AD

UM-P-94/22
OZ-94/09

A Non-commutative Geometric Model with Horizontal Family Symmetry

B. E. HANLON AND G. C. JOSHI
Research Centre for High Energy Physics
School of Physics
University of Melbourne
Parkville, Victoria 3052
Australia

Abstract

We investigate the important role of multiple fermionic families in noncommutative geometry by introducing a horizontal symmetry into the model building scheme. We demonstrate that this can be done consistently.

Keywords: non-commutative geometry

1991 MSC: 81T13, 81R40

PACS: 02.40.+m, 12.10.-g

LAMEL - P 94-22

fu 9422

1 Introduction

The standard model remains as the central edifice of particle physics phenomenology resisting, thus far, attempts to reduce the number of free parameters or give some clarification to its origin. So it is remarkable that within the framework of non-commutative geometry[1, 2, 3] it is possible not only to construct the standard model but also to provide parameter restrictions that are absent in phenomenological model building. This is the driving notion behind all attempts at setting the standard model within a broader framework. While a consistent quantum formulation is still lacking it remains compelling that the standard model can be constructed within this algebraic-geometric scheme.

Non-commutative geometry may be thought of in terms of the Kaluza-Klein idea, where now the additional internal dimensions are discrete rather than continuous. Such geometrical considerations emerge from applications of gauge theory beyond Riemannian spaces. The notion of a manifold is generalized to be the product of a continuous manifold by a discrete set of points described algebraically by the algebra $\mathcal{A}_t$. Gauge fields now arise from appropriately chosen fibre bundles along the continuous directions, while Higgs scalars result from gauging the discrete directions. In this way gauge and Higgs fields arise on the same level. However, unlike with Kaluza-Klein models, there are no unwanted order parameters arising at the Planck scale. These bosonic fields emerge as bounded operators on the Hilbert space of spinors $\mathcal{H}$. In this way spinor fields are the fundamental fields in non-commutative geometry allowing the fermionic action to be introduced in a simple way. Relevant models then arise by inputting the appropriate fermionic data.

The differential calculus constructed on $\mathcal{A}_t$ is not respected when represented on $\mathcal{H}$ unless certain offending forms are divided out of the differential algebra[3]. As a result of this division the derived Higgs potential will no longer be in a symmetry breaking form. This can be overcome by demanding that there be more than one fermionic family in the model, an intriguing correspondence with phenomenology.

Since the fermionic information plays such a crucial role in model building it appears natural to investigate this fermionic family replication more closely. While a unique mechanism for replication is by no means obvious we choose in this paper to introduce an

horizontal family symmetry thereby making replication a dynamical part of the model. We will demonstrate that such models can be consistently constructed by considering a toy model with $U(2) \otimes U(1)$ vertical symmetry and $U(2)$ horizontal symmetry. Interestingly we find that Cabbibo like mixing may be a necessary requirement for a realistic model.

2 Non-commutative Gauge Theory

To generalize the Riemannian metric the notion of geodesic distance must be consistently incorporated and this is encoded in the concept of a K-cycle. A K-cycle over the involutive algebra, $\mathcal{B}$ say, is a $*$ -action of $\mathcal{B}$ by bounded operators on a Hilbert space $\mathcal{H}$, denoted by ρ , and a possibly unbounded, self adjoint, operator D , denoted Dirac operator, such that $[D, f]$ is a bounded operator $\forall f \in \mathcal{B}$ and $(1 + D^2)^{-1}$ is compact. Let X be a compact Riemannian spin manifold, $\mathcal{F}$ the algebra of functions on X and $(\mathcal{H}_{\mathcal{F}}, \not{D})$ the Dirac K-cycle with $\mathcal{H}_{\mathcal{F}} = L^2(x, \sqrt{g}d^d x)$ of $\mathcal{F}$. Denote by γ_5 the fifth anticommuting Dirac gamma matrix, the chirality operator, defining a Z_2 grading on $\mathcal{H}_{\mathcal{F}}$. Similarly a K-cycle can be defined on the discrete set representing the internal space. Let $\mathcal{A}$ be given by $\mathcal{A} = M_n(\mathbb{C}) \oplus M_p(\mathbb{C}) \oplus \dots$ corresponding to the Hilbert spaces $\mathbb{C}^n, \mathbb{C}^p, \dots$ respectively. For a two point space the Dirac operator takes the form

$$D = \begin{pmatrix} 0 & M \\ M^* & 0 \end{pmatrix}, \quad (1)$$

M being a mass matrix of size $\dim \mathcal{H}_L \times \dim \mathcal{H}_R$, where $\mathcal{H} = \mathcal{H}_L \oplus \mathcal{H}_R$ decomposes under the action of a suitable chirality operator

$$\chi = \begin{pmatrix} 1_L & 0 \\ 0 & -1_R \end{pmatrix}. \quad (2)$$

For a product geometry $\mathcal{A}_t = \mathcal{F} \otimes \mathcal{A}$ a product K-cycle is naturally defined with the generalized Dirac operator

$$D_t = \not{D} \otimes 1 + \gamma_5 \otimes D. \quad (3)$$

For any C^* -algebra a K-cycle will define a metric d on the state space of $\mathcal{B}$ by

$$d(p, q) = \sup\{|p(f) - q(f)| : f \in \mathcal{B}, ||[D, f]|| \leq 1\}. \quad (4)$$

To make contact with gauge theories a differential algebra must be constructed on $\mathcal{B}$. The space of all differential forms $\hat{\Omega}^*(\mathcal{B}) = \bigoplus_{p \in \mathbb{N}} \hat{\Omega}^p(\mathcal{B})$ is a graded differential algebra equipped with a differential operator δ such that

$$\delta : \hat{\Omega}^p(\mathcal{B}) \rightarrow \hat{\Omega}^{p+1}(\mathcal{B}), \quad (5)$$

along with nilpotency

$$\delta^2 = 0. \quad (6)$$

The space of p -forms $\hat{\Omega}^p(\mathcal{B})$ is generated by finite sums of terms of the form

$$\hat{\Omega}^p(\mathcal{B}) = \left\{ \sum_j a_0^j \delta a_1^j \dots \delta a_p^j, a_p^j \in \mathcal{B} \right\}, \quad (7)$$

which follows from the relations

$$\begin{aligned} \delta 1 &= 0 \\ \delta(ab) &= (\delta a)b + a\delta b, \end{aligned} \quad (8)$$

with the differential δ defined by

$$\delta(a_0 \delta a_1 \dots \delta a_p) = \delta a_0 \delta a_1 \dots \delta a_p. \quad (9)$$

Extending the representation of $\mathcal{B}$ on $\mathcal{H}$ to its universal differential envelope $\hat{\Omega}^*(\mathcal{B})$ is achieved via the map

$$\pi : \hat{\Omega}^*(\mathcal{B}) \rightarrow B(\mathcal{H}), \quad (10)$$

$B(\mathcal{H})$ being the algebra of bounded operators on $\mathcal{H}$ defined by

$$\pi(a_0 \delta a_1 \dots \delta a_p) = (-i)^p \rho(a_0) [D, \rho(a_1)] \dots [D, \rho(a_p)]. \quad (11)$$

The crucial aspect which must be considered is that the representation π is ambiguous, with the correct space of forms actually given by

$$\Omega^*(\mathcal{B}) = \hat{\Omega}^*(\mathcal{B})/J, \quad (12)$$

J being given by the differential ideal[4]

$$J = \ker \pi + \delta \ker \pi = \bigoplus_p J^p, \quad (13)$$

where

$$J^p = (\ker \pi)^p + \delta(\ker \pi)^{p-1} . \quad (14)$$

Degree by degree the correct space of forms becomes

$$\begin{aligned} \Omega^0(\mathcal{B}) &= \hat{\Omega}^0(\mathcal{B}) \cong \rho(\mathcal{B}) , \\ \Omega^1(\mathcal{B}) &= \hat{\Omega}^1(\mathcal{B})/(\ker \pi)^1 \cong \pi(\hat{\Omega}^1(\mathcal{B})) \end{aligned}$$

and for degree $p \geq 2$

$$\Omega^p(\mathcal{B}) = \pi(\hat{\Omega}^p(\mathcal{B}))/\pi(\delta(\ker \pi)^{p-1}) . \quad (15)$$

When $\mathcal{B} = \mathcal{F}$, $\Omega^*(\mathcal{B})$ recovers deRham's differential algebra of differential forms on X .

The simplest means by which the quotient may be considered as a subspace is in the presence of a scalar product. This is naturally defined on the internal space by the trace on matrices. In the infinite dimensional case the inner product is defined by the Dixmier trace

$$\text{tr}_w(Q|\not{D}|^{-d}) = \lim_{N \rightarrow \infty} \frac{1}{\log N} \sum_{n=1}^N \lambda_n , \quad (16)$$

where Q is a bound operator on $\mathcal{H}_{\mathcal{F}}$, $d = \dim X$ and λ_n are eigenvalues of $Q|\not{D}|^{-d}$ arranged in a decreasing sequence discarding the Dirac zero modes. By correspondence with the deRham complex on X this reduces to the usual scalar product on X ,

$$(\phi, \psi) = 1/8\pi^2 \int_X \phi^* * \psi \quad \phi, \psi \in \Omega^p(X) \quad (17)$$

incorporating the Hodge star. Since the fermionic fields are the fundamental fields the spinor action can be simply written down.

3 Introducing an Horizontal Symmetry

Determining the correct space of forms in the case of a product geometry is complicated by mixing of component terms i.e.

$$\hat{\Omega}^2(\mathcal{A}_t) = [\hat{\Omega}^2(\mathcal{F}) \otimes \rho(\mathcal{A}) + \mathcal{F} \otimes \hat{\Omega}^2(\mathcal{A})] \oplus \hat{\Omega}^1(\mathcal{F}) \otimes \hat{\Omega}^1(\mathcal{A}) . \quad (18)$$

The sum in brackets is not direct because space-time zero and two forms mix before division by J . As a result it is not possible to simply decompose the projection operator on the product space, $P_t \neq P_{\mathcal{F}} \otimes P_{\mathcal{A}}$. The final potential modifies to

$$V' = V - V_0 = \text{tr}[(C - \alpha C)^2] , \quad (19)$$

V_0 being the modification on V , $C \in \Omega^2 \mathcal{A}$ the curvature calculated on the internal space and α the map

$$\alpha : \Omega^2 \mathcal{A} \rightarrow [\rho(\mathcal{A}) + \pi(\delta(\ker \pi)^1_{\mathcal{A}})]^C . \quad (20)$$

Thus tensoring produces the additional contribution to the potential which in the case of a single fermionic family results in a final potential V' which is no longer in a symmetry breaking form.

As is clear from the map (20) the determination of V' rests on the internal structure, that is, it is not space-time dependent. That symmetry breaking is lost in the single family case is thus related to the form of C when projection to $\Omega^2(\mathcal{A})$ is trivial as we are projecting out a one-dimensional space. Demanding more than one fermionic family enlarges the vector space upon which this projection is performed. Consequently this now need not be trivial, a fact imposed by non-degeneracy conditions on the masses and replication.

While playing a vital role in this kind of model building the origin of family replication remains a mystery. Introducing family symmetries is one approach to this problem. In this paper we wish to investigate this possibility as an intrinsic part of the non-commutative model building prescription. To this extent we do not address ourselves to distinct phenomenological problems which must be overcome, in particular the suppression of flavour changing neutral currents. This would require an extended Higgs sector which can be accommodated in non-commutative geometric schemes[5] but is here left as an issue for particular models. As such we will investigate a toy model with $U(2) \otimes U(1)$ vertical symmetry and $U(2)$ horizontal symmetry. This choice is already sufficiently restrictive due to the need to accommodate appropriate quantum numbers on the Higgs and fermions.

In introducing an horizontal symmetry we wish to extend the number of quantum numbers associated with given spinor representations of the fermions. Thus it is not

natural to simply add new symmetry breaking scales to the model associated with Higgs fields which are scalars under the vertical symmetry. So we will consider a Riemannian spin manifold extended by two points described by the algebra

$$\mathcal{A}_V = M_2(\mathbb{C}) \oplus \mathbb{C} , \quad (21)$$

and add an auxiliary algebra corresponding to the horizontal space

$$\mathcal{A}_H = M_2(\mathbb{C}) . \quad (22)$$

Thus the internal structure is described by the algebra

$$\mathcal{A}_V \otimes \mathcal{A}_H = (M_2(\mathbb{C}) \oplus \mathbb{C}) \otimes M_2(\mathbb{C}) , \quad (23)$$

with elements $(a, b) \otimes A$, a and A being 2×2 matrices and $b \in \mathbb{C}$, which may be described as a bi-module on $\mathcal{H}$. This choice of $\mathcal{A}_H$ corresponds to introducing two fermionic families, that is the fermions transform as doublets under the corresponding $U(2)_H$ symmetry. We define a K-cycle by

$$\mathcal{H}_L = \mathbb{C}^2 \otimes \mathbb{C}^{N_G} , \quad \mathcal{H}_R = \mathbb{C}^2 \otimes \mathbb{C}^{N_G} , \quad (24)$$

so that

$$\mathcal{H} = \mathcal{H}_L \oplus \mathcal{H}_R = \mathbb{C}^4 \otimes \mathbb{C}^{N_G} , \quad N_G = 2 \quad (25)$$

and

$$\rho_L((a, b) \otimes A) = a \otimes A , \quad \rho_R((a, b) \otimes A) = B \otimes A , \quad (26)$$

with

$$B = \begin{pmatrix} b & 0 \\ 0 & \bar{b} \end{pmatrix} . \quad (27)$$

The components of the internal Dirac operator, M , are correspondingly generalized

$$M = m_V \otimes m_H , \quad (28)$$

such that

$$m_V = \begin{pmatrix} m_1 & 0 \\ 0 & m_2 \end{pmatrix} , \quad m_1, m_2 \in \mathbb{C} \quad (29)$$

with $|m_1| \neq |m_2|$ and

$$m_H = \begin{pmatrix} M_1 & M_3 \\ M_3^* & M_2 \end{pmatrix} , \quad (30)$$

where $m_H^* = m_H$ and $M_1 \neq M_2$. This hermiticity requirement follows from the necessity of yielding gauge singlets in Yukawa couplings with fermions, which in both left and right sectors are doublets under $U(2)_H$. M_1 and M_2 thus describe mass splitting between families while M_3 describes inter-family mixing.

We wish now to define an appropriate connection one-form on the internal space. The physically relevant one-forms are a subset of the space of all possible one-forms which distinguish the vertical from the horizontal symmetries. Since the internal geometry is described in terms of matrices we may make the simple choice of identifying the vector bundle with the algebra and take the connection one-form to be

$$w = \sum_i (a_0^i, b_0^i) \delta(a_1^i, b_1^i) \otimes 1 + 1 \otimes \sum_j A_0^j \delta A_1^j , \quad (31)$$

which is a subset of the general one-form

$$\sum_i ((a_0^i, b_0^i) \otimes A_0^i) \delta((a_1^i, b_1^i) \otimes A_1^i) . \quad (32)$$

We have thus generalized the non-commutative geometric framework to include an horizontal symmetry. However, it should be immediately clear that this is still inadequate. With the fermionic data entered in $m_V \otimes m_H$ as above mass splitting and mixing will be identical in both the up and down sector of a “quark” doublet. This is too restrictive to allow for non-trivial projections, as in the usual case with non-dynamical replication.

4 Multiple Higgs Fields

To allow for different mass splittings and mixings we modify the internal Dirac operator components to read

$$M = \begin{pmatrix} m_1 & 0 \\ 0 & 0 \end{pmatrix} \otimes \begin{pmatrix} M_1 & M_3 \\ M_3^* & M_2 \end{pmatrix} + \begin{pmatrix} 0 & 0 \\ 0 & m_2 \end{pmatrix} \otimes \begin{pmatrix} M_1' & M_3' \\ M_3'^* & M_2' \end{pmatrix}$$

$$= m_1^V \otimes m_1^H + m_2^V \otimes m_2^H . \quad (33)$$

This implies the existence of two Higgs fields acting as connection between the copies of space-time. Actually, this is consistent with the previous choice of M (28) which corresponds to two Higgs fields, one for each column of m_V (29). Leptonically reducing the vertical components in (33) recovers this interpretation.

We define the internal Dirac operator:

$$D = \begin{pmatrix} 0 & m_1^V \\ m_1^{V*} & 0 \end{pmatrix} \otimes m_1^H + \begin{pmatrix} 0 & m_2^V \\ m_2^{V*} & 0 \end{pmatrix} \otimes m_2^H \\ = D_1 + D_2 , \quad (34)$$

and represent the physical theory defined by the connection (31) on the Hilbert space $\mathcal{H}$:

$$\pi(w) = (-i) \times \\ \left\{ \sum_i \begin{pmatrix} 0 & a_0^i m_1^V b_1^i - a_0^i a_1^i m_1^V \\ b_0 m_1^{V*} a_1^i - b_0^i b_1^i m_1^{V*} & 0 \end{pmatrix} \otimes m_1^H \right. \\ \left. + \sum_j \begin{pmatrix} 0 & m_1^V \\ m_1^{V*} & 0 \end{pmatrix} \otimes (A_0^j m_1^H A_1^j - A_0^j A_1^j m_1^H) \right. \\ \left. + \sum_i \begin{pmatrix} 0 & a_0^i m_2^V b_1^i - a_0^i a_1^i m_2^V \\ b_0 m_2^{V*} a_1^i - b_0^i b_1^i m_2^{V*} & 0 \end{pmatrix} \otimes m_2^H \right. \\ \left. + \sum_j \begin{pmatrix} 0 & m_2^V \\ m_2^{V*} & 0 \end{pmatrix} \otimes (A_0^j m_2^H A_1^j - A_0^j A_1^j m_2^H) \right\} \\ = i \left\{ \begin{pmatrix} 0 & h_1 \\ \tilde{h}_1 & 0 \end{pmatrix} \otimes m_1^H + \begin{pmatrix} 0 & m_1^V \\ m_1^{V*} & 0 \end{pmatrix} \otimes H_1 + \begin{pmatrix} 0 & h_2 \\ \tilde{h}_2 & 0 \end{pmatrix} \otimes m_2^H + \begin{pmatrix} 0 & m_2^V \\ m_2^{V*} & 0 \end{pmatrix} \otimes H_2 \right\} \quad (35)$$

This is an element of $(\ker \pi)^1$ if

$$h_1 \otimes m_1^H + m_1^V \otimes H_1 = 0 , \quad \tilde{h}_1 \otimes m_1^H + m_1^{V*} \otimes H_1 = 0 , \quad (36)$$

and similarly with $h_1 \rightarrow h_2$ and $H_1 \rightarrow H_2$ since by assumption the mass matrix contributions are not proportional. Calculating $\pi(\delta w)$ we get, in terms of h 's and H 's:

$$\pi(\delta w) = \\ \left\{ \begin{pmatrix} m_1^V \tilde{h}_1 + h_1 m_1^{V*} - \sum_i (a_0^i a_1^i m_1^V m_1^{V*} - a_0^i m_1^V m_1^{V*} a_1^i) & 0 \\ 0 & m_1^{V*} h_1 + \tilde{h}_1 m_1^V \end{pmatrix} \otimes m_1^H m_1^H \right. \\ \left. + \begin{pmatrix} 0 & 0 \\ 0 & m_1^{V*} h_2 + \tilde{h}_1 m_1^V \end{pmatrix} \otimes m_1^H m_2^H + \begin{pmatrix} 0 & 0 \\ 0 & m_2^{V*} h_1 + \tilde{h}_2 m_1^V \end{pmatrix} \otimes m_2^H m_1^H \right. \\ \left. + \begin{pmatrix} m_2^V \tilde{h}_2 + h_2 m_2^{V*} - \sum_i (a_0^i a_1^i m_2^V m_2^{V*} - a_0^i m_2^V m_2^{V*} a_1^i) & 0 \\ 0 & m_2^{V*} h_2 + \tilde{h}_2 m_2^V \end{pmatrix} \otimes m_2^H m_2^H \right. \\ \left. + \begin{pmatrix} m_1^V m_1^{V*} & 0 \\ 0 & m_1^{V*} m_1^V \end{pmatrix} \otimes (m_1^H H_1 + H_1 m_1^H - \sum_j (A_0^j A_1^j m_1^H m_1^H - A_0^j m_1^H m_1^H A_1^j)) \right. \\ \left. + \begin{pmatrix} m_2^V m_2^{V*} & 0 \\ 0 & m_2^{V*} m_2^V \end{pmatrix} \otimes (m_2^H H_2 + H_2 m_2^H - \sum_j (A_0^j A_1^j m_2^H m_2^H - A_0^j m_2^H m_2^H A_1^j)) \right\} \quad (37)$$

where use has been made of the observation $m_1^V m_2^V = 0$. We can thus determine the elements of $\pi((\delta \ker \pi)^1)$ from those elements which cannot be expressed in terms of the fields h_i and H_i :

$$\pi((\delta \ker \pi)^1) = (-1) \times \\ \left\{ \begin{pmatrix} \sum_i [(a_0^i a_1^i m_1^V m_1^{V*} - a_0^i m_1^V m_1^{V*} a_1^i) \otimes m_1^H m_1^H & 0 \\ 0 & 0 \end{pmatrix} \right. \\ \left. + \begin{pmatrix} \sum_i [(a_0^i a_1^i m_2^V m_2^{V*} - a_0^i m_2^V m_2^{V*} a_1^i) \otimes m_2^H m_2^H & 0 \\ 0 & 0 \end{pmatrix} \right. \\ \left. + \begin{pmatrix} m_1^V m_1^{V*} & 0 \\ 0 & m_1^{V*} m_1^V \end{pmatrix} \otimes \sum_j (A_0^j A_1^j m_1^H m_1^H - A_0^j m_1^H m_1^H A_1^j) \right. \\ \left. + \begin{pmatrix} m_2^V m_2^{V*} & 0 \\ 0 & m_2^{V*} m_2^V \end{pmatrix} \otimes \sum_j (A_0^j A_1^j m_2^H m_2^H - A_0^j m_2^H m_2^H A_1^j) \right\} \quad (38)$$

The appearance of mixing terms in the matrices m_1^H and m_2^H greatly complicates an explicit calculation of the true differential algebra. As such we will make the simplifying assumption that there are no inter-family mixing terms by setting $M_3 = M'_3 = 0$. As has been pointed out previously, a subset of $\pi(\delta(\ker\pi)^1)$ consists of those elements for which

$$\sum_i a_0^i a_1^i = 0, \quad (39)$$

which we extend also to the horizontal sector

$$\sum_j A_0^j A_1^j = 0. \quad (40)$$

A general element in $\pi(\delta(\ker\pi)^1)$ then becomes

$$\begin{pmatrix} \sum_i a_0^i (m_1^V m_1^{V*} \otimes m_1^H m_1^H + m_2^V m_2^{V*} \otimes m_2^H m_2^H) a_1^i & 0 \\ 0 & 0 \end{pmatrix} + \begin{pmatrix} \sum_j A_0^j (m_1^V m_1^{V*} \otimes m_1^H m_1^H + m_2^V m_2^{V*} \otimes m_2^H m_2^H) A_1^j & 0 \\ 0 & \sum_j A_0^j (m_1^V m_1^{V*} \otimes m_1^H m_1^H + m_2^V m_2^{V*} \otimes m_2^H m_2^H) A_1^j \end{pmatrix} \quad (41)$$

We simplify this further by decomposing the mass matrix contributions:

$$\begin{aligned} & m_1^V m_1^{V*} \otimes m_1^H m_1^H + m_2^V m_2^{V*} \otimes m_2^H m_2^H \\ &= \begin{pmatrix} m_1 m_1^* m_1^H m_1^H & 0 \\ 0 & m_2 m_2^* m_2^H m_2^H \end{pmatrix} = 1 \otimes \Sigma + \sigma_3^V \otimes \Delta, \end{aligned} \quad (42)$$

where

$$\begin{aligned} \Sigma &= \frac{1}{2}(m_1 m_1^* m_1^H m_1^H + m_2 m_2^* m_2^H m_2^H) \\ \Delta &= \frac{1}{2}(m_1 m_1^* m_1^H m_1^H - m_2 m_2^* m_2^H m_2^H) \end{aligned} \quad (43)$$

with $\sigma_3^V = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$ being a matrix in the vertical sector. This can also be decomposed in an horizontal sense

$$m_1^V m_1^{V*} \otimes m_1^H m_1^H + m_2^V m_2^{V*} \otimes m_2^H m_2^H$$

$$\begin{aligned} &= \begin{pmatrix} m_1^V m_1^{V*} M_1 M_1 & 0 \\ 0 & m_1^V m_1^{V*} M_2 M_2 \end{pmatrix} + \begin{pmatrix} m_2^V m_2^{V*} M'_1 M'_1 & 0 \\ 0 & m_2^V m_2^{V*} M'_2 M'_2 \end{pmatrix} \\ &= \Lambda \otimes 1 + \Xi \otimes \sigma_3^H, \end{aligned} \quad (44)$$

where

$$\begin{aligned} \Lambda &= \frac{1}{2}(m_1^V m_1^{V*} (M_1 M_1 + M_2 M_2) + m_2^V m_2^{V*} (M'_1 M'_1 + M'_2 M'_2)) \\ \Xi &= \frac{1}{2}(m_1^V m_1^{V*} (M_1 M_1 - M_2 M_2) + m_2^V m_2^{V*} (M'_1 M'_1 - M'_2 M'_2)) \end{aligned} \quad (45)$$

and $\sigma_3^H = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$ is a matrix in the horizontal sector. $\pi(\delta(\ker\pi)^1)$ is thus given by

$$\begin{aligned} \pi(\delta(\ker\pi)^1) &= \left\{ \begin{pmatrix} k \otimes \Delta & 0 \\ 0 & 0 \end{pmatrix} + \begin{pmatrix} \Xi \otimes K & 0 \\ 0 & \Xi \otimes K \end{pmatrix}, k \in M_2(\mathbb{C}), K \in M_2(\mathbb{C}) \right\} \\ &= \pi(\delta(\ker\pi)^1)_V + \pi(\delta(\ker\pi)^1)_H \end{aligned} \quad (46)$$

Note that although the fermionic representations in the horizontal sector are vector-like the general fermionic representations are not. As such the horizontal contribution to $\pi(\delta(\ker\pi)^1)$ does not vanish. This is reflected in the generalized constraints (36) which cannot be used to eliminate the vector-like part[4].

To make contact with the space of gauge potentials we require that the fields defined in (35) define a vector space of anti-hermitian one-forms, a general element of which will thus take the form

$$H = i \left\{ \begin{pmatrix} 0 & h_1 \\ h_1^* & 0 \end{pmatrix} \otimes m_1^H + \begin{pmatrix} 0 & m_1^V \\ m_1^{V*} & 0 \end{pmatrix} \otimes H_1 + \begin{pmatrix} 0 & h_2 \\ h_2^* & 0 \end{pmatrix} \otimes m_2^H + \begin{pmatrix} 0 & m_2^V \\ m_2^{V*} & 0 \end{pmatrix} \otimes H_2 \right\} \quad (47)$$

We shift these fields, the Higgs fields, away from the broken phase by addition of the term $-iD = -iD_1 - iD_2$, where upon we get

$$\Phi = i \left\{ \begin{pmatrix} 0 & h_1 \\ h_1^* & 0 \end{pmatrix} \otimes m_1^H + \begin{pmatrix} 0 & m_1^V \\ m_1^{V*} & 0 \end{pmatrix} \otimes H_1 - \begin{pmatrix} 0 & m_1^V \\ m_1^{V*} & 0 \end{pmatrix} \otimes m_1^H \right\}$$

$$\begin{aligned}
& +i \left\{ \begin{pmatrix} 0 & h_2 \\ h_2^* & 0 \end{pmatrix} \otimes m_2^H + \begin{pmatrix} 0 & m_2^V \\ m_2^{V*} & 0 \end{pmatrix} \otimes H_2 - \begin{pmatrix} 0 & m_2^V \\ m_2^{V*} & 0 \end{pmatrix} \otimes m_2^H \right\} \\
& = i(\phi_1 + \phi_2) , \tag{48}
\end{aligned}$$

which transform homogenously under a general gauge transformation. The components of H thus represent individual fluctuations in the vertical and horizontal directions about the vacuum expectation value defined by D .

The curvature on the internal space can now be calculated which in terms of ϕ_1 and ϕ_2 becomes

$$\begin{aligned}
\pi(\theta) &= \pi(\delta w + w^2) \\
&= \begin{pmatrix} m_1^V m_1^{V*} & 0 \\ 0 & m_1^{V*} m_1^V \end{pmatrix} \otimes m_1^H m_1^H + \begin{pmatrix} m_2^V m_2^{V*} & 0 \\ 0 & m_2^{V*} m_2^V \end{pmatrix} \otimes m_2^H m_2^H - \begin{pmatrix} 0 & \phi_1 + \phi_2 \\ \phi_1^* + \phi_2^* & 0 \end{pmatrix}^2 \tag{49}
\end{aligned}$$

This expression incorporates (37) with the auxiliary terms modded out. However to determine the correct form of $\pi(\theta)$, which we write as C , we must project this onto the orthocomplement of $\pi(\delta(\ker\pi)^1)$. We find

$$\begin{aligned}
C &= \\
& \begin{pmatrix} m_1^V m_1^{V*} & 0 \\ 0 & m_1^{V*} m_1^V \end{pmatrix} \otimes m_1^H m_1^H + \begin{pmatrix} m_2^V m_2^{V*} & 0 \\ 0 & m_2^{V*} m_2^V \end{pmatrix} \otimes m_2^H m_2^H - \begin{pmatrix} 0 & \phi_1 + \phi_2 \\ \phi_1^* + \phi_2^* & 0 \end{pmatrix}^2 \\
& - \begin{pmatrix} y_1 & 0 \\ 0 & 0 \end{pmatrix} \otimes \frac{\Delta}{\text{tr}\Delta^2} - \frac{\Xi}{\text{tr}\Xi^2} \otimes \begin{pmatrix} y_2 & 0 \\ 0 & y_3 \end{pmatrix} \\
& + \begin{pmatrix} \text{tr}(y_1\Xi)\text{tr}(y_2\Delta)/\text{tr}(\Xi^2)\text{tr}(\Delta^2) & 0 \\ 0 & 0 \end{pmatrix} \Xi \otimes \Delta \tag{50}
\end{aligned}$$

where

$$y_1 = m_1^V m_1^{V*} \text{tr}(m_1^H m_1^H \Delta) + m_2^V m_2^{V*} \text{tr}(m_2^H m_2^H \Delta) - \text{tr}(\phi_1 \phi_1^* \Delta) - \text{tr}(\phi_2 \phi_2^* \Delta)$$

$$\begin{aligned}
y_2 &= \text{tr}(m_1^V m_1^{V*} \Xi) m_1^H m_1^H + \text{tr}(m_2^V m_2^{V*} \Xi) m_2^H m_2^H - \text{tr}(\phi_1 \phi_1^* \Xi) - \text{tr}(\phi_2 \phi_2^* \Xi) \\
y_3 &= \text{tr}(m_1^{V*} m_1^V \Xi) m_1^H m_1^H + \text{tr}(m_2^{V*} m_2^V \Xi) m_2^H m_2^H - \text{tr}(\phi_1^* \phi_1 \Xi) - \text{tr}(\phi_2^* \phi_2 \Xi) \\
& - \text{tr}(\phi_1^* \phi_2 \Xi) - \text{tr}(\phi_2^* \phi_1 \Xi) . \tag{51}
\end{aligned}$$

Unlike in previous models the fields ϕ_1 and ϕ_2 are intimately involved in the projection procedure. Note that traces in (51) involving these fields are understood to be over the appropriate sector i.e. horizontal in y_1 and vertical in y_2 and y_3 . We thus have the curvature on our generalized internal space $C \in \Omega^2 \mathcal{A}$. However the full algebra includes the tensor product with $\mathcal{F}$ so that we must also accomodate the modification of C , described by the map α , which is determined by the equations[4]:

$$\begin{aligned}
\langle R, C - \alpha C \rangle &= 0 \quad \forall R \in \rho(\mathcal{A})^C \\
\langle N, \alpha C \rangle &= 0 \quad \forall N \in \pi(\delta(\ker\pi)^1)^C \tag{52}
\end{aligned}$$

These conditions cannot be simultaneously satisfied on the vertical and horizontal sectors and so must be addressed separately. We therefore consider the conditions

$$\begin{aligned}
\langle \tilde{\rho}(\mathcal{A}_V) \otimes 1, C - \alpha C \rangle &= 0 \\
\langle 1 \otimes \tilde{\rho}(\mathcal{A}_H), C - \alpha C \rangle &= 0 , \tag{53}
\end{aligned}$$

which in order to differentiate the two sectors we require

$$\text{tr}(\tilde{\rho}(\mathcal{A}_V) \Xi) = \text{tr}(\tilde{\rho}(\mathcal{A}_H) \Delta) = 0 . \tag{54}$$

We have the additional requirement that

$$\text{tr}(\tilde{\rho}(\mathcal{A}_V)) = \text{tr}(\tilde{\rho}(\mathcal{A}_H)) = 0 . \tag{55}$$

Such a condition is necessary to avoid elements which are proportional to $1 \otimes 1$ and thus can not be uniquely associated with the horizontal or vertical sector. Similarly we require

$$\begin{aligned}
\langle \pi(\delta(\ker\pi)^1)_V, \alpha C \rangle &= 0 \\
\langle \pi(\delta(\ker\pi)^1)_H, \alpha C \rangle &= 0 , \tag{56}
\end{aligned}$$

with $k \in \tilde{\rho}(\mathcal{A}_V)$ and $K \in \tilde{\rho}(\mathcal{A}_H)$. αC therefore has the form

$$\alpha C = \begin{pmatrix} x_1 \otimes \beta_1 + \beta_2 \otimes x_2 & 0 \\ 0 & \beta_2 \otimes x_3 + x_4/N_H \end{pmatrix} \quad (57)$$

where

$$\begin{aligned} x_1 &= m_1^V m_1^{V*} \text{tr}(m_1^H m_1^H) + m_2^V m_2^{V*} \text{tr}(m_2^H m_2^H) - \text{tr}((\phi_1 + \phi_2)(\phi_1^* + \phi_2^*)) \\ &\quad - y_1 \text{tr} \Delta / \text{tr} \Delta^2 \\ x_2 &= \text{tr}(m_1^V m_1^{V*}) m_1^H m_1^H + \text{tr}(m_2^V m_2^{V*}) m_2^H m_2^H - \text{tr}((\phi_1 + \phi_2)(\phi_1^* + \phi_2^*)) \\ &\quad - y_2 \text{tr} \Xi / \text{tr} \Xi^2 \\ x_3 &= \text{tr}(m_1^V m_1^{V*}) m_1^H m_1^H + \text{tr}(m_2^V m_2^{V*}) m_2^H m_2^H - \text{tr}((\phi_1^* + \phi_2^*)(\phi_1 + \phi_2)) \\ &\quad - y_3 \text{tr} \Xi / \text{tr} \Xi^2 \\ x_4 &= m_1^{V*} m_1^V \text{tr}(m_1^H m_1^H) + m_2^{V*} m_2^V \text{tr}(m_2^H m_2^H) - \text{tr}((\phi_1^* + \phi_2^*)(\phi_1 + \phi_2)) \end{aligned} \quad (58)$$

$N_V = N_H = 2$ and

$$\beta_1 = \frac{(1 - (\text{tr} \Delta / \text{tr}(\Delta^2)) \Delta)}{(N_H - (\text{tr}(\Delta))^2 / \text{tr}(\Delta^2))} \quad (59)$$

$$\beta_2 = \frac{(1 - (\text{tr} \Xi / \text{tr}(\Xi^2)) \Xi)}{(N_V - (\text{tr}(\Xi))^2 / \text{tr}(\Xi^2))} \quad (60)$$

The final potential can be expressed as

$$\begin{aligned} V' &= V - V_0 = \text{tr}[(C - \alpha C)^2] \\ &= \text{tr}((m_1^V m_1^{V*} \otimes m_1^H m_1^H + m_2^V m_2^{V*} \otimes m_2^H m_2^H - (\phi_1 + \phi_2)(\phi_1^* + \phi_2^*) \\ &\quad - y_1 \otimes \Delta / \text{tr}(\Delta^2) - \Xi / \text{tr}(\Xi^2) \otimes y_2 - x_1 \otimes \beta_1 - \beta_2 \otimes x_2 + \frac{\text{tr}(y_1 \Xi) \text{tr}(y_2 \Delta)}{\text{tr}(\Xi^2) \text{tr}(\Delta^2)} \Xi \otimes \Delta)^2 \\ &\quad + (m_1^{V*} m_1^V \otimes m_1^H m_1^H + m_2^{V*} m_2^V \otimes m_2^H m_2^H - (\phi_1^* + \phi_2^*)(\phi_1 + \phi_2) \\ &\quad - \Xi / \text{tr}(\Xi^2) \otimes y_3 - \beta_2 \otimes x_3 - x_4 / N_H)^2), \end{aligned} \quad (61)$$

which, barring a numerical coincidence, is in a symmetry breaking form and is minimized when

$$\begin{aligned} \langle \phi_1 \rangle &= m_1^V \otimes m_1^H \\ \langle \phi_2 \rangle &= m_2^V \otimes m_2^H. \end{aligned} \quad (62)$$

We have thus demonstrated that it is possible to consistently incorporate an horizontal symmetry into the model building scheme.

5 Conclusion

Clearly the derived potential does not lend itself to simple analysis. Nevertheless, given the recent interest in determining the quantum behaviour of non-commutative geometric models it is significant that an additional symmetry may be introduced, which could impact on the renormalization group flow[6].

It has been argued that Higgs fields contributions in self-adjoint representations, with diagonal mass matrices, will yield VeV's which are singlets under the gauge group[4]. In this case it would be crucial to have family mixing terms in the horizontal matrices to avoid such a reduction. Interestingly, the necessity of such Cabibbo angle mixing would be an important correspondence with phenomenology, related now to suppressing the horizontal gauge bosons.

To whatever extent family symmetries may be considered as solutions of the family replication problem it remains crucial to test these hypothesis within new conceptual frameworks. In this way we seek to find connections between diverse phenomena. Such was the motivation in this case. We found that family symmetry could indeed be incorporated but it is yet to be seen if this yields greater insight or if a more exotic replication scheme is required.

6 Acknowledgements

It is with much warmth that I thank V. Sivanathan for much crucial input to this work.

References

- [1] A. Connes and J. Lott, *Particle Models and Noncommutative Geometry*, Nucl. Phys. B (proc. suppl.) **18**, 29 (1991). See also R. Coquereaux, G. Espisito-Farese and G.

Vaillant, Nucl. Phys. B 353 (1991) 689; M. Dubois-Violette, R. Kerner & J. Madore, Phys. Lett. B 217 (1989), Class. Quant. Grav. 6 (1989) 1709.

- [2] A. Connes and J. Lott to appear in *Proceedings of 1991 Cargese Summer Conference*.
- [3] A. Connes, *Essay on Physics and Non-commutative Geometry* in *The Interface of Mathematics and Particle Physics*, Edited by D. Quillen, G. Segal and S. Tsou (Oxford University Press, 1990); *Non-commutative Differential Geometry*. Publ. Math IHES 62, 41 (1981).
- [4] T. Schücker and J. M. Zylinski, CNRS Preprint CPT-93/P.2960; B. Iochum and T. Schücker, CNRS Preprint CPT-93/P.2973.
- [5] A. H. Chamseddine, G. Felder, and J. Fröhlich, Phys. Lett. B 296, 109 (1993) and Nucl. Phys. B 395 (1993) 672, A. H. Chamseddine and J. Fröhlich, *SO(10) Unification in Non-commutative Geometry*, Zurich preprint ZU-TH-10/93, B.E. Hanlon and G. C. Joshi *A Partial Unification Model in Non-Commutative Geometry* UM-P-93/54, to appear in J. Geom and Physics, B. E. Hanlon and G. C. Joshi, Lett. Math. Phys. 27 (1993) 105.
- [6] E. Alvarez, J. M. Gracia-Bondia and C. P. Martin, Phys. Lett. B 306, 55 (1993), Universidad Autonoma Preprint FTUAM 94/2.