedding F obtained after the first step will be complex, but after the second step we obtain again a real F .

Remark 1. Formulas (3.13), (3.16) for BDT are the same for all subclasses of Bianchi surfaces: B_0 , B_1 and B_2 . However, as we shall see, the one- and two-soliton solutions (which we obtain applying the BDT to the simplest seed solution) have in these three cases different form due to different analytical properties of the simplest seed function Φ_0 .

Remark 2. Notice the following important property of transformation (3.16). Since the new surface F may be obtained as a shift of the old surface F_0 along some tangent line, surface F_0 may be considered as a focal surface of the two-parametric family (congruence) of these tangent lines. Conversely, we can always consider the inverse BDT, which leads to F_0 starting from F ; this inverse transformation will be along tangent lines of surface F . Thus the congruence of straight lines along which we transform F to F_0 and back is a Bianchi congruence with the focal surfaces F and F_0 .

This agrees with the original point of view of Bianchi who introduced the Bianchi congruence just as a way to transform one surface (F) to another (F_0) preserving the Gaussian curvature. So the theorem 2.1 means that the surfaces obeying condition (4.10) are *all* surfaces possessing one-soliton BDT preserving the Gaussian curvature. The two- (and more) soliton dressing transformations include (in generic situation) a shift in normal direction to surface F_0 and may be described by two (or more) congruences corresponding to one-soliton transformations.

For other two cases of integrable surfaces considered below (Bianchi surfaces with positive curvature and surfaces with harmonic inverse mean curvature) only even-soliton BDT exist, and, therefore, the congruence defining such a transformation is not tangent to surfaces described by F and F_0 .

Now consider the simplest soliton surfaces for B_0 , B_1 and B_2 -case separately.

3.2 Two-soliton B_2 -surface

Here $f(x) = x$, $g(y) = y$. Choose

$$\Phi_0 = I$$

Formally $\mathcal{E}_0 = \infty$; function Φ_0 solves (2.26) with vanishing r.h.s. and obeys conditions of Th.3.1. Taking into account that now $\beta_j = \alpha_j$, $j = 1, 2$, we obtain from (3.5), (3.10):

$$\Phi = T = C \begin{pmatrix} q_0 + q_1 \lambda & \omega \\ -\omega & \bar{q}_0 + \bar{q}_1 \lambda \end{pmatrix}$$

$$\mathcal{E} = q_1 = \frac{1}{\gamma_1 - \gamma_2} [\alpha_2 \sqrt{(\gamma_2 + x)(\gamma_2 - y)} - \alpha_1 \sqrt{(\gamma_1 + x)(\gamma_1 - y)}] \quad (3.17)$$

$$q_0 = \frac{1}{\gamma_2 - \gamma_1} [\gamma_1 \alpha_2 \sqrt{(\gamma_2 + x)(\gamma_2 - y)} - \gamma_2 \alpha_1 \sqrt{(\gamma_1 + x)(\gamma_1 - y)}]$$

Formulae (3.17) give solution of linear system (2.26) and B_2 -equation (2.34) if parameters γ_j, α_j , $j = 1, 2$ satisfy one of the conditions (3.8) or (3.9). Embedding of the surface is given by (3.13) with $F_0 = 0$. Thus the one-soliton B_2 -surface doesn't exist: if we put $\gamma_2 = -x$, the σ_3 -component of embedding F is absent and we obtain just a straight line.

If $\gamma_{1,2} \in \mathbf{R}$, the two-soliton surface described by (3.17) has the following structure. It is everywhere non-singular, has no intersections and looks like a plane with one handle (so topologically, considering infinity of the plane as one point, we have a torus).

Case $\gamma = \bar{\gamma}_2$ corresponds to a surface with boundary; extending the surface analytically outside, we obtain a surface having the following structure. Asymptotically this is again a plane, but in finite domain it has a part with positive Gaussian curvature, self-intersections and branch points.

3.3 One-soliton B_1 -surface

Here $f(x) = 0$, $g(y) = y$. We choose

$$\Phi_0 = \frac{1}{2} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} \begin{pmatrix} e_1 & 0 \\ 0 & e_2 \end{pmatrix} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} \quad (3.18)$$

where

$$e_1 = \exp \left\{ \frac{ix}{2} \sqrt{\frac{\lambda - y}{\lambda y}} \right\}, \quad e_2 = e_1^{-1} = \exp \left\{ -\frac{ix}{2} \sqrt{\frac{\lambda - y}{\lambda y}} \right\}$$

It turns out that here, in contrast to B_2 -case, the one-soliton solution describes some non-trivial surface. Embedding of this one-soliton surface is given by (3.16) where, according to (3.11),

$$\beta_1 = \frac{1 - i\delta e^{i\frac{x}{\sqrt{y}}\mu_1}}{1 + i\delta e^{i\frac{x}{\sqrt{y}}\mu_1}} \quad \mu_1 \equiv \sqrt{\frac{\gamma_1 - y}{\gamma_1}} \in i\mathbf{R} \quad i\delta \equiv \frac{1 + \alpha_1}{1 - \alpha_1} \quad \delta \in \mathbf{R}$$

$$\gamma_1 \in]0, y[\quad \omega_1 = \sqrt{\gamma_1(\gamma_1 - y)} \in i\mathbf{R}$$

From (3.18) we obtain

$$F_0 = -2\Phi_0^{-1}\Phi_{01/\lambda}|_{\lambda=\infty} = -x\sqrt{y}\frac{\sigma_1}{2i}$$

As a result we obtain the following 1-soliton B_1 -surface:

$$F = \{-x\sqrt{y} + 2i\omega_1 \operatorname{Re}\beta_1\} \frac{\sigma_1}{2i} + \{2i\omega_1 \operatorname{Im}\beta_1 \cos \frac{x}{\sqrt{y}}\} \frac{\sigma_2}{2i} + \{2i\omega_1 \operatorname{Im}\beta_1 \sin \frac{x}{\sqrt{y}}\} \frac{\sigma_3}{2i} \quad (3.19)$$

Coefficients in front of matrices $\frac{\sigma_i}{2i}$ give embedding of this surface in $\mathbf{R}^3$. This surface depends on two arbitrary real constants: $\gamma_1 \in]0, y[$ and δ . It may be approximately described as a two-side infinite "screw" around σ_1 - axis.

3.4 One-soliton B_0 -surface (Pseudosphere)

Here $f(x) = 0$, $g(y) = 1$; this is a well-known case of sine-Gordon equation.

We choose the seed solution in the form (3.18) where

$$e_1 = \exp\left\{\frac{i}{2}\left(\mu x + \frac{y}{\mu}\right)\right\} \quad e_2 = e_1^{-1} \quad \mu = \sqrt{\frac{\lambda - 1}{\lambda}}$$

Parameters which have to be substituted in the general formula (3.16) now look as follows:

$$\beta_1 = \frac{1 - i\delta e^{\frac{y}{q} - qx}}{1 + i\delta e^{\frac{y}{q} - qx}} \quad q \equiv -i\mu_1 = -i\sqrt{\frac{\gamma_1 - 1}{\gamma_1}} \in \mathbf{R} \quad \delta = -i\frac{1 + \alpha_1}{1 - \alpha_1} \in \mathbf{R}$$

$$\gamma_1 \in]0, 1[\quad \omega_1 = \sqrt{\gamma_1(\gamma_1 - 1)} \in i\mathbf{R}$$

We have

$$F_0 = \Phi_0^{-1} \Phi_{0\mu}|_{\mu=1} = (y - x) \frac{\sigma_3}{2i}$$

and

$$F = \{y - x - \operatorname{Re}\beta_1 \frac{2q}{1 + q^2}\} \frac{\sigma_1}{2i} - \left\{\frac{2q}{1 + q^2} \operatorname{Im}\beta_1 \cos(x + y)\right\} \frac{\sigma_2}{2i} - \left\{\frac{2q}{1 + q^2} \operatorname{Im}\beta_1 \sin(x + y)\right\} \frac{\sigma_3}{2i} \quad (3.20)$$

This is a pseudosphere surface defined by two real parameters: q and δ .

3.5 Multisoliton BDT of Bianchi surfaces

Iterating 1-soliton BDT we can obtain N -soliton transformations. As in other integrable systems, it turns out to be possible to write down expressions for N -soliton solutions in terms of $N \times N$ determinants. It is convenient, in analogy to the cases $N = 1, 2$ to write down the explicit formulae for $N = 2n$; then $2n - 1$ -soliton solutions may be obtained by the formal replacement $\gamma_{2n} = -f(x)$ as in subsection 3.1.

Let $\mathcal{E}_0$ be some seed solution of (2.22) and Φ_0 -related solution of (2.26) obeying conditions of Th.3.1. To obtain $2n$ -soliton BDT we have to construct matrix T obeying conditions of Th.3.2 which may be obtained by iteration of n two-soliton transformations. Taking this into account, choose the following ansatz for $T(P)$:

$$T = C(P, x, y) \tilde{T} \quad \tilde{T} = \begin{pmatrix} Q_n(\lambda) & \omega P_{n-1}(\lambda) \\ -\omega \overline{P_{n-1}(\bar{\lambda})} & \overline{Q_n(\bar{\lambda})} \end{pmatrix} \quad (3.21)$$

where

$$P_{n-1}(\lambda) = \lambda^{n-1} + \sum_{j=0}^{n-2} p_j \lambda^j \quad Q_n(\lambda) = \sum_{j=0}^n q_j$$

are two polynomials; $C = (\det \tilde{T})^{-1/2}$.

In analogy to the two-soliton case, choose $2n$ (x, y) -independent zeros of $\det \tilde{T}$ on $\mathcal{L}_0$: $\gamma_1, \dots, \gamma_{2n}$ and $2n$ (x, y) -independent parameters $\alpha_1, \dots, \alpha_{2n}$. Regularity of $\Phi_x \Phi^{-1}$ and $\Phi_y \Phi^{-1}$ (where $\Phi = T \Phi_0$) everywhere on $\mathcal{L}_0$ outside of the poles assigned by the structure of (2.26) is provided by the system of $4n$ linear equations for $2n$ variables $p_0, \dots, p_{n-2}, q_0, \dots, q_n$:

$$\tilde{T}(\gamma_j)\Phi_0(\gamma_j)\begin{pmatrix} 1 \\ \alpha_j \end{pmatrix} = 0 \quad j = 1, \dots, 2n \quad (3.22)$$

To provide compatibility of the system (3.22) we have to impose on (γ_j, α_j) the following conditions: $\forall j = 1, \dots, n$

$$\gamma_{2j-1}^{\tau\sigma} = \gamma_{2j-1} \quad \gamma_{2j}^{\tau\sigma} = \gamma_{2j} \quad |\alpha_{2j-1}| = |\alpha_{2j}| = 1 \quad (3.23)$$

or

$$\gamma_{2j-1}^{\tau\sigma} = \gamma_{2j} \quad \alpha_{2j-1} = \overline{\alpha_{2j}}^{-1} \quad (3.24)$$

Defining, as before,

$$\beta_j = \frac{(\Phi_0(\gamma_j))_{12} + \alpha_j(\Phi_0(\gamma_j))_{11}}{(\Phi_0(\gamma_j))_{22} + \alpha_j(\Phi_0(\gamma_j))_{21}} \quad j = 1, \dots, 2n$$

we see that $|\beta_{2j-1}| = |\beta_{2j}| = 1$ in the case of (3.23) and $\beta_{2j-1} = \bar{\beta}_{2j}^{-1}$ in the case of (3.24).

To find function $\mathcal{E}(x, y)$ we need only q_n ; to find related embedding we need also q_{n-1} and p_{n-2} . According to Kramer formulae we have:

$$q_n = \frac{\Delta_1}{\Delta} \quad q_{n-1} = \frac{\Delta_2}{\Delta} \quad p_{n-2} = \frac{\Delta_3}{\Delta}$$

where

$$\begin{aligned} \Delta &= \det \begin{pmatrix} A_1 \\ \vdots \\ A_{2n} \end{pmatrix} & \Delta_1 &= (-1)^n \det \begin{pmatrix} B_1 \\ \vdots \\ B_{2n} \end{pmatrix} \\ \Delta_2 &= (-1)^{n-1} \det \begin{pmatrix} C_1 \\ \vdots \\ C_{2n} \end{pmatrix} & \Delta_3 &= - \det \begin{pmatrix} D_1 \\ \vdots \\ D_{2n} \end{pmatrix} \end{aligned}$$

and $A_j, \dots, D_j, j = 1, \dots, 2n$ are $1 \times 2n$ matrices:

$$A_j = (1, \gamma_j, \dots, \gamma_j^n, \omega_j \beta_j, \omega_j \beta_j \gamma_j, \dots, \omega_j \beta_j \gamma_j^{n-2})$$

$$B_j = (1, \gamma_j, \dots, \gamma_j^{n-1}, \omega_j \beta_j, \omega_j \beta_j \gamma_j, \dots, \omega_j \beta_j \gamma_j^{n-1})$$

$$C_j = (1, \gamma_j, \dots, \gamma_j^{n-2}, \gamma_j^n, \omega_j \beta_j, \omega_j \beta_j \gamma_j, \dots, \omega_j \beta_j \gamma_j^{n-1})$$

$$D_j = (1, \gamma_j, \dots, \gamma_j^n, \omega_j \beta_j, \omega_j \beta_j \gamma_j, \dots, \omega_j \beta_j \gamma_j^{n-3}, \omega_j \beta_j \gamma_j^{n-1})$$

For new function $\mathcal{E}$ we have:

$$\mathcal{E} = \frac{q_n \mathcal{E}_0 - 1}{q_n + \mathcal{E}_0}$$

Embedding of transformed surface is according to (2.27) defined by

$$F - F_0 = -2\Phi_0^{-1}T^{-1}T_{1/\lambda}\Phi_0|_{\lambda=\infty} = \frac{-4}{1+|q_n|^2}\{\text{Im}[q_n\bar{q}_{n-1} + p_{n-2}]N_0 - \text{Im}[q_{n-1} - q_n(\frac{f-g}{2} + \bar{p}_{n-2})]A_n + \text{Re}[q_{n-1} - q_n(\frac{f-g}{2} + \bar{p}_{n-2})]B_0\}$$

where basis (N_0, A_0, B_0) is defined by (3.14),(3.15). For $n = 1$, when $p_{n-2} = 0$, this formula coincides with (3.13).

To get $2n - 1$ -soliton solution we have to put simply $\gamma_{2n} = -f(x)$, $\gamma_{2n-1} \in] -f(x), g(y)[$, $|\alpha_{2n-1}| = 1$.

4 Bianchi surfaces in terms of theta-functions

Here it will be more convenient to work in terms of function Ψ solving the linear system (2.23),(2.24). Taking into account the gauge equivalence of the systems (2.23) and (2.26) we can reformulate Th.3.1 for function Ψ as follows

Theorem 4.1 *Let function $\Psi(P)$ ($P \in \mathcal{L}_0$) obey the following conditions:*

1. *Logarithmic derivatives $\Psi_x \Psi^{-1}$ and $\psi_y \Psi^{-1}$ are holomorphic everywhere on $\mathcal{L}_0$ except the branch points $\lambda = -f(x)$ and $\lambda = g(y)$ respectively where they have simple poles (in the local parameters $\sqrt{\lambda + f(x)}$ and $\sqrt{\lambda - g(y)}$ respectively)*

2. *Reduction*

$$\Psi(\lambda^\tau) = \sigma_3 \Psi(\lambda) \sigma_1 \quad (4.1)$$

3. *'Unitarity' and normalization:*

$$\Psi_{11}(P)\bar{\Psi}_{21}(P^\tau) + \Psi_{12}(P)\bar{\Psi}_{22}(P^\tau) = 0 \quad (4.2)$$

$$\Psi_{12}(P = \infty^+) = -\Psi_{21}(P = \infty^+) = 1 \quad (4.3)$$

where ∞^+ is those of two infinite points on $\mathcal{L}_0$, where $\omega/\lambda \rightarrow 1$ as $\lambda \rightarrow \infty$.

Then $\Psi(P)$ solves linear system (2.23) with $\mathcal{E}(x, y) = \Psi_{11}(P = \infty^+)$.

Proof. We have to comment only on (4.2) and (4.3). These two conditions entail condition (2.24) with $\mathcal{E} = \Psi_{11}(P = \infty^+)$. $\square$

Consider Riemann surface $\hat{\mathcal{L}}$ - the two-sheeted covering of $\mathcal{L}_0$ with (x, y) -independent branch points. Let $*$: $\hat{\mathcal{L}} \rightarrow \hat{\mathcal{L}}$ be an involution interchanging the copies of this covering. Now choose the following ansatz for function $\Psi(P, x, y)$ ($P \in \mathcal{L}_0$):

$$\Psi(P) = \begin{pmatrix} \psi_1(P) & \psi_1(P^*) \\ \psi_2(P) & \psi_2(P^*) \end{pmatrix} \quad (4.4)$$

where $\psi_{1,2}$ are two scalar-valued functions on $\hat{\mathcal{L}}$. Suppose that function $\Psi(P)$ obeys items 1.-3. of Th.4.1. Then (4.1) is equivalent to the following relations:

$$\psi_1(P) = \psi_1(P^{\tau*}) \quad \psi_2(P) = -\psi_2(P^{\tau*}) \quad (4.5)$$

These equations imply the existence of involution τ on $\hat{\mathcal{L}}$ and, therefore, the invariance of the set of branch points of the covering $\hat{\mathcal{L}} \rightarrow \mathcal{L}$ with respect to involution τ . Denote these branch points by

$$E_1, F_1, \dots, E_g, F_g, E_1^\tau, F_1^\tau, \dots, E_g^\tau, F_g^\tau$$

The Hurvitz diagram of $\hat{\mathcal{L}}$ is shown in Fig.2a (horizontal lines denote the sheets of $\hat{\mathcal{L}}$; vertical ones denote the positions of branch points).

Taking into account reduction (4.5) introduce the auxiliary hyperelliptic curve $\mathcal{L}$ consisting of sheets 1 and 2 of $\hat{\mathcal{L}}$ (Fig.2b) with the branch points at $-f(x)$, $g(y)$, E_j , F_j , $j = 1, \dots, g$. Curve $\mathcal{L}$ has genus g ; it defined by the following equation:

$$\omega^2 = (\lambda + f(x))(\lambda - g(y)) \prod_{j=1}^g (\lambda - E_j)(\lambda - F_j) \quad (4.6)$$

As we see, curve $\mathcal{L}$ is (x, y) -independent in B_0 -case $f = 0$, $g = 1$ (as it should be, because this is in fact the case of SG equation). In B_1 -case ($f = 0$, $g = y$) one of the branch points depends on y ; in B_2 -case one branch point depends on x , another one depends on y .

Due to (4.5) we can construct $\psi_{1,2}(P)$ first on $\mathcal{L}$. Then we can extend $\psi_{1,2}$ from $\mathcal{L}$ to $\hat{\mathcal{L}}$ by means of (4.5) as shown in Fig.2. To provide the continuity of this extension on the closed contour l (Fig.2b) which consists of two banks of branch cut $[-f(x), g(y)]$ we have to construct $\psi_{1,2}(P)$ on $\mathcal{L}$ such that $\psi_1(P)$ is single-valued on l and ψ_2 changes on l its sign.

In order to provide condition (4.2) which in terms of $\psi_{1,2}$ may be rewritten as

$$\psi_1(P)\overline{\psi_2(P^\sigma)} + \psi_1(P^*)\overline{\psi_2(P^{\sigma*})} = 0 \quad (4.7)$$

we have to assume existence on $\mathcal{L}$ of an anti-holomorphic involution σ which acts in the neighbourhoods of two infinite points $\infty^{1,2}$ as $\lambda \rightarrow \bar{\lambda}$ on every sheet. Hence impose the following restrictions on E_j , F_j :

$$E_j, F_j \in \mathbf{R}, j = 1, \dots, k; \quad E_j = \bar{F}_j, j = k+1, \dots, g$$

and choose the branch cuts as shown in Fig.3.

Introduce on $\mathcal{L}$ the canonical basis of cycles (a_j, b_j) with intersection indexes $a_i \circ b_j = \delta_{ij}$, $a_i \circ a_j = b_i \circ b_j = 0$ as shown in Fig.3. Let $dU_j(P)$, $P \in \mathcal{L}$, $j = 1, \dots, g$ be the basis of holomorphic differentials on $\mathcal{L}$ with the normalization

$$\oint_{a_j} dU_j = \delta_{ij}, \quad i, j = 1, \dots, g$$

Let P_0 be some fixed point of $\mathcal{L}$. Vector-valued function

$$U(P) = \left(\int_{P_0}^P dU_1, \dots, \int_{P_0}^P dU_g \right)$$

defines a map $\mathcal{L} \rightarrow \mathbb{C}^g$ (more precisely, to Jacobi variety $J(\mathcal{L})$) which is called Abel map on $\mathcal{L}$.

Introduce the matrix of b -periods by

$$\mathbf{B}_{ij} = \oint_{b_i} dU_j \quad i, j = 1, \dots, g$$

Matrix $\mathbf{B}$ is symmetric and has positively-determined imaginary part. Related theta-function is defined by

$$\theta(x) = \sum_{m \in \mathbb{Z}^g} \exp\{\pi i \langle \mathbf{B}m, m \rangle + 2\pi i \langle x, m \rangle\} \quad x \in \mathbb{C}^g \quad (4.8)$$

Function $\theta(x)$ possesses the following properties:

$$\theta(-x) = \theta(x)$$

$$\theta(x + e_j) = \theta(x) \quad (4.9)$$

$$\theta(x + \mathbf{B}e_j) = \theta(x) \exp\{-\pi i \mathbf{B}_{jj} - 2\pi i x_j\}$$

where $e_j = (0, \dots, 1, \dots, 0)$ (1 on the j th place).

The theta-function is a universal tool to construct functions on $\mathcal{L}$ with fixed analytical properties. Besides (4.9) we shall need the following standard facts [17, 6]:

Statement 4.1 *Let $D = D_1 + \dots + D_g$, $D_j \in \mathcal{L}$ be some non-special divisor on $\mathcal{L}$ and K be vector of Riemann constants. Then function $\theta(U(P) - U(D) - K)$ has g zeros on $\mathcal{L}$ at the points $D_1, \dots, D_g$.*

(Notice that according to (4.9) function $\theta(U(P) - U(D) - K)$ is non-single-valued on $\mathcal{L}$, but its transformation with respect to the tracing along any of the basic cycles doesn't change positions of the zeros.)

The previous statement is valid for arbitrary algebraic curve; the following two facts concern only hyperelliptic curves:

Statement 4.2 *If the initial point P_0 of the Abel map on hyperelliptic curve $\mathcal{L}$ (4.6) is equal to the branch point $\lambda = -f(x)$ and the basic cycles are chosen according to Fig.3 then vector of Riemann constants K may be represented as follows*

$$K_j = \frac{1}{2} \sum_{i=1}^g \mathbf{B}_{ij} + \frac{j}{2} \quad j = 1, \dots, g \quad (4.10)$$

Statement 4.3 *Divisor $D = D_1 + \dots + D_g$, where $D_j \neq D_k$, $k \neq j$, on hyperelliptic curve $\mathcal{L}$ is non-special if and only if points D_j have different projections on λ -plane.*

Now consider B_0 -, B_1 - and B_2 - surfaces separately.

4.1 B_0 -surfaces

Here $f = 0$, $g = 1$ and curve $\mathcal{L}$ is (x, y) -independent. Formulate the following

Theorem 4.2 *Let functions ψ_1 and ψ_2 be holomorphic on $\mathcal{L}$ everywhere outside of the points mentioned below in items 1.-3.*

1. *Functions $\psi_{1,2}$ have exponential essential singularities at the points $\lambda = 0$ and $\lambda = 1$ with the following local expansion:*

$$\psi_j \sim (\psi_j^0 + \psi_j^1 \sqrt{\lambda} + \dots) \exp \left\{ \frac{i}{2} \sqrt{\frac{\lambda-1}{\lambda}} \right\} \quad \text{as } \lambda \sim 0, \quad j = 1, 2 \quad (4.11)$$

$$\psi_j \sim (\tilde{\psi}_j^0 + \tilde{\psi}_j^1 \sqrt{\lambda-1} + \dots) \exp \left\{ \frac{i}{2} \sqrt{\frac{\lambda}{\lambda-1}} \right\} \quad \text{as } \lambda \sim 1, \quad j = 1, 2$$

2. *Functions $\psi_{1,2}$ have poles of degree $\frac{1}{2}$ at E_j , F_j , $j = 1, \dots, k$ connected by cuts $l_j = [E_j, F_j]$ where $\psi_{1,2}$ change sign (Fig. 3), and simple poles at E_j , $j = k+1, \dots, g$.*

3. *Functions $\psi_{1,2}$ are invariant with respect to the tracing along a -cycles and possess the following monodromy properties with respect to the tracing along b -cycles:*

$$M_{b_j}[\psi_1] = e^{2\pi i \kappa_j} \psi_1 \quad (4.12)$$

$$M_{b_j}[\psi_2] = -e^{2\pi i \kappa_j} \psi_2 \quad (4.13)$$

where κ_j are some constants with half-integer real parts: $\text{Re} \kappa_j = \frac{N_j}{2}$, $j = 1, \dots, g$; M_{b_j} , $j = 1, \dots, g$ are monodromy operators of the tracing along b -cycles. [We exclude in (4.12), (4.13) the change of the sign on contours l_j from item 2] To provide (4.12), (4.13) we can equivalently require that $\psi_{1,2}$ have jumps on the contours a_j and the boundary values on the different banks of a_j differ by the factor $e^{2\pi i \kappa_j}$; in addition ψ_2 changes sign on the contour l .

4. *Normalization condition*

$$\psi_1(\infty^2) = -\psi_2(\infty^1) = 1 \quad (4.14)$$

Then function $\Psi(P)$ defined by (4.4) (where $\psi_{1,2}$ are extended on $\hat{\mathcal{L}}$ from $\mathcal{L}$ according to (4.5)) obeys all items 1.-3. of Th.4.1 and, therefore, solves linear system (2.23), (2.24) with $\mathcal{E} = \psi_1(\infty^1)$.

Proof. Fulfillment of all items of Th.4.1 except item 1. and relation (4.2) is obvious. Consider item 1. Existence of simple poles of $\Psi_x \Psi^{-1}$ and $\Psi_y \Psi^{-1}$ at $\lambda = 0$ and $\lambda = 1$ respectively follows from (4.11). Regularity of $\Psi_x \Psi^{-1}$ and $\Psi_y \Psi^{-1}$ at E_j , F_j , $j = 1, \dots, g$ may be easily derived considering the local representations of Ψ at these points. It may also be easily verified that $\Psi_x \Psi^{-1}$ and $\Psi_y \Psi^{-1}$ are single-valued on the contours a_j and l_j , where $\psi_{1,2}$ have jumps. It remains to check that $\Psi_x \Psi^{-1}$ and $\Psi_y \Psi^{-1}$ have no extra singularities arising from zeros of $\det \Psi$: the only poles of $\det \Psi$ are simple poles at $E_{k+1}, \dots, E_g$. We know also $g - k$ zeros at $F_{k+1}, \dots, F_g$ which are regular. So the zeros outside of $F_{k+1}, \dots, F_g$ are absent. We see that item 1 of Th.4.1 is fulfilled.

It remains to check (4.2). Denote

$$G(P) = \psi_1 \overline{\psi_2(P^\sigma)} + \psi_1(P^*) \overline{\psi_2(P^{\sigma*})} \quad (4.15)$$

Due to condition 1 essential singularities of ψ_1 and ψ_2 at $\lambda = 0, 1$ cancel in (4.15) each other; due to the change of the sign of ψ_2 on contour l according to item 3 we obtain $G(0) = G(1) = 0$. Besides that, we can easily check that $G(P)$ is non-singular at $P = E_j, F_j$, $j = 1, \dots, g$. For example, in the neighbourhood of E_j , $j \geq k+1$ we have:

$$\psi_1(P) = \tau^{-1}(a_0 + O(\tau)) \quad \psi_1(P^*) = -\tau^{-1}(a_0 + O(\tau))$$

$$\overline{\psi_2(P^\sigma)} = b_0 + O(\tau) \quad \overline{\psi_2(P^{\sigma*})} = b_0 + O(\tau)$$

$$\tau = \sqrt{\lambda - E_j}$$

and

$$G = \tau^{-1}a_0b_0 - \tau^{-1}a_0b_0 + O(1) = O(1)$$

is holomorphic at $P = E_j$. Analogously we can verify that G is holomorphic at $P = E_j$, $j \leq k$. Besides that, we can easily see that G is single-valued on contours l_j between E_j and F_j , $j < k$ where $\psi_{1,2}$ change sign (this follows from the fact that contours l_j are invariant with respect to anti-involution σ and, therefore, the change of sign of $\psi_{1,2}$ on l_j cancels in G).

We have to verify also that $G(P)$ is single-valued on contours a_j where according to item.3 the boundary values of $\psi_{1,2}$ are related by

$$\psi_j(\lambda^+) = e^{2\pi i \kappa_j} \psi_j(\lambda^-) \quad \text{Re} \kappa_j = \frac{N_j}{2}, \quad N_j \in \mathbf{Z}, \quad j = 1, \dots, g$$

Thus

$$G(\lambda^+) = e^{2\pi i \kappa_j} \psi_1(\lambda^-) e^{-2\pi i \kappa_j} \overline{\psi_2(\lambda^{\sigma-})} + e^{-2\pi i \kappa_j} \psi_1(\lambda^{*-}) e^{-2\pi i \kappa_j} \overline{\psi_2(\lambda^{\sigma*-})} = G(\lambda^-)$$

Remark. Carrying out this calculation we understood contour a_j not as an element of homological group $H^1(\mathcal{L}, \mathbf{Z})$ satisfying relation $a_j^* = -a_j$ but as the set of points of $\mathcal{L}$ satisfying relation $*(a_j) = a_j$ (sign minus here is senseless). The latter relation is, precisely speaking, not true for the contours a_j shown in Fig.3, but may always be achieved by 'little stirring' of a_j without changing related element a_j of $H^1(\mathcal{L}, \mathbf{Z})$.

As a result we see that function G is holomorphic everywhere on $\hat{\mathcal{L}}$ and has simple zeros at $\lambda = 0$ and $\lambda = 1$. Therefore, $G \equiv 0$. $\square$

Conditions 1-4 of Th.4.2 completely define $\psi_{1,2}$. Explicit expressions for $\psi_{1,2}$ may be written in terms of theta-functions (4.8) as follows.

Introduce on $\mathcal{L}$ two abelian integrals of the second kind $\Omega_{0,1}(P)$ which are holomorphic on $\mathcal{L}$ outside of the points $\lambda = 0$ and $\lambda = 1$ respectively where they have the following asymptotical expansion:

$$\Omega_0(P) \sim \frac{1}{2} \sqrt{\frac{\lambda-1}{\lambda}} + o(1) \quad \text{as } \lambda \sim 0, \quad \Omega_1(P) \sim \frac{1}{2} \sqrt{\frac{\lambda}{\lambda-1}} + o(1) \quad \text{as } \lambda \sim 1 \quad (4.16)$$

Assume that $\Omega_{0,1}$ are normalized by

$$\oint_{a_j} d\Omega_0 = \oint_{a_j} d\Omega_1 = 0 \quad , \quad j = 1, \dots, g$$

Their vectors of b -periods $2\pi iV_0$ and $2\pi iV_1$ are defined by

$$(2\pi iV_l)_j \equiv \oint_{b_j} d\Omega_l \quad l = 0, 1 \quad j = 1, \dots, g$$

Now we can express $\psi_{1,2}$ as follows:

$$\begin{aligned} \psi_m(P) = C_m(x, y) \times \\ \times \frac{\theta(U(P) - \frac{1}{2} \sum_{j=1}^k (U(E_j) + U(F_j)) - \sum_{j=k+1}^g U(E_j) + ixV_0 + iyV_1 - K + \kappa + (m-1)\frac{\Delta}{2})}{\{\theta(U(P) - \sum_{j=1}^g U(E_j) - K)\theta(U(P) - \sum_{j=1}^k U(F_j) - \sum_{j=k+1}^g U(E_j) - K)\}^{1/2}} \times \\ \times \exp\{ix\Omega_0(P) + iy\Omega_1(P)\} \quad m = 1, 2 \end{aligned} \quad (4.17)$$

where $C_m(x, y)$ are normalization constants defined by (4.14); $\Delta = \frac{1}{2}(1, \dots, 1)$; all other objects were defined above. Contours $[E_j, F_j]$, $j = 1, \dots, k$ which have to be chosen to completely define denominator of (4.17) we take equal to l_j , $j = 1, \dots, k$.

Item 1. of Th.4.2 is fulfilled due to (4.16); conditions 2. and 3. may be verified using periodicity properties of theta-function (4.9) and Statements 4.1 and 4.3. Condition 4 coincides with definition of C_m .

So functions $\psi_{1,2}$ defined by (4.17) obey all conditions of Th.4.2, and, therefore, define by means of (4.4) some solution of linear system (2.23), (2.24). Relates solution of equation (2.30) may be obtained according to Th.4.2 as

$$\mathcal{E} = \psi_1(\infty^1)$$

Choosing in (4.17) initial point of Abel map P_0 equal to $\lambda = 0$, using expression for K given by Statement 2 and taking into account our choice of the basic cycles (Fig.3) we obtain after simple transformations

$$\mathcal{E} = \frac{\theta(U|_0^{\infty^1} + ixV_0 + iyV_1 + \kappa_0)}{\theta(U|_0^{\infty^2} + ixV_0 + iyV_1 + \kappa_0)} \exp\{ix\Omega_0|_{\infty^2}^{\infty^1} + it\Omega_1|_{\infty^2}^{\infty^1}\} \quad (4.18)$$

where κ_0 is an arbitrary constant vector obeying the following condition:

$$\text{Re}\kappa_0 = \frac{\Delta_1}{4} - \frac{N}{2} \quad , \quad \Delta_1 \equiv (1, \dots, 1, 0, \dots, 0) \quad (4.19)$$

(1 stands in Δ_1 on the first k places); N is an arbitrary integer vector. ($\kappa_0 - \frac{\Delta_1}{4}$ differs from vector κ in (4.17) by some half-integer vector).

Expression (4.18) solves equation (2.30); let's see how to obtain related (a, b, ϕ) solving (2.31). From (2.35) we obtain in the neighbourhood of $\lambda = 0$ and $\lambda = 1$:

$$\Psi_x \Psi^{-1} = \frac{i}{2} \sqrt{\frac{\lambda-1}{\lambda}} \begin{pmatrix} 0 & r \\ r^{-1} & 0 \end{pmatrix} + O(1) \quad \text{as } \lambda \sim 0$$

$$\Psi_y \Psi^{-1} = \frac{i}{2} \sqrt{\frac{\lambda}{\lambda-1}} \begin{pmatrix} 0 & s \\ s^{-1} & 0 \end{pmatrix} + O(1) \quad \text{as } \lambda \sim 1$$

Comparing with (2.25) we obtain

$$\frac{ir}{2} = \frac{\mathcal{E}_x}{\mathcal{E} + \bar{\mathcal{E}}^{-1}} \quad \frac{ir^{-1}}{2} = \frac{(\bar{\mathcal{E}}^{-1})_x}{\mathcal{E} + \bar{\mathcal{E}}^{-1}} \quad \frac{is}{2} = \frac{\mathcal{E}_y}{\mathcal{E} + \bar{\mathcal{E}}^{-1}} \quad \frac{is^{-1}}{2} = \frac{(\bar{\mathcal{E}}^{-1})_y}{\mathcal{E} + \bar{\mathcal{E}}^{-1}} \quad (4.20)$$

Now from general expressions (2.16) for a, b and ϕ we have

$$a \equiv 2 \frac{|\mathcal{E}_x|}{1 + |\mathcal{E}|^2} = 1 \quad b \equiv 2 \frac{|\mathcal{E}_y|}{1 + |\mathcal{E}|^2} = 1 \quad (4.21)$$

and

$$\phi = \pi + \arg \frac{s}{r}$$

Using (4.17) and (4.20) we obtain after simple transformations:

$$\phi = \pi - 2i \log \frac{\theta(ixV_0 + iyV_1 + \kappa_0 + \frac{\Delta}{2})}{\theta(ixV_0 + iyV_1 + \kappa_0)} \quad (4.22)$$

where as before $\Delta_j = \frac{1}{2}$, $j = 1, \dots, g$; κ_0 is defined by (4.19). Relations (4.21) obtained due to our initial choice of asymptotical expansions (2.35) imply that function ϕ defined by (4.22) is a solution of the SG equation (where $x' = x$, $y' = y$).

Remark 1. Let $g = 0$. Then the integrals Ω_0 and Ω_1 have the following form:

$$\Omega_0(\lambda) = \frac{1}{2} \sqrt{\frac{\lambda-1}{\lambda}} \quad , \quad \Omega_1(\lambda) = \frac{1}{2} \sqrt{\frac{\lambda}{\lambda-1}} \quad ;$$

theta-functions in (4.18) are absent and

$$\mathcal{E} = e^{i(x+y)}$$

This solution obviously satisfies (2.30) since in the case $|\mathcal{E}| = 1$ (2.30) reduces to

$$(\log \mathcal{E})_{xy} = 0$$

Remark 2. Expression (4.22) is a well-known solution of SG equation obtained first in [14]. Notice that from (2.30) we automatically obtain only real solutions of SG. Arbitrary parameter κ_0 obeying

(4.19) arises in our construction in pure deductive way from the analytical properties of functions $\psi_{1,2}$. This is a new element of our construction in comparison with the canonical one (see [4]).

Now consider the equations for B_1 - and B_2 - surfaces for which the presented construction of finite-gap solutions is new.

4.2 B_1 -surfaces

Here $f(x) = 0$, $g(y) = y$ and equation (4.6) of curve $\mathcal{L}$ depends on y :

$$\omega^2 = \lambda(\lambda - y) \prod_{j=1}^g (\lambda - E_j)(\lambda - F_j)$$

Formulate the following

Theorem 4.3 *Let functions $\psi_{1,2}$ be holomorphic on $\mathcal{L}$ everywhere outside of the points mentioned below in items 1-4:*

1. *Functions $\psi_{1,2}$ are holomorphic and non-vanishing at $\lambda = y$ and have essential singularity of the following form at $\lambda = 0$:*

$$\psi_j \sim (\psi_j^0 + \psi_j^1 \sqrt{\lambda} + \dots) \exp\left\{\frac{ix}{2\sqrt{\lambda}}\right\} \quad (4.23)$$

2. *Coincides with item 2 of Th.4.2.*

3. *Function ψ_2 changes sign on contour l consisting of two banks of branch cut $[-f(x), g(y)]$. Besides that, functions $\psi_{1,2}$ change sign on contours a_k i.e.*

$$\psi_j^+ = e^{\pi i N_k} \psi_j^-, \quad j = 1, 2 \quad k = 1, \dots, g$$

where $N_k \in \mathbb{Z}$; $\psi_j^\pm$ are boundary values of ψ_j on different banks of a_j .

4. *Functions ψ_j may have arbitrary additional to items 2 and 3 (the same for ψ_1 and ψ_2) set of (x, y) -independent (also the same for $j = 1, 2$) singular parts. These singularities should not spoil "reality condition" (4.2).*

5. *Coincides with item 4 of Th.4.2.*

Then function $\Psi(P)$ defined by (4.4) where $\psi_{1,2}$ are extended on $\hat{\mathcal{L}}$ from $\mathcal{L}$ solves linear system (2.23), (2.24) with $\mathcal{E} = \psi_1(\infty^1)$.

Proof. Taking into account the proof of Th.4.2 we have to comment only on items 1, 3 and 4. Proper analytical structure of $\Psi_x \Psi^{-1}$ at $\lambda = 0$ follows from (4.23). To show that $\Psi_y \Psi^{-1}$ has simple pole at $\lambda = y$ consider the local expansion of Ψ at $\lambda = y$:

$$\Psi = \Psi_0 + \sqrt{\lambda - y} \Psi_1 + \dots, \quad \Psi_0 \equiv \begin{pmatrix} \psi_1(\lambda = y) & \psi_1(\lambda = y) \\ \psi_2(\lambda = y) & -\psi_2(\lambda = y) \end{pmatrix}$$

As we assumed that $\psi_{1,2}(\lambda = y) \neq 0$, we have $\det \Psi_0 \neq 0$ and

$$\Psi_y \Psi^{-1} = \frac{-1}{2\sqrt{\lambda - y}} \Psi_1 \Psi_0^{-1} + O(1) \quad \text{as } \lambda \sim y \quad (4.24)$$

Item 3 here is a partial case of item 3 of Th.4.2 (now $\text{Im} \kappa_j = 0$).

Consider item 4. Requirement of (x, y) -independence of additional to items 2,3 singularities and related singular parts of $\psi_{1,2}$ and the requirement that these singularities and singular parts are the same for $\psi_{1,2}$ provide regularity of $\Psi_x \Psi^{-1}$ and $\Psi_y \Psi^{-1}$ therein. These singularities don't spoil (4.2) according to conditions of the theorem. [This follows from the invariance of the set of additional singularities and related singular parts with respect to anti-involution σ]. $\square$

To construct explicitly functions $\psi_{1,2}$ satisfying conditions of Th.4.3 consider the integral of the second kind $\Omega_0(P)$ normalized by $\oint_{a_j} d\Omega_0 = 0$, $j = 1, \dots, g$ and having the only pole at $\lambda = 0$ with the following local behaviour:

$$\Omega_0(P) = \frac{1}{2\sqrt{-\lambda}} + o(\lambda) \quad \text{as } \lambda \rightarrow 0$$

Let

$$(2\pi i V_0)_j = \oint_{b_j} d\Omega_0$$

be vector of b -periods of $\Omega_0(P)$.

Define functions $\psi_{1,2}$ as follows:

$$\begin{aligned} \psi_m(P) = C_m(x, y) \times \\ \times \frac{\theta(U(P) - \sum_{j=1}^k \frac{1}{2}(U(E_j) + U(F_j)) - \sum_{j=k+1}^g U(E_j) + ixV_0 - K + \frac{N}{2} + V + (m-1)\frac{\Delta}{2})}{\{\theta(U(P) - \sum_{j=1}^g U(E_j) - K)\theta(U(P) - \sum_{j=1}^k U(F_j) - \sum_{j=k+1}^g U(E_j) - K)\}^{1/2}} \times \\ \times \exp\{ix\Omega_0(P) + \Omega(P)\} \quad m = 1, 2 \end{aligned} \quad (4.25)$$

where integer vector N is defined in item 3 of Th.4.3.

Remark 1. Notice that in contrast to (4.17) explicit dependence on y in (4.25) is absent. This dependence becomes 'hidden'. As curve $\mathcal{L}$ itself now depends on y , all the objects in (4.25) corresponding to curve $\mathcal{L}$: matrix of b -periods $\mathbf{B}$, Abel map U and so on depend on y .

Remark 2. The only new object in (4.25) in comparison with (4.17) is an abelian integral $\Omega(P)$ having vector of b -periods

$$(2\pi i V)_j = \oint_{b_j} d\Omega \quad (4.26)$$

Integral $\Omega(P)$ is an arbitrary linear combination of the integrals of the second and third kind obeying the following conditions:

1. All poles and related singular parts of $\Omega(P)$ are (x, y) - independent.
2. Normalization:

$$\oint_{a_j} d\Omega = 0 \quad (4.27)$$

3. 'Reality':

$$\overline{\Omega(P)^{\sigma*}} = \Omega(P) \quad (4.28)$$

(we understand this relation as a relation between two abelian integrals i.e. up to an arbitrary constant). Taking into account the normalization condition (4.27) and relations

$$b_j^\sigma = -b_j + \{\text{combination } a_j\}$$

following from our choice of basic cycles on $\mathcal{L}$ we obtain

$$V = -\overline{V} \quad (4.29)$$

Integral Ω corresponds to item 4 of Th.4.3 which allows to insert new singularities of $\psi_{1,2}$ in comparison with B_0 -case. The (x, y) -independence of poles and singular parts of Ω provide the non-singularity of $\Psi_x \Psi^{-1}$ and $\Psi_y \Psi^{-1}$ at the poles of Ω ; condition (4.28) shows that the presence of $\Omega(P)$ doesn't spoil 'reality' condition (4.2).

Remark 3. Certainly we could introduce the integral $\Omega(P)$ in B_0 -case also; however, then it would be (x, y) -independent together with imaginary vector V . So it would not give anything new in addition to an arbitrary constant vector $\text{Im}\kappa$ in (4.17). Now situation is in some sense the reverse: vector $\text{Im}\kappa$ may be accumulated by V (this explains why we deleted $\text{Im}\kappa$ from item 3 of Th.4.3 in comparison with item 3 of Th.4.2) in the following way. Component κ_j arises in (4.17) due to the jump $2\pi i \kappa_j$ of $\log \psi_{1,2}$ on contour a_j . Take in (4.25) $\Omega = \Omega^j$ where normalized integral Ω^j has only two poles: $Q = Q^\sigma$ and $R = R^\sigma$ with the following local expansion:

$$\Omega^j(P) \sim \kappa_j \log(\lambda - Q) \quad \text{as } \lambda \sim Q$$

$$\Omega^j(P) \sim -\kappa_j \log(\lambda - R) \quad \text{as } \lambda \sim R$$

Then functions $\log \psi_{1,2}$ have jump $2\pi i \kappa_j$ on some contour l_j between Q and R . Now if we consider limit $Q \rightarrow R$ and choose contour l_j in such a way that in this limit $l_j = a_j$, we see that the singularities of Ω^j at Q and R disappear, but the jump $2\pi i \kappa_j$ of $\psi_{1,2}$ on a_j remains and gives contribution $(0, \dots, \kappa_j, \dots, 0)$ in vector V .

Remark 4. We see that due to the presence of Ω a number of parameters in (4.25) is in fact infinite in contrast to (4.17): even if the branch points E_j, F_j , $j = 1, \dots, g$ are fixed, we can take an arbitrary (even infinite) number of poles of $\Omega(P)$ with arbitrary constant singular parts.

In analogy with B_0 -case, taking into account the previous remarks and the properties of theta-function, we obtain after simple transformations:

$$\mathcal{E} = \frac{\theta(U|_0^{\infty^1} + ixV_0 + V + \frac{\Delta_1}{4} + \frac{N_0}{2})}{\theta(U|_0^{\infty^2} + ixV_0 + V + \frac{\Delta_1}{4} + \frac{N_0}{2})} \exp\{ix\Omega_0|_{\infty^2}^{\infty^1} + \Omega|_{\infty^2}^{\infty^1}\} \quad (4.30)$$

where as before $\Delta_1 = (1, \dots, 1, 0, \dots, 0)$ (units on the first k places); N_0 is an arbitrary integer vector (in general case it doesn't coincide with vector N in (4.25)).

Formula (4.30) gives an algebraic-geometrical solution of (2.33). In analogy with B_0 -case, here we can easily find coefficient a in (2.16): it follows from (4.23) that now

$$\Psi_x \Psi^{-1} \sim \frac{i}{2\sqrt{-\lambda}} \begin{pmatrix} 0 & \frac{\psi_1^0}{\psi_2^0} \\ \frac{\psi_2^0}{\psi_1^0} & 0 \end{pmatrix} + O(1) \quad \text{as } \lambda \sim 0$$

So

$$\sqrt{y} \frac{\mathcal{E}_x}{\mathcal{E} + \bar{\mathcal{E}}^{-1}} = \frac{i}{2} \frac{\psi_1^0}{\psi_2^0} \quad ; \quad \sqrt{y} \frac{(\bar{\mathcal{E}}^{-1})_x}{\mathcal{E} + \bar{\mathcal{E}}^{-1}} = \frac{i}{2} \frac{\psi_2^0}{\psi_1^0}$$

and

$$a = 2 \frac{|\mathcal{E}_x|}{1 + |\mathcal{E}|^2} = \frac{1}{\sqrt{y}}$$

How to find coefficients b and ϕ in this case not in terms of the derivatives of theta-functions, but in terms of theta-functions themselves, we don't know. Stress once more that the dynamics in variable x in (4.30) is defined in analogy with B_0 -case on the Jacobi variety of x -independent algebraic curve; deformation of this curve in y defines the y -dependence of our solution.

For $|\mathcal{E}| = 1$ equation (2.33) takes the form

$$(\log \mathcal{E})_{xy} + \frac{(\log \mathcal{E})_x}{2y} = 0$$

The general solution of this equation is

$$\mathcal{E} = \exp \left\{ i \left(\frac{C_1(x)}{\sqrt{y}} + C_2(y) \right) \right\} \quad (4.31)$$

where $C_1(x)$ and $C_2(y)$ are two arbitrary real-valued functions. If $g = 0$, integral $\Omega_0(P)$ may be written explicitly as follows: $\Omega_0(\lambda) = \frac{1}{2\sqrt{y}} \sqrt{\frac{\lambda-y}{\lambda}}$; term with theta-functions is equal to 1 and (4.30) turns into

$$\mathcal{E} = \exp \left\{ \frac{ix}{\sqrt{y}} + \Omega(\lambda)|_{\infty^2}^{\infty^1} \right\} \quad (4.32)$$

where $\Omega|_{\infty^2}^{\infty^1}$ is an imaginary function depending on y by means of the branch point $\lambda = y$. The freedom to choose analytical properties of $\Omega(P)$ on $\mathcal{L}$ is equivalent for $g = 0$ to the freedom to choose function $C_2(y)$. An arbitrary function $C_1(x)$ corresponds simply to the re-definition of variable x ; our choice $C_1(x) = x$ is fixed by the choice of coefficient ix in front of $\Omega_0(P)$ in formula (4.25).

4.3 B_2 -surfaces

Here $f(x) = x$, $g(y) = y$ and equation of curve $\mathcal{L}$ takes the following form:

$$\omega^2 = (\lambda + x)(\lambda - y) \prod_{j=1}^g (\lambda - E_j)(\lambda - F_j) \quad (4.33)$$

The theorem analogous to Th.4.2 for B_0 -surfaces and Th.4.3 for B_1 -surfaces may now be formulated as follows

Theorem 4.4 *Let functions $\psi_{1,2}$ be holomorphic on $\mathcal{L}$ everywhere outside of the points mentioned below in items 2.-4 and obeys the following conditions:*

1. Functions $\psi_{1,2}$ are holomorphic and non-vanishing at $\lambda = y$ and $\lambda = -x$.
- 2.-5. See items 2.-5. of Th.4.3.

Then function $\Psi(P)$ defined by (4.4), where $\psi_{1,2}$ are extended on $\hat{\mathcal{L}}$ from $\mathcal{L}$, solves linear system (2.23), (2.24) with $\mathcal{E} = \psi_1(\infty^1)$.

Proof. Follows from the proofs of Th.4.2 and Th.4.3; the only difference with Th.4.3 is that expression like (4.24) should be obtained not only at $\lambda = y$, but also at $\lambda = -x$. $\square$

An explicit formula for ψ_j may be written as follows:

$$\begin{aligned} \psi_m(P) &= C_m(x, y) \times \\ &\times \frac{\theta(U(P) - \sum_{j=1}^k \frac{1}{2}(U(E_j) + U(F_j)) - \sum_{j=k+1}^g U(E_j) - K + \frac{N}{2} + V + (m-1)\frac{\Delta}{2})}{\{\theta(U(P) - \sum_{j=1}^g U(E_j) - K)\theta(U(P) - \sum_{j=1}^k U(F_j) - \sum_{j=k+1}^g U(E_j) - K)\}^{1/2}} \times \\ &\times \exp\{\Omega(P)\} \quad m = 1, 2 \end{aligned} \quad (4.34)$$

where all objects are the same as in (4.17) and (4.25) (but corresponding to curve $\mathcal{L}$ defined by (4.33)).

From (4.34) we can calculate related function $\mathcal{E} = \psi_1(\infty^1)$:

$$\mathcal{E} = \frac{\theta(U|_0^{\infty^1} + V + \frac{\Delta_1}{4} + \frac{N_0}{2})}{\theta(U|_0^{\infty^2} + V + \frac{\Delta_1}{4} + \frac{N_0}{2})} \exp\{\Omega|_{\infty^2}^{\infty^1}\} \quad (4.35)$$

where as in (4.30) integer vector N_0 doesn't coincide in general case with integer vector N in (4.34) (however, since N is arbitrary, N_0 is arbitrary too).

Remark 1. In algebraic-geometrical B_0 -solution (4.18) dependence on (x, y) is defined by the straight winding of the Jacobi variety of (x, y) -independent curve. Case of B_1 -equation is intermediate: dependence on y is defined by the deformation of basic curve having one y -dependent branch point; dependence on x is in analogy to B_0 -case defined by the trajectory on the Jacobi variety of this curve. In algebraic-geometrical solutions (4.35) dynamics in both variables x and y is defined by the deformation of basic curve (4.33)

Remark 2. For $|\mathcal{E}| = 1$ equation (2.34) reduces to

$$(\log \mathcal{E})_{xy} + \frac{(\log \mathcal{E})_x + (\log \mathcal{E})_y}{2(x+y)} = 0 \quad (4.36)$$

Taking in (4.35) $g = 0$ we obtain:

$$\mathcal{E} = \exp\{\Omega|_{\infty^2}^{\infty^1}\} \quad (4.37)$$

Since $\Omega|_{\infty^2}^{\infty^1}$ is always imaginary, we see that the extra parameters introduced by $\Omega(P)$ in this case in comparison with B_0 -case correspond to the freedom to choose different 'background' solutions of (4.36) (however, here, as in B_1 -case, there is an open question whether all solutions of (4.36) may be represented in the form (4.37)).

5 Bianchi surfaces with positive Gaussian curvature

Here we consider a class of surfaces which is analytically closely related to Bianchi surfaces: the surfaces with positive Gaussian curvature

$$\mathcal{K} = \frac{1}{(h(z) + \overline{h(z)})^2} \quad (5.1)$$

where $(z, \bar{z})$ are conformal coordinates of the second quadratic form ($II = f(z, \bar{z})dzd\bar{z}$); $h(z)$ is an arbitrary holomorphic function of z .

Taking into account an analogy with Bianchi surfaces we shall call the surfaces obeying condition (5.1) 'Bianchi-plus' surfaces (or B^+ -surfaces).

Consider first an arbitrary surface S of positive Gaussian curvature. If the curvature of surface S is equal to

$$\mathcal{K} = \frac{1}{\sigma^2}, \sigma > 0$$

the fundamental forms may be parametrised as follows:

$$I = \langle dF, dF \rangle = \sigma^2(c^2 dz^2 + 2c\bar{c} \cosh \chi dzd\bar{z} + \bar{c}^2 d\bar{z}^2)$$

$$II = - \langle dF, dN \rangle = 2\sigma c\bar{c} \sinh \chi dzd\bar{z} \quad \chi \in \mathbf{R}$$

On the level of general equations the present case is related to the case of negative Gaussian curvature by the formal replacement [5]:

$$\begin{aligned} \rho &\rightarrow i\sigma & x &\rightarrow z & y &\rightarrow \bar{z} & -ia &\rightarrow c \\ -ib &\rightarrow \bar{c} & i\phi &\rightarrow \chi & i\phi_1 &\rightarrow \chi_0 & i\phi_2 &\rightarrow \bar{\chi}_0 \end{aligned}$$

In analogy with (2.5), defining matrix $\tilde{\Phi}(z, \bar{z}) \in SU(2)$ by

$$F_z = \frac{1}{2i}\sigma \tilde{\Phi}^{-1} \begin{pmatrix} 0 & e^{-\chi_0} \\ e^{\chi_0} & 0 \end{pmatrix} \tilde{\Phi} \quad F_{\bar{z}} = \frac{1}{2i}\sigma \bar{c} \tilde{\Phi}^{-1} \begin{pmatrix} 0 & e^{\bar{\chi}_0} \\ e^{-\bar{\chi}_0} & 0 \end{pmatrix} \tilde{\Phi} \quad (5.2)$$

$$N = \frac{1}{2i}\tilde{\Phi}^{-1}\sigma_3\tilde{\Phi} \quad \chi_0 \in \mathbf{C} \quad \chi_0 + \bar{\chi}_0 = \chi$$

we obtain for $\tilde{\Phi}$ the following Gauss-Weingarten system

$$\tilde{\Phi}_z = U\tilde{\Phi} \quad \tilde{\Phi}_{\bar{z}} = V\tilde{\Phi}$$

$$U = \frac{c}{2} \begin{pmatrix} 0 & e^{-\chi_0} \\ e^{\chi_0} & 0 \end{pmatrix} + \frac{1}{2}(\bar{\chi}_{0z} + \frac{1}{2}\frac{c}{\bar{c}}\frac{\sigma_{\bar{z}}}{\sigma} \sinh(\chi_0 + \bar{\chi}_0))\sigma_3 \quad (5.3)$$

$$V = -\frac{c}{2} \begin{pmatrix} 0 & e^{\bar{\chi}_0} \\ e^{-\bar{\chi}_0} & 0 \end{pmatrix} - \frac{1}{2}(\chi_{0\bar{z}} + \frac{1}{2}\frac{\bar{c}}{c}\frac{\sigma_z}{\sigma} \sinh(\chi_0 + \bar{\chi}_0))\sigma_3$$

Compatibility condition of (5.3) gives the Gauss-Codazzi system

$$\chi_{z\bar{z}} + \frac{1}{2}\left(\frac{\sigma_{\bar{z}}}{\sigma}\frac{c}{\bar{c}} \sinh \chi\right)_{\bar{z}} + \frac{1}{2}\left(\frac{\sigma_z}{\sigma}\frac{\bar{c}}{c} \sinh \chi\right)_z + c\bar{c} \sinh \chi = 0$$

$$c_{\bar{z}} + \frac{\sigma_{\bar{z}}}{2\sigma} c - \frac{\sigma_z}{2\sigma} \bar{c} \cosh \chi = 0 \quad (5.4)$$

$$\bar{c}_{\bar{z}} + \frac{\sigma_z}{2\sigma} \bar{c} - \frac{\sigma_{\bar{z}}}{2\sigma} c \cosh \chi = 0$$

In analogy to the case $\mathcal{K} < 0$ the Gauss-Codazzi system (5.4) may always be equivalently rewritten as one differential equation of the second order in terms of the function

$$\mathcal{E} = \tilde{\Phi}_{11}(\tilde{\Phi}_{12})^{-1}$$

as follows

$$(\mathcal{E} + \bar{\mathcal{E}}^{-1})(\mathcal{E}_{z\bar{z}} + \frac{\sigma_z}{2\sigma} \mathcal{E}_{\bar{z}} + \frac{\sigma_{\bar{z}}}{2\sigma} \mathcal{E}_z) = 2\mathcal{E}_z \mathcal{E}_{\bar{z}} \quad (5.5)$$

Coefficients c and χ may be expressed in terms of $\mathcal{E}$ as

$$c = 2 \frac{(-\mathcal{E}_z \bar{\mathcal{E}})^{1/2}}{1 + |\mathcal{E}|^2} \quad \chi = \frac{1}{2} \log \frac{\bar{\mathcal{E}}_z \mathcal{E}_{\bar{z}}}{\mathcal{E}_z \bar{\mathcal{E}}}$$

The mean curvature, second quadratic form and Beltrami coefficient may be written in terms of $\mathcal{E}$ as

$$\mathcal{H} = \frac{1}{\rho} \coth \chi = \frac{1}{\rho} \frac{\mathcal{E}_{\bar{z}} \bar{\mathcal{E}}_z + \mathcal{E}_z \bar{\mathcal{E}}_{\bar{z}}}{\mathcal{E}_{\bar{z}} \bar{\mathcal{E}}_z - \mathcal{E}_z \bar{\mathcal{E}}_{\bar{z}}}$$

$$II = \frac{-4\sigma}{(1 + |\mathcal{E}|^2)^2} (\mathcal{E}_z \bar{\mathcal{E}}_{\bar{z}} + \mathcal{E}_{\bar{z}} \bar{\mathcal{E}}_z) dz d\bar{z}$$

$$\mu_{\bar{z}}^z \equiv \frac{g_{z\bar{z}} - \sqrt{(g_{z\bar{z}})^2 - g_{zz} g_{\bar{z}\bar{z}}}}{g_{zz}} = -\frac{\mathcal{E}_{\bar{z}}}{\mathcal{E}_z}$$

where g_{ij} are elements of the first fundamental form.

The first fundamental form and moving frame $(F_z, F_{\bar{z}}, N)$ may be obtained from formulae (2.18) and (2.20) in the case of negative $\mathcal{K}$ by the standard replacement $x \rightarrow z$ $y \rightarrow \bar{z}$ $\rho \rightarrow -i\sigma$.

If $\sigma = h(z) + \bar{h}(\bar{z})$ where $h(z)$ is an arbitrary holomorphic function, equation (5.5) takes the following form:

$$(\mathcal{E} + \bar{\mathcal{E}}^{-1})(\mathcal{E}_{z\bar{z}} + \frac{h_z \mathcal{E}_{\bar{z}} + \bar{h}_{\bar{z}} \mathcal{E}_z}{2(h + \bar{h})}) = 2\mathcal{E}_z \mathcal{E}_{\bar{z}} \quad (5.6)$$

In analogy with the case $\mathcal{K} < 0$, equation (5.6) turns out to admit the zero curvature representation with the spectral parameter:

$$\Psi_z = U \Psi \quad \Psi_{\bar{z}} = V \Psi \quad (5.7)$$

$$U = \begin{pmatrix} A & 0 \\ 0 & B \end{pmatrix} + \sqrt{\frac{\lambda + i\bar{h}(z)}{\lambda - i\bar{h}(z)}} \begin{pmatrix} 0 & A \\ B & 0 \end{pmatrix} \quad V = \begin{pmatrix} C & 0 \\ 0 & D \end{pmatrix} + \sqrt{\frac{\lambda - i\bar{h}(z)}{\lambda + i\bar{h}(z)}} \begin{pmatrix} 0 & C \\ D & 0 \end{pmatrix}$$

where as in (2.23) expressions for $A, \dots, D$

$$A = \frac{\mathcal{E}_z}{\mathcal{E} + \bar{\mathcal{E}}^{-1}} \quad B = \frac{(\bar{\mathcal{E}}^{-1})_z}{\mathcal{E} + \bar{\mathcal{E}}^{-1}} \quad C = \frac{\mathcal{E}_{\bar{z}}}{\mathcal{E} + \bar{\mathcal{E}}^{-1}} \quad D = \frac{(\bar{\mathcal{E}}^{-1})_{\bar{z}}}{\mathcal{E} + \bar{\mathcal{E}}^{-1}} \quad (5.8)$$

follow from the normalization condition (2.24).

In contrast to B -surfaces, here there are only two essentially different cases:

1. B_0^+ -surfaces: $h = \text{const}$; without loss of generality we can take $h = 1$; then $\mathcal{K} = \frac{1}{4}$ and (5.6) reduces to

$$(\mathcal{E} + \bar{\mathcal{E}}^{-1})\mathcal{E}_{z\bar{z}} = 2\mathcal{E}_z\bar{\mathcal{E}}_{\bar{z}} \quad (5.9)$$

The Gauss-Codazzi system (5.4) looks as follows:

$$\chi_{z\bar{z}} + c\bar{c}\sinh\chi = 0$$

$$c_{\bar{z}} = 0$$

and reduces to sinh-Gordon equation

$$\chi_{z'\bar{z}'} + \sinh\chi = 0$$

after change of variable $dz' = c(z)dz$.

2. B_2^+ -surfaces: $h \neq \text{const}$. Here by holomorphic coordinate transformation we can always reduce $h(z)$ to $h(z) = -iz$. Equation (5.6) takes the following form

$$(\mathcal{E} + \bar{\mathcal{E}}^{-1})(\mathcal{E}_{z\bar{z}} - \frac{\mathcal{E}_z - \mathcal{E}_{\bar{z}}}{2(z - \bar{z})}) = 2\mathcal{E}_z\bar{\mathcal{E}}_{\bar{z}} \quad (5.10)$$

This equation turns into the Ernst equation for stationary axis-symmetric space-time if we substitute $\bar{\mathcal{E}}$ instead of $\bar{\mathcal{E}}^{-1}$.

Remark. It is well-known [19] that the Ernst equation describes axis-symmetric harmonic maps from $\mathbf{R}^3$ into upper half-plane $\mathbf{H}$ with the standard metric

$$ds^2 = \frac{d\mathcal{E}d\bar{\mathcal{E}}}{(\mathcal{E} + \bar{\mathcal{E}})^2}$$

In the same way we can see that B_2^+ - equation(5.10) describes axisymmetric harmonic maps from $\mathbf{R}^3$ into the Riemann sphere S^2 with the standard metric

$$ds^2 = \frac{d\mathcal{E}d\bar{\mathcal{E}}}{(1 + |\mathcal{E}|^2)^2}$$

This harmonic map coincides with the Gauss map

$$N(z, \bar{z}) = \frac{1}{2i} \frac{1}{1 + |\mathcal{E}|^2} \begin{pmatrix} |\mathcal{E}|^2 - 1 & 2\bar{\mathcal{E}} \\ 2\mathcal{E} & 1 - |\mathcal{E}|^2 \end{pmatrix}$$

of our surface. In terms of N equation (5.10) takes the following form:

$$((z - \bar{z})N_z N^{-1})_{\bar{z}} + ((z - \bar{z})N_{\bar{z}} N^{-1})_z = 0$$

Cylindrical coordinates in $\mathbf{R}^3$ are given by $\text{Im}z$ (the radius) and $\text{Re}z$.

So we obtain the following alternative characterization of B_2^+ - surfaces:

Theorem 5.1 *Surface S has Gaussian curvature*

$$\mathcal{K} = \frac{-1}{(z - \bar{z})^2}$$

where z is a conformal coordinate of the second fundamental form if and only if Gauss map $N(z, \bar{z})$ of surface S defines axi-symmetric harmonic map $\mathbf{R}^3 \rightarrow S^2$, where cylindrical coordinates in $\mathbf{R}^3$ are $\text{Im}z$ (radius) and $\text{Re}z$.

We see that from the point of view of harmonic maps equation (5.10) differs from the Ernst equation by the change of the sign of the curvature of target space.

5.1 Bäcklund-Darboux transformations and soliton solutions for B^+ -surfaces

The BDT for (5.6) almost coincides with BDT for equation (2.22). The only essential difference is a restriction on parameters $\gamma_{1,2}$ and $\alpha_{1,2}$ which entails existence of N -soliton solutions only for even $N = 2n$. Choose on the curve $\mathcal{L}_0$

$$\omega^2 = (\lambda - ih(z))(\lambda + ih(\bar{z}))$$

branch cut between $ih(z)$ and $-ih(\bar{z})$ along the segment $[ih(z), -ih(\bar{z})]$ and denote by σ an anti-holomorphic involution on $\mathcal{L}_0$ acting on every sheet of $\mathcal{L}_0$ as $\lambda \rightarrow \bar{\lambda}$. Then the reality restrictions on parameters γ_j , α_j are the following:

$$\gamma_1 = \gamma_2^\sigma \quad \alpha_1 = \frac{1}{\alpha_2} \quad \gamma_1 \in \mathbf{C} \quad \alpha_1 \in \mathbf{C} \quad (5.11)$$

Substituting these parameters into (3.10),(3.11),(3.12) and (3.13) and taking into account that now

$$\omega_j = [(\gamma_j - ih(z))(\gamma_j + ih(\bar{z}))]^{1/2}$$

we get BDT for B^+ -surfaces.

Two-soliton B_2^+ -surface ($h(z) = -iz$) is defined by (3.17) where we should make replacement $x \rightarrow -z$, $y \rightarrow \bar{z}$ and choose parameters γ_j , α_j according to (5.11). Embedding of the surface may as before be calculated using (2.27),(2.28). Computer plotting ² shows that this surface is topologically a

²I thank Ulrich Eitner who plotted this surface

sphere with two 'punctures' where $z \in \mathbf{R}$ ($\mathcal{K} = \infty$); domain $z \rightarrow \infty$ ($\mathcal{K} \rightarrow 0$) corresponds to a straight segment between the 'punctures'. If $|\alpha_1| \rightarrow \infty$, the surface takes almost spheroidal form (the punctures glue together); in another limit $|\alpha_1| \rightarrow \infty$ the surface looks like a long "sausage" with punctures on the ends and degenerates into straight line as $|\alpha_1| = \infty$ (parameter $\arg \alpha_1$ is non-essential; it corresponds to global rotations in $\mathbf{R}^3$).

Multisoliton solutions for B^+ -surfaces may be obtained from formulae of Sect.2.5 by the standard replacement $x \rightarrow -z$ $y \rightarrow \bar{z}$ imposing condition (5.11) on all pairs γ_{2j-1} , γ_{2j} $j = 1, \dots, g$.

5.2 B^+ -surfaces in terms of theta-functions

Construction presented here mostly differs from B -case by reality conditions which should be imposed on the parameters of solutions. We shall point out only the features which are different in these cases.

Equation of curve $\mathcal{L}$ for B^+ -surfaces is

$$\omega^2 = (\lambda - ih(z))(\lambda + i\overline{h(z)}) \prod_{j=1}^g (\lambda - E_j)(\lambda - F_j) \quad (5.12)$$

where $E_j = \bar{F}_j$, $j = 1, \dots, g$ - only the "vertical" branch cuts are admissible. Canonical basis of cycles on $\mathcal{L}$ is chosen as shown in Fig.3. Anti-involution σ acts as $\lambda \rightarrow \bar{\lambda}$ on every sheet of (5.12).

Theorem 3.2 is applicable to the present case after replacement $f(x) \rightarrow -ih(z)$, $g(y) \rightarrow -i\overline{h(z)}$. All other facts from the general part of Sect.3 don't suffer any modifications.

Consider first the B_2^+ -case.

5.2.1 B_2^+ -surfaces

Here $h(z) = -iz$, and curve $\mathcal{L}$

$$\omega^2 = (\lambda - z)(\lambda - \bar{z}) \prod_{j=1}^g (\lambda - E_j)(\lambda - F_j), \quad F_j = \bar{E}_j \quad (5.13)$$

depends on $(z, \bar{z})$. The theorem analogous to Th.4.4 is the following

Theorem 5.2 *Let functions $\psi_{1,2}$ be holomorphic on $\mathcal{L}$ everywhere outside the points mentioned below in items 2-4 and obeys the following conditions:*

1. Functions $\psi_{1,2}$ are holomorphic and non-vanishing at $\lambda = z$ and $\lambda = \bar{z}$.
2. Functions $\psi_{1,2}$ have simple poles at $\lambda = E_j$, $j = 1, \dots, g$.
3. Function ψ_2 changes sign on contour l consisting of two banks of branch cut $[z, \bar{z}]$
- 4., 5. - see items 4, 5 of Th.4.3

Then function $\Psi(P)$ defined by (4.4), where $\psi_{1,2}$ are extended on $\hat{\mathcal{L}}$ from $\mathcal{L}$, solves linear system (5.7), (2.24) with $\mathcal{E} = \psi_1(\infty^1)$

Proof. May be easily combined from the proofs of Th.4.2, Th.4.3 and Th.4.4.

Functions $\psi_{1,2}$ may be represented as follows:

$$\begin{aligned} \psi_m(P) = C_m(z, \bar{z}) & \frac{\theta(U(P) - \sum_{j=1}^g U(E_j) + V - K + \frac{N}{2} + (m-1)\frac{\Delta}{2})}{\theta(U(P) - \sum_{j=1}^g U(E_j) - K)} \times \\ & \times \exp\{\Omega(P)\} \quad m = 1, 2 \end{aligned} \quad (5.14)$$

where all objects are the same as in (4.17) and (4.34), but correspond to curve (5.13); $\Delta_j = 1$, $j = 1, \dots, g$. Choosing initial point of Abel map $P_0 = z$ we find from (5.14):

$$\mathcal{E} = \frac{\theta(U|_z^{\infty^1} + V)}{\theta(U|_z^{\infty^2} + V)} \exp\{\Omega|_{\infty^2}^{\infty^1}\} \quad (5.15)$$

Notice that now as before $\overline{\Omega(P^{\sigma^*})} = \Omega(P)$. Taking into account that on our curve $\mathcal{L}$ (Fig 3)

$$b_j^\sigma = b_j + \{\text{combination } a_j\}$$

and normalization condition of $\Omega(P)$ ($\oint_{a_j} d\Omega(P) = 0$, $j = 1, \dots, g$) we see that

$$V = \overline{V}$$

in contrast to (4.29) (due to this fact we accumulated in (5.15) an arbitrary half-integer vector from (5.14) by V).

According to the procedure described in *Remark 3* after Th.4.3 we can consider a partial case of solution (5.15), when $\Omega|_{\infty^2}^{\infty^1} = 0$; $V = \kappa_0$ is an arbitrary real vector;

$$\mathcal{E} = \frac{\theta(U|_z^{\infty^1} + \kappa_0)}{\theta(U|_z^{\infty^1} - \kappa_0)} \quad \kappa_0 \in \mathbf{R}^g \quad (5.16)$$

Certainly it is very difficult to investigate analytically the structure of the constructed surfaces. However, we can obtain some simple partial result illustrating the role of constant κ_0 in (5.16) for the elliptic case $g = 1$.

If $g = 1$, functions $\psi_{1,2}$ corresponding to (5.16) may be written as follows

$$\psi_m(P) = C_m \frac{\theta(U|_z^P + \kappa_0 + \frac{m-1}{2})}{\theta(U|_z^P + \frac{1}{2})} \quad m = 1, 2 \quad (5.17)$$

In terms of $\psi_m(P)$ we can describe embedding (2.27): $F = -2\Phi^{-1}\Phi_{1/\lambda}$ where function Φ differs from Ψ (4.4) by the normalization of ψ_m , $m = 1, 2$. However, in formula for F constants C_m cancel out; so we can put in (5.17) $C_m = 1$. Obviously solution (5.16) may possess some monodromy property in z -plane when branch point $\lambda = z$ traces around the branch point $\lambda = E$. Denote by M_E the monodromy operator of the tracing of z around E in positive direction and by $\lambda^{(j)}$ the point lying on the j th sheet of $\mathcal{L}$ having projection λ on λ -plane.

We have

$$U|_z^{\lambda^{(2)}} = -U|_z^{\lambda^{(1)}} \quad M_E[U|_z^{\lambda^{(1)}}] = U|_z^{\lambda^{(1)}} + B$$

Taking into account the periodicity properties of theta-function (4.9) we obtain:

$$M[\Psi(\lambda)] = -\sigma_3 \Psi(\lambda) \begin{pmatrix} e^{-2\pi i \kappa_0} & 0 \\ 0 & e^{2\pi i \kappa_0} \end{pmatrix}$$

and

$$M[F_1] = \cos(4\pi\kappa_0)F_1 + \sin(4\pi\kappa_0)F_2$$

$$M[F_2] = -\sin(4\pi\kappa_0)F_1 + \cos(4\pi\kappa_0)F_2$$

$$M[F_3] = F_3$$

This is a rotation of vector F around σ_3 - (i.e. $\vec{k}$ -) axis.

We see that if κ_0 is non-rational, surface S turns out to be a dense subset of some space domain. If κ_0 is rational then several tracings of z about E in parameter domain returns F back. Moreover, the denominator of κ_0 is obviously equal to degree of rotational symmetry of the surface defined by (5.2), (5.17) with respect to $\vec{k}$ -axis. It seems probable that this denominator gives also a number of 'punctures' on surface S .: according to experience with two-soliton solutions, the 'puncture' arises if $\mathcal{K} = \infty$ i.e. $\text{Im}z = 0$; a number of disconnected lines $\text{Im}z = 0$ coincides in our case with denominator of κ_0 (i.e. the number of z -sheets glued at point $z = E$).

As a result we see that even in the simplest case of solution (5.16) with $g = 1$ the properties of finite-gap solutions are much more non-trivial then of related two-soliton solutions described in previous subsection. [Notice that solution (3.17) (with replacement $x \rightarrow -z$, $y \rightarrow \bar{z}$ and conditions $\gamma_1 = \bar{\gamma}_2$, $\alpha_1 = \alpha_2^{-1}$) may be obtained by degeneration $E_1 \rightarrow E_2$ of solution (5.2) in $g = 2$ case.]

5.2.2 B_0^+ -surfaces

Now $h(z) = 1$ and curve $\mathcal{L}$

$$\omega^2 = (\lambda - i)(\lambda + i) \prod_{j=1}^g (\lambda - E_j)(\lambda - F_j) \quad F_j = \bar{E}_j$$

doesn't depend on z .

The theorem stating analytical properties of $\psi_{1,2}$ on $\mathcal{L}$ now coincides with Th.5.2 where instead of items 1., 3. and 4. we have to put the following items

1'. Functions $\psi_{1,2}$ have essential singularities at $\lambda = \pm i$ with the following local expansion:

$$\psi_m(\lambda) \sim (\psi_m^0 + \psi_m^1 \sqrt{\lambda - i} + \dots) \exp \left\{ \sqrt{\frac{i}{2(\lambda - i)}} i z \right\} \quad \text{as} \quad \lambda \sim i, \quad m = 1, 2 \quad (5.18)$$

$$\psi_m(\lambda) \sim (\tilde{\psi}_m^0 + \tilde{\psi}_m^1 \sqrt{\lambda + i} + \dots) \exp \left\{ \sqrt{\frac{-i}{2(\lambda + i)}} i \bar{z} \right\} \quad \text{as} \quad \lambda \sim -i, \quad m = 1, 2$$

3'. Item 3. of Th.4.2, where κ_j are arbitrary real constants.

To express functions $\psi_{1,2}$ in terms of theta-functions consider normalized (all a -periods are zero) integral of the second kind $\Omega_0(P)$ with pole at $\lambda = i$ with the following local behaviour:

$$\Omega_0(P) \sim \sqrt{\frac{i}{2(\lambda - i)}} \quad \text{as} \quad \lambda \sim i$$

Then

$$\overline{\Omega_0(P^\sigma)} \sim \sqrt{\frac{-i}{2(\lambda+i)}} \quad \text{as } \lambda \sim -i$$

Define vector of b -periods of $\Omega_0(P)$ by

$$(2\pi i V_0)_j = \oint_{b_j} d\Omega_0$$

Taking into account that $a_j^\sigma = -a_j$; $b_j^\sigma = b_j + \{\text{comb. } a_j\}$ we see that integral $\overline{\Omega_0(P^\sigma)}$ is also normalized and has vector of b -periods $-2\pi i \bar{V}_0$.

Functions $\Psi_{1,2}$ may now be defined by

$$\begin{aligned} \psi_m(P) &= C_m(z, \bar{z}) \times \\ &\times \frac{\theta(U(P) - \sum_{j=1}^g U(E_j) + izV_0 - i\bar{z}\bar{V}_0 + \kappa - K + (m-1)\frac{\Delta}{2})}{\theta(U(P) - \sum_{j=1}^g U(E_j) - \kappa)} \times \\ &\times \exp\{iz\Omega_0(P) + i\bar{z}\overline{\Omega_0(P^\sigma)}\} \quad m = 1, 2 \end{aligned} \quad (5.19)$$

where real constant vector κ_0 differs from κ by $\frac{\Delta}{2}$. Then

$$\mathcal{E} = \frac{\theta(U|_i^{\infty^1} + i(zV_0 - \bar{z}\bar{V}_0) + \kappa_0)}{\theta(U|_i^{\infty^2} + i(zV_0 - \bar{z}\bar{V}_0) + \kappa_0)} \exp\{iz\Omega_0|_{\infty^2}^{\infty^1} + i\bar{z}\overline{\Omega_0}|_{\infty^2}^{\infty^1}\}$$

Now, taking into account the local expansion of $\Omega_0(P)$ at $\lambda = i$ we can find from (5.19) related coefficients c , χ in analogy with SG case (4.21), (4.22):

$$\begin{aligned} c &= 2 \frac{(\mathcal{E}_z \bar{\mathcal{E}}_z)^{1/2}}{1 + |\mathcal{E}|^2} = 1 \\ \chi &= \frac{1}{2} \log \frac{\bar{\mathcal{E}}_z \mathcal{E}_{\bar{z}}}{\mathcal{E}_z \bar{\mathcal{E}}_{\bar{z}}} = 2 \log \frac{\theta(i(zV - \bar{z}\bar{V}) + \kappa_0 + \frac{\Delta}{2})}{\theta(i(zV - \bar{z}\bar{V}) + \kappa_0)} \end{aligned} \quad (5.20)$$

So χ is a well-known finite-gap solution of sinh-Gordon equation

$$\chi_{z\bar{z}} + \sinh \chi = 0$$

6 Surfaces with harmonic inverse mean curvature

First, show how to rewrite the Gauss-Codazzi system as one second order equation if the mean curvature $H(z, \bar{z}) \neq 0$ is fixed and the coordinate z is a conformal coordinate of the first fundamental form.

Parametrize the fundamental forms as follows

$$I = e^u dz d\bar{z} \quad (6.1)$$

$$II = Q dz^2 + \mathcal{H} e^u dz d\bar{z} + \bar{Q} d\bar{z}^2 \quad (6.2)$$

where $\mathcal{H}(z, \bar{z}) \in \mathbf{R}$ is a mean curvature, $Q(z, \bar{z}) \in \mathbf{C}$, $u(z, \bar{z}) \in \mathbf{R}$.

Parametrize the moving frame (F_x, F_y, N) ($z = x + iy$) according to (we have now $\langle F_x, F_y \rangle = 0$; $|F_x| = |F_y| = e^{u/2}$):

$$F_x = e^{u/2} \tilde{\Phi}^{-1} \left(\frac{\sigma_1}{2i} \cos \beta + \frac{\sigma_2}{2i} \sin \beta \right) \tilde{\Phi} \quad F_y = e^{u/2} \tilde{\Phi}^{-1} \left(-\frac{\sigma_1}{2i} \sin \beta + \frac{\sigma_2}{2i} \cos \beta \right) \tilde{\Phi} \quad (6.3)$$

$$N = \tilde{\Phi}^{-1} \frac{\sigma_3}{2i} \tilde{\Phi} \quad \tilde{\Phi} \in SU(2)$$

where $\beta(z, \bar{z}) \in \mathbf{R}$ is an arbitrary "phase" (β plays the same role as in (2.5) parameter $\phi_1 - \phi_2$). Then

$$F_z = \frac{1}{2i} e^{u_0} \tilde{\Phi}^{-1} \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \tilde{\Phi} \quad F_{\bar{z}} = \frac{1}{2i} e^{u_0} \tilde{\Phi}^{-1} \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \tilde{\Phi} \quad (6.4)$$

where $u_0 = \frac{u}{2} + i\beta$; $u = u_0 + \bar{u}_0$.

Using relations $\langle F_{zz}, N \rangle = Q$ and $\langle F_{z\bar{z}}, N \rangle = \frac{1}{2} \mathcal{H} e^u$ which follow from (6.12) and taking into account the integrability condition $F_{z\bar{z}} = F_{\bar{z}z}$ we can express the logarithmic derivatives of $\tilde{\Phi}$: $\tilde{\Phi}_z \tilde{\Phi}^{-1}$ and $\tilde{\Phi}_{\bar{z}} \tilde{\Phi}^{-1}$ in terms of u_0 , Q and H :

$$\tilde{\Phi}_z = \begin{pmatrix} \frac{1}{2} u_{0z} & -Q e^{-u_0} \\ \frac{1}{2} \mathcal{H} e^{u_0} & -\frac{1}{2} \bar{u}_{0z} \end{pmatrix} \tilde{\Phi} \quad \tilde{\Phi}_{\bar{z}} = \begin{pmatrix} -\frac{1}{2} u_{0\bar{z}} & -\frac{1}{2} \mathcal{H} e^{u_0} \\ \bar{Q} e^{-\bar{u}_0} & \frac{1}{2} u_{0\bar{z}} \end{pmatrix} \tilde{\Phi} \quad (6.5)$$

(as before, $u_0 + \bar{u}_0 = u$). This is a Gauss-Weingarten system in our case. Compatibility condition of (6.5) gives Gauss-Codazzi system:

$$u_{z\bar{z}} + \frac{1}{2} \mathcal{H}^2 e^u - 2Q\bar{Q} e^{-u} = 0$$

$$Q_{\bar{z}} = \frac{1}{2} \mathcal{H}_z e^u \quad (6.6)$$

$$Q_z = \frac{1}{2} \mathcal{H}_{\bar{z}} e^u$$

Assuming, as before, that function $\tilde{\Phi}$ takes values in the coset space $SU(2)/U(1)$:

$$\tilde{\Phi} = \frac{1}{(1 + |\mathcal{E}|^2)^{1/2}} \begin{pmatrix} \mathcal{E} & 1 \\ -1 & \bar{\mathcal{E}} \end{pmatrix} \quad (6.7)$$

we fix an arbitrary function $\beta(z, \bar{z}) = \text{Im} u_0$ according to (6.5), (6.7):

$$e^{u_0} = \frac{2\bar{\mathcal{E}}_z}{\mathcal{H}(1 + |\mathcal{E}|^2)} \quad (6.8)$$

i.e. $\beta = \arg\{\bar{\mathcal{E}}_z\}$.

For e^u and Q we have:

$$e^u = \frac{4}{\mathcal{H}^2} \frac{\bar{\mathcal{E}}_z \mathcal{E}_{\bar{z}}}{(1 + |\mathcal{E}|^2)^2} \quad Q = \frac{2}{\mathcal{H}} \frac{\mathcal{E}_z \bar{\mathcal{E}}_z}{(1 + |\mathcal{E}|^2)^2} \quad (6.9)$$

Substituting (6.8), (6.9) and (6.7) into (6.5) and considering the compatibility condition of (6.5) we can rewrite the Gauss-Codazzi system (6.6) in terms of $\mathcal{E}$ as follows

Theorem 6.1 *Let function $\mathcal{E}(z, \bar{z})$ be solution of the following equation:*

$$(\mathcal{E} + \bar{\mathcal{E}}^{-1})(\mathcal{E}_{z\bar{z}} + \frac{\delta_z}{\delta} \mathcal{E}_{\bar{z}}) = 2\mathcal{E}_z \mathcal{E}_{\bar{z}} \quad \delta(z, \bar{z}) = \frac{1}{\mathcal{H}} \in \mathbf{R} \quad (6.10)$$

Then functions Q and u defined by (6.9) solve the Gauss-Codazzi system (6.6).

Proof. May alternatively be obtained by direct substitution of (6.9) into (6.6) using (6.10). $\square$

Fundamental forms, Gaussian curvature and moving frame may be expressed in terms of $\mathcal{E}$ as follows:

$$I = \frac{4}{\mathcal{H}^2} \frac{\bar{\mathcal{E}}_z \mathcal{E}_{\bar{z}}}{(1 + \mathcal{E}\bar{\mathcal{E}})^2} dz d\bar{z} \quad (6.11)$$

$$II = \frac{2}{\mathcal{H}(1 + \mathcal{E}\bar{\mathcal{E}})^2} \{ \mathcal{E}_z \bar{\mathcal{E}}_z dz^2 + 2\bar{\mathcal{E}}_z \mathcal{E}_{\bar{z}} dz d\bar{z} + \mathcal{E}_{\bar{z}} \bar{\mathcal{E}}_{\bar{z}} d\bar{z}^2 \} \quad (6.12)$$

$$\mathcal{K} = \mathcal{H}^2 \left\{ 1 - \frac{\mathcal{E}_z \bar{\mathcal{E}}_{\bar{z}}}{\bar{\mathcal{E}}_z \mathcal{E}_{\bar{z}}} \right\} \quad (6.13)$$

$$F_z = \frac{1}{2i} \frac{2}{\mathcal{H}} \frac{\bar{\mathcal{E}}_z}{\bar{\mathcal{E}} + \mathcal{E}^{-1}} \begin{pmatrix} -1 & -\mathcal{E}^{-1} \\ \mathcal{E} & 1 \end{pmatrix} \quad F_{\bar{z}} = -F_z^\dagger \quad (6.14)$$

$$N = \frac{1}{2i} \frac{1}{1 + |\mathcal{E}|^2} \begin{pmatrix} |\mathcal{E}|^2 - 1 & 2\bar{\mathcal{E}} \\ 2\mathcal{E} & 1 - |\mathcal{E}|^2 \end{pmatrix}$$

Remark. In spite of we assumed from the beginning that $\mathcal{H} \neq 0$, we can see from (6.10) that in the limit $\mathcal{H} \rightarrow 0$ function $\mathcal{E}$ becomes arbitrary holomorphic function:

$$\mathcal{E}_{\bar{z}} = 0$$

In this case $\mathcal{E}(z)$ coincides with function $g(z)$ which is used in Weierstrass representation of minimal surfaces (see [5]). (if $\mathcal{H} \neq 0$, holomorphic solutions of (6.10) lead to vanishing fundamental forms).

Consider some partial cases of equation (6.10).

1. Let in addition to (6.10) function $\mathcal{E}$ satisfies condition

$$\mathcal{E}_z \bar{\mathcal{E}}_{\bar{z}} = \mathcal{E}_{\bar{z}} \bar{\mathcal{E}}_z \quad (6.15)$$

Then the Gaussian curvature $\mathcal{K}$ (6.13) is equal to zero and surface S is an isometrical deformation of a plane i.e. developable surface (surface of tangent lines to some space curve), cone or cylinder. At this moment we don't know whether condition (6.15) compatible with (6.10) for arbitrary $\mathcal{H}$ or imposes on $\mathcal{H}$ some restrictions.

2. Partial case of (6.15) is $|\mathcal{E}| = 1$ i.e.

$$\mathcal{E} = e^{i w(z, \bar{z})} \quad w \in \mathbf{R}$$

Then (6.10) takes the form

$$\left(\frac{w_{\bar{z}}}{\mathcal{H}} \right)_z = 0 \quad w \in \mathbf{R} \quad (6.16)$$

Solvability condition of this equation imposes already some restrictions on $H(z, \bar{z})$ due to additional requirement $w \in \mathbf{R}$. Namely, (6.16) implies

$$w_{\bar{z}} = c(\bar{z}) \mathcal{H}$$

where $c(\bar{z})$ is an arbitrary holomorphic function. Let $c(\bar{z}) = c_0 + i c_1$ $c_{0,1} \in \mathbf{R}$. Then

$$\frac{w_x}{2} = \mathcal{H} c_0 \quad c_{0x} = -c_{1y} \quad (6.17)$$

$$\frac{w_y}{2} = \mathcal{H} c_1 \quad c_{1x} = c_{0y} \quad z \equiv x + iy$$

Condition $w_{xy} = w_{yx}$ implies

$$\frac{\mathcal{H}_x}{\mathcal{H}_y} = \frac{c_0}{c_1}$$

i.e. function $\mathcal{H}_x/\mathcal{H}_y$ should be equal to ratio of two arbitrary harmonic functions. In particular, this condition is fulfilled if $\mathcal{H}$ depends only on x or only on y (then without loss of generality we can choose $c_0 + i c_1 = 1$ or $c_0 + i c_1 = i$ respectively). Let, for example, $\mathcal{H} = \mathcal{H}(x)$. Then we have from (6.17)

$$w_y = 0 ; \quad w = 2 \int^x \mathcal{H}(x') dx' + C \quad (6.18)$$

where $C \in \mathbf{R}$ is an arbitrary (x, y) -independent constant.

From the formulae for moving frame (6.14) we obtain

$$F_y = -4 \frac{\sigma_3}{2i} \quad F_x = 4 \left(\sin w \frac{\sigma_1}{2i} - \cos w \frac{\sigma_2}{2i} \right)$$

and

$$F_1 = 4 \int^x \sin w(x') dx' \quad F_2 = -4 \int^x \cos w(x') dx' \quad F_3 = -4y \quad (6.19)$$

This is a cylinder parallel to $\vec{k}$ -axis with a 'profile' defined by (6.18),(6.19).

3. Constant mean curvature $\mathcal{H} = \text{const.}$ Now (6.10) takes the form

$$(\mathcal{E} + \bar{\mathcal{E}}^{-1})\mathcal{E}_{z\bar{z}} = 2\mathcal{E}_z\mathcal{E}_{\bar{z}} \quad (6.20)$$

which coincides with equation for surfaces of constant $\mathcal{K} > 0$. Coefficient $Q(z, \bar{z})$ (6.9) in this case is holomorphic: $Q_{\bar{z}} = 0$; holomorphic quadratic differential Qdz^2 is called the Hopf differential on S . In the neighbourhood of non-ombilic point ($\mathcal{K} \neq \mathcal{H}^2$, or, equivalently, (6.13), $\mathcal{E}_{\bar{z}} \neq 0$) we can always choose coordinate z such that locally $Q = \frac{\mathcal{H}}{2}$ and (6.20) reduces to sinh-Gordon equation

$$u_{z\bar{z}} + \sinh u = 0 \quad (6.21)$$

4. Harmonic inverse mean curvature.

It turns out [5] that the zero curvature representation of (6.21) may be generalized to the case of arbitrary $\mathcal{H}$ satisfying equation

$$\left(\frac{1}{\mathcal{H}}\right)_{z\bar{z}} = 0$$

i.e. case of harmonic inverse mean curvature $\delta = \mathcal{H}^{-1}$. Then

$$\mathcal{H} = \frac{1}{h(z) + \bar{h}(z)}$$

where $h(z)$ is an arbitrary holomorphic function. Equation (6.10) takes the form

$$(\mathcal{E} + \mathcal{E}^{-1})(\mathcal{E}_{z\bar{z}} + \frac{h_z(z)}{h(z) + \bar{h}(z)}\mathcal{E}_{\bar{z}}) = 2\mathcal{E}_z\mathcal{E}_{\bar{z}}$$

and may be represented as a compatibility condition of the linear system (5.7) with spectral parameter λ and the following matrices U and V :

$$U = \begin{pmatrix} A & 0 \\ 0 & B \end{pmatrix} + \begin{pmatrix} 0 & A \\ \frac{\lambda + i\bar{h}(z)}{\lambda - i\bar{h}(z)}B & 0 \end{pmatrix} \quad V = \begin{pmatrix} C & 0 \\ 0 & D \end{pmatrix} + \begin{pmatrix} 0 & \frac{\lambda + i\bar{h}(z)}{\lambda - i\bar{h}(z)}C \\ D & 0 \end{pmatrix} \quad (6.22)$$

Coefficients $A, \dots, D$ are defined by the condition (2.24):

$$\Psi(\lambda = \infty) = \begin{pmatrix} \mathcal{E} & 1 \\ \bar{\mathcal{E}}^{-1} & -1 \end{pmatrix}$$

and may be expressed in terms of $\mathcal{E}$ as in (5.8).

Notice that here the form of $\Psi(\lambda = \infty)$ and coefficients $A, \dots, D$ are the same as in linear system (5.7) which arises for surfaces with positive Gaussian curvature $(h(z) + \bar{h}(z))^{-2}$ (variable z in these

two cases is certainly different). The new and essential feature of (6.22) is an analytical behaviour in λ -plane.

An alternative form of (6.22) may be obtained by the gauge transformation

$$\Psi_1 = \begin{pmatrix} \left(\frac{\lambda + i\overline{h(z)}}{\lambda - ih(z)}\right)^{1/4} & 0 \\ 0 & \left(\frac{\lambda - ih(z)}{\lambda + i\overline{h(z)}}\right)^{1/4} \end{pmatrix} \Psi$$

Then

$$\Psi_{1z} = \left[\frac{ih_z}{4(\lambda - ih(z))} \sigma_3 + \begin{pmatrix} A & 0 \\ 0 & B \end{pmatrix} + \sqrt{\frac{\lambda + i\overline{h(z)}}{\lambda - ih(z)}} \begin{pmatrix} 0 & A \\ B & 0 \end{pmatrix} \right] \Psi_1 \quad (6.23)$$

$$\Psi_{1\bar{z}} = \left[\frac{i\overline{h_z}}{4(\lambda + i\overline{h})} \sigma_3 + \begin{pmatrix} C & 0 \\ 0 & D \end{pmatrix} + \sqrt{\frac{\lambda - ih(z)}{\lambda + i\overline{h(z)}}} \begin{pmatrix} 0 & C \\ D & 0 \end{pmatrix} \right]$$

with the same condition on the infinity: $\Psi_1(\lambda = \infty) = \Psi(\lambda = \infty)$. Notice that matrices $\Psi_{1z}\Psi_1^{-1}$ and $\Psi_{1\bar{z}}\Psi_1^{-1}$ differ from the case of positive Gaussian curvature (5.7) only by terms with σ_3 ; if $h = \text{const}$ (i.e. the surface has constant mean curvature), these terms vanish and (6.23) coincides with (5.7): the cases $\mathcal{H} = \text{const}$ and $\mathcal{K} = \text{const} > 0$ reduce to the same equation (sinh-Gordon).

If $\mathcal{H} \neq \text{const}$, systems (5.7) and (6.23) are essentially different. However, it turns out that the Bäcklund-Darboux transformations may formally be written for these two systems in the same way. To make this analogy complete, rewrite system (6.23) in the form similar to (2.26):

$$\begin{aligned} \Phi_z &= \left[\frac{ih_z}{4(\lambda - ih(z))} \sigma_3 + \begin{pmatrix} * & 0 \\ 0 & * \end{pmatrix} + \sqrt{\frac{\lambda + i\overline{h(z)}}{\lambda - ih(z)}} \begin{pmatrix} 0 & * \\ * & 0 \end{pmatrix} \right] \Phi \\ \Phi_{\bar{z}} &= \left[\frac{i\overline{h_z}}{4(\lambda + i\overline{h(z)})} \sigma_3 + \begin{pmatrix} * & 0 \\ 0 & * \end{pmatrix} + \sqrt{\frac{\lambda - ih(z)}{\lambda + i\overline{h(z)}}} \begin{pmatrix} 0 & * \\ * & 0 \end{pmatrix} \right] \Phi \end{aligned} \quad (6.24)$$

where coefficients denoted by asterisk are defined by the condition

$$\Phi(\lambda = \infty) = \tilde{\Phi}$$

i.e. coincide with coefficients of (6.5).

The BDT for (6.24) may already be written in precisely the same way as for B^+ -surfaces (with the same form of matrix T , the same restrictions on parameters $\gamma_{1,2}$ and $\alpha_{1,2}$ and so on). In the case $h = \text{const}$ these BDT are in fact BDT for sinh-Gordon equation.

However, if $\mathcal{H} \neq \text{const}$ it turns out to be impossible to write down explicitly for example, the multisoliton solutions due to absence of simple non-trivial 'seed' solution of (6.24), (6.23) or (6.22). Namely, if we take a holomorphic solution of (6.10) $\mathcal{E} = \mathcal{E}(z)$ then related matrices $\Psi_z\Psi^{-1}$ and $\Psi_{\bar{z}}\Psi^{-1}$ from (6.22) are λ -independent and it is easy to check that BDT leads again to holomorphic solutions of (6.10). The simplest solution leading to non-trivial Ψ for $h \neq \text{const}$ is (we put without loss of generality $h = iz$)

$$\mathcal{E} = \exp\{iC \log(iz - i\bar{z})\} \quad C = \text{const} \in \mathbf{R} \quad (6.25)$$

(this is a partial case of solutions with $|\mathcal{E}| = 1$ corresponding to cylindrical surfaces). Then (6.22) takes the form

$$\Psi_{0z} = \frac{ic}{2(z - \bar{z})} \begin{pmatrix} 0 & 1 \\ \frac{\lambda - \bar{z}}{\lambda - z} & 0 \end{pmatrix} \Psi_0 \quad \Psi_{0\bar{z}} = \frac{-ic}{2(z - \bar{z})} \begin{pmatrix} 0 & \frac{\lambda - z}{\lambda - \bar{z}} \\ 1 & 0 \end{pmatrix} \Psi_0$$

$$\Psi_0 \equiv \frac{1}{(\det \Psi)^{1/2}} \Psi$$

In terms of components of Ψ_0 this system reduces to some partial case of hypergeometric equation which can not be resolved explicitly (in spite of solution (6.25) itself is quite explicit and we can easily find embedding of related surface in $\mathbf{R}^3$). Since the BDT involves values of Ψ_0 at some fixed points $(\gamma_{1,2})$ of $\mathbf{C}$, the multisoliton solutions can not already be expressed in elementary functions.

The complicated character of essential singularity of Ψ_0 at the singular points $\lambda = z$ and $\lambda = \bar{z}$ is also an obstacle to construct algebro-geometric solutions in this case (in the cases of Bianchi surfaces the behaviour of Ψ at the poles of matrices U and V was exponential or rational).

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Fig. 1

Fig 2 Branch cuts and canonical basis of cycles (a_j, b_j) on the spectral curve L for Bianchi surfaces. Continuous contours lie on the first sheet of L ; dotted - on the second one.

Fig. 3 Curve Z for Bianchi-plus surfaces
with canonical basis of cycles (a_j, b_j) , $j=1, \dots, g$