Bremsstrahlung in the gravitational field of a cosmic string

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ABSTRACT:

In the framework of QED we investigate the bremsstrahlung process for an electron passing by a straight static cosmic string. This process is precluded in empty Minkowski space-time by energy and momentum conservation laws. It happens in the presence of the cosmic string as a consequence of the conical structure of space, in spite of the flatness of the metric. The cross section and emitted electromagnetic energy are computed and analytic expressions are found for different energies of the incoming electron. The energy interval is divided in three parts depending on whether the energy is just above electron rest mass $M$, much larger than $M$, or exceeds $M/\delta$, with $\delta$ the string mass per unit length in Planck units. We compare our results with those of scalar QED and classical electrodynamics and also with conic pair production process computed earlier.

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1. Introduction

The bremsstrahlung process is a rather common process well known since earlier times of QED [1]. It happens usually when a charged particle changes its momentum in collision with obstacles such as other particles or due to an acceleration in electromagnetic fields. This quantum process has its transparent classical counterpart permitting to compare classical and QED calculations. Charged particles moving freely in flat space-time do not radiate.

A quite different situation occurs in curved space-time. As was shown in [2], a charged particle moving on geodesics does radiate. The radiation occurs not so much because of the non-zero curvature, but rather due to the falldown of the Huygens principle in curved space-time. It was manifested recently in [3] by direct calculation of the classical bremsstrahlung radiation of an electric charge in the gravitational field of a straight static cosmic string and also for scalar and gravitational radiation [4,5]. These authors have given analytic expressions for the total radiation energy emitted by classical point charge moving on geodesics of cosmic string space-time. The quantum bremsstrahlung process for a scalar field model was treated in details in [6,7]. The aim of this paper is to evaluate the total cross section as well as the radiation energy emitted by an electron passing by the cosmic string and to compare our results with those found early for classical and scalar quantum electrodynamics.

The space-time of a straight static cosmic string is locally flat except for the string itself where the Riemann tensor is concentrated. The metric around the string that lies along the z-axis reads, in cylindrical coordinates [8,9]:

$$ds^2 = dt^2 - d\rho^2 - \rho^2 d\theta^2 - dz^2.$$  \hspace{1cm} (1)

The metric is the same as in Minkowski space, but here the periodicity of the angular coordinate is within the range

$$0 \leq \theta \leq \frac{2\pi}{\nu}, \quad \text{with} \quad \nu = (1 - 4G\mu)^{-1}. $$  \hspace{1cm} (2)

$\mu$ is the mass per unit length of string, and $G$ is Newton's constant. The space-like sections around the string have the topology of a cone with the vertex at the core and with deficit angle $8\pi G\mu$. This dimensionless quantity measures the strength of the gravitational effects of the string.

Without any Newtonian gravitational field around, the string can produce physical effects in a hidden way, via the topological structure of the space-time as well as massive particles do this in 2+1 dimensional gravity [10,11,12]. It bears some resemblance with the Aharonov-Bohm effect [13] at least in its topological aspects. The close analogy between these effects, though they are of a different nature, was noticed by many authors [14,15,16,17,18,19]. Recently the Aharonov-Bohm interaction of fermions with the pure gauge potentials around cosmic strings has been shown to lead to their significantly large interaction with matter fields concentrated in the cosmic string core [16,17,18,19]. We notice that it does not happen due to gravitational interaction alone.
Classically, the gravitational effects of a straight cosmic string become manifest when two or more particles move along opposite sides of a string. Initially parallel trajectories converge as they move past the string, which acts as a gravitational lens \cite{20,8}. When moving thorough surrounded matter, a string produces wakes besides itself \cite{8}, and its relative motion introduces a steplike discontinuity in the observed microwave background temperature \cite{9,24}. Other classical effects may affect a single particle: the conical structure of the space-time induces a self-force, both gravitational as well as electrostatic, on a test particle around a string \cite{22,23}. Being dependent on the distance from the string it is the effect which leads to the classical bremsstrahlung radiation from the particle passing by the string \cite{3,4}. Also, a charged particle radiates when it suffers Aharonov-Bohm scattering \cite{24}. Among its quantum effects, a string polarizes the vacuum that surrounds it, in a way analogous to the Casimir effect between two plane conductors forming a wedge \cite{25,26,27,28}.

The conical structure of the cosmic string space-time prevents the invariance under translations in the plane perpendicular to the string and is the source of non-conservation for transverse momentum. It allows for another type of effects on quantum fields. Firstly it was noticed in \cite{29} for pair production in the context of a simplified model based on a scalar field theory. This process was analyzed in more details in \cite{30,31}. In the previous paper \cite{32} the cross section for pair production by a single photon in the cosmic string background was evaluated and some of its potential consequences were discussed in the framework of QED. An heuristic, semiclassical explanation for the mechanism underlying different quantum effects with free particles around the string follows. The vacuum is full of virtual particles which are continuously created and quickly disappear. A free particle is not able to make them real in empty Minkowski space-time, even if it has sufficient energy, since otherwise momentum conservation law would be violated. The presence of the string allows to take off the momentum excess and make them real if outgoing particles move along opposite sides of the string. This picture proves to be helpful to understand qualitatively some of the quantitative results for processes in study.

The paper is organized as follows: In section 2 we review briefly the Dirac and Maxwell equations in the cosmic string space-time with reference for details to the previous paper \cite{32}. In section 3 we evaluate the first order matrix element for the bremsstrahlung process in the cosmic string gravitational background. The analytic expressions for differential probability and partial cross section are found and analyzed. The radiated energy emitted by a free electron is computed in section 4 where it is also compared with the classical one. We present analytic approximations valid at different energy regimes. We conclude in section 5 with a discussion about the obtained results.
2. Dirac and Maxwell fields in the cosmic string space-time

The Dirac and Maxwell fields in the space-time of a straight cosmic string with the metric (1) have been treated in a previous paper [32]. In this section we review briefly their main properties and present in more details the physical states which describe transverse, physical photons.

The electron-positron field operator $\psi(x, t)$ obeys the free Dirac equation written in cylindrical coordinates for the cosmic string metric (1) as

$$\left(i \left[ \gamma^0 \partial_t + \gamma^3 \partial_z + \gamma^\rho (\partial_\rho - \frac{\nu}{2\rho}) + \gamma^\theta \partial_\theta \right] - M \right) \psi = 0 .$$

It differs from the usual one by the spin connection term $\gamma^\rho \frac{\nu}{2\rho}$. The equation (3) was investigated early in 2+1 dimensions, with two-component spinors, both in a conical space [33] as well as in the Aharonov-Bohm external field [16,17,33,19,34,35,36]. To specify the solution to the Dirac equation (3) we introduce a complete set of commuting operators

$$\hat{H}\psi = E\psi, \quad \hat{p}_3\psi = p_3\psi,$$

$$\hat{J}_3\psi = (-i\partial_\theta + \frac{\nu}{2} \Sigma_3)\psi = j_3\psi, \quad j_3 = \nu j, \quad j = l + \frac{1}{2},$$

$$\hat{S}_i\psi = s\psi, \quad \hat{\Sigma}_i = \frac{1}{\sqrt{E^2 - M^2}} \Sigma_i p_i.$$

and use their eigenvalues to label the quantum states of electrons. Here $\hat{H}, \hat{p}_3, \hat{J}_3$ and $\hat{\Sigma}_i$ correspond to energy, $z$-momentum, $z$-projection of total angular momentum and helicity operators, respectively.

We need for the following calculations the electron wave functions only. They are presented by the cylindrical modes:

$$\psi_\nu(j, x) = \frac{\sqrt{\nu}}{2\pi} N_\nu \exp(-iEt + ip_3z) \exp\left(i\frac{\pi}{2} |l| \right) \cdot \begin{pmatrix} u \\ v \end{pmatrix}$$

with the two-component spinors $u, v$ given by

$$u = \frac{1}{\sqrt{E - M}} \cdot \left( \frac{i\nu p_3}{p^+sp_3} J_{\alpha_+}(p_+\rho) \exp(i\nu l) \right)$$

$$v = \frac{1}{\sqrt{E + M}} \cdot \left( \frac{i\nu p_3}{p^+sp_3} J_{\alpha_+}(p_+\rho) \exp(i\nu(l + 1)) \right)$$

and the normalization constant $N_\nu = \frac{\sqrt{p^+p_3}}{2\sqrt{E}}$. The index of the Bessel function is $\alpha_+ = |\nu j \mp 1/2|$ and its argument depends on the transverse momentum $p_\perp = \sqrt{p^2 - p_3^2} =$
\[ \sqrt{E^2 - M^2 - p_3^2}, \quad \epsilon_l = \text{sign}(l) \] and \( s = \pm 1 \). We added a phase factor to (4) to simplify the calculations.

The normalization condition for these normal modes is

\[
\int dx \psi_\xi^\dag (j, x) \psi_\xi (j', x) = \delta_{j,j'} = \delta_{s,s'} \delta_l \nu \delta(p_3 - p'_3) \frac{\delta(p_\perp - p'_\perp)}{\sqrt{p_\perp p'_\perp}}. \tag{5}
\]

We denote collectively by \( j \) or \( j' \) the quantum numbers of a given state, and integration is also collectively denoted as

\[
\int d\mu_j = \sum_{l=-\infty}^{\infty} \int_{-\infty}^{\infty} dp_3 \int_0^\infty p_\perp dp_\perp.
\]

In order to make a comparison with classical and scalar calculations possible we will discuss these modes now. This can be done in analogy with the case of scalar quantum fields as it was developed in [6]. The modes (4) give a particle density at \( p_3 = 0 \),

\[
j^0(\rho) = \frac{\nu}{8\pi^2} [J_{\alpha-}^2 (p_\perp \rho) + J_{\alpha+}^2 (p_\perp \rho)],
\]

which behaves for \( j \nu \ll p_\perp \rho \) like \( j^0(\rho) \approx \nu/(4\pi^3 p_\perp \rho) \) and for \( j \nu \gg p_\perp \rho \) like \( (\nu/8\pi^2) \cdot (p_\perp \rho/2)^2 \Gamma^{-2}(j + 1) \). The transition between these regimes takes place around \( \rho_{\text{min}} = \nu j/p_\perp \), which is the radius of closest approach of a classical test particle of radial momentum \( p_\perp \) and \( z \)-angular momentum \( j \). For \( \rho_{\text{min}} < \rho \) the particle density is very small so that one may speak of a localized absence [6]. The classical counterpart of the modes (4) is the following: From all directions particles with the impact parameter \( \rho_{\text{min}} \) move towards the string. Due to the symmetry this leads to a zero net radial flux but nonvanishing \( \theta \)-flux. The particle density is then zero for \( \rho < \rho_{\text{min}} \) and \( j^0(\rho) \approx N_0(\nu E_\rho/\pi p_\perp \rho) \) for large distances where \( N_0 \) is the total number of particles. From this we can read off that the modes (4) correspond to a number of \( N_0 = 1/4\pi^2 E_\rho \) particles.

The Maxwell equations in the cosmic string space-time with metric (1), in the Lorentz gauge, take a decoupled form

\[
\Box \xi A_\xi = 0, \quad \text{with} \quad \Box \xi = \Delta \xi - \partial^2_{tt}, \tag{6}
\]

\[
\Delta \xi = \frac{1}{\rho} \partial_\rho (\rho \partial_\rho) - \frac{1}{\rho^2} L_3^2 + \partial^2_z, \quad L_3 = -i \partial_\theta + \xi \tag{7}
\]

for independent spin-weighted components of the vector potential [4],

\[
A_\xi = \frac{1}{\sqrt{2}} (A_\rho + \frac{i \xi}{\rho} A_\theta) \quad \text{if} \quad \xi = \pm 1 \quad \text{for} \quad A_z, A_t \quad \xi = 0. \tag{8}
\]

In these terms the Maxwell field operator is

\[
A_\xi (t, x) = \int d\mu_j \left( f_\xi (j, x) c_\xi + f^*_\xi (j, x) c_{-\xi} \right). \tag{9}
\]
where the coefficients $c_{\xi}(j)$, $c_{\xi}^\dagger(j)$ are annihilation and creation operators for a photon with quantum numbers $j = (k_\perp, k_3, m, \xi)$ with commutation relations

$$[c_{\xi}(j), c_{\xi}^\dagger(j')] = \delta_{j,j'}.$$  \hspace{1cm} (10)

The photon normal modes are given by

$$f_{\xi}(j, x) = \frac{\sqrt{\nu}}{2\pi} \exp(i\nu m\theta + ik_3 z) J_{\nu m + \xi}(k_\perp \rho) \exp \left( i \frac{\pi}{2} |m + \xi| \right) \frac{1}{\sqrt{2\omega_k}} \exp(-i\omega_k t), \hspace{1cm} (11)$$

$\xi$ being a polarization state. They are the eigenfunctions for the set of operators $\hat{k}_z$, $L_3 = i\hat{\partial}_\theta + \xi$ and are labeled by their eigenvalues $k_3$, $l_3 = \nu m + \xi$. Notice that we add a phase factor too. The modes (11) are normalized according to

$$\int dx f_{\xi}^* (j, x)(i \frac{\partial}{\partial t}) f_{\xi}(j', x) = \delta_{j,j'}.$$  \hspace{1cm} (12)

Now we discuss what the physical photon states are. To fix them we make use of the Lorentz gauge condition which translates into

$$\left( \frac{k_\perp}{\sqrt{2}} (c_+ + c_-) + k_3 c_3 - \omega_k c_0 \right) |\text{phys.state} >= 0.$$  \hspace{1cm} (13)

One can see that the operator

$$c_l = \frac{k_\perp c_+ + c_-}{\omega_k \sqrt{2}} + \frac{k_3}{\omega_k} c_3$$  \hspace{1cm} (14)

corresponds to a polarization vector directed along $\vec{k}$. It is the annihilation operator for longitudinal photons. With this operator the Lorentz condition (13) takes the simple form

$$(c_l - c_0) |\text{phys.state} >= 0$$  \hspace{1cm} (15)

where $c_0$ is the annihilation operator for scalar photons. Annihilation operators for transverse, physical photons can be defined up to a rotation around $\vec{k}$. We fix them as follows

$$c_\sigma = -i \frac{(c_+ - c_-)}{\sqrt{2}}, \quad c_\pi = -\frac{(c_+ + c_-)}{\sqrt{2}} \frac{k_3}{\omega_k} + c_3 \frac{k_\perp}{\omega_k}. \hspace{1cm} (16)$$

Notice that operators $c_l$, $c_\sigma$ and $c_\pi$ obey the same commutation relations (10).
3. Cross section for bremsstrahlung process $\epsilon \rightarrow \epsilon + \gamma$

3.1. Matrix elements for the bremsstrahlung process

In this section we evaluate the first order $S$-matrix element for the bremsstrahlung of an electron moving freely in the flat but conical space-time around the string. This was already done for a simplified model based on scalar QED [7]. Then we present the general expression for the partial cross section for the bremsstrahlung process.

The QED interaction Lagrangian is

$$L_{\text{int}} = -e \sqrt{-g} \bar{\psi}(x)A_\mu(x)\gamma^\mu(x)\psi(x)$$  \hspace{1cm} (17)

where

$$A_\mu(x)\gamma^\mu(x) = \sqrt{2} \left[ A_+ \exp(i\nu \theta)\gamma^+ + A_- \exp(-i\nu \theta)\gamma^- \right] + A_z(x)\gamma^3$$  \hspace{1cm} (18)

with

$$\gamma^\pm = \frac{1}{2} \left( \gamma^1 \mp i \gamma^2 \right) = \begin{pmatrix} 0 & \sigma^\pm \\ -\sigma^\pm & 0 \end{pmatrix}, \quad \gamma^3 = \begin{pmatrix} 0 & \sigma_3 \\ -\sigma_3 & 0 \end{pmatrix},$$

$$\sigma_+ = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad \sigma_- = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad \sigma_3 = \begin{pmatrix} 0 & 1 \\ 0 & -1 \end{pmatrix}$$  \hspace{1cm} (19)

Let an ingoing electron with quantum numbers $j_p = (p_\perp, p_3, l, s)$ emit a photon with quantum numbers $j_k = (k_\perp, k_3, m, \lambda)$ and an outgoing electron take quantum numbers $j_q = (q_\perp, q_3, n, r)$. The matrix element of this bremsstrahlung process for physical, transverse photon states $\lambda = \sigma, \pi$ can be calculated more easily in terms of matrix elements $M_\lambda$ for photon states with the polarization $\lambda = \pm, 3$,

$$M_\lambda = -i < j_q, j_k | S^{(1)} | j_p > = -e \int d^4 x \bar{\psi}_c(j_q, x) [c_\lambda, A_\mu(\gamma^\mu)] \psi_c(j_p, x)$$  \hspace{1cm} (20)

where

$$[c_\pm, A_\mu(\gamma^\mu)] = \sqrt{2} \gamma^\mp \exp(\mp i\nu \theta) f^*_\pm(j_k, x), \quad [c_3, A_\mu(\gamma^\mu)] = \gamma^3 f^*_3(j_k, x).$$

After a simple integration on $t, z$ and $\theta$ it becomes

$$M_\lambda = -e \sqrt{\gamma^\perp \gamma^3} \frac{\sqrt{p_\perp q_\perp}}{4 \sqrt{2\omega_k E_q E_p}} \exp(i\eta) \delta(E_p - E_q - \omega_k) \delta(p_3 - q_3 - k_3) \delta_l, m_+ n, m_\lambda$$  \hspace{1cm} (21)

where $\eta = i(\pi/2)(|l| - |n| - |m|)$ and

$$m_+ = \epsilon_l \epsilon_m s r \sqrt{2(p - s p_3)(q + r q_3)} R J_{+-},$$

$$m_- = \epsilon_n \epsilon_m s R \sqrt{2(p + s p_3)(q - r q_3)} R J_{+-},$$

$$m_3 = \left[ \sqrt{(p + s p_3)(q + r q_3)} R J_{-0} - \epsilon_l \epsilon_n \sqrt{2(p - s p_3)(q - r q_3)} s R J_{+0} \right]$$
with
\[ R = \frac{1}{\sqrt{E_p - M \sqrt{E_q + M}}} + \frac{\sqrt{s}}{\sqrt{E_q - M \sqrt{E_p + M}}} \]

Here we denote
\[ J(\alpha, \beta) = \int_0^\infty dp \rho J_{\alpha}(q \rho) J_{\beta}(k \rho) J_{\alpha + \beta}(p \rho) \]

and
\[ J_{\pm \pm} = J(\alpha_+, \beta_-), \quad J_{-+} = J(\alpha_-, \beta_+), \quad J_{+0} = J(\alpha_+, \beta_0), \quad J_{00} = J(\alpha_-, \beta_0) \]

where
\[ \alpha_\pm = \nu (n + \frac{1}{2}) \pm \frac{1}{2}, \quad \beta_\pm = \nu m \pm 1, \quad \beta_0 = \nu m. \]

The ingoing electron can emit the bremsstrahlung radiation when \( p_\perp > q_\perp + k_\perp \). For this case we obtain \(^{37}\)

\[ J(\alpha, \beta) = \frac{2}{\pi p_\perp^2 \cos(A + B) \cos(A - B)} d(\alpha, \beta), \]

\[ d(\alpha, \beta) = \Theta(-\alpha \beta) \sin[\pi \min(|\alpha|, |\beta|)] \left( \frac{\sin A}{\cos B} \right)^{|\alpha|} \left( \frac{\sin B}{\cos A} \right)^{|\beta|} \]

where
\[ q_\perp = p_\perp \sin A \cos B, \quad k_\perp = p_\perp \sin B \cos A. \]

We notice in (21) that energy as well as linear momentum along the string direction are of course conserved. The condition \( l = n + m \) is the conservation law for the total angular momentum projection along the string direction.

The matrix elements (21) contains the step function \( \Theta(-\alpha \beta) = \Theta(-n \cdot m) \) from (23). This is also the case for scalar field models \(^{29,6,30}\) and for pair production in conic QED \(^{32}\). The outgoing particles carry out total angular momentum projections of opposite signs. This common feature for quantum processes around cosmic strings can be explained in the framework of the semiclassical picture presented in section 1. Opposite signs for angular momentum projections means that virtual particles created from the vacuum move along opposite sides of the string. In this case they can give their momentum excess to the string and become real. The process is concentrated near the string, and this assumes a localization mechanism for quantum processes in the neighborhood of the string core, as it was discussed in \(^{6,7}\). Also, one can easily see that the matrix element (21) is zero for the photon state with \( m = 0 \).

From the matrix element (21) we evaluate the partial cross section per unit length of string for the bremsstrahlung process for the physical states \( \sigma, \pi \):

\[ d\sigma_\lambda = W_\lambda q_\perp dk_\perp dq_\perp dk_\perp dq_3 dk_3 \]

(24)
where

\[
W_\lambda = \frac{\nu e^2 pq}{32(2\pi)^2 \omega_k E_q E_p} \delta(E_p - E_q - \omega_k) \delta(p_3 - q_3 - k_3) \delta_{l, n+m} |m_\lambda|^2,
\]  

(25)

and \( \lambda = \sigma, \pi \),

\[
m_\sigma = -\frac{i}{\sqrt{2}}(m_+ - m_-), \quad m_\pi = -\frac{m_+ + m_-}{\sqrt{2}} \frac{k_3}{\omega_k} + m_3 \frac{k_\perp}{\omega_k},
\]

The expressions (24) and (25) describe the distributions of intensity for outgoing electron and photon over their quantum numbers, transverse and longitudinal momenta, angular momentum projections and polarizations.

3.2. Partial cross sections for the bremsstrahlung process

In particular, we are interested in the partial cross sections and radiated energy for the process in study. To evaluate them, we need to integrate over final states. Now we will assume that the ingoing electron moves perpendicular to the string, and we put \( p_3 = 0 \) to facilitate calculations. Due to the invariance of the cosmic string metric (1) under boost transformation along the string direction we can easily recover the general case.

Summing over polarizations for the outgoing electron and averaging over them for the ingoing electron yields

\[
\frac{1}{2} \sum_{\sigma, \pi} |m_\sigma|^2 = \frac{2}{pq} \left( (p_\perp^2 + q_\perp^2 - k_\perp^2)(J_{\perp+}^2 + J_{\perp-}^2) - 4\epsilon_\perp \epsilon_n p_\perp q_\perp J_{\perp+} J_{\perp-} \right),
\]

(26)

\[
\frac{1}{2} \sum_{\sigma, \pi} |m_\pi|^2 = \frac{2}{pq} \left( \frac{k_3^2}{\omega_k^2} \left( (p_\perp^2 + q_\perp^2 - k_\perp^2)(J_{\perp+}^2 + J_{\perp-}^2) \right. \right.
\]

\[
\left. + 4\epsilon_\perp \epsilon_n p_\perp q_\perp J_{\perp+} J_{\perp-} + 4\epsilon_\perp \epsilon_n p_\perp q_\perp J_{\perp+} J_{\perp-} \right) + \frac{k_\perp^2}{\omega_k^2} \left( (p_\perp^2 + q_\perp^2 - k_\perp^2)(J_{\perp+}^2 + J_{\perp-}^2) - 4\epsilon_\perp \epsilon_n p_\perp q_\perp J_{\perp+} J_{\perp-} \right) \}
\]

(27)

With the \( n, m \)-dependent part \( d(\alpha, \beta) \) of integral (23) we calculate the sums over angular momentum quantum numbers of outgoing particles, and we find (see Appendix):

\[
\sum_{n, m} \delta_{l,n+m}(d_{-+}^2 + d_{+-}^2) = \frac{1 + ab^2}{b\sqrt{a}} F_\nu(a, b), \quad \sum_{n, m} \delta_{l,n+m}(d_{-0}^2 + d_{+0}^2) = \frac{1 + a}{\sqrt{a}} F_\nu(a, b),
\]

\[
\sum_{n, m} \delta_{l,n+m} \epsilon_\perp \epsilon_n (d_{-+}^2 + d_{+-}^2) = \sum_{n, m} \delta_{l,n+m} \epsilon_\perp \epsilon_n (d_{-0}^2 + d_{+0}^2) = F_\nu(a, b),
\]

\[
\sum_{n, m} \delta_{l,n+m} \epsilon_\perp \epsilon_m (d_{-+} d_{-0} + d_{+-} d_{+0}) = \frac{1 + ab}{\sqrt{ab}} F_\nu(a, b)
\]

(28)
where

\[ F_\nu(a, b) = \sum_{l=0}^{\infty} \frac{a^{l+\frac{1}{2}}}{l!} + (ab)^{-\frac{1}{2}} \sum_{l=0}^{\infty} \frac{b^{l+\frac{1}{2}}}{l!}, \]

\[ \Sigma_1 = \sum_{m=1}^{\infty} \sin^2(\pi \nu m) (ab)^m = \frac{c(1 + c) \sin^2 \pi \nu}{(1 - c)[(1 - c)^2 + 4c \sin^2 \pi \nu]}, \]

\[ \Sigma_2 = \sum_{m=1}^{\infty} \cos^2 \pi \nu (m - \frac{1}{2}) (ab)^m = \frac{c[(1 - c)^2 \cos^2 \frac{\pi \nu}{2} + 2c \sin^2 \pi \nu]}{(1 - c)[(1 - c)^2 + 4c \sin^2 \pi \nu]} \]

and

\[ a = \frac{\sin^2 A}{\cos^2 B}, \quad b = \frac{\sin^2 B}{\cos^2 A}, \quad c = (ab)^\nu. \]

Inserting this in (25) we get

\[ \frac{1}{2} \sum_{r, n, m} W_\lambda = \frac{\nu e^2}{16\pi^4 \omega k E_q E_p} \frac{\delta(E_p - E_q - \omega_k) \delta(p_3 - q_3 - k_3)}{\left| p_1^2 - 2p_1^2(q_1^2 + k_1^2) + (q_1^2 - k_1^2) \right|} \]

\[ w_\lambda F_\nu(a, b); \]

where

\[ w_\sigma = (p_1^2 + q_1^2 - k_1^2 - 1 + \frac{ab^2}{b\sqrt{a}} - 4p_\perp q_\perp, \]

\[ w_\pi = (p_1^2 + q_1^2 - k_1^2 - 1 + \frac{ab^2}{b\sqrt{a}} + 4p_\perp q_\perp + 4p_\perp k_\perp - \frac{1 + ab}{\sqrt{ab}} + \frac{k_2^2}{\omega_k^2}((p_1^2 + q_1^2 - k_1^2)^{1+\alpha} + \frac{(p_1^2 + q_1^2 - k_1^2)^{1+\alpha} - 8p_\perp q_\perp - 4p_\perp k_\perp}{b\sqrt{a}}). \]

Next we need to integrate on \( dq_3dk_3 \). One can easily calculate for \( p_3 = 0 \)

\[ I(f) = \int_{-\infty}^{\infty} dq_3 \int_{-\infty}^{\infty} dk_3 \frac{f(\omega_k)}{E_q \omega_k} \delta(E_p - E_q - \omega_k) \delta(p_3 - q_3 - k_3) = I_0 f(p_\perp \omega) \]

where

\[ I_0 = \frac{4\Theta(s)}{p_1^2 \sqrt{s}}, \quad s = 1 - \frac{q_1^2 + k_1^2}{p_1^2} + \frac{(q_1^2 - k_1^2)^2}{p_1^2} - 4\epsilon \frac{k_2^2}{p_1^2} \]

and

\[ \omega = \frac{p_1^2 - q_1^2 + k_1^2}{2p_\perp \sqrt{p_\perp^2 + M^2}}, \quad \epsilon = \frac{M^2}{p_1^2}. \]

The final step would be the integration on \( q_\perp \) and \( k_\perp \), which can not be done analytically for arbitrary ingoing electron energy. Thus we now introduce a convenient parametrization that facilitates analytic approximations in different energy regimes.

The partial cross section is

\[ \sigma_\lambda^\nu = \frac{\nu e^2}{4\pi^4 E_p} \int q_\perp dq_\perp \int k_\perp dk_\perp \frac{\Theta(s) A^\lambda(p_\perp, q_\perp)}{\left| p_1^2 - 2p_1^2(q_1^2 + k_1^2) + (q_1^2 - k_1^2)^2 \right| \sqrt{s}} F_\nu(a, b) \]
where

\[
p_{\perp}^2 A^\nu(p_{\perp}, q_{\perp}) = \left( p_{\perp}^2 + q_{\perp}^2 - k_{\perp}^2 \right) \frac{1 + ab^2}{b\sqrt{a}} - 4 p_{\perp} q_{\perp},
\]

\[
p_{\perp}^2 A^\nu(p_{\perp}, q_{\perp}) = \left( p_{\perp}^2 + q_{\perp}^2 - k_{\perp}^2 \right) \frac{1 + ab^2}{b\sqrt{a}} + 4 p_{\perp} q_{\perp} + 4 p_{\perp} k_{\perp} \frac{1 + ab}{\sqrt{ab}}
\]

\[
+ \frac{4k_{\perp}^2(p_{\perp}^2 + M^2)}{(p_{\perp}^2 - q_{\perp}^2 + k_{\perp}^2)^2} \left[ (p_{\perp}^2 + q_{\perp}^2 - k_{\perp}^2) \frac{1 + ab}{\sqrt{a}} - (p_{\perp}^2 + q_{\perp}^2 - k_{\perp}^2) \frac{1 + ab}{\sqrt{a}} \right] - 8 p_{\perp} q_{\perp} - 4 p_{\perp} k_{\perp} \frac{1 + ab}{\sqrt{a}}.
\]

The convenient variables to the problem are \( \omega \) and \( x \),

\[
\omega = \frac{p_{\perp}^2 - q_{\perp}^2 + k_{\perp}^2}{2p_{\perp}\sqrt{p_{\perp}^2 + M^2}}, \quad x = \frac{2k_{\perp}\sqrt{p_{\perp}^2 + M^2}}{p_{\perp}^2 - q_{\perp}^2 + k_{\perp}^2}.
\]

Here \( p_{\perp} \omega \) is the photon energy \( \omega_k \) and \( x = \sin \theta_k \) where \( \theta_k \) is the angle between the photon momentum vector and the string direction. In these variables we write the final closed expression for the partial cross section

\[
\sigma^\lambda = \frac{\nu e^2}{32\pi^4 E_p} \cos^2 \frac{\pi \nu}{2} \int_0^1 dx \int_0^{\omega_{\text{max}}} d\omega \frac{x}{\omega(1 - x^2 v^2)^{1/2}} B^\lambda(x, \omega) f_\nu(a, b)
\]

where \( \omega_{\text{max}} = v \left( 1 + \sqrt{1 - x^2 v^2} \right)^{-1} \), \( v = \frac{p_{\perp}}{\sqrt{p_{\perp}^2 + M^2}} \) is the velocity of the ingoing electron and

\[
B^\sigma(x, \omega) = \frac{4(1 - x^2 v^2)}{x^2 \sqrt{1 - 2\frac{v}{2} + \omega^2 x^2}} \left[ 2(1 - \frac{\omega}{v}) + \omega^2 x^2 \right],
\]

\[
B^\tau(x, \omega) = \frac{4(1 - x^2 v^2)}{x^2 \sqrt{1 - 2\frac{v}{2} + \omega^2 x^2}} \left[ 2(1 - \frac{\omega}{v})(1 - x^2) + \omega^2 x^2 \right]
\]

\[
+ 8(1 - x^2) \frac{1 - \frac{\omega}{v}}{\sqrt{1 - 2\frac{v}{2} + \omega^2 x^2}},
\]

\[
f_\nu(a, b) = \frac{c}{(1 - c)(1 - c)^2 + 4c \sin^2 \frac{\pi \nu}{2}} \left\{ 4(1 + c) a^{\nu |+\frac{1}{2}|} \sin^2 \frac{\pi \nu}{2}
\]

\[
+ \frac{1}{\sqrt{c}} b^{\nu |+\frac{1}{2}|} \left[ (1 - c)^2 + 8c \sin^2 \frac{\pi \nu}{2} \right] \right\}
\]

with

\[
c = (ab)^\nu, \quad a = \frac{v - \omega(1 + \sqrt{1 - x^2 v^2})}{v - \omega(1 - \sqrt{1 - x^2 v^2})}, \quad b = \frac{1 - \sqrt{1 - x^2 v^2}}{1 + \sqrt{1 - x^2 v^2}}.
\]

The expression (36) describes the energy and angular distributions for the intensity of the bremsstrahlung radiation. It allows us to analyze correlations between the energy and the direction of radiation. We extract the factor \( \cos^2 \frac{\pi \nu}{2} \) from \( F_\nu(a, b) \) to stress that the partial
cross section vanishes, as it should be, for \( \nu = 1 \), when there is no deficit angle. But it is different from the analogous factor \( \sin^2 \pi \nu \) that appears in the case of scalar particles. This difference may be understood as the influence of the spin connection on the Dirac equation in curved metrics.

We now analyze the expression (36) under different approximations. We will always consider the realistic case of GUT cosmic strings

\[
\nu \approx 1 + \delta, \quad \text{with} \quad \delta = 4G\mu \ll 1. \tag{39}
\]

Since \( \delta \) is of order of the mass per unit length in Planck units, it is reasonable to assume it to be small (for instance it is of order \( 10^{-6} \) for GUT cosmic strings).

Under this approximation the partial cross section becomes

\[
\sigma_i^\lambda = \frac{e^2 \delta^2}{128\pi^2 E_p} \int_0^1 dx \int_0^{\omega_{\text{max}}} d\omega \frac{x}{\omega(1-x^2v^2)\sqrt{1-x^2}} B^\lambda(x, \omega)f_\nu(a, b) \tag{40}
\]

with

\[
f_1(a, b) = \frac{ab}{(1-ab)[(1-ab)^2 + 4ab\pi^2 \delta^2]}, \quad \left\{ 4(1+ab) d^{l+\frac{1}{2}} + \frac{1}{\sqrt{ab}} d^{l+\frac{1}{2}} \right\} [(1-ab)^2 + 8ab]. \tag{41}
\]

### 3.3. Approximations at different energy regimes

Now we discuss the behavior of the partial cross section (40) at low, high and ultrahigh energies. The low energy case means \( v \to 0 \). At high energies there exist two different regimes. For one of them, when \( 1 \ll \gamma \ll 1/\delta \) where \( \gamma \) is the Lorentz factor \( \gamma = (1-v^2)^{-1/2} \), we can drop the term with \( \delta^2 \) in denominator (41). On the contrary, at ultrahigh energies, \( \gamma \gg 1/\delta \), this term becomes dominant.

We first consider the low energy case. At \( v \to 0 \) the interval of integration over \( \omega \) is very small since \( \omega_{\text{max}} \approx v/2 \ll 1 \). Then we have

\[
a \approx (1 - 2\frac{\omega}{v}), \quad b \approx \frac{x^2v^2}{4} \ll 1, \quad ab \ll 1,
\]

\[
B^\sigma \approx \frac{8(1 - \frac{\omega}{v})}{\sqrt{1 - 2\frac{\omega}{v}}} \frac{1}{x^2}, \quad B^\pi \approx \frac{8(1 - \frac{\omega}{v})}{\sqrt{1 - 2\frac{\omega}{v}}} \frac{1 - x^2}{x^2}, \tag{42}
\]

\[
f_1(a, b) \approx (ab) \left( 4a^{l+\frac{1}{2}} + \frac{1}{\sqrt{ab}} b^{l+\frac{1}{2}} \right).
\]
One can see that the second term in $f_{1}(a, b)$ contributes at $l = 0$ only. At $l > 0$ we find

$$
\sigma_{\lambda}^{\gamma} = \frac{e^{2} \delta_{2} v^{2}}{16 \pi^{2} M} \int_{0}^{1} dx \frac{(1 - x^{2})}{(1 - x^{2} v^{2})^{\frac{1}{2}} - x^{2}} \int_{0}^{\omega_{\min}} d\omega \frac{(1 - \omega) v^{2}}{\omega},
$$

$$
\sigma_{\pi}^{\gamma} = 3 \sigma_{\pi}^{\gamma} = \frac{e^{2} \delta_{2} v^{2}}{32 \pi^{2} M} \left[ \psi(1) - \psi(l + 2) - \ln \eta_{\min} - \frac{1}{2(l + 2)} \right],
$$

$$
\sigma_{\pi + \pi}^{\gamma} = \frac{e^{2} \delta_{2} v^{2}}{24 \pi^{2} M} \left[ \psi(1) - \psi(l + 2) - \ln \eta_{\min} - \frac{1}{2(l + 2)} \right],
$$

where $\psi(x)$ is the psi-function. We cut off the infrared divergent integral at low frequency $\omega_{\min} = v^{2} \eta_{\min}/2$. An infrared singularity of the same type arises in the first radiative correction to $S$-matrix elements for scattering of an electron by the cosmic string. These singular terms cancel each other in the electron scattering cross section with the emission of a number of soft photons. We will not discuss this interesting problem now, and we plan to return to it for detailed investigation in future. We are interested here in the energy behavior of the bremsstrahlung process only. Note that the partial cross sections increase quadratically in velocity independently on values of angular momentum projection $l$. This universal energy behavior differs in this case from pair the production one and can be naturally explained by the fact that there is no energy threshold for the bremsstrahlung process.

At high energy, $\gamma \gg 1$, (and $v \approx 1$) both $a$ and $b \approx 1$, $B^{\sigma}$, $B^{\pi} \approx (1 - x^{2} v^{2})$ and

$$
f_{1}(a, b) \approx \frac{16}{(1 - ab)((1 - ab)^{2} + 4 \pi^{2} \delta^{2})}
$$

with

$$
(1 - ab) \approx \frac{2 \sqrt{1 - x^{2} v^{2}}}{1 - \omega}.
$$

It means that the main contribution in the integrals arises from values of $x \approx 1$.

In this case one needs to be careful in finding the asymptotic behavior. We dropped factors $a^l$, $b^l$ considering $l$ is finite. It is necessary to take them into account at infinite $l$ if one is interested in the classical consideration of bremsstrahlung process.

We still need to distinguish two different regimes at high energy: $1 \ll \gamma \ll 1/\delta$ and $\gamma \gg 1/\delta$. The term between square brackets in the denominator of (44) plays different roles whether $(1 - ab)$ can be smaller than $\delta$ or not. If the electron energy is not too high, $\gamma \ll 1/\delta$, we can neglect the term proportional $\delta^{2}$ and then we find

$$
\sigma_{\lambda}^{\gamma} \sim \frac{e^{2} \delta^{2}}{M \gamma},
$$

Finally, at ultrahigh electron energy, $\gamma \gg 1/\delta$, the term proportional $\delta^{2}$ dominates and then we obtain

$$
\sigma_{\lambda}^{\gamma} \sim \frac{e^{2}}{M \gamma} \ln \gamma.
$$
Notice that in both high energy regimes the cross section does not depend on the electron quantum number $l$ that determines its angular momentum projection along the string axis. This is actually only true up to some large value of $l$, of order $p_\perp / M$, after which the approximations we were making break down, and the cross section decreases with $l$. An heuristic, semiclassical explanation for this property may arise from the following observation. The classical analog of the cylindrical modes with given $l$ may be imagined as a flux of particles incident upon the string from all directions with radius of closest approach of order $r_{\text{min}} \approx l/p_\perp$. [6,7] This will be smaller than the Compton wavelength $1/M$ of outgoing particles if $l < p_\perp / M$. In that case virtual outgoing particles that move along opposite sides of the string, and thus are able to extract momentum from it, will get hit by the electron. All values of $l$ smaller than $p_\perp / M$ should then be equally efficient at bremsstrahlung, while larger values should be less efficient. This seems to be a common feature for all quantum processes around cosmic strings.

The obtained energy dependence for partial cross section should not continue unbounded, and actually a cutoff is expected in a more realistic model, with the conical singularity at the string core smoothed out. Also, we believe that the perturbation theory will break down at ultrahigh energies when second approximation can exceed the first one.

A striking feature is the independence of the cross section at ultrahigh energies on $\delta$, it being a measure of the conical deficit angle. This fact can also arouse a suspicion that the perturbation theory will break down at ultrahigh energies. Again, an heuristic explanation may be found in the fact that the string transfers momentum to ingoing particles of order $\delta p_\perp$, and it is an effective parameter which determines the intensity of quantum processes around the cosmic string [8]. At ultrahigh energy, when $p_\perp \gg M/\delta$, this effective parameter becomes independent on the value of $\delta$.

4. The radiated energy

4.1. The partial radiated energy

It has been shown early [8] that a charged particle moving uniformly in the gravitational field of the cosmic string produces electromagnetic radiation. In the framework of classical electrodynamics the radiated energy reads [3,4]

$$E_{\text{rad}} = - \int d^4 x \sqrt{-g} j^\nu(x) A^\nu_{\text{rad}}$$

$$= \frac{e^2 \delta^2}{64 \rho_{\text{min}}} \gamma^2 \left[ v(6v^2 - 1) + \gamma(4v^2 + 1) \arctan(v \gamma) \right]$$

(47)

Quantum field theoretical calculation of bremsstrahlung process for scalar electrodynamics was made in [7].

The averaged energy per unit length of the string carried off by a photon with polarization $\lambda$ can be calculated by formula

$$E_{\text{rad}}^\lambda = \int \omega_k d\omega^\lambda.$$
Here the differential probability for the bremsstrahlung process for states that correspond to 1 particle is given by \( dw = 4\pi^2 E_p d\sigma \). Thus we get for the radiated energy of one electron moving in the cosmic string space-time

\[
E_{\text{rad}}^\lambda = 4\pi^2 E_p \int \omega_k d\sigma^\lambda.
\]  
(48)

After integration over \( dq_3 dk_3 \) the photon energy \( \omega_k \) is replaced by its effective value (35),

\[
p_{\perp} \omega = \frac{p_{\perp}^2 - q_{\perp}^2 + k_{\perp}^2}{2\sqrt{p_{\perp}^2 + M^2}}.
\]

So we have for the radiated energy

\[
E_{\text{rad}}^\lambda = \frac{\nu e^2 p_{\perp} \cos \frac{\pi \nu}{2}}{8\pi^2} \int_0^1 dx \int_0^{\omega_{\text{max}}} d\omega \frac{x}{(1 - x^2 v^2)^{\nu/2}} B^\lambda(x, \omega) f_{1\nu}(a, b)
\]  
(49)

or, in GUT approximation,

\[
E_{\text{rad}}^\lambda = \frac{e^2 \delta^2 p_{\perp}}{32} \int_0^1 dx \int_0^{\omega_{\text{max}}} d\omega \frac{x}{(1 - x^2 v^2)^{\nu/2}} B^\lambda(x, \omega) f_1(a, b).
\]  
(50)

At low electron energies, \( v \to 0 \) one can easily obtain the expression for the radiated energy from (43)

\[
E_{\text{rad}}^\sigma = 3E_{\text{rad}}^\pi = \frac{e^2 \delta^2 M v^4}{16} \frac{2l + 5}{(l + 2)(l + 3)},
\]

\[
E_{\text{rad}}^{\sigma+\pi} = \frac{e^2 \delta^2 M v^4}{12} \frac{2l + 5}{(l + 2)(l + 3)}
\]  
(51)

To compare with results of [3, 4] for classical electrodynamics we use the radius of closest approach [6] \( \rho_{\text{min}} = l/p_{\perp} \) and find from (51) at large \( l \)

\[
E_{\text{rad}}^\sigma = 3E_{\text{rad}}^\pi = \frac{e^2 \delta^2 v^3}{8} \frac{1}{\rho_{\text{min}}},
\]

\[
E_{\text{rad}}^{\sigma+\pi} = \frac{e^2 \delta^2 v^3}{6} \frac{1}{\rho_{\text{min}}},
\]

(52)

At not too high electron energy, \( 1 \ll \gamma \ll 1/\delta \), the radiated energy increases rapidly with electron energy,

\[
E_{\text{rad}}^\lambda \sim e^2 \delta^2 M \gamma^3,
\]  
(53)

and then, at ultrahigh energy, \( \gamma \gg 1/\delta \), it goes asymptotically to

\[
E_{\text{rad}}^\lambda \sim e^2 M \gamma \ln \gamma.
\]  
(54)

In both high energy regimes the radiated energy does not depend on the electron angular momentum quantum number \( l \). It is valid till not extremely large fixed values \( l \).
4.2. Classical radiated energy

In previous sections we have considered the bremsstrahlung process from quantum electron in the eigenstate with given energy and fixed angular momentum quantum number $j$. It is quite possible to derive the known expression (47) for radiated energy from the general quantum formula (49). To do this it is necessary to consider the situation when the Compton wave length of the electron is much less than the radius of closest approach, $1/M \ll \rho_{\text{min}} = v|j|/p_\perp$, or $v|j| \gg v\gamma$. Under this approximation the main contribution to the integral (49) will be determined by values of variables with $a, b \approx 1$. It happens if $\omega \approx 0$ and in this case we can neglect $\delta |j|$. At small $\omega$ one can put $a \approx 1 - 2\omega \sqrt{1 - x^2 v^2}$,

$$B^\sigma \approx \frac{8(1-x^2v^2)}{x^2}, \quad B^\pi \approx \frac{8(1-x^2)}{x^2},$$

$$f_1(a, b) \approx \frac{x^2 v^2 \delta |j|}{\sqrt{1 - x^2 v^2 [1 - x^2 v^2 + \pi^2 \delta^2 x^2 v^2]}}$$

and

$$a |j| = \exp(|j| \ln a) \approx \exp \left( - |j| \frac{2\omega}{v} \sqrt{1 - x^2 v^2} \right)$$

at $j \to \infty$. Then we obtain as final expression for the classical radiated energy

$$E_{\text{rad}}^{\sigma+\pi} = \frac{e^2 \delta^2 v^3}{8\rho_{\text{min}}} \int_0^1 dx \frac{x[2 - x^2(1 + v^2)]}{\sqrt{1 - x^2 (1 - x^2 v^2)^2 [1 - x^2 v^2 + \pi^2 \delta^2 x^2 v^2]}}$$

which interpolates between different energy regimes. At not too high electron energy, $1 \ll \gamma \ll 1/\delta$ we find the known result (47),

$$E_{\text{rad}} = \frac{e^2 \delta^2}{64\rho_{\text{min}}} \gamma^2 \left[ v(6v^2 - 1) + \gamma(4v^2 + 1) \arctg(v\gamma) \right].$$

At low energy we find from here

$$E_{\text{rad}} \approx \frac{e^2 \delta^2 v^3}{6\rho_{\text{min}}}$$

and at high energy

$$E_{\text{rad}} \approx \frac{5\pi e^2 \delta^2 \gamma^3}{128\rho_{\text{min}}}.$$ 

Taking into account the term with $\delta^2$ we obtain the radiated energy at ultrahigh electron energy

$$E_{\text{rad}} \approx \frac{3e^2 \gamma}{32\pi \rho_{\text{min}}}.$$
5. CONCLUSIONS

We have shown that the lack of global linear momentum conservation in the plane perpendicular to a cosmic string, which is a consequence of its conical topology, permits bremsstrahlung from a single electron passing by the cosmic string as well as others quantum processes although there is no local gravitational field. Expressions (40), (43), (45), (46) for the cross sections of the bremsstrahlung process and (50), (51), (53), (54) and (55) for emitted energy contain our quantitative results for this process at different energy regimes and for alternative quantum states of the ingoing electron and emitted photon.

Previous results of a similar nature were already obtained for a simplified model based on scalar fields [29,31,6,7,30]. Their extension to QED, though, turned out to be not so straightforward, both in its technical details as well as in the energy dependence of the resulting cross sections. The scalar model was based on a Lagrangian \( \lambda \phi \psi^2 \), with \( \lambda \) being a coupling constant with the dimension of mass, \( \phi \) being a massless and \( \psi \) a massive scalar field. The QED result corresponds to the scalar case with \( \lambda \) replaced by \( e p_\perp \). In the high energy regime this makes a significant difference.

The quantum processes for particles in the space-time of a cosmic string can be regarded as a consequence of a kind of gravitational analog of the Aharonov-Bohm effect. We mention that there is also a proper Aharonov-Bohm interaction of fermions with the gauge potential around a cosmic string, in models where the string carries non-integer fluxes [16,33,19].

It is very striking that the cross sections for all quantum processes for scalar as well as for electromagnetic fields at ultrahigh energies, eq. (46), are independent on the angular momentum quantum number \( l \) and on the cosmic string gravitational parameter \( \delta \). This has two consequences: Firstly, it means that the cross section of the process for “plane wave” electron states [29] will have quite different high energy behavior than the partial cross sections have. This behavior should be cut off at energies that probe the core of the string, \( p_\perp \approx \sqrt{\mu} \), where the metric is not well approximated by that of eq. (1), which has a conical singularity at the origin. A real cosmic string has a smooth core, and its metric approaches a flat, regular metric at the origin. The metric is that of a snub-nosed cone [35,39]. It was shown, for instance, that the \( 1/\rho \) self-force that a charged particle experiences in the conical space-time around a string [22,23] is cut off at a distance of order the core radius in the snub-nosed cone metric [40]. The falldown of the thin-tube approximation was noticed in [41]. Secondly, the disappearance of the gravitational parameter from expressions for cross sections of processes allows to question about the validness of the perturbation theory for ultrahigh energies. This makes the calculation of second order approximations for quantum processes around cosmic strings desirable.
Appendix

We calculate here the sums over angular momentum quantum numbers of outgoing particles (28). Let us to bring the expression (23) for \( d(\alpha, \beta) \) to a more suitable form

\[
d(\alpha, \beta) = \Theta(-\alpha \beta) \sin[\pi \min(|\alpha|, |\beta|)] x^{\alpha} y^{\beta} \\
= \Theta(\alpha + \beta) \left[ -\Theta(-\beta) \sin(\pi \beta) x^{\alpha} y^{-\beta} - \Theta(-\alpha) \sin(\pi \alpha) x^{-\alpha} y^{\beta} \right] \\
+ \Theta(-\alpha - \beta) \left[ \Theta(\alpha) \sin(\pi \alpha) x^{\alpha} y^{-\beta} + \Theta(\beta) \sin(\pi \beta) x^{-\alpha} y^{\beta} \right],
\]

where we denote \( x = \sqrt{a} = \sin A/\cos B, \ y = \sqrt{b} = \sin B/\cos A. \)

For all possible combinations \( \alpha_\pm \) and \( \beta_\pm \) from (22) the inequality \( \alpha + \beta > 0 \) means \( l \geq 0 \) and \( \alpha + \beta < 0 \) means \( l \leq 0 \). We use \( \Theta(\pm \beta) \) and \( \Theta(\pm \alpha) \) to set limits for \( m \) and \( n \), respectively.

Then we obtain

\[
d_{-+} = d(\alpha_-, \beta_+) \\
= \Theta(l \geq 0) \left[ \Theta(m \leq -1) \sin \pi \nu m (xy)^{-\nu m} x^{\nu(l+\frac{1}{2})} y^{-\frac{1}{2}} \\
+ \Theta(n \leq -1) \cos \pi \nu (n + \frac{1}{2}) (xy)^{-\nu n} (xy)^{-\frac{1}{2}} y^{\nu(l+\frac{1}{2})} x^{\frac{1}{2}} \right] \\
+ \Theta(l \leq -1) \left[ -\Theta(n \geq 0) \cos \pi \nu (n + \frac{1}{2}) (xy)^{\nu n} (xy)^{-\frac{1}{2}} y^{-\nu(l+\frac{1}{2})} x^{-\frac{1}{2}} y^{-1} \\
- \Theta(m \geq 1) \sin \pi \nu m (xy)^{\nu m} x^{-\nu(l+\frac{1}{2})} y^{-\frac{1}{2}} \right],
\]

\[
d_{+-} = d(\alpha_+, \beta_-) \\
= \Theta(l \geq 0) \left[ \Theta(m \leq -1) \sin \pi \nu m (xy)^{-\nu m} x^{\nu(l+\frac{1}{2})} y^{-\frac{1}{2}} \\
- \Theta(n \leq -1) \cos \pi \nu (n + \frac{1}{2}) (xy)^{-\nu n} (xy)^{-\frac{1}{2}} y^{\nu(l+\frac{1}{2})} x^{-\frac{1}{2}} y^{-1} \right] \\
+ \Theta(l \leq -1) \left[ \Theta(n \geq 0) \cos \pi \nu (n + \frac{1}{2}) (xy)^{\nu n} (xy)^{-\frac{1}{2}} y^{-\nu(l+\frac{1}{2})} x^{-\frac{1}{2}} y^{-1} \\
- \Theta(m \geq 1) \sin \pi \nu m (xy)^{\nu m} x^{-\nu(l+\frac{1}{2})} y^{-\frac{1}{2}} \right],
\]

\[
d_{-0} = d(\alpha_-, \beta_0) \\
= \Theta(l \geq 0) \left[ -\Theta(m \leq -1) \sin \pi \nu m (xy)^{-\nu m} x^{\nu(l+\frac{1}{2})} y^{-\frac{1}{2}} \\
+ \Theta(n \leq -1) \cos \pi \nu (n + \frac{1}{2}) (xy)^{-\nu n} (xy)^{-\frac{1}{2}} y^{\nu(l+\frac{1}{2})} x^{\frac{1}{2}} \right] \\
+ \Theta(l \leq -1) \left[ -\Theta(n \geq 0) \cos \pi \nu (n + \frac{1}{2}) (xy)^{\nu n} (xy)^{-\frac{1}{2}} y^{-\nu(l+\frac{1}{2})} x^{-\frac{1}{2}} y^{-1} \\
+ \Theta(m \geq 1) \sin \pi \nu m (xy)^{\nu m} x^{-\nu(l+\frac{1}{2})} x^{\frac{1}{2}} y \right].
\]
\[ d_{+0} = d(\alpha_+, \beta_0) \]
\[ = \Theta(l \geq 0) \left[ -\Theta(m \leq -1) \sin \pi \nu m (xy)^{-\nu m} x^{\nu(l + \frac{1}{2})} + y^{\frac{1}{2}} \right] 
\[ - \Theta(n \leq -1) \cos \pi \nu (n + \frac{1}{2}) (xy)^{-\nu n} (xy)^{\frac{1}{2}} x^{\nu(l + \frac{1}{2})} y^{\frac{1}{2}} \] 
\[ + \Theta(l \leq -1) \left[ \Theta(n \geq 0) \cos \pi \nu (n + \frac{1}{2}) (xy)^{\nu n} (xy)^{\frac{1}{2}} y^{\nu(l + \frac{1}{2})} x^{\frac{1}{2}} \right. \]
\[ + \Theta(m \geq 1) \sin \pi \nu m (xy)^{-\nu m} x^{-\nu(l + \frac{1}{2})} y^{\frac{1}{2}} \right] . \]

Now we can easily compute the sums (28). We obtain, for example,
\[
\sum_{n, m} \delta_{l, n+m}(d_{-+}^2 + d_{++}^2) = 
\sum_{n, m} \delta_{l, n+m} \left\{ \Theta(l \geq 0) \left[ \Theta(m \leq -1) \sin^2 \pi \nu m (ab)^{-\nu m} a^{\nu(l + \frac{1}{2})} \left( \frac{1}{b \sqrt{a}} + b \sqrt{a} \right) \right. \right.
\[ + \Theta(n \leq -1) \cos^2 \pi \nu (n + \frac{1}{2}) (ab)^{-\nu n} (ab)^{-\frac{1}{2}} b^{\nu(l + \frac{1}{2})} \left( \frac{1}{b \sqrt{a}} + b \sqrt{a} \right) \right] 
\[ + \Theta(l \leq -1) \left[ \Theta(n \geq 0) \cos^2 \pi \nu (n + \frac{1}{2}) (ab)^{\nu n} (ab)^{-\frac{1}{2}} b^{-\nu(l + \frac{1}{2})} \left( \frac{1}{b \sqrt{a}} + b \sqrt{a} \right) \right. \right. \]
\[ + \Theta(m \geq 1) \sin^2 \pi \nu m (ab)^{-\nu m} a^{-\nu(l + \frac{1}{2})} \left( \frac{1}{b \sqrt{a}} + b \sqrt{a} \right) \right] \right\} . \]

The sums over \( n \) and \( m \) are computed independently and we obtain the first line in (28). The other sums are calculated analogously.

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