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A Relaxation of Local Energy Conditions due to Asymptotic Flatness

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ABSTRACT

One needs a local energy condition sufficient to show the positivity of gravitational mass. Roughly speaking, the usual condition requires the positivity of local energy density for all observers. On the other hand, there exist quantum effects such that the above condition is not satisfied. As one cannot believe that our universe becomes unstable by these quantum effects, one desires a new weaker local energy condition. So, this paper is intended as a proposal of a new local energy condition. We restrict our consideration to static asymptotically flat space-times.

1. Introduction

It is an important problem whether gravitational mass is positive or not. Systems have a tendency to change to lower energy states. If the mass is not bounded below, then catastrophes will happen and space-times will be unstable. Properly speaking, this problem should be treated by quantum gravity. However, at present, quantum gravity has not yet been accomplished. Hence one must investigate the problem in classical relativity. This problem was first resolved under a geometrical condition and a local energy condition by Schoen and Yau^[1]. Furthermore, they later removed the geometrical condition^[2] and Witten has proved positivity using spinors^[3]. Recently, Penrose et al have proven a positive energy theorem by investigating the behavior of congruences of null geodesics^[4]. All the above proofs have been performed in asymptotically flat space-times, because gravitational mass is naturally defined there, the so called ADM mass^[5]. On the other hand, there is much interest in asymptotically non-flat spaces. Fortunately, in asymptotically deSitter and anti deSitter space-times, one can define a good gravitational mass^[6]. In such space-times, under a no-momentum condition, the positivity of mass has been proven^{[7][8]}. In these theorems one must assume that the local energy density for all observers is non-negative, roughly. At a glance, the condition seems reasonable. But, we know situations which do not satisfy the condition by considering quantum effects. If such a phenomenon happens somewhere in the universe, these theorems cannot be applied and then one cannot conclude whether the universe is stable or not. However, we cannot believe that the universe becomes unstable, because we can set up easily Casimir effects^[9] in simple laboratories and we have not yet observed the catastrophic phenomenon. Hence, we want to propose a local energy condition which some quantum effects satisfy.

Another reason for stability exists. The gravitational energy contains global information and is written as an integral of an energy density over a spacelike hypersurface Σ . In fact, in Witten's proof the ADM mass is

$$M_{ADM} = \frac{1}{8\pi\psi_0^\dagger\psi_0} \int_{\Sigma} \left[{}^{(4)}\hat{\nabla}\psi^\dagger {}^{(4)}\hat{\nabla}\psi + 4\pi\psi^\dagger (T_{ab}n^a n^b + T_{ab}\gamma^b{}_{\hat{a}} n^{\hat{a}})\psi \right] d\Sigma, \quad (1.1)$$

where ${}^{(4)}\hat{\nabla}$ is the super-covariant derivative, ψ is the Witten spinor, ψ_0 is the

constant spinor which coincides with the asymptotic value of ψ , n^a is the unit normal vector of the spacelike hypersurface Σ , and γ^a is the Dirac matrix. Here, one might think that there is no necessity to consider relaxations of *local* energy conditions. But we insist on the need, because people have *local* existences and have knowledge of physics only in their neighborhood. Owing to the local energy condition, we will live with a sense of relief in our universe.

In this paper, we attempt to relax the local energy condition of the positive energy theorems. There exist two main motivations which convince us that there exists a more relaxed local energy condition. The first motivation is the energy condition in Penrose et al's recent positive energy theorem. The condition is that all null geodesic congruences have a pair of conjugate points. Translating this to local energy conditions, it becomes the null convergence condition and the generic condition^[10]. If the pressure is positive, the density can be negative under the null convergence condition. So a relaxation has been achieved, although the generic condition is extra. But this cannot be applied to some quantum effects. The second motivation is the possibility that the asymptotic condition restricts physical quantities. In all physics, the laws and equations for physical quantities appear. To solve the equations concretely, one must impose boundary conditions to get the unique solution which corresponds to the boundary conditions. In the same way, physical quantities are restricted by the asymptotic condition on space-time. In fact, we shall show that the pressure must be positive somewhere in *static* asymptotically flat space-times without any local energy condition. Hence we shall treat static asymptotically flat space-times and consider the possibility of relaxation there.

Furthermore, we assume that the space-time has no Killing horizons. Such restrictions are not strong, because it is reasonable to think that perfect fluids etc. satisfying an energy condition (except for gauge fields and Higgs fields) cannot exist in a static space-time containing black holes. However, this has not yet been proven, although it is almost clear. The gauge fields and Higgs fields will be left as hair of black holes. Apart from this, the uniqueness theorem^[11] will hold—a static space-time must be spherical—allowing one to use exact solutions. As

one knows a lot about exact solutions, the answer will be trivial. For example, in Schwarzschild space-time, the positivity of gravitational mass is guaranteed by global hyperbolicity.

This paper is organized as follows. In Sec. 2, we shall review previous positive energy theorems, as these theorems will be powerful tools in our study. In Sec.3, we discuss the influence of asymptotic flatness on pressure. In Sec. 4, we perform the relaxation of the local energy condition and in Sec. 5, we give a summary and discussion. In this paper we use units in which $G = c = \hbar = 1$.

2. Previous Positive Energy Theorems

We first recall Schoen and Yau's theorem^[1], because we shall consider a relaxation of local energy conditions using it.

Theorem 2.1(Schoen and Yau): Let (Σ, q_{ab}) be an asymptotically flat three dimensional surface. If $^{(3)}R \geq 0$, the ADM mass is non-negative.

Here, the ADM mass is defined by

$$M_{ADM} := \frac{1}{16\pi} \int_{S_\infty} (\partial^i q_{ij} - \partial_j q_i^j) dS^j, \quad (2.1)$$

asymptotically flatness means

$$q_{IJ} = \delta_{IJ} + h_{IJ}, \quad (2.2)$$

where I, J represent Cartesian coordinates adopted near spatial infinity, and $|h_{IJ}| = O(1/r)$, $|\partial_K h_{IJ}| = O(1/r^2)$ and $|\partial_K \partial_L h_{IJ}| = O(1/r^3)$.

Furthermore, they have gotten the topological restriction on (Σ, q_{ab}) when $M_{ADM} = 0$.

Theorem 2.2(Schoen and Yau): Let (Σ, q_{ab}) be an asymptotically flat three dimensional surface with $M_{ADM} = 0$. If $^{(3)}R \geq 0$, (Σ, q_{ab}) is Euclidean space.

In these theorems, the geometrical assumption $^{(3)}R \geq 0$ connects with the local energy condition through the Hamiltonian constraint:

$$^{(3)}R - K_{ab}K^{ab} + K^2 = 16\pi\rho.$$

If one adopts a maximal hypersurface Σ , the above equation yields

$$^{(3)}R = K_{ab}K^{ab} + 16\pi\rho. \quad (2.3)$$

Hence, if one assume $\rho \geq 0$, one gets $^{(3)}R \geq 0$.

From the above argument one must prepare a maximal hypersurface to connect the geometrical condition with the local energy condition. Unfortunately, at present, the existence of a maximal hypersurface has not yet been proven in general asymptotically flat space-times*. Hence, there exists the necessity to prove the positivity of the ADM mass only by local energy conditions without the geometrical assumption in order to give a completely general theorem. In 1981, Schoen and Yau removed the assumption of the existence of a maximal hypersurface from their first theorem and obtained the theorem that if the dominant energy condition holds in space-time, its ADM mass is non-negative. In the same year, Witten proved using spinors that if the dominant energy condition holds in space-time, the ADM 4-momentum is a causal 4-vector. In their paper, the topology of the hypersurface on which the ADM mass is defined was restricted to $\mathbf{R}^3$. Gibbons et al have studied the case which has holes (black holes) in the hypersurface, and the asymptotically anti-deSitter cases.

The most recent positive energy theorem has been proved by Penrose et al. The focusing of null congruences around gravitational sources plays an important role in their proof. Then, they have gotten the theorem that we stated in the Introduction. The original idea for this proof comes from the physical considerations of Geroch and Horowitz^[13].

* The existence have been proved by Bartnik in 1984.^[12]

3. Positive Pressure Theorem in Static Space-Times

As we stated in the Introduction, the physical quantities are restricted by boundary or asymptotic conditions. For simplicity, we consider static states. In a static space-time, the metric is

$$ds^2 = -V^2 dt^2 + q_{ij} dx^i dx^j, \quad (3.1)$$

where $i, j = 1, 2, 3$, $V = V(x^1, x^2, x^3)$ and $q_{ij} = q_{ij}(x^1, x^2, x^3)$. We do not assume spherical symmetry, because it has not been proven that static space-times have spherical symmetry in general. Furthermore, note that we do not make any assumptions about the matter. The Einstein equation are

$$D^a D_a V = 4\pi(\rho + 3p)V, \quad (3.2)$$

and

$$^{(3)}R = 16\pi\rho, \quad (3.3)$$

where $\rho := \xi^a \xi^b T_{ab}/V^2$, $p := (1/3)T_{ab}q^{ab}$ and T_{ab} is the energy-momentum tensor. D_a is the covariant derivative on time-symmetric surfaces (Σ, q_{ab}) orthogonal to the Killing vector ξ . The objects which we shall treat here are 'stars' whose components are perfect fluids and complex scalar fields. The Lagrangian density of the scalar field is

$$\mathcal{L} = -\frac{1}{2}\nabla_a \phi^* \nabla^a \phi - U(\phi), \quad (3.4)$$

where $U(\phi)$ is the potential term. We get the following theorem using Theorem 2.2.

Theorem 3.1 (Positive pressure theorem): Let (M, g_{ab}) be a static space-time. The pressure, defined by $p := (1/3)T_{ab}q^{ab}$, of either a perfect fluid or a scalar field with non-negative potential, must be positive on at least one point in the space-time.

Remark: Note that we did not assume any symmetry nor local energy conditions. This proof will be performed in the next section.

This theorem supports an intuition that pressure should be positive to balance with the gravitational force, without any assumptions. This result is remarkable. This gives the corollary that boson stars composed by only real scalar fields cannot exist, because the pressure is negative everywhere.

4. Positive Energy Theorem in Static Space-Times

In the previous section, asymptotic flatness gave a constraint on the pressure, although it is merely geometrical. Hence, it is natural that the local energy condition can be relaxed by asymptotic flatness. In this section, we consider a positive energy theorem in static space-times. From eq.(3.2) one obtains the expression for the ADM mass:

$$M_{ADM} = \int_{\Sigma} (\rho + 3p) d\Sigma. \quad (4.1)$$

Thus, one can show its positivity if one assumes $\rho + 3p \geq 0$. Applying Schoen and Yau's positive energy theorem, we can show its positivity assuming $\rho \geq 0$. Also, from the original positive energy theorem^{[1][2][3]}, it is guaranteed by the dominant energy condition, $\rho \geq |p|$, for a perfect fluid. According to Penrose *et al*'s recent positive energy theorem^[4], one needs the weak energy condition, $\rho + p \geq 0$, for a perfect fluid, and generic conditions for all null geodesics. Although one does not understand clearly the relationship between this relaxation of energy conditions and global information, its existence is obvious. In other words, one does not need the energy conditions to hold everywhere in order to show its positivity, because the ADM mass as an integral contains global information. For example, it is in general possible that ρ and p are negative on a compact region if the total integral is positive. Thus, we guess that its positivity can be shown under more relaxed local energy conditions and so go on to propose new local energy conditions. Such relaxations are important, because some quantum effects do not satisfy ordinary local energy conditions and yet one cannot believe that our universe is unstable by violating these conditions on a compact region of space-time.

In static space-times, one can relax the local energy condition using a conformal transformation on the time-symmetric surface.

Theorem 4.1(Positive energy theorem in static space-times): Let (M, g_{ab}) be a static asymptotically flat space-time. If a real number $\alpha > 2$ exists such that the space-time satisfies

$$(\alpha - 2)\rho - 6p \geq 0, \quad (4.2)$$

then the ADM mass is non-negative.

Proof: From asymptotic flatness, the behavior of the three metric q_{IJ} and V near infinity are

$$q_{IJ} = (1 + \frac{2M}{r})\delta_{IJ} + b_{IJ}, \quad (4.3)$$

and

$$V = 1 - \frac{M}{r} + c, \quad (4.4)$$

respectively, where $|b_{IJ}| = O(1/r^2)$, $|\partial_K b_{IJ}| = O(1/r^3)$, $c = O(1/r^2)$, $|\partial_I c| = O(1/r^3)$ and $|\partial_I \partial_J c| = O(1/r^4)$. Next, we perform a conformal transformation on the time-symmetric surface (Σ, q_{ab}) ;

$$\tilde{q}_{ab} = \Omega^4 q_{ab}, \quad (4.5)$$

where

$$\Omega = V^{1/\alpha}. \quad (4.6)$$

The asymptotic behavior of the metric and the Ricci scalar after the transformation become

$$\tilde{q}_{IJ} = (1 + \frac{2\tilde{M}}{r})\delta_{IJ} + O(\frac{1}{r^2}), \quad (4.7)$$

and

$$\begin{aligned} {}^{(3)}\tilde{R} &= 16\pi\Omega^{-4} \left[\rho - \frac{2}{\Omega} \frac{d\Omega}{dV} (\rho + 3p)V \right] - 8\Omega^{-5} \frac{d^2\Omega}{dV^2} D^a V D_a V \\ &= \frac{16\pi}{\alpha V^{4/\alpha}} \left[(\alpha - 2)\rho - 6p \right] + \frac{8}{\alpha^2} (\alpha - 1) \frac{1}{V^{4/\alpha+2}} D_a V D^a V, \end{aligned} \quad (4.8)$$

respectively, where $\tilde{M} := (1 - 2/\alpha)M$. Thus, assuming the energy condition (4.2), one can show positivity of $\tilde{M}$ from Theorem 2.1. As $1 - 2/\alpha > 0$, $\tilde{M}$ is non-negative also. *QED.*

In some sense this relaxes the energy conditions. The dominant energy condition imposes positivity of the density ρ . In the case of the null convergence condition, the density can be negative if the pressure is positive. Condition (4.2) allows the density to be negative when the pressure is negative, and so the existence of a region in which density and pressure are negative is possible.

In the limit $\alpha \rightarrow \infty$, the above theorem yields Schoen and Yau's positive energy theorem^[1]. On the other hand, in the case of $\alpha = 2$, the hypersurface $(\tilde{\Sigma}, \tilde{g}_{ab})$ becomes an asymptotic massless space and then one gets the positive pressure theorem that pressure must be positive at some point^[14].

Proof of positive pressure theorem: Let us assume that the pressure is negative everywhere. In the above proof, put $\alpha = 2$, then the asymptotic behavior of the metric and three curvature are

$$q_{IJ} = \delta_{IJ} + O\left(\frac{1}{r^2}\right), \quad (4.9)$$

and

$${}^{(3)}\tilde{R} = -48\pi \frac{p}{V^2} + \frac{2}{V^4} D^a V D_a V, \quad (4.10)$$

respectively. Then, as ${}^{(3)}\tilde{R} \geq 0$, the three-surface $(\tilde{\Sigma}, \tilde{q}_{ab})$ must be Euclidean space from Theorem 2.2, and one gets $p = D_a V = 0$. First from eq. (3.2), one gets $\rho = 0$. In the case of perfect fluids as the three Ricci scalar curvature of (Σ, q_{ab}) is

$${}^{(3)}R_{ab} = \frac{1}{V} D_a D_b V + 4\pi(\rho - p)q_{ab}, \quad (4.11)$$

one gets ${}^{(3)}R_{ab} = 0$. Thus, (Σ, q_{ab}) is Euclidean space. In the case of scalar fields, the density and the three Ricci scalar are

$$\rho = \frac{\omega^2}{2V^2} \Phi^2 + \frac{1}{2} D^a \Phi^* D_a \Phi + U(\Phi), \quad (4.12)$$

and

$${}^{(3)}R_{ab} = \frac{1}{V} D_a D_b V + 4\pi[D_a \Phi^* D_b \Phi + D_a \Phi D_b \Phi^* + 2U(\Phi)q_{ab}], \quad (4.13)$$

where we assumed $\phi = e^{-i\omega t} \Phi$ for scalar fields ϕ . From $\rho = 0$, one gets $\Phi = 0$ and then ${}^{(3)}R_{ab} = 0$. *QED.*

5. Summary and Discussions

First, let us pick up Casimir effects^{[9][17]} as famous effects which do not satisfy the usual local energy conditions and check whether these phenomena satisfy the new local energy condition or not. Prepare two parallel reflecting planes and calculate the vacuum energy of the electromagnetic field. The Casimir effect has negative density

$$\rho = \langle T_{tt} \rangle = -\frac{\pi^2}{720a^4}, \quad (5.1)$$

and pressure

$$p_x = \langle T_{xx} \rangle = p_y = \langle T_{yy} \rangle = \frac{\pi^2}{720a^4}, \quad (5.2)$$

$$p_z = \langle T_{zz} \rangle = -3\frac{\pi^2}{720a^4}, \quad (5.3)$$

where a is the separation distance between the two planes. One can see easily that these do not satisfy the weak nor dominant energy condition. Does this satisfy our local energy condition (4.2)?

$$(\alpha - 2)\rho - 6 \cdot \frac{1}{3}(p_x + p_y + p_z) = (4 - \alpha)\frac{\pi^2}{720a^4}. \quad (5.4)$$

For selected α such that $4 > \alpha(> 2)$, eq. (4.2) holds! Note that if the region outside the Casimir effect is vacuum, the ADM mass is negative from eq. (4.1). However, as matter with positive density and negligible pressure is distributed around the Casimir region, the ADM mass will be positive.

Let us summarize. We considered positive energy theorems in *static* asymptotically flat space-times and could relax the usual local energy condition due to asymptotic flatness. The situation, that Casimir effects happen, belongs to the category of our positive energy theorem. But there are unsettled problems, yet. Hawking radiation^{[18][17]} is a famous phenomenon which does not satisfy the usual local energy conditions. Unfortunately, our theorem cannot be applied to this, because we assumed non-existence of black holes. Besides, when one considers Casimir effects, one should investigate the back reaction on space-time. Then, the space-time might be non-static and one cannot use the theorem.

We guess that one can relax the local energy conditions of the positive energy theorems in general non-static asymptotically flat space-times, although our result is only suggestive. Furthermore, one might similarly relax the local energy condition for the existence of conjugate points of null geodesics which plays an important role in some crucial theorems (singularity theorem^[16], area theorem^[18], etc). The grounds for this guess are that the local energy condition can be changed into a global integral condition^[19]. The global information is related to boundary or asymptotic conditions. Thus, we expect that the origin for this relaxation of local energy conditions might also come from asymptotic conditions, that is, we expect if some asymptotic conditions are assumed then it might be possible to show the appearance of conjugate points under weaker local energy conditions.

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