Functional Relations in Solvable Lattice Models I: Functional Relations and Representation Theory

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Abstract. We study a system of functional relations among a commuting family of row-to-row transfer matrices in solvable lattice models. The role of exact sequences of the finite dimensional quantum group modules is clarified. We find a curious phenomenon that the solutions of those functional relations also solve the so-called thermodynamic Bethe ansatz equations in the high temperature limit for $sl(r+1)$ models. Based on this observation, we propose possible functional relations for models associated with all the simple Lie algebras. We show that these functional relations certainly fulfill strong constraints coming from the fusion procedure analysis. The application to the calculations of physical quantities will be presented in the subsequent publication.

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1. Introduction

Studies of statistical systems on lattice greatly concern the spectra of transfer matrices. Solvable lattice models in two dimensions are especially interesting examples where one can actually compute them exactly [1]. At the heart of such a solvability resides, of course, the Yang-Baxter equation (YBE), which implies the local Boltzmann weights (or microscopic interactions) are governed by a certain symmetry algebra. The relevant algebras are quantum groups such as the quantized affine algebras $U_q(\hat{g})$ [2,3] or their rational degenerations, Yangian $Y(g)$ [2,4] and their central extensions. Then a fundamental question would be: how does the symmetry algebra at such a microscopic level control the transfer matrix spectra which is macroscopic?

The purpose of the present paper (Part I) and the subsequent one (Part II) [5] is to develop an approach for obtaining the spectra that resolves the above issue and possesses the applicability no less wide than the Bethe ansatz. It is based on the functional relations (FRs) among the row-to-row transfer matrices. We shall propose a family of FRs for the solvable lattice models associated to any simple Lie algebra $X_r$. It is a systematic extension of the earlier examples for $X_r = sl(2)$ [1, 6-12], $sl(n)$ [13] and $o(n)$ [14] cases. The main idea in these works was to exploit the FRs with a few additional information on the analyticity to determine the spectra itself. We shall show that this is indeed possible also with our FRs and derive correlation lengths of vertex models and central charges of critical restricted solid-on-solid (RSOS) models. (Part II). The calculations generalize the earlier results [10-12], demonstrating the efficiency of our FRs.

The origin of those FRs lies directly in the symmetry algebra and its finite dimensional representations (FDRs). To sketch it roughly, consider the family of solvable vertex models generated from the fundamental rational $R$-matrices through the fusion procedure [15]. The symmetry algebra in this case is a central extension $\tilde{Y}(X_r)$ of the Yangian $Y(X_r)$, which is generated by the $L$-operators [4]. Take any irreducible FDR (IFDR) $W$ and fix one FDR $H$ of the $\tilde{Y}(X_r)$. Then one builds, by definition, a row-to-row transfer matrix $T_W \in \text{End} H$ as the trace of some operator (monodromy matrix) over the auxiliary space $W$ corresponding to the periodic boundary condition. The $T_W$ itself acts on the quantum space $H$, and carries the fusion type labeled by $W$. The transfer matrices $T_{W_i}, T_{W_i}, \ldots \in \text{End} H$ so obtained for various IFDRs are commutative due to the fusion $R$-matrix $\in \text{Hom}(W_i \otimes W_j, W_j \otimes W_i)$. Now consider the product $T_{W_i} T_{W_i} \in \text{End} H$, which
by construction is again a trace of a certain operator over $W_0 \otimes W_1$. Thus, if there exists an exact sequence of the $\tilde{\mathcal{Y}}(X_r)$-modules, say,

$$0 \to W_0 \otimes W_1 \to W_2 \otimes W_3 \to W_4 \otimes W_5 \to 0,$$

(1.1)

connected by $\tilde{\mathcal{Y}}(X_r)$-homomorphisms, it follows that the FR

$$0 = T_{W_5}T_{W_1} - T_{W_2}T_{W_3} + T_{W_4}T_{W_5} \in \text{End} H$$

(1.2)

must hold for any choice of $H$. By regarding $T_{W_i}$'s as the eigenvalues, (1.2) exhibits how the “microscopic data” (1.1) controls the spectra. As an example, for $X_r = sl(2)$, the FR in [12]

$$T_m(u - \frac{1}{2})T_m(u + \frac{1}{2}) = T_{m+1}(u)T_{m-1}(u) + g_m(u)\text{Id.}$$

(1.3)

can be reproduced in this manner and its representation theoretical background is thereby clarified. In (1.3), $u$ denotes the spectral parameter that naturally enters IFDRs $W_i$ of $\tilde{\mathcal{Y}}(X_r)$, $T_m(u)$ stands for the transfer matrix for $m$-fold fusion model, and $g_m(u)$ is a scalar function satisfying $g_m(u + \frac{1}{2})g_m(u - \frac{1}{2}) = g_{m+1}(u)g_{m-1}(u)$ and specified unambiguously from $H$. See (2.14) for the explicit formula. We shall call the FR among the commuting family of the transfer matrices like (1.3) as the $T$-system in this paper. To fully work out such $T$-system however requires the knowledge as (1.1) on a family of IFDRs of the symmetry algebra and we have only been able to derive it rigorously for $X_r = A_r$ so far. Thus, it appeared a challenge to proceed further to postulate a general form of the $T$-system for all the classical simple Lie algebras $X_r$. Nevertheless, we seem to manage it by a blend of ideas from another ingredient, namely, the thermodynamic Bethe ansatz (TBA) as we shall see below.

The subject has a long history going back to [16] but there was a renewed interest recently both in the lattice model context [7,9,13,17] and the perturbed conformal field theory (CFT) context [18–20] as some fascinating structures were emerging. Let us recall two of them here exclusively for the case $X_r = sl(2)$ for simplicity. The first structure is the universal form of the so called TBA equation in the high temperature limit [20–23]:

$$Y_m(u - \frac{1}{2})Y_m(u + \frac{1}{2}) = (1 + Y_{m+1}(u))(1 + Y_{m-1}(u)),$$

(1.4)
where the function \( Y_m(u) \) corresponds to the ratio of \( m \)-string and hole density functions in the context of lattice models [17,22]. We call systems such as (1.4) the \( Y \)-system borrowing the naming in [23]. To see the second structure, recall the dilogarithm identity [7,24]

\[
\frac{6}{\pi^2} \sum_{m=1}^{\ell} L\left( \frac{1}{Q_m^2} \right) = \frac{3\ell}{\ell + 2},
\]

where \( L(\cdot) \) is the Rogers dilogarithm and \( \ell \) is any positive integer. The rhs is the well known central charge of the level \( \ell \) \( \hat{sl}(2) \) WZW model [25]. The quantity \( Q_m \) is given by

\[
Q_m = \frac{\sin \left( \frac{(m+1)\pi}{\ell+2} \right)}{\sin \frac{\pi}{\ell+2}},
\]

which is the character of the \( m + 1 \)-dimensional representation of \( sl(2) \) specialized at a rational point. The second structure relevant for us is the following character identity among them [24,26]:

\[
Q_m^2 = Q_{m+1} Q_{m-1} + 1.
\]

We call such relations among the characters as the \( Q \)-system. Having these structures at hand, we make the crucial observation; the \( T \)-system (1.3) is the Yang-Baxterization (i.e., spectral parameter dependent version) of the \( Q \)-system (1.7) such that the combination

\[
y_m(u) = \frac{T_{m+1}(u)T_{m-1}(u)}{g_m(u)}
\]

solves the \( Y \)-system (1.4) as follows:

\[
y_m(u) - \frac{1}{2} y_m(u + \frac{1}{2}) = \frac{T_{m+1}(u - \frac{1}{2})T_{m+1}(u + \frac{1}{2})T_{m-1}(u - \frac{1}{2})T_{m-1}(u + \frac{1}{2})}{g_m(u - \frac{1}{2})g_m(u + \frac{1}{2})} = \frac{(T_{m+2}(u)T_m(u) + g_{m+1}(u))(T_{m-2}(u)T_m(u) + g_{m-1}(u))}{g_m(u - \frac{1}{2})g_m(u + \frac{1}{2})} = (y_{m+1}(u) + 1)(y_{m-1}(u) + 1).
\]

Though the meaning of this phenomenon is yet to be clarified, it opens a route to guess a general form of the \( T \)-system. Namely, “Yang-Baxterize the \( Q \)-system so as to solve the \( Y \)-system”, which can be tested because these systems are now explicitly known for all \( X_r \)'s in [26] and [22], respectively. See section 3 and Appendix B for details. We have found that there certainly exist such a Yang-Baxterization as listed in (3.20). Moreover, it is essentially unique for each \( X_r \) up to some freedom responsible for the arbitrariness of choosing \( H \) and elementary redefinitions of the transfer matrices.

We must then ask whether the \( T \)-system so obtained fits the representation theoretical scheme as (1.1-3). It turns out that the existence of exact sequences like (1.1) imposes
strong constraints for a possible form of the $T$-system. Based on reasonable assumptions on the fusion procedure, we will show that our solution (3.20) indeed satisfies those constraints.

Supported by these backgrounds we propose them as the $T$-system for $X_r$ (main result of Part I). Its unrestricted and restricted versions (see section 2) are to hold for the vertex and the RSOS type solvable models, respectively. Part II will be devoted to further studies where we apply the $T$-system to those models and recover the known results or even generalize them. It is our hope that further studies unveil a deep structure connecting $T$, $Q$ and $Y$-systems and lead to a greater understanding of the related subjects, representation theories of quantum groups, dilogarithm identities, TBA, analytic Bethe ansatz and so forth.

The outline of Part I is as follows. In section 2, we review the simplest example of the transfer matrix FRs for $sl(2)$ related models, and explain the connection to the exact sequences of the quantum group. We also include a proof of the $T$-system for $X_r = A_r$ and thereby establish the same relation among $Q$, $T$ and $Y$-systems as the $sl(2)$ case discussed above. This turns out to be a simple exercise using the FRs in [13]. In section 3, we propose the $T$-system for every $X_r$ from the condition that it solves the $Y$-system as explained above. In section 4, we describe how our $T$-system is consistent with the representation theoretical viewpoint. Due to the lack of the knowledge of the fusion procedure, however, we leave the justification of the cases $E_7$, $E_8$ as a future problem. Section 5 presents interesting conjectures on certain determinant formulas in our $T$-system for $X_r = B_r$, $C_r$ and $D_r$. In section 6, we give a summary and discussion. Appendices A and B provide additional information on the $Q$ and $Y$-systems, respectively. The former contains a few new results on the special values of the quantity $Q_m^{(a)}$.

Before closing the introduction, let us include one remark. Most of our arguments will be given upon the Yangian symmetry $Y(X_r)$ hence they are relevant to the models associated with rational $R$ matrices. However for $X_r = A_r$, they are also valid in trigonometric and even elliptic cases. We suppose this to be true for all the other $X_r$’s as well. Despite the absence of quasi-classical elliptic $R$-matrices for $X_r \neq A_r$ [27], the elliptic SOS model will still exist [28,29] in general, on which we expect our $T$-system.

2. Functional Relations of Transfer Matrices

This is a warm-up section for getting familiar with the FRs and some background ideas. We will exclusively take examples from solvable models related to $A_r$. 


2.1. A simplest example

We begin by reviewing an extremely simple case of the McGuire-Yang $R$-matrix [30,31]

$$\hat{R}(u) = 1 + uP \in \text{End}(W \otimes W), \quad W = \mathbb{C}^2,$$

(2.1)

where $u$ is the spectral parameter and $P$ is the transposition operator $P(x \otimes y) = y \otimes x$. It satisfies the YBE

$$(1 \otimes \hat{R}(u))(\hat{R}(u + v) \otimes 1)(1 \otimes \hat{R}(v)) = (1 \otimes \hat{R}(v))(\hat{R}(u + v))(1 \otimes \hat{R}(u) \otimes 1),$$

(2.2)

and defines a solvable 2-state model (a rational limit of the 6-vertex model) on a 2D square lattice in the usual way. We depict the $\hat{R}(u)$ graphically as

$$\begin{array}{c}
\includegraphics[width=0.2\textwidth]{monodromy_matrix}
\end{array}$$

wherein the assigned little arc is also specifying that it acts from the SW to the NE direction. The $R$-matrix (2.1) has the spectral decomposition

$$\hat{R}(u) = (1 + u)P_2 \oplus (1 - u)P_0,$$

(2.3)

where $P_{1\pm 1}$ are the projectors $\frac{1}{2}\hat{R}(\pm 1)$ onto the spaces of symmetric and antisymmetric tensors. Regarding $W$ as the simplest non-trivial IFDR $W_1$ of $sl(2)$, we put $W_{1\pm 1} = \text{Im}P_{1\pm 1}$. Thus $W_1 \otimes W_1 = W_2 \oplus W_0$ as the $sl(2)$-module ($\dim W_m = m + 1$) and $\hat{R}(u)$ reduces to the projectors on the irreducible components at its singularities $u = \pm 1$.

Consider the (homogeneous) transfer matrix $T_1(u) \in \text{End}(W^\otimes N)$ defined graphically as

$$\begin{array}{c}
\includegraphics[width=0.35\textwidth]{transfer_matrix}
\end{array}$$

where the sum is implied for each internal horizontal edge as usual. The object inside the trace is the so called monodromy matrix: $W_1 \otimes W^\otimes N \to W^\otimes N \otimes W_1$ and the trace is over the auxiliary space $W_1 = W$ to account for the periodic boundary condition. The suffix 1 in $T_1(u)$ is to signify this fact, namely, its fusion degree is 1. The $R$-matrix $\hat{R}(u - v)$ assures the
commutativity \([T_1(u), T_1(v)] = 0\) as is well known [1]. Take the product \(T_1(u - \frac{1}{2}) T_1(u + \frac{1}{2})\)
corresponding to the picture

\[
\text{Tr}_{W_1 \otimes W_1} \left( \begin{array}{c}
\begin{array}{c}
\hline
\vdots \\
\hline
u+1/2
\end{array}
\begin{array}{c}
\hline
\vdots \\
\hline
u-1/2
\end{array}
\end{array} \right),
\]

which is a trace over \(W_1 \otimes W_1\). By inserting the operator \(\text{Id}_{W_1 \otimes W_1} = \frac{1}{2} \tilde{R}(1) + \frac{1}{2} \tilde{R}(-1)\) at
the left end of the monodromy matrix, this is equal to

\[
\text{Tr}_{W_1 \otimes W_1} \left( \begin{array}{c}
\begin{array}{c}
\hline
\vdots \\
\hline
u+1/2
\end{array}
\begin{array}{c}
\hline
\vdots \\
\hline
u-1/2
\end{array}
\end{array} \right) + \text{Tr}_{W_1 \otimes W_1} \left( \begin{array}{c}
\begin{array}{c}
\hline
\vdots \\
\hline
u+1/2
\end{array}
\begin{array}{c}
\hline
\vdots \\
\hline
u-1/2
\end{array}
\end{array} \right),
\]
due to the cyclicity of the trace. With the aid of the YBE (2.1), the inserted pieces can
now be slid to any position. By remembering that they are the projectors onto \(W_{1 \pm 1}\), the
rhs can be written as \(\text{Tr}_{W_2}(\cdots) + \text{Tr}_{W_0}(\cdots)\). Obviously the two terms are the transfer
matrices with fusion degrees 2 and 0, with the latter being a scalar matrix. Thus we get

\[
T_1(u + \frac{1}{2}) T_1(u - \frac{1}{2}) = T_2(u) + (u - \frac{1}{2})^N (u + \frac{3}{2})^N \text{Id},
\]

(2.4)

wherein the last factor is an easy exercise. The commutativity \([T_m(u), T_{m'}(v)] = 0\) again
holds owing to the fusion \(R\)-matrices [15]. See Fig. 1.

Figure 1. The fusion \(R\)-matrices of the fusion degrees \((m, m') = (2, 1), (1, 2), (2, 2)\) (from
the left to the right) along with the projectors.
Eq. (2.4) is the simplest example of the (rational limit of) FR due to many authors [6–9]. There are already a lot of lessons gained from it. (i) The decomposition (2.4) looks analogous to the classical Clebsh-Gordan rule $W_1 \otimes W_1 = W_2 \oplus W_0$ but carries the spectral parameter $u$ non-trivially. This is a typical situation showing that the problem is in fact relevant to FDRs of the Yangian $Y(sl(2))$ and its central extension rather than $sl(2)$ itself. We will come to this point more later. (ii) The shift $u_0 = \pm 1$ in $T_1(u \mp \frac{1}{2})T_1(u \mp \frac{1}{2} + u_0)$ had to be a singularity of the $R$-matrix $\det R(u_0) = 0$ where it reduces to projectors onto invariant subspaces in $W_1 \otimes W_1$. Nothing interesting would have happened for generic shifts $u_0$. (iii) The essential structure of the FR is governed by the fusion procedure for the $R$-matrix. It uses the singularities at $u = \pm u_0$ and automatically guarantees $[T_m(u), T_{m'}(v)] = 0$. (iv) The fusion in question concerns the $R$-matrix acting on the auxiliary spaces $W_1, W_2$, etc and not the quantum space $W^\otimes N$ which enters (2.4) rather trivially only through the factor $(u - \frac{1}{2})^N(u + \frac{3}{2})^N$.

One may proceed further to directly work out the FRs for the transfer matrices with higher fusion degrees. Such calculations are indeed possible [7,9,12] leading to the forthcoming result (2.14) we want. However we will seek, in the next subsection, a more intrinsic understanding from the representation theoretical viewpoint of our symmetry algebra, Yangian $Y(sl(2))$ and its central extension. As it turns out, it provides a natural framework in which all the observations (i)–(iv) can be understood most elegantly. (Parallel argument holds for $U_q(s\hat{l}(2))$ hence for the trigonometric and even elliptic cases.)

2.2. Representation theoretical viewpoint

Yangian and its central extension

The Yangian $Y(X_r)$ is an important class of quantum groups [2] such that to every its finite dimensional representation is associated a rational solution of the YBE (2.1). It is generated by the elements $\{x, J(x) \mid x \in X_r\}$ under certain commutation relations and contains the universal enveloping algebra $U(X_r)$ as a subalgebra. Let us recall a few results [2,32,33] on its FDR theory for $X_r = sl(2)$, which are related to our FRs.

Fact 1 ([33] Proposition 2.5). There is an algebra homomorphism $\phi_u : Y(sl(2)) \to U(sl(2))$ depending on a complex parameter $u$ such that $\phi_u(x) = x$, $\phi_u(J(x)) = ux$. 

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Suppose \( \rho \) is a FDR of \( sl(2) \), i.e., \( \rho : U(sl(2)) \to \text{End}W \). From the Fact 1, one can then form a one parameter family of FDRs of \( Y(sl(2)) \) via the composition

\[
Y(sl(2)) \xrightarrow{\rho} U(sl(2)) \xrightarrow{\rho} \text{End}W.
\]  

(2.5)

The resulting \( Y(sl(2)) \)-module will be denoted by \( W(u) \) and called the evaluation representation. Let \( W_m \) be the \((m+1)\)-dimensional irreducible \( sl(2) \)-module and \( W_m(u) \) be the associated evaluation representation which is also irreducible. Remarkably, every IFDR of \( Y(sl(2)) \) is known to be isomorphic to \( W_{m_1}(u_1) \otimes \cdots \otimes W_{m_k}(u_k) \) for some choice of \( m_i \in \mathbb{Z}_{\geq 0} \) and \( u_i \in \mathbb{C} \) [33]. Note however that this statement is not asserting that such tensor products are always irreducible. In fact one has

**Fact 2** ([33] Corollary 4.7, Proposition 4.2, Section 5). The tensor product \( W_m(u) \otimes W_{m'}(v) \) is reducible if and only if

\[
|u - v| = \frac{m + m'}{2} - j + 1
\]

(2.6)

for some \( 1 \leq j \leq \min(m, m') \). For all the other values of \( u \) and \( v \) than (2.6), the IFDRs \( W_m(u) \otimes W_{m'}(v) \) and \( W_{m'}(v) \otimes W_m(u) \) are isomorphic. The isomorphism is given by the unique intertwiner \( \tilde{R}_{m,m'}(u - v) \in \text{Hom}_{Y(sl(2))}(W_m(u) \otimes W_{m'}(v), W_{m'}(v) \otimes W_m(u)) \) which agrees with the rational fusion \( R \)-matrices in [15].

There is a precise description of the reducibility content in the case (2.6) as follows.

**Fact 3** ([33] Proposition 4.5). When (2.6), there are following exact sequences of the \( Y(sl(2)) \)-modules:

\[
0 \to W_{m-j}(u + \frac{j}{2}) \otimes W_{m'-j}(v - \frac{j}{2}) \to W_m(u) \otimes W_{m'}(v) \\
\to W_{j-1}(u - \frac{m-j+1}{2}) \otimes W_{m+m'-j+1}(v + \frac{m-j+1}{2}) \to 0
\]

(2.7a)

for \( u - v = \frac{m+m'}{2} - j + 1 \).

\[
0 \to W_{j-1}(u + \frac{m-j+1}{2}) \otimes W_{m+m'-j+1}(v - \frac{m-j+1}{2}) \to W_m(u) \otimes W_{m'}(v) \\
\to W_{m-j}(u - \frac{j}{2}) \otimes W_{m'-j}(v + \frac{j}{2}) \to 0
\]

(2.7b)

for \( v - u = \frac{m+m'}{2} - j + 1 \).
The first spaces of these sequences are the proper subrepresentations of $W_m(u) \otimes W_{m'}(v)$ and (2.7) may be viewed as a quantum version (or Yang-Baxterization) of the classical Clebsh-Gordan rule. Note however that the reducible space $W_m(u) \otimes W_{m'}(v)$ is no longer decomposable here in general, which is a distinct feature from the classical case.

To discuss the FRs, we actually need a central extension $\hat{Y}$ of $Y(sl(2))$ [4], which is most naturally seen in the alternative (and original) realization of the latter using $L$-operators. See the lecture note [34] and the references therein for further information. Let $L^{(k)}_{ij}$ be noncommutative elements, $(1 \leq i, j \leq 2, k \geq 1)$, and let

$$L_{ij}(u) = \delta_{ij} + \sum_{k \geq 1} L^{(k)}_{ij} u^{-k},$$

$$L(u) = \sum_{i,j=1}^{2} L_{ij}(u) E_{ij},$$

where $E_{ij} \in \text{End} W$ is $(E_{ij})_{k\ell} = \delta_{ik} \delta_{j\ell}$ for a fixed basis of $W$. Consider the associative noncommutative algebra $\hat{Y}$ generated by the elements $L^{(k)}_{ij}$ with the following relations

$$\hat{R}(u) (L(u + v) \otimes 1)(1 \otimes L(v)) = (L(v) \otimes 1)(1 \otimes L(u + v)) \hat{R}(u).$$

Then,

**Fact 4.** $\hat{Y}$ is a Hopf algebra in a weak sense without the antipode, and $Y(sl(2))$ is isomorphic to the quotient of $\hat{Y}$ by the ideal generated by the central element $\Delta(u)$ called the *quantum determinant* [2]. It is possible to construct all the IFDR of $\hat{Y}$ using the elementary $R$-matrix (2.1) as a building block [32].

Any IFDR of $\hat{Y}$ is naturally viewed as an IFDR of $Y(sl(2))$. On the other hand, a lift of a given IFDR of $Y(sl(2))$ to an IFDR of $\hat{Y}$ is always possible, though not unique. Thus, it will be possible to lift the exact sequences (2.7) to exact sequences of $\hat{Y}$-modules which are described completely in terms of the elementary $R$-matrix. The FRs are consequences of the latter exact sequences. Below we shall explain this lifting procedure in detail.

**IFDRs of $\hat{Y}$**

To describe IFDRs of $\hat{Y}$, it is convenient to assign a spectral parameter to each line of the $R$-matrix (2.1) which determines its argument by the following rule:

\[
\begin{array}{c}
\begin{array}{ccc}
\text{ } & u_1 & u_2 \\
\downarrow & & \\
\text{ } & u_1-u_2 & \\
\end{array}
\end{array}
\]
We introduce the IFDR of $\tilde{Y}$ on the space $W_m \subset W^\otimes m$ [15]. Namely, the projection operator $W^\otimes m \to W_m$ is given by the product of the $R$-matrix as

$$u+(m-1)/2 \quad \cdots \quad u-(m-1)/2$$
$$u+(m-3)/2 \quad \cdots \quad u-(m-3)/2$$
$$u-(m-3)/2$$
$$u-(m-1)/2$$

We remind the reader that operators here are considered to act from the left to the right space. The action (representation) of the generator $L_{ij}(v)$ (see (2.8)) is defined by the action of the operator

$$u-(m-1)/2$$
$$u-(m-3)/2$$
$$\vdots$$
$$u+(m-3)/2$$
$$u+(m-1)/2$$

on $W_m$, where the top and the bottom states are fixed as basis vectors of $W$ corresponding to $i$ and $j$.\footnote{This defines only the $\tilde{Y}$-action on the auxiliary space. The $\tilde{Y}$-action on the quantum space is defined similarly by reflecting the above diagrams along the SW-NE diagonal line.} Since this IFDR is isomorphic to $W_m(u)$ as a $Y(sl(2))$ representation, we also write it as $W_m(u)$. Keeping these definitions in mind, we also use the graphical abbreviation of the fusion $R$-matrix $\tilde{R}_{m,m'}(u-v)$ as

$$W_m(u) \quad \begin{array}{c} {u-v} \\ {u-v} \\ \vdots \\ u-(m-3)/2 \\ u-(m-1)/2 \end{array} \quad W_{m'}(v)$$

The tensor product $W_m(u) \otimes W_{m'}(v)$ of IFDRs is naturally defined from (2.9) by aligning the spaces $W_m(u)$ and $W_{m'}(v)$ from the top to the bottom (resp. from the right to the left) for the $\tilde{Y}$-action on the auxiliary (resp. quantum) space.

Set the quantum space as $H = W_{m_1}(v_1) \otimes \cdots \otimes W_{m_N}(v_N)$, where the choice of $m_i$’s and $v_i$’s is arbitrary. It is natural to define the transfer matrix $T_m(u) \in \text{End}H$ with
fusion degree $m$ by $T_m(u) = \text{Tr}_{W_m(u)}L$, where the monodromy matrix $L : W_m(u) \otimes H \to H \otimes W_m(u)$ is defined by

$$
W_m(u) \xrightarrow{\text{fusion}} \xrightarrow{H} W_m(u)
$$

$$
\begin{array}{c}
\xrightarrow{u-v_1} \cdots \xrightarrow{u-v_2} \xrightarrow{u-v_W} W_m(v_0) \quad W_m(v_1) \quad W_m(v_2)
\end{array}
$$

We have abbreviated the quantum space $H$ by a single wavy line. The commutativity $[T_m(u), T_{m'}(v)] = 0$ is valid owing to $\check{R}_{m,m'}(u)$. The situation (2.6) corresponds to a singularity of the intertwiner $\check{R}_{m,m'}(u - v)$ hence it is intimately connected to our FRs as noted in the lessons (ii) and (iii) previously.

**Exact sequences of $\tilde{Y}$-modules**

We consider a lift of the exact sequence (2.7b) to an exact sequence of $\tilde{Y}$-modules in the case $m' = j = m$ in the following way:

$$
0 \longrightarrow \mathcal{A}_1 \xrightarrow{\varepsilon_1} \mathcal{A}_0 \xrightarrow{\varepsilon_0} \mathcal{A}_{-1} \longrightarrow 0,
$$

$$
\mathcal{A}_1 = W_{m-1}(u) \otimes W_{m+1}(u),
$$

$$
\mathcal{A}_0 = W_m(u - \frac{1}{2}) \otimes W_m(u + \frac{1}{2}).
$$

(2.10)

One must be careful for the third space $\mathcal{A}_{-1}$. As a $Y(sl(2))$ representation it is trivial, but as a $\tilde{Y}$ representation it is not necessary so because the center $\Delta(u)$ may act non-trivially. In order to define $\mathcal{A}_{-1}$, it is necessary to introduce an additional IFDR $\bar{W}_0(u)$ of $\tilde{Y}$ on the antisymmetric subspace in $W^{\otimes 2}$. As a $Y(sl(2))$ representation, $\bar{W}_0(u)$ is trivial. The projection operator for $\bar{W}_0(u)$, together with the specification of the spectral parameters on the lines, is given by

$$
\begin{array}{c}
\xrightarrow{u-1/2} \xrightarrow{u+1/2}^{-1}
\end{array}
$$

Then, $\mathcal{A}_{-1}$ is defined by

$$
\mathcal{A}_{-1} = \bar{W}_0(u - \frac{m-1}{2}) \otimes \bar{W}_0(u - \frac{m-3}{2}) \otimes \cdots \otimes \bar{W}_0(u + \frac{m-1}{2}).
$$

(2.11)
The homomorphisms $\varepsilon_i$'s are given by successive multiplications of the $R$-matrices. The following example in the case $m = 3$

clarifies the construction of $\varepsilon$'s for a general $m$ as well, where in the figure for $\varepsilon_1$ straight lines mean the identity operators on the corresponding tensor components, and $\varepsilon_0$ is defined as the leading coefficient of the operator $\varepsilon_0(z)$ in the expansion $\varepsilon_0(z) = z^{m-1}\varepsilon_0 + \sigma(z^{m-1})$. (See Lemma A.1 and Proposition A.2 of [35], and also [36].)

Let us check that the above constructions of $\varepsilon_i$'s indeed have desired properties. First, the fact that $\varepsilon_i$'s are $\mathcal{Y}$-homomorphisms (i.e., commute with $\mathcal{Y}$-actions) is a consequence of the YBE. For example, for $\varepsilon_1$ the statement is equivalent to the following graphical equality:

Second, to see the exactness we first note that $A_1$ and $A_{-1}$ are irreducible by (2.6). Thus, by Schur’s lemma, if $\varepsilon_i$'s are nonvanishing, then they should coincide with those in (2.7b) as $\mathcal{Y}(\mathfrak{sl}(2))$-homomorphisms, from which the exactness follows. To check the nonvanishment is straightforward. Thus, we conclude that (2.10–11) is a $\mathcal{Y}$ lift of (2.7b).

**Functional relations**

To derive the FR from (2.10), we introduce the composite monodromy matrices $L^{(i)}$, $i = 0, \pm 1$ defined by the figures

$$
L^{(1)} = \begin{array}{c}
W_{m-1}(u) \\
W_{m+1}(u)
\end{array}
\begin{array}{c}
W_m(u+1/2) \\
H
\end{array}
\begin{array}{c}
E_1 \\
W_m(u-1/2)
\end{array}
\begin{array}{c}
W_m(u-1/2) \\
W_m(u+1/2)
\end{array} = \begin{array}{c}
W_{m-1}(u) \\
W_{m+1}(u)
\end{array}
\begin{array}{c}
W_m(u+1/2) \\
H
\end{array}
\begin{array}{c}
E_1 \\
W_m(u-1/2)
\end{array}
\begin{array}{c}
W_m(u-1/2) \\
W_m(u+1/2)
\end{array}

$$

$$
L^{(0)} = \begin{array}{c}
W_{m-1}(u) \\
W_{m+1}(u)
\end{array}
\begin{array}{c}
W_m(u+1/2) \\
H
\end{array}
\begin{array}{c}
W_m(u+1/2) \\
W_m(u-1/2)
\end{array} = \begin{array}{c}
W_{m-1}(u) \\
W_{m+1}(u)
\end{array}
\begin{array}{c}
W_m(u+1/2) \\
H
\end{array}
\begin{array}{c}
W_m(u+1/2) \\
W_m(u-1/2)
\end{array}

$$

$$
L^{(-1)} = \begin{array}{c}
\tilde{W}_0(u-(m-1)/2) \\
\tilde{W}_0(u+(m-1)/2)
\end{array} = \begin{array}{c}
\tilde{W}_0(u-(m-1)/2) \\
\tilde{W}_0(u+(m-1)/2)
\end{array}
$$

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wherein the arrangement of the horizontal lines for $L^{(i)}$ corresponds to the auxiliary space $A_i$ in (2.10-11) hence $L^{(i)} : A_i \otimes H \to H \otimes A_i$. Next we fix a basis $\{\alpha, \beta, \ldots\}$ of $H$ and consider the operators $L^{(i)}_{\alpha \beta} \in \text{End}A_i$. For fixed $\alpha$ and $\beta$, $L^{(i)}_{\alpha \beta}$ are the representations of a common element of $Y$ on each space $A_i$. Therefore, the exact sequence (2.10) of $Y$-modules implies the commutative diagram

$$
\begin{array}{cccc}
0 & \longrightarrow & A_1 & \xrightarrow{\varepsilon_1} & A_0 & \xrightarrow{\varepsilon_0} & A_{-1} & \longrightarrow & 0 \\
\downarrow L^{(1)}_{\alpha \beta} & & \downarrow L^{(0)}_{\alpha \beta} & & \downarrow L^{(-1)}_{\alpha \beta} & & & \downarrow L^{(-1)}_{\alpha \beta} & & \downarrow L^{(-1)}_{\alpha \beta} \\
0 & \longrightarrow & A_1 & \xrightarrow{\varepsilon_1} & A_0 & \xrightarrow{\varepsilon_0} & A_{-1} & \longrightarrow & 0
\end{array}
$$

(2.12)

From the exactness of (2.12) it follows that

$$
0 = \sum_{i=0;\pm 1} (-)^i \text{Tr}_{A_i}(L_{\alpha \beta}^{(i)}),
$$

(2.13)

This is nothing but the $(\alpha, \beta)$-matrix element of the FR $(m \geq 1)$:

$$
0 = T_{m-1}(u)T_{m+1}(u) - T_m(u - \frac{1}{2})T_m(u + \frac{1}{2}) + g_m(u)\text{Id}, \quad (2.14a)
$$

$$
g_m(u) = \text{Tr}_{A_{-1}}(L_{\alpha \alpha}^{(-1)}) \quad (\text{for any } \alpha) \quad (2.14b)
$$

$$
= \prod_{k=1}^N \left( \prod_{i=0}^{m_k-1} \prod_{j=0}^{m_k-1} g(u_j - v_{k,i}) \right), \quad (2.14c)
$$

$$
g(u) = (u - \frac{1}{2})(u + \frac{3}{2}), \quad (2.14d)
$$

$$
u_j = u - \frac{m - 1}{2} + j, \quad v_{k,i} = v_k - \frac{m_k - 1}{2} + i, \quad (2.14e)
$$

where we define $T_0(u) = 1$. This is the $T$-system we have mentioned in (1.3), which generalizes (2.4). Note that from (2.14b) and (2.11) one has

$$
g_m(u + \frac{1}{2})g_m(u - \frac{1}{2}) = g_{m+1}(u)g_{m-1}(u) \quad (2.15)
$$

irrespective of the choice of the quantum space $H$.

Although (2.14) is derived here for the rational vertex models, it is equally valid for the trigonometric and even the elliptic vertex models [37] if $g(u)$ in (2.14d) is replaced by its analogues in those functions. This is because the $R$-matrices in the trigonometric and elliptic cases still have the common features in their singularities and fusion procedures so that analogous construction of the sequences like (2.12) remains valid. For the trigonometric case, a parallel result to (2.7) for $U_q(\widehat{sl}(2))$ is also known as Proposition 4.9 in [38].

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The trigonometric and elliptic versions of the $T$-system (2.14) also apply to the critical and off-critical fusion RSOS models [39], respectively. In the trigonometric case, this follows in principle from the relation of these models to the $U_q(\widehat{sl}(2))$ $R$-matrix at $q = \exp(\frac{2\pi i}{\ell + \frac{1}{4}})$ [40]. Here $\ell \in \mathbb{Z}_{\geq 1}$ denotes the level which is a characteristic label of the RSOS models. There is however one fundamental difference between the $T$-systems for the vertex and the level $\ell$ RSOS model. In the former, the index $m$ in (2.14) runs over the whole set of positive integers, while in the latter, the truncation

$$T_{\ell+1}(u) = 0$$

occurs hence (2.14) closes among finitely many $T_m(u)$’s with $1 \leq m \leq \ell$. We shall call these $T$-systems the unrestricted and (level $\ell$) restricted $T$-systems accordingly. Eq.(2.16) is due to the vanishing of corresponding symmetrizers observed in [13]. Plainly, one cannot fuse more than level times when $q = \exp(\frac{2\pi i}{\ell + \frac{1}{4}})$. The transfer matrix $T_\ell(u)$ especially corresponds to a trivial model in which the configuration is frozen. See the remark at the bottom of pp235 in [39] where $L - 2$ and $N$ therein are the level and the fusion degree, respectively.

**Remark 2.1.** The same argument using (2.7a) with $m' = j = m$ also leads to (2.14). In fact, more FRs follow from general cases of (2.7). However, we remark that (2.14) is a set of generating relations of the commutative algebra defined by the relations from (2.7). Namely, repeated applications of (2.14) with various $m$ allow one to express all the $T_m(u)$’s as polynomials in $T_1(u + \text{shifts})$ and the scalar operators whereby the other $T$-systems will simply reduce to identities. See also [7] and the Remark 2.2 below. Such recursive “integrations” of the $T$-systems lead to curious determinant expressions which will be discussed in section 2.3 and section 5.

**Remark 2.2.** The $T$-system (2.14) was firstly obtained in [12] for the fusion RSOS models [39] when $m_k$ and $v_k$ are $k$-independent. Their derivation is a direct computation based on the simpler FR eq.(3.19) in [9]. The later FR corresponds, in the sense of Remark 2.1 above, to the case $m' = j = 1$ of (2.7a), which is also a set of generating relations of the commutative algebra mentioned in Remark 2.1. Our argument here clarifies the role of the exact sequence of the symmetry algebra and the idea is applicable in principle to all the other algebras $X_r$.

### 2.3. $T$-system for $\widehat{sl}(r + 1)$
Here we shall prove the following $X_r = A_r$ version of the $T$-system:

$$T^{(a)}_m(u + \frac{1}{2})T^{(a)}_m(u - \frac{1}{2}) = T^{(a)}_{m+1}(u)T^{(a)}_{m-1}(u) + T^{(a+1)}_m(u)T^{(a-1)}_m(u) \quad 1 \leq a \leq r,$$  (2.17)

where $T^{(a)}_m(u) = 0$ for $m < 0$, $T^{(a)}_0(u) = T^{(a)}_0(u) = 1$. The symbol $T^{(a)}_m(u)$ denotes the row-to-row transfer matrix with fusion type $(a, m)$. By this we mean that the auxiliary space carries the $m$-fold symmetric tensor of the $a$-th fundamental representation of $sl(r + 1)$. It is the irreducible $sl(r + 1)$-module associated with the $a \times m$ rectangular Young diagram corresponding to the highest weight $m\Lambda_a$ ($\Lambda_a : a$-th fundamental weight) and will be denoted by $W^{(a)}_m$, $1 \leq a \leq r$. There is the associated evaluation representation $W^{(a)}_m(u)$ of $Y(sl(r + 1))$ analogous to the $sl(2)$ case [33]. Let $W = C^{r+1}$ be the vector representation of $sl(r + 1)$. We consider the lift of $W^{(a)}_m(u)$ to the $Y(sl(r + 1))$ representation on the invariant subspace of $W^{\otimes a}$, and it will be also denoted by $W^{(a)}_m(u)$. The transfer matrix $T^{(a)}_m(u)$ $(1 \leq a \leq r)$ is visualized in Fig. 2.

$$\text{Tr}_{W^{(a)}_m} \left( W^{(a)}_m(u) \begin{array}{c} \vdots \vdots \ \vdots \end{array} \right)$$

Figure 2. The graphical representation of the transfer matrices $T^{(a)}_m(u)$.

To define the extra matrix $T^{(r+1)}_m(u)$ in (2.17), we introduce the additional $Y(sl(r + 1))$ representation $W^{(0)}_1(u)$ on the totally antisymmetric subspace of $W^{(0)}$. This is trivial as a $Y(sl(r + 1))$-module and analogous to $W_0(u)$ defined for $sl(2)$ in section 2.2. See section 4.1,(or (4.5)) for a more description. The matrix $T^{(r+1)}_m(u)$ is then defined by Fig. 2 with the auxiliary space replaced by the following one:

$$W^{(0)}_1(u - \frac{m - 1}{2}) \otimes W^{(0)}_1(u - \frac{m - 3}{2}) \otimes \cdots \otimes W^{(0)}_1(u + \frac{m - 1}{2}),$$  (2.18)

which is analogous to (2.11). As (2.14), $T^{(r+1)}_m(u)$ is a scalar operator. The explicit form is irrelevant here and we will later need only the property

$$T^{(r+1)}_m(u + \frac{1}{2})T^{(r+1)}_m(u - \frac{1}{2}) = T^{(r+1)}_{m+1}(u)T^{(r+1)}_{m-1}(u),$$  (2.19)

which is immediate from (2.18) and is a special case of (2.17) with “$a = r + 1$”. The commutativity $[T^{(a)}_m(u), T^{(a')}_{m'}(v)] = 0$ is valid due to the intertwining $R$-matrix as in the
sl(2) case. We shall henceforth treat the $T_m^{(a)}(u)$’s as commutative variables (or they may be regarded as eigenvalues). The underlying elliptic vertex and face models have actually been constructed in [37] and [35], respectively.

Our proof of (2.17) is based on the FR due to Bazhanov–Reshetikhin (eqs.(3.10,11) in [13]), which can be derived from the exact sequences (resolutions) by Cherednik [41] in the same way as (2.14) was derived from (2.7) in the previous subsection. Its relevant case to our problem here reads, in our notation, as follows:

$$T_m^{(a)}(u) = \det(T_{m-i+j}(u + \frac{i + j - 1 - a}{2}))_{1 \leq i, j \leq a} \quad (2.20a)$$

$$= \det(T_{1-i+j}(u + \frac{m - i - j + 1}{2}))_{1 \leq i, j \leq m}. \quad (2.20b)$$

The original FR of [13] provides determinant formulas for more general transfer matrices $T_Y(u)$ with fusion types parametrized by general Young diagrams $Y$ and (2.20) is the specialization to the case $Y = a \times m$ rectangular shape. \(^2\) The FR may be viewed as a quantum analogue of the second Weyl character formula (or the Jacobi–Trudi formula) [42] as noted in [13]. Put $t_k^a = T_1^{(a)}(u + \frac{m+1-k}{2})$ and introduce the determinant $D$ of the $m+1$-dimensional matrix

$$
\begin{pmatrix}
t_1^a & t_2^{a+1} & \cdots & t_{m+1}^{a+m} \\
t_2^{a-1} & t_3^a & \cdots & t_{m+2}^m \\
\vdots & \vdots & \ddots & \vdots \\
t_{m+1}^{a-m} & t_{m+2}^{a-m+1} & \cdots & t_{2m+1}^a
\end{pmatrix}.
$$

(2.21)

Denoting by $D_{[i_1,j_1;i_2,j_2;\ldots]}$ its minor removing $i_k$’s rows and $j_k$’s columns, we find

$$T_m^{(a)}(u + \frac{1}{2}) = D_{[m+1,m+1];1}, \quad T_m^{(a)}(u - \frac{1}{2}) = D_{[1,1];m+1}, \quad T_{m+1}^{(a)}(u) = D_{[1,m+1];1}, \quad T_{m-1}^{(a)}(u) = D_{[1,1];m+1},$$

$$T_m^{(a+1)}(u) = D_{[m+1,1];1}, \quad T_m^{(a-1)}(u) = D_{[1,m+1];1}, \quad (2.22)$$

by virtue of (2.20b). Substituting these expressions into (2.17) one finds the result reduces to the so-called Jacobi identity among determinants (or an example of the Plücker relation)

\(^2\) We thank I. Cherednik for his communication about the relation between his result and the FR in [13].

\(^3\) We thank V.V. Bazhanov for communicating the misprints contained in the published version, which have been corrected here in our convention of the definition of the transfer matrix.

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thereby completing the proof. A similar proof is also possible by using (2.20a). Although our proof relied on the FR (2.20), we expect there are the exact sequences of \( Y(sl(r+1)) \) and \( U_q(sl(r+1)) \)-modules analogous to (2.10) that directly lead to (2.17) in the same sense as in the previous subsection.

The restricted \( T \)-system relevant to the level \( \ell \mathbb{A}^{(1)}_r \) RSOS models [35] obeys a similar truncation to (2.16) as

\[
T_{\ell+1}^{(a)}(u) = 0 \quad \text{for all } a, \tag{2.23}
\]

which may be deduced from eq.(3.14) in [13].

Having established the \( T \)-system (2.17), let us check if the curious phenomenon observed in the introduction is happening. Namely, does (2.17) \textbf{Yang-Baxterize the} \( Q \)-system \textbf{so as to satisfy the} \( Y \)-system? From (3.8) and (B.30) the latter two are given by

\[
\begin{align*}
Q_m^{(a)} &= Q_{m+1}^{(a)}Q_{m-1}^{(a)} + Q_{m+1}^{(a+1)}Q_{m-1}^{(a-1)}, \\
Q_m^{(0)} &= Q_m^{(r+1)} = 1, \tag{2.24a}
\end{align*}
\]

\[
Y_m^{(a)}(u + \frac{1}{2})Y_m^{(a)}(u - \frac{1}{2}) = \frac{(1 + Y_{m+1}^{(a)}(u))(1 + Y_{m-1}^{(a)}(u))}{(1 + Y_{m+1}^{(a+1)}(u)(u-1))(1 + Y_{m-1}^{(a-1)}(u)(u-1))}, \tag{2.24b}
\]

wherin the \( Y \)-system we have made a notational change \( Y_m^{(a)}(u + \frac{k}{2}) \) here is \( Y_m^{(a)}(u+k)^{-1} \) in (B.30)) so as to suite (1.4). The \( T \)-system (2.17) is a Yang-Baxterization of (2.24a) in the obvious sense and it is elementary to see that the combination

\[
y_m^{(a)}(u) = \frac{T_{m+1}^{(a)}(u)T_{m-1}^{(a)}(u)}{T_{m+1}^{(a+1)}(u)T_{m-1}^{(a-1)}(u)} \tag{2.25}
\]

solves (2.24b) by means of (2.17) and (2.19). Thus we have established the same connection among the \( T \), \( Q \) and \( Y \)-systems also for \( X_r = sl(r+1) \).

3. \( T \)-system for a general \( X_r \)

Let us proceed to the \( T \)-systems for the other \( X_r \)'s. We introduce them through the curious connection to the \( Q \) and \( Y \)-systems which has been already established for \( X_r = sl(r+1) \) in section 2.

3.1. Preliminaries
We firstly fix our notations. Let $X_r$ denote one of the classical simple Lie algebras $A_r(r \geq 1), B_r(r \geq 2), C_r(r \geq 2), D_r(r \geq 4), E_{6,7,8}, F_4$ and $G_2$. Let $\{a_a \mid 1 \leq a \leq r\}, \{\Lambda_a \mid 1 \leq a \leq r\}$ and $(\cdot \mid \cdot)$ be the set of the simple roots, the fundamental weights and the invariant form on $X_r$, respectively. We employ the convention $\Lambda_0 = 0$. We identify the Cartan subalgebra $\mathcal{H}$ and its dual $\mathcal{H}^*$ via the form $(\cdot \mid \cdot)$. We employ the normalization $|\text{long root}|^2 = 2$ and put

$$t_a = \frac{2}{(a_a | a_a)}, \quad t_{ab} = \max(t_a, t_b), \quad (3.1a)$$

$$C_{ab} = \frac{2(a_a | a_b)}{(a_a | a_a)}, \quad B_{ab} = \frac{t_b}{t_{ab}} C_{ab} = B_{ba}, \quad I_{ab} = 2\delta_{ab} - B_{ab}, \quad (3.1b)$$

$$\alpha_a^\vee = t_a a_a, \quad (3.1c)$$

$$g = \text{dual Coxeter number of } X_r, \quad (3.1d)$$

$$\rho = \Lambda_1 + \cdots + \Lambda_r. \quad (3.1e)$$

Here $C_{ab}, B_{ab}$ and $I_{ab}$ are the Cartan, the symmetrized Cartan and the incidence matrix, respectively. By definition $(a_a | \Lambda_b) = \delta_{ab}/t_a$ and $t_a = 1$ if $a_a$ is a long root and otherwise $t_a = 2$ for $B_r, C_r, F_4$ and $t_a = 3$ for $G_2$. One may equivalently define $t_i$ by $t_i = a_i/a_i^\vee$ with the Kac and dual Kac labels $a_i$ and $a_i^\vee$, respectively. In Table 1, we list the Dynkin diagram with the numeration of its nodes $1 \leq a \leq r$, $\dim X_r$ and $g$ for each $X_r$ along with those $t_a \neq 1$. In this paper and the subsequent Part II, the letters $a, b \ldots$ will mainly be used to denote the indices corresponding to the Dynkin diagram nodes. We shall also use the standard notations

$$Q = \bigoplus_{a=1}^r \mathbb{Z}a_a, \quad Q^\vee = \bigoplus_{a=1}^r \mathbb{Z}a_a^\vee, \quad (3.2)$$

$$P = (Q^\vee)^* = \bigoplus_{a=1}^r \mathbb{Z}\Lambda_a, \quad P_+ = \bigoplus_{a=1}^r \mathbb{Z}_{\geq 0}\Lambda_a.$$

These are the root, coroot, weight lattices and the set of dominant weights, respectively.

In order to discuss the restricted $T$-systems, we fix an integer $\ell \in \mathbb{Z}_{\geq 1}$ and set

$$\ell_a = t_a \ell, \quad (3.3a)$$

$$G = \{(a, m) \mid 1 \leq a \leq r, 1 \leq m \leq \ell_a - 1\}. \quad (3.3b)$$

Let

$$P_\ell = \{\Lambda \in P_+ | (\Lambda | \text{maximal root}) \leq \ell\}, \quad (3.4)$$
which is the classical projection of the set of the dominant integral weights of the affine Lie algebra $X_r^{(1)}$ at level $\ell$.

3.2. IFDR of Yangian and $Q$-system

Let us recall some basic properties of FDRs of the Yangian $Y(X_r)$ for a general $X_r$ [2,26,43]. Firstly, $Y(X_r)$ contains $U(X_r)$ as a subalgebra hence any FDR of the former is decomposed into a direct sum of the IFDRs of the latter. A practical way of finding IFDRs of $Y(X_r)$ would be to seek a surjection $Y(X_r) \to X_r$ which pulls back any IFDR of $X_r$ to that of $Y(X_r)$. This is indeed known to be possible for $X_r = A_r$, yielding the so-called evaluation representations as noted for $sl(2)$ in section 2.2. However, for $X_r \neq A_r$, there is no obvious analogue of such an evaluation map that the restriction on $X_r$ is an identity map [2,43]. In general, IFDRs of $Y(X_r)$ are no longer irreducible and possess nontrivial reducibility contents under the decomposition as the $X_r$-modules. Secondly, the Yangian $Y(X_r)$ is equipped with an automorphism depending on a complex parameter. Thus, any given IFDR $W$ of $Y(X_r)$ can be pulled back by it to a one parameter family of the IFDRs, which will be denoted by $W(u)$, $u \in \mathbb{C}$. As an $X_r$-module, $W(u) \simeq W$ holds and one may regard $W = W(0)$ for example. The parameter $u$ is related to the spectral parameter of $R$-matrices as in section 2.2.

Hereafter we concentrate on the family of IFDRs $W_m^{(a)}(u)$ $(1 \leq a \leq r, 1 \leq m)$ of $Y(X_r)$ introduced in [26] that are relevant to the $T$-system. (though there is no written proof of the irreducibility in [26]). Let $W_m^{(a)}$ denote $W_m^{(a)}(u)$ as an $X_r$-module, and $V_\Lambda$ be the IFDR of $X_r$ with highest weight $\Lambda$. Then, its reducibility content is given as [26]

$$W_m^{(a)} = \sum_{\Lambda} Z(a, m, m\Lambda_a - \Lambda)V_\Lambda, \quad (3.5a)$$

$$Z(a, m, \sum_{b=1}^r n_b a_b) = \sum_{\nu} \prod_{b=1}^r \prod_{k=1}^\infty \left(\mathcal{P}_k^{(b)} + \nu_k^{(b)}\right), \quad (3.5b)$$

$$\mathcal{P}_k^{(b)} = \delta_{ab} \min(m, k) - \sum_{c=1}^{r} \sum_{j=1}^{\infty} (a_b|a_c)\min(t_c k, t_b j)\nu_j^{(c)}. \quad (3.5c)$$

Here in (3.5a) $\Lambda$-sum extends over those $\Lambda$ such that $m\Lambda_a - \Lambda \in \sum_{a=1}^r Z_{\geq 0} a_a$ hence $\forall n_b \in Z_{\geq 0}$ is implied in the lhs of (3.5b). In (3.5b) the symbol $\binom{a}{b}$ means the binomial coefficient and $\nu$-sum extends over all possible decompositions $\nu_k^{(b)} = \sum_{k=1}^{\infty} k^{(b)}$, $\nu_k^{(b)} \in Z_{\geq 0}$, $1 \leq b \leq r, k \geq 1$ such that $\mathcal{P}_k^{(b)} \geq 0$ for all $1 \leq b \leq r$ and $k \geq 1$. More explicit formulas of
(3.5) are also known [17, 26] in many cases. See Appendix A.1 for a further information on (3.5).

The case $X_r = A_r$ is simplest in that $W^{(a)}_m = V_{m\Lambda_a}$ holds, which is consistent to the notation in section 2.3. In general, the decomposition of $W^{(a)}_m$ always contains the module $V_{m\Lambda_a}$ as the highest term w.r.t the root system with multiplicity one. $W^{(a)}_1(u)$'s are identified with the fundamental representations of $Y(X_r)$ [43] whose characterization by the Drinfel’d polynomial [45] is known, and $W^{(a)}_m(u)$ is a quantum analogue of the $m$-fold symmetric tensor of $W^{(a)}_1(u)$.

There are remarkable polynomial relations among the characters of $W^{(a)}_m$ as we describe below. Let $Q^{(a)}_m = Q^{(a)}_m(z)$ $(z \in \mathcal{H}^a)$ be the character of $W^{(a)}_m$, namely,

$$ Q^{(a)}_m(z) = \sum_{\Lambda} Z(a, m, m\Lambda_a - \Lambda) \chi_{\Lambda}(z), \quad (3.6a) $$

$$ \chi_{\Lambda}(z) = \text{Tr}_{V_{m\Lambda_a}} \exp \left(-\frac{2\pi i}{\ell + g}(z + \rho)\right), \quad (3.6b) $$

where $\ell \in \mathbb{Z}_{\geq 0}$ has been fixed in (3.3), and $Q^{(a)}_0 = 1$, $Q^{(a)}_{-1} = 0$ for all $a$. The following relation holds [24,26], which we call the unrestricted $Q$-system:

*Unrestricted $Q$-system*

$$ Q^{(a)}_m = Q^{(a)}_{m-1} Q^{(a)}_{m+1} + Q^{(a)}_m \prod_{b=1}^r \prod_{k=0}^{\infty} Q^{(b)-2,J_{bk}^m} \quad \text{for } 1 \leq a \leq r, \ m \geq 1, \quad (3.7) $$

where $2J_{ba}^m$ is given by (B.22b) and all the powers of $Q^{(a)}_m$'s are nonnegative integers. The $k$-product in (3.7) actually extends over only finite support of $J_{ba}^m$. The second term on the rhs of (3.7) contains 0 (i.e., equal to 1), 1, 2 or 3 $Q$-factors depending on the choice of $a, m$ and $X_r$. For simply-laced algebras $X_r = A_r, D_r$ and $E_{6,7,8}$, (3.7) takes an especially simple form due to (B.32b) as

$$ Q^{(a)}_m = Q^{(a)}_{m-1} Q^{(a)}_{m+1} + \prod_{b=1}^r Q^{(b)}_m I_{ab}, \quad (3.8) $$

where $I_{ab}$ is the incidence matrix (3.1).

An interesting feature emerges when one specializes $z$ to 0 $P$. In this case, all $Q^{(a)}_m(0) (a, m) \in G$ are real positive. Moreover, admitting (A.6a) of [17] and (A.7), we see that

$$ Q^{(a)}_{\ell+a+1}(0) = 0, \quad (3.9a) $$

$$ Q^{(a)}_{\ell+a}(0) = 1. \quad (3.9b) $$
Therefore (3.7) closes among \( Q_m^{(a)}(0) \) for \((a, m) \in G\), which we call the restricted \( Q \)-system, i.e.,

**Restricted \( Q \)-system**

\[
Q_m^{(a)}(0)^2 = Q_{m-1}^{(a)}(0)Q_{m+1}^{(a)}(0) + Q_m^{(a)}(0)^2 \prod_{(b,k) \in G} Q_k^{(b)}(0)^{-2J_{b,m}^{k}} \quad \text{for} \ (a, m) \in G. \quad (3.10)
\]

In section 3.4 we shall consider the Yang-Baxterization of (3.7) and (3.10). See Appendix A for more general restricted systems. Those restricted \( Q \)-systems are related to the dilogarithm identities \([7,22,24]\), on which further analyses will be given in Part II.

The reducibility content (3.5) and the \( Q \)-system (3.7) among the characters are the basic features in the IFDRs \( W_m^{(a)} \) of the Yangian \( Y(X_r) \). However, those properties are also known to be common in the quantum affine algebra \( U_q(X_r^{(1)}) \) for \( X_r = A_1 \) [38]. We assume this for all \( X_r \)'s in this paper and Part II. Namely, for each \( W_m^{(a)}(u) \), we postulate the existence of its natural \( q \)-analogue having the properties (3.5) and (3.7). We shall write the IFDR of \( U_q(X_r^{(1)}) \) so postulated also as \( W_m^{(a)}(u) \).

### 3.3. Models with fusion type \( W_m^{(a)} \)

Having identified the spaces \( W_m^{(a)} \), we can now describe a set of models associated with them. Though our description here might seem formal, most known examples of the trigonometric vertex and RSOS weights obey this scheme (cf. [46]). See also remark 2.5 in [17].

Let \( W_m^{(a)}(u) \) and \( W_{m'}^{(a')} (v) \) be IFDRs of \( U_q(X_r^{(1)}) \) as described in section 3.2. For generic \( u - v \) and \( q \), their tensor products are irreducible and isomorphic in both order, which implies the existence of the unique \( U_q(X_r^{(1)}) \)-isomorphism \( \hat{R}(u - v) \in \text{Hom}_{U_q(X_r^{(1)})}(W_m^{(a)}(u) \otimes W_{m'}^{(a')} (v), W_{m'}^{(a')} (v) \otimes W_m^{(a)}(u)) \) solving the Yang-Baxter equation. Regarding its matrix elements as the Boltzmann weights, one has the vertex model with fusion type labeled by the pair \((W_m^{(a)}, W_{m'}^{(a')})\). Each horizontal (vertical) edge of the vertex therein carries the IFDR \( W_m^{(a)}(W_{m'}^{(a')}) \) as in section 2.2.

From the vertex model, one can in principle construct the level \( \ell \, U_q(X_r^{(1)}) \) RSOS models with the same fusion type when \( q = e^{2\pi i / \ell} \) through the vertex-IRF correspondence [40], which we shall briefly sketch below. Let \( V_{\lambda} (\lambda \in P_\ell) \) and \( W_m^{(a)} \) be the \( U_q(X_r) \)-modules.
which are uniquely lifted from the corresponding $X_r$-modules. When $q$ is not a root of unity, the tensor product decomposes as an $U_q(X_r)$-module as

$$\quad V_\lambda \otimes W_m^{(a)} \otimes W_m^{(a')} = V_\lambda \otimes W_m^{(a')} \otimes W_m^{(a)} = \bigoplus_{\mu \in P_+} \Omega(\lambda)_{\mu} \otimes V_\mu, \quad (3.11)$$

where $\Omega(\lambda)_{\mu}$ is the space of highest weight vectors with weight $\mu$. Since $\hat{R}(u)$ commutes with $U_q(X_r)$, the space $\Omega(\lambda)_{\mu}$ is invariant under the action of $\text{id} \otimes \hat{R}(u)$: $V_\lambda \otimes W_m^{(a)} \otimes W_m^{(a')} \rightarrow V_\lambda \otimes W_m^{(a')} \otimes W_m^{(a)}$. When $q = e^{\frac{i2\pi i}{r}}$, the decomposition (3.11) no longer holds [47,48]. However, in the case $A_1$ it is known [40] that the action of $\text{id} \otimes \hat{R}(u)$ remains well-defined on the quotient of the rhs of (3.11) divided by the Type I modules [49,50] (i.e., indecomposable modules with quantum dimension zero). It is natural to assume the same prescription holds for any $X_r$. Then, the RSOS Boltzmann weights should be defined by the matrix elements of the operator $\text{id} \otimes \hat{R}(u)$ acting on the quotient space of $\Omega(\lambda)_{\mu}$ and they satisfy the YBE.

The transfer matrices $T_m^{(a)}(u)$ are defined as in Fig. 2 by using the above vertex (or RSOS) weights. For vertex models, we consider fusion type (of auxiliary space) $W_m^{(a)}$ with $1 \leq a \leq r, m \geq 1$, while for RSOS models with $1 \leq a \leq r, 1 \leq m \leq \ell_a$ (cf. (3.9)). The commutativity of transfer matrices with the same quantum space is assured by the YBE.

The RSOS model described above has the admissibility matrix which is remarkably consistent with the TBA analyses [9,17] as we shall explain below. The fluctuation variables of the RSOS model are attached to both the sites and the edges [35]. In our model, the site variable belongs to $P_\ell$. To describe the edge variable, we consider the decomposition

$$\quad V_\lambda \otimes W_m^{(a)} = \bigoplus_{\lambda' \in P_+} \overline{\Omega}_{\lambda'\lambda} \otimes V_{\lambda'}, \quad (3.12)$$

at generic $q$. When $q = e^{\frac{i2\pi i}{r}}$, we need to take the quotient of the rhs by the Type I modules, and this induces the quotient $\Omega_{\lambda'\lambda}$ of $\overline{\Omega}_{\lambda'\lambda}$. Then, the edge variable associated to the $W_m^{(a)}$ fusion belongs to the space $\Omega_{\lambda'\lambda}$. Let $M_{\lambda'\lambda} = \dim \Omega_{\lambda'\lambda}$ and $\overline{M}_{\lambda'\lambda} = \dim \overline{\Omega}_{\lambda'\lambda}$. We call the matrix $M$ the admissibility matrix of fusion type $W_m^{(a)}$. Following the standard argument of the fusion algebra in conformal field theories [51–55], the matrix $M$ is related to $\overline{M}$ as

$$\quad M_{\lambda'\lambda} = \sum_{w \in \mathcal{W}} \det w \cdot \overline{M}_{\hat{w}((\lambda'),\lambda)}, \quad \lambda', \lambda \in P_\ell, \quad (3.13)$$
where $W$ is the Weyl group of $X_r^{(1)}$ at level $\ell$ and $\hat{w}(\lambda) = w(\lambda + \rho) - \rho$. Then, from the pseudo-Weyl invariance of the specialized characters [54], it follows that

$$M \cdot \vec{v} = Q_m^{(a)}(0) \vec{v},$$

(3.14)

where $(\vec{v})_\lambda = \chi_\lambda(0)$. Since $M$ and $\vec{v}$ are nonnegative matrix and vector, $Q_m^{(a)}(0)$ is nothing but the maximum eigenvalue of $M$. Thus, for the quantum space $H_N$ with lattice size $N$, we have

$$\lim_{N \to \infty} (\dim H_N)^{1/N} = \lim_{N \to \infty} (\text{Tr} M^N)^{1/N} = Q_m^{(a)}(0),$$

(3.15)

where we have assumed the periodic boundary condition for the first equality. This is quite consistent with the result of the TBA analysis on a corresponding quantum spin chain model that the high temperature limit of entropy per edge is equal to $\log Q_m^{(a)}(0)$ [9,17]. Note in particular from (3.9b) that the fusion type $W_{r_a}^{(a)}$ corresponds to a “frozen model”, which is consistent to the comments after (2.16).

3.4. T-system for arbitrary classical Lie algebras

Let us proceed to the main problem of this paper. What are functional relations among the transfer matrices described in section 3.3? Our strategy here is to apply the working hypothesis that the T-system Yang-Baxterizes the $Q$-system in such a way that particular combinations of transfer matrices solve the $Y$-system. This has been established for $X_r = A_r$ in section 2.

We start with the consideration on the $T$-systems that correspond to the unrestricted $Q$-systems (3.7). A general relation in (3.7) has the following form:

$$Q_m^{(a)2} = Q_{m+1}^{(a)}Q_{m-1}^{(a)} + Q_k^{(b)} \cdots Q_j^{(c)}.$$  

(3.16)

Accordingly, we take the following ansatz for the $T$-system:

$$T_m^{(a)}(u + \frac{1}{2t_a})T_m^{(a)}(u + \frac{1}{2t_a}) = T_{m+1}^{(a)}(u)T_{m-1}^{(a)}(u) + g_m^{(a)}(u)T_k^{(b)}(u + \alpha) \cdots T_j^{(c)}(u + \beta),$$

(3.17)

where all $T_m^{(a)}$’s may be treated as commutative variables. The functions $g_m^{(a)}$ play a similar role to those in (2.14), (2.19), and they are related to a central extension of $Y(X_r)$. Below its explicit form is not needed but only the property

$$g_m^{(a)}(u - \frac{1}{2t_a})g_m^{(a)}(u + \frac{1}{2t_a}) = g_{m-1}^{(a)}(u)g_{m+1}^{(a)}(u),$$

(3.18)
which will be justified in section 4 in many cases. In (3.17) \( \alpha, \beta \) etc. are parameters to be determined in the following way. Let \( y^{(a)}_m(2iu) \) be the combination:

\[
y^{(a)}_m(2iu) = \frac{g^{(a)}_m(u)T_k^{(b)}(u + \alpha) \cdots T_j^{(c)}(u + \beta)}{T_{m+1}(u)T_{m-1}(u)}. \tag{3.19}
\]

We then demand that this \( y^{(a)}_m(2iu) \) solves the unrestricted \( Y \)-system (cf.(B.6) and remarks following it). Remarkably, this fixes the parameters \( \alpha, \beta \) etc. uniquely up to trivial freedoms which can be absorbed into the redefinition of the transfer matrices by \( T^{(a)}_m(u) \rightarrow T^{(\alpha)}_m(u + \gamma^{(a)}_m) \). We note that the presence of \( g^{(a)}_m(u) \) is irrelevant to fixing \( \alpha, \beta \), etc. as long as (3.18) is satisfied.

In this way, we reach the following conjecture of the functional equations for the transfer matrices on the vertex models associated to the simple Lie algebras. We call them the unrestricted \( T \)-systems.

\( X_r = A_r, D_r, E_6, E_7, E_8 : \)

\[
T^{(a)}_m(u - \frac{1}{2})T^{(a)}_m(u + \frac{1}{2}) = T^{(a)}_{m+1}(u)T^{(a)}_{m-1}(u) + g^{(a)}_m(u)\prod_{b=1}^r T^{(b)}_m(u)^{I_{ab}}. \tag{3.20a}
\]

\( X_r = B_r : \)

\[
T^{(a)}_m(u - \frac{1}{2})T^{(a)}_m(u + \frac{1}{2}) = T^{(a)}_{m-1}(u)T^{(a)}_{m+1}(u) + g^{(a)}_m(u)T^{(a-1)}_m(u)T^{(a+1)}_m(u),
\quad (1 \leq a \leq r - 2),
\]

\[
T^{(r-1)}_m(u - \frac{1}{2})T^{(r-1)}_m(u + \frac{1}{2}) = T^{(r-1)}_{m-1}(u)T^{(r-1)}_{m+1}(u) + g^{(r-1)}_m(u)T^{(r-2)}_m(u)T^{(r)}_{2m}(u),
\]

\[
T^{(r)}_{2m}(u - \frac{1}{4})T^{(r)}_{2m}(u + \frac{1}{4}) = T^{(r)}_{2m-1}(u)T^{(r)}_{2m+1}(u) + g^{(r)}_m(u)T^{(r-1)}_m(u - \frac{1}{4})T^{(r-1)}_m(u + \frac{1}{4}),
\]

\[
T^{(r)}_{2m+1}(u - \frac{1}{4})T^{(r)}_{2m+1}(u + \frac{1}{4}) = T^{(r)}_{2m}(u)T^{(r)}_{2m+2}(u) + g^{(r)}_m(u)T^{(r-1)}_m(u)T^{(r-1)}_{m+1}(u). \tag{3.20b}
\]

\(^4\) Our normalization of the spectral parameter here differs from the one in Appendix B by the factor \( 2i \). For a notational simplification, the \( y^{(a)}_m \)'s in (1.8) and (2.25) have been taken to be the inverse of (3.19).

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\[ X_r = C_r : \]
\[ T_m^{(a)}(u - \frac{1}{4})T_m^{(a)}(u + \frac{1}{4}) = T_{m-1}^{(a)}(u)T_{m+1}^{(a)}(u) + g_m^{(a)}(u)T_m^{(a-1)}(u)T_m^{(a+1)}(u), \quad (1 \leq a \leq r - 2), \]
\[ T_{2m}^{(r-1)}(u - \frac{1}{4})T_{2m}^{(r-1)}(u + \frac{1}{4}) = T_{2m-1}^{(r-1)}(u)T_{2m+1}^{(r-1)}(u) + g_{2m}^{(r-1)}(u)T_{2m}^{(r-2)}(u)T_m^{(r)}(u - \frac{1}{4})T_m^{(r)}(u + \frac{1}{4}), \]
\[ T_{2m+1}^{(r-1)}(u - \frac{1}{4})T_{2m+1}^{(r-1)}(u + \frac{1}{4}) = T_{2m-1}^{(r-1)}(u)T_{2m+2}^{(r-1)}(u) + g_{2m+1}^{(r-1)}(u)T_{2m+1}^{(r-2)}(u)T_m^{(r)}(u), \]
\[ T_m^{(r)}(u - \frac{1}{2})T_m^{(r)}(u + \frac{1}{2}) = T_{m-1}^{(r-1)}(u)T_{m+1}^{(r-1)}(u) + g_m^{(r)}(u)T_{2m-1}^{(r-1)}(u). \quad (3.20c) \]

\[ X_r = F_4 : \]
\[ T_m^{(1)}(u - \frac{1}{2})T_m^{(1)}(u + \frac{1}{2}) = T_{m-1}^{(1)}(u)T_{m+1}^{(1)}(u) + g_m^{(1)}(u)T_m^{(2)}(u), \]
\[ T_m^{(2)}(u - \frac{1}{2})T_m^{(2)}(u + \frac{1}{2}) = T_{m-1}^{(2)}(u)T_{m+1}^{(2)}(u) + g_m^{(2)}(u)T_m^{(1)}(u)T_m^{(3)}(u), \]
\[ T_{2m}^{(3)}(u - \frac{1}{4})T_{2m}^{(3)}(u + \frac{1}{4}) = T_{2m-1}^{(3)}(u)T_{2m+1}^{(3)}(u) + g_{2m}^{(3)}(u)T_m^{(2)}(u)T_m^{(4)}(u), \]
\[ T_m^{(4)}(u - \frac{1}{4})T_m^{(4)}(u + \frac{1}{4}) = T_{m-1}^{(4)}(u)T_{m+1}^{(4)}(u) + g_m^{(4)}(u)T_m^{(3)}(u). \quad (3.20d) \]

\[ X_r = G_2 : \]
\[ T_m^{(1)}(u - \frac{1}{2})T_m^{(1)}(u + \frac{1}{2}) = T_{m-1}^{(1)}(u)T_{m+1}^{(1)}(u) + g_m^{(1)}(u)T_m^{(2)}(u), \]
\[ T_{3m}^{(2)}(u - \frac{1}{6})T_{3m}^{(2)}(u + \frac{1}{6}) = T_{3m-1}^{(1)}(u)T_{3m+1}^{(1)}(u) + g_{3m}^{(2)}(u)T_m^{(1)}(u)T_m^{(1)}(u + \frac{1}{3}), \]
\[ T_{3m+1}^{(2)}(u - \frac{1}{6})T_{3m+1}^{(2)}(u + \frac{1}{6}) = T_{3m}^{(1)}(u)T_{3m+2}^{(1)}(u) + g_{3m+1}^{(2)}(u)T_m^{(1)}(u)T_m^{(1)}(u + \frac{1}{6}), \]
\[ T_{3m+2}^{(2)}(u - \frac{1}{6})T_{3m+2}^{(2)}(u + \frac{1}{6}) = T_{3m+1}^{(1)}(u)T_{3m+3}^{(1)}(u) + g_{3m+2}^{(2)}(u)T_m^{(1)}(u)T_m^{(1)}(u + \frac{1}{6}). \quad (3.20e) \]
In the above the subscripts of the transfer matrices in the lhs extend over positive integers, and we adopt the conventions $T_m^{(0)} = T_0 = 1$.

For the level $\ell$ RSOS models, we suppose the same system (3.20) is valid except that the truncation

$$T_{\ell_a}^{(a)}(u) = 0 \quad \text{for all } a$$  \hspace{1cm} (3.21)

takes place based on the $X_r = A_r$ case (2.16), (2.23) and the fact (3.9a). Then (3.20) reduces to a finite set of functional relations among $T_m^{(a)}$ for $1 \leq a \leq r, 1 \leq m \leq \ell_a$. We call it the (level $\ell$) restricted $T$-system. Since (3.21) leads to $y_{\ell_a}^{(a)} - 1 = 0$ in (3.19), the restricted $T$-system is a Yang-Baxterization of the restricted $Q$-system (3.10) solving the restricted $Y$-system (B.6) in the parallel sense to the unrestricted case. Notice that $T_{\ell_a}^{(a)}$ ($1 \leq a \leq r$) satisfy closed relations among themselves:

$$T_{\ell_a}^{(a)}(u - \frac{1}{2t_a})T_{\ell_a}^{(a)}(u + \frac{1}{2t_a}) = g_{\ell_a}^{(a)}(u) \prod_{b \neq a}^{r} \prod_{k=1}^{r} T_{\ell_b}^{(b)}(u + \frac{2k - 1 + C_{ab}}{2t_a}).$$  \hspace{1cm} (3.22)

As remarked in the end of section 3.3, the $T_{\ell_a}^{(a)}$'s are the transfer matrices for the frozen RSOS models. The restricted $T$-system is invariant under the simultaneous transformations

$$T_{\ell_a}^{(a)}(u) \rightarrow Z_a T_{\ell_a-m}^{(a)}(u + \text{constant}),$$  \hspace{1cm} (3.23)

for all $a$ if the operators $Z_a$ satisfy

$$[Z_a, Z_b] = [Z_a, T_{k}^{(b)}(u)] = 0 \quad \text{for all } a, b, k,$$

$$\prod_{a=1}^{r} (Z_a)^{C_{ka}} = \text{Id.}$$  \hspace{1cm} (3.24)

This may be viewed as a generalization of the property found in [9] for the $sl(2)$ case.

Finally we remark that the level 1 restricted $T$-systems for non-simply laced algebras $X_r = B_r, C_r, F_4, G_2$ reduce to those for $A_1, A_{r-1}, A_2, A_1$ of levels 2, 2, 2, 3, respectively. These reductions are consistent to the equivalences of the associated RSOS models indicated from the TBA analysis. See eq.(3.12) in [17].

4. Fusion contents and $T$-system
In this section we examine our $T$-system (3.20) from the the representation theoretical scheme as in section 2.2. For definiteness, we restrict our description to the unrestricted $T$-system of the rational vertex models throughout the section.

4.1. Fusion contents

In this subsection we reconsider the $T$-system (2.17) for $A_r$. We begin by recalling the fusion procedure to build the space $W_m^{(a)}(u)$ and the data encoding it. The rational $R$-matrix (2.1) of the vector representation of $Y(A_r)$ has the following spectral decomposition form:

$$\hat{R}(u) = \hat{R}_{\Lambda_1, \Lambda_1}(u) = (1 + u)P_{\Lambda_1} \oplus (1 - u)P_{\Lambda_2}, \quad (4.1)$$

where $P_{\Lambda}$ is the projection operator on the space $V_{\Lambda}$. From (4.1), $\text{Im} \hat{R}(1) = W_2^{(1)}$ and $\text{Im} \hat{R}(-1) = W_1^{(2)}$. Thus, one can define the IFDRs $W_2^{(1)}(u)$ and $W_1^{(2)}(u)$ through the embeddings

$$W_2^{(1)}(u) \leftrightarrow W_1^{(1)}(u - \frac{1}{2}) \otimes W_1^{(1)}(u + \frac{1}{2}),$$

$$W_1^{(2)}(u) \leftrightarrow W_1^{(1)}(u + \frac{1}{2}) \otimes W_1^{(1)}(u - \frac{1}{2}), \quad (4.2)$$

which are the simplest cases of the fusion procedures [15]. The fusion procedures for $W_2^{(1)}(u)$ and $W_1^{(2)}(u)$ above are compactly summarized by the sequences of the spectral parameter shifts $(-\frac{1}{2}, \frac{1}{2})$ and $(\frac{1}{2}, -\frac{1}{2})$, respectively. For a sequence of complex numbers $(u_1, \ldots, u_n)$, the permutation of the adjacent elements, say, $u_i$ and $u_{i+1}$ will be called nonsingular if $u_i - u_{i+1} \neq \pm 1$. The product of two nonsingular permutations is also called nonsingular. Two sequences $(u_1, \ldots, u_n)$ and $(v_1, \ldots, v_n)$ are equivalent if one is obtained from the other by applying a nonsingular permutation. For a sequence $(u_1, \ldots, u_n)$, let $[u_1, \ldots, u_n]$ denote its equivalence class by the above equivalence relation. Then, we call the equivalence class $[u_1, \ldots, u_n]$ the fusion content of $W(u)$ if there is an embedding $W(u) \leftrightarrow W_1^{(1)}(u + u_1) \otimes \cdots \otimes W_1^{(1)}(u + u_n)$. The product of two fusion contents is naturally defined as

$$[k_1, \ldots, k_n] \cdot [j_1, \ldots, j_m] = [k_1, \ldots, k_n, j_1, \ldots, j_m], \quad (4.3)$$

Thus, the fusion content of the tensor product of two FDRs is the product of their fusion contents with the same ordering. We say two fusion contents are similar if all the elements of both sequences coincide ignoring the ordering.
It is known that the IFDR $W_m^{(a)}(u)$ has the following fusion content \[15,35,36\]

\[W_m^{(a)}(u + \frac{a + m}{2} - 1) : [a - 1, \cdots, 1, 0] \cdot [a, \cdots, 2, 1] \cdots [a + m - 2, \cdots, m, m - 1]. \tag{4.4}\]

We also define the IFDR $\bar{W}_m^{(0)}(u)$ to be the unique one dimensional module having the fusion content

\[[\frac{r}{2}, \frac{r}{2} - 1, \cdots, -\frac{r}{2}]. \tag{4.5}\]

Let us consider the implication of the fusion contents to the Yang-Baxterization of the $Q$-system (2.24a) such as

\[
T_m^{(a)}(u + c_1)T_m^{(a)}(u + c_2) = T_m^{(a)}(u + c_3)T_m^{(a)}(u + c_4)
+ T_m^{(a+1)}(u + c_5)T_m^{(a-1)}(u + c_6), \tag{4.6}
\]

where the meaning of $T_m^{(r+1)}(u)$ is undefined for a while.

We first consider the case $1 \leq a \leq r - 1$ of (4.6). Our basic assumption here is the existence of the exact sequences of $\bar{Y}(A_r)$-modules

\[0 \longrightarrow \mathcal{A}_1(u) \longrightarrow \mathcal{A}_2(u) \longrightarrow \mathcal{A}_3(u) \longrightarrow 0 \tag{4.7}\]

such that

\[
\mathcal{A}_1(u) = W_{m+1}^{(a)}(u + c_3) \otimes W_{m-1}^{(a)}(u + c_4) \text{ or } W_{m-1}^{(a)}(u + c_4) \otimes W_{m+1}^{(a)}(u + c_3),
\]

\[
\mathcal{A}_2(u) = W_m^{(a)}(u + c_1) \otimes W_m^{(a)}(u + c_2) \text{ or } W_m^{(a)}(u + c_2) \otimes W_m^{(a)}(u + c_1),
\]

\[
\mathcal{A}_3(u) = W_{m+1}^{(a+1)}(u + c_5) \otimes W_{m-1}^{(a-1)}(u + c_6)
\text{ or } W_{m-1}^{(a-1)}(u + c_6) \otimes W_{m+1}^{(a+1)}(u + c_5). \tag{4.8}\]

Then, (4.6) is a consequence of (4.7–8) as explained in section 2.2. In section 2.2, we saw that a $\bar{Y}(A_r)$-homomorphism only interchanges the ordering of the elements of the fusion contents. Thus, a necessary condition for (4.7) to exist is that the fusion contents of all the three representations $\mathcal{A}_i(u)$ ($i = 1, 2, 3$) are similar. This condition, in fact, almost determines the possible form of the Yang-Baxterization. To see it, we let $c_1 = 0$, $c_2 \geq 0$ without losing the generality, then the above condition is satisfied if and only if

\[c_2 = 1, \quad c_3 = c_4 = c_5 = c_6 = \frac{1}{2}. \tag{4.9}\]
or
\[ c_2 = a, \quad (c_3, c_4) = \left( \frac{1}{2}, a + \frac{1}{2} \right) \text{ or } \left( a - \frac{1}{2}, -\frac{1}{2} \right), \]
\[ (c_5, c_6) = \left( \frac{1}{2}, a + \frac{1}{2} \right) \text{ or } \left( a - \frac{1}{2}, -\frac{1}{2} \right), \quad \text{if } a = m \geq 2. \]  

(4.10)

The solution (4.9) reproduces our \( T \)-system (2.17) for \( a \leq r - 1 \). On the other hand, the known formula (2.20) tells that there are no exact sequences of the form (4.7) corresponding to the solutions (4.10), thus, they are irrelevant to the \( T \)-system.

In the case \( a = r \), the numbers of the elements in the fusion contents of \( \mathcal{A}_2(u) \) and \( \mathcal{A}_3(u) \) can be matched by replacing \( \mathcal{A}_3(u) \) in (4.8) with
\[ \mathcal{A}_3(u) = W_1^{(0)}(u + d_1) \otimes \cdots \otimes W_1^{(0)}(u + d_m) \otimes W_m^{(r-1)}(u + c_6). \]  

(4.11)

The solution analogous to (4.9) is
\[ c_2 = 1, \quad c_3 = c_4 = c_6 = \frac{1}{2}, \quad d_k = -\frac{m}{2} + k, \]  

(4.12)

which agrees with the definition of the scalar factor \( T_m^{(r+1)}(u) \) defined through (2.18). Again, there is an extra solution when \( m = r \) as in (4.10), to which we cannot associate an exact sequence.

To summarize, the assumption of the existence of the exact sequences (4.7) together with the fusion contents data imposes a strong constraint on the possible form of the \( T \)-system and the scalar functions \( g_m^{(a)}(u) \).

4.2. General cases

Let us study (3.20) for other \( X_r \)'s by the method described in the previous subsection. We consider the algebra \( \hat{Y}(X_r) \) of the \( L \)-operators analogously defined as (2.8-9). The algebra \( \hat{Y}(X_r) \) is a central extension of \( Y(X_r) \) [2]. As in the case \( A_r \), the IFDRs of \( \hat{Y}(X_r) \) are constructed through the fusion procedure using the \( R \)-matrix of the “minimal” representation as building blocks [26,56,43]. The list of the minimal representations is

\[
\begin{align*}
A_r & : \quad W_1^{(1)}(u), \\
B_r & : \quad W_1^{(r)}(u), \\
C_r & : \quad W_1^{(r-1)}(u), W_1^{(r)}(u), \\
D_r & : \quad W_1^{(r-1)}(u), W_1^{(r)}(u), \\
\end{align*}
\]

\[
\begin{align*}
E_6 & : \quad W_1^{(1)}(u), W_1^{(5)}(u), \\
E_7 & : \quad W_1^{(6)}(u), \\
E_8 & : \quad W_1^{(1)}(u), \\
F_4 & : \quad W_1^{(4)}(u), \\
G_2 & : \quad W_1^{(2)}(u). \\
\end{align*}
\]  

(4.13)

We consider the fusion contents with respect to these minimal representations. For that purpose, in the definition of the nonsingular exchanges described after (4.2), the condition
\( u_i - u_{i+1} \neq \pm 1 \) should be replaced by \( u_i - u_{i+1} \notin \{ u | \det \tilde{R}(u) = 0 \} \) for the \( R \)-matrix \( \tilde{R}(u) \) of the minimal representation. Then, we call the equivalence class \([u_1, \ldots, u_n] \) the fusion content of \( W(u) \) if there is an embedding \( W(u) \hookrightarrow W_{\text{min}}(u + u_1) \otimes \cdots \otimes W_{\text{min}}(u + u_n) \) with \( W_{\text{min}}(u) \) being the minimal representation. (In the cases \( D_r, E_6 \), each factor \( W_{\text{min}}(u + u_k) \) in the above is considered to be one of two minimal representations, and we distinguish the spectral shifts of these two representations. See (4.31c) and (4.33c). The definition of the equivalence classes also needs to be modified adequately.) We remark that as \( X_\mathfrak{s} \)-modules, the minimal representations in (4.13) are just \( V_{\lambda_a} \) with the only exception for \( E_8 \) where \( W_1^{(1)} = V_{\lambda_1} \oplus V_0 \). The analysis as in section 4.1 is straightforward once the fusion contents of all \( W_m^{(a)}(u) \)'s are known. In our normalization of the spectral parameter here, the higher representations are assumed to be obtained by the fusion

\[
W_m^{(a)}(u) \hookrightarrow W_1^{(a)}(u - \frac{m-1}{2t_a}) \otimes W_1^{(a)}(u - \frac{m-3}{2t_a}) \otimes \cdots \otimes W_1^{(a)}(u + \frac{m-1}{2t_a}). \tag{4.14}
\]

Though we lack a general proof of (4.14), in most cases the assumption (4.14) itself is a necessary condition for the exact sequences questioned to exist. From (4.14), it is enough to know the fusion contents of the fundamental representations.

Let us illustrate the analysis in the case \( G_2 \) in some detail. By explicitly calculating the coefficients in (3.5a), we have [17]

\[
W_1^{(1)} = V_{\lambda_1} \oplus V_0, \quad W_1^{(2)} = V_{\lambda_2}, \quad W_2^{(2)} = V_{2\lambda_2} \oplus V_{\lambda_2}. \tag{4.15}
\]

The \( R \)-matrix for the minimal representation \( W_1^{(2)}(u) \) is [57]

\[
\tilde{R}(u) = (1 + 3u)(4 + 3u)(6 + 3u)P_{2\lambda_2} \oplus (1 - 3u)(4 + 3u)(6 - 3u)P_0 \\
\oplus (1 - 3u)(4 + 3u)(6 + 3u)P_{\lambda_1} \oplus (1 + 3u)(4 - 3u)(6 + 3u)P_{\lambda_2}. \tag{4.16}
\]

By comparing (4.15) and (4.16), \( W_2^{(2)}(u) \) and \( W_1^{(1)}(u) \) can be defined through the fusion procedures

\[
W_2^{(2)}(u) \hookrightarrow W_1^{(2)}(u - \frac{1}{6}) \otimes W_1^{(2)}(u + \frac{1}{6}),
\]

\[
W_1^{(1)}(u) \hookrightarrow W_1^{(2)}(u + \frac{1}{6}) \otimes W_1^{(2)}(u - \frac{1}{6}). \tag{4.17}
\]

In addition, we define \( W_1^{(2)}(u) \) by the fusion

\[
W_1^{(2)}(u) \hookrightarrow W_1^{(2)}(u + \frac{2}{3}) \otimes W_1^{(2)}(u - \frac{2}{3}). \tag{4.18}
\]

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where $W^{(2)}_1(u)$ and $W^{(2)}_1(u)$ are isomorphic as $Y(G_2)$-modules, but distinguished as $\bar{Y}(G_2)$-modules. By (4.16–17), we have the fusion contents

$$W^{(1)}_1(u): \left[ \frac{1}{6}, -\frac{1}{6} \right], \quad W^{(2)}_1(u): \left[ 0, \frac{2}{3}, -\frac{2}{3} \right]. \quad (4.19)$$

Now let us consider the necessary condition for the existence of (4.7) corresponding to the relation

$$Q^{(1)}_1 = Q^{(1)}_2 + Q^{(2)}_3 \quad (4.20)$$
in (3.7). We first assume the forms of $\mathcal{A}_1(u)$ and $\mathcal{A}_2(u)$ as (we shall ignore the precise ordering of the tensor products in this argument),

$$\mathcal{A}_1(u) = W^{(1)}_2(u),$$
$$\mathcal{A}_2(u) = W^{(1)}_1(u + c_1) \otimes W^{(1)}_1(u + c_2), \quad (c_1 \leq c_2). \quad (4.21)$$

In order for (4.7) to exist, their fusion contents must be similar (denoted by $\sim$), therefore we have

$$\left[ -\frac{1}{3}, -\frac{2}{3}, \frac{2}{3}, \frac{1}{3} \right] \sim \left[ c_1 + \frac{1}{6}, c_1 - \frac{1}{6} \right] \cdot \left[ c_2 + \frac{1}{6}, c_2 - \frac{1}{6} \right], \quad (4.22)$$

by (4.14) and (4.19). It is easy to see that the unique solution of (4.22) is

$$c_1 = -\frac{1}{2}, \quad c_2 = \frac{1}{2}. \quad (4.23)$$

The third space $\mathcal{A}_3(u)$ requires more consideration. If we simply assume

$$\mathcal{A}_3(u) = W^{(2)}_3(u + c_3), \quad (4.24)$$
then, its fusion content becomes $[c_3 - \frac{1}{3}, c_3, c_3 + \frac{1}{3}]$ and there is no way to match it with (4.22). A trick is to consider the fusion

$$\tilde{W}^{(2)}_3(u) \mapsto W^{(2)}_1(u - \frac{1}{3}) \otimes \tilde{W}^{(2)}_1(u) \otimes W^{(2)}_1(u + \frac{1}{3}) \quad (4.25)$$
instead of (4.14). The fusion content of the rhs is then $[-\frac{1}{3}, \frac{2}{3}, -\frac{2}{3}, \frac{1}{3}]$, which is similar to (4.22). So we may identify $\mathcal{A}_3(u) = \tilde{W}^{(2)}_3(u)$. It follows that the only possible $T$-system for (4.20) based on the assumption of the exact sequence of type (4.7) is

$$T^{(1)}_1(u - \frac{1}{2})T^{(1)}_1(u + \frac{1}{2}) = T^{(1)}_2(u) + g(u)T^{(2)}_3(u), \quad (4.26)$$
and the function $g(u)$ is defined through the factorization

$$T_1^{(2)}(u) = g(u) T_1^{(2)}(u),$$  \hspace{1cm} (4.27)

where $T_1^{(2)}(u)$ is the transfer matrix associated to $\tilde{W}_1^{(2)}(u)$. Repeating these arguments, one finds that in the case $G$, our $T$-system (3.20e) is the only possible solution under the assumptions (4.7) and (4.14). The scalar functions in (3.20e) are fixed as

$$g_m^{(1)}(u) = g(u - \frac{m - 1}{2}) g(u - \frac{m - 3}{2}) \cdots g(u + \frac{m - 1}{2}), \quad g_m^{(2)}(u) = 1,$$  \hspace{1cm} (4.28)

where $g(u)$ is given in (4.27). In particular, (3.18) is automatically satisfied.

Similar analyses have been done for all $X_r$'s except $E_7$ and $E_8$. In all the cases, (3.20) has certainly a structure consistent with the fusion procedure, and furthermore the only possible solution in most cases.

Below we only present the necessary data of (a) the $R$-matrix, (b) the explicit form of (3.5a) for most fundamental representations [26,56], (c) the fusion contents of fundamental representations and (d) the nontrivial scalar functions $g_m^{(a)}(u) \neq 1$. All the spectral decomposition formulas of the rational $R$-matrices in data (a) are found in the references for $B_r$, $D_r$ [58], $C_r$ [59], $E_6$, $F_4$ [56]. Additional comments are included when necessary. Some of the fusion contents of the fundamental representations in $E_6$, $F_4$ are not directly proved yet as described in the remark for $F_4$. For $D_r$ and $E_6$, there is an additional type of isomorphisms by the conjugate transformation. Those are indicated in data (c) by the symbol $\sim$. In data (d), $T_1^{(a)}(u)$ is the transfer matrix associated to $\tilde{W}_1^{(a)}(u)$ which is found in data (c).

**$X_r = B_r$:**

$$((2r - 1 + 2u)(2r - 3 + 2u) \cdots (1 + 2u))^{-1} \tilde{R}(u)$$

$$= P_{2\Lambda_r} \oplus \bigoplus_{i=2, \text{even}}^{r} \frac{(2i - 1 - 2u)(2i - 5 - 2u) \cdots (3 - 2u)}{(2i - 1 + 2u)(2i - 5 + 2u) \cdots (3 + 2u)} P_{\Lambda_r - i}$$

$$\oplus \bigoplus_{i=1, \text{odd}}^{r} \frac{(2i - 1 - 2u)(2i - 5 - 2u) \cdots (1 - 2u)}{(2i - 1 + 2u)(2i - 5 + 2u) \cdots (1 + 2u)} P_{\Lambda_r - i}. \hspace{1cm} (4.29a)$$

$$W_1^{(a)} = V_{\Lambda_a} \oplus V_{\Lambda_{a-2}} \oplus \cdots, \quad (1 \leq a \leq r - 1), \quad W_1^{(r)} = V_{\Lambda_r}. \hspace{1cm} (4.29b)$$

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\[ W_1^{(a)}(u) : [\frac{2r-2a-1}{4}, -\frac{2r-2a-1}{4}], \quad (1 \leq a \leq r-1), \]
\[ W_1^{(r)}(u) : [0], \quad W_1^{(0)}(u) : [\frac{2r-1}{4}, -\frac{2r-1}{4}]. \quad (4.29c) \]

\[ g_m^{(1)}(u) = \prod_{k=1}^{m} g(u - \frac{m-2k+1}{2}), \quad T_1^{(0)}(u) = g(u) \text{Id.} \quad (4.29d) \]

\[ X_r = C_r: \]
\[ \tilde{R}(u) = (1 + 2u)(r + 1 + 2u)P_{2\Lambda_1} \oplus (1 + 2u)(r + 1 - 2u)P_0 \]
\[ \oplus (1 - 2u)(r + 1 + 2u)P_{\Lambda_2}. \quad (4.30a) \]

\[ W_1^{(a)} = V_{\Lambda_a}, \quad (1 \leq a \leq r). \quad (4.30b) \]
\[ W_1^{(a)}(u) : [\frac{a-1}{4}, -\frac{a-1}{4}], \quad (1 \leq a \leq r), \quad W_1^{(0)}(u) : [\frac{r+1}{4}, -\frac{r+1}{4}]. \quad (4.30c) \]

\[ g_m^{(r)}(u) = \prod_{k=1}^{m} g(u - \frac{m-2k+1}{2}), \quad T_1^{(0)}(u) = g(u) \text{Id.} \quad (4.30d) \]

\[ X_r = D_r: \] We have two minimal representations \( W_1^{(r-1)}(u) \) (conjugate spin reps.) and \( W_1^{(r)}(u) \) (spin reps.). Thus, we need two \( R \)-matrices \( \tilde{R}_{\Lambda_{r-1}, \Lambda_r}(u), \tilde{R}_{\Lambda_r, \Lambda_r}(u) \) below as building blocks of the \( Y(D_r) \) representation. We distinguish the fusion contents relevant to \( W_1^{(r-1)}(u+j) \) and \( W_1^{(r)}(u+j) \) as \([j]\) and \([j]\), respectively. There are isomorphisms of IFDRs for the “vector” type representations \( W_m^{(a)}(u) \) (1 \leq a \leq r - 2) as in (4.31c). The spectral decomposition of \( \tilde{R}_{\Lambda_r, \Lambda_{r-1}}(u) \) coincides with that of \( \tilde{R}_{\Lambda_{r-1}, \Lambda_r}(u) \). The spectral decomposition of \( \tilde{R}_{\Lambda_{r-1}, \Lambda_{r-1}}(u) \) is obtained by that of \( \tilde{R}_{\Lambda_r, \Lambda_r}(u) \) with \( P_{2\Lambda_r} \) replaced with \( P_{2\Lambda_{r-1}} \):

\[ (2[2(r-1)/2] + u) \cdots (2 + u)(2 + u)^{-1} \tilde{R}_{\Lambda_{r-1}, \Lambda_r}(u) \]
\[ = P_{\Lambda_{r-1} + \Lambda_r} \bigoplus \bigoplus_{i=1}^{[r-1]/2} \frac{(2i - u) \cdots (4 - u)(2 - u)}{(2i + u) \cdots (4 + u)(2 + u)} P_{\Lambda_{r-i-1}}, \]
\[ (2[r/2] - 1 + u) \cdots (3 + u)(1 + u)^{-1} \tilde{R}_{\Lambda_r, \Lambda_r}(u) \]
\[ = P_{2\Lambda_r} \bigoplus \bigoplus_{i=1}^{[r/2]} \frac{(2i - 1 - u) \cdots (3 - u)(1 - u)}{(2i + 1 - u) \cdots (3 + u)(1 + u)} P_{\Lambda_{r-2i}}. \quad (4.31a) \]

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$W_1^{(a)} = V_{\Lambda_a} \oplus V_{\Lambda_{a-2}} \oplus \cdots$ , $(1 \leq a \leq r - 2)$, $W_1^{(r-1)} = V_{\Lambda_{r-1}}$, $W_1^{(r)} = V_{\Lambda_r}$. \hspace{1cm} (4.31b)

$$W_1^{(a)}(u) : \begin{cases} \left[\frac{r-a-1}{2}, -\frac{r-a-1}{2}\right] \simeq \left[\frac{r-a-1}{2}, -\frac{r-a-1}{2}\right] , & r - a \text{ : even,} \\ \left[\frac{r-a-1}{2}, -\frac{r-a-1}{2}\right] \simeq \left[\frac{r-a-1}{2}, -\frac{r-a-1}{2}\right] , & r - a \text{ : odd,} \\ (1 \leq a \leq r - 2) , & \end{cases}$$

$$W_1^{(r-1)}(u) : [0], \quad W_1^{(r)}(u) : [0],$$

$$W_1^{(0)}(u) : \begin{cases} \left[\frac{r-1}{2}, -\frac{r-1}{2}\right] \simeq \left[\frac{r-1}{2}, -\frac{r-1}{2}\right] , & r \text{ : even,} \\ \left[\frac{r-1}{2}, -\frac{r-1}{2}\right] \simeq \left[\frac{r-1}{2}, -\frac{r-1}{2}\right] , & r \text{ : odd.} \\ \end{cases} \hspace{1cm} (4.31c)$$

$$g_m^{(1)}(u) = \prod_{k=1}^{m} g(u - \frac{m - 2k + 1}{2}) , \quad T_1^{(0)}(u) = g(u) \text{Id.} \hspace{1cm} (4.31d)$$

$X_r = E_1$: The fusion content of $W_1^{(2)}(u)$ cannot be determined immediately from the $R$-matrix. However, the assumption of the existence of the $T$-system corresponding to the relation $Q_1^{(2)} = Q_2^{(2)} + Q_1^{(1)} Q_2^{(3)}$ requires that the elements of the fusion content of $W_1^{(2)}(u)$ should be $\frac{1}{2}, 0, -\frac{1}{2}$. By the double square bracket $[[ \cdots ]]$ in the data, we indicate that the ordering of these elements are yet to be examined.

$$\tilde{R}(u) = (1 + 2u)(4 + 2u)(6 + 2u)(9 + 2u)P_{2\Lambda_4} \oplus (1 + 2u)(4 - 2u)(6 + 2u)(9 - 2u)P_0$$
$$\oplus (1 + 2u)(4 - 2u)(6 + 2u)(9 + 2u)P_{\Lambda_1} \oplus (1 - 2u)(4 + 2u)(6 + 2u)(9 + 2u)P_{\Lambda_2}$$
$$\oplus (1 - 2u)(4 + 2u)(6 - 2u)(9 + 2u)P_{\Lambda_3}.$$ \hspace{1cm} (4.32a)

$$W_1^{(1)}(u) = V_{\Lambda_1} \oplus V_{\Lambda_0}, \quad W_1^{(3)}(u) = V_{\Lambda_3} \oplus V_{\Lambda_4}, \quad W_1^{(4)}(u) = V_{\Lambda_4}. \hspace{1cm} (4.32b)$$

$$W_1^{(1)}(u) : [1, -1], \quad W_1^{(2)}(u) : [[\frac{1}{2}, 0, -\frac{1}{2}]], \quad W_1^{(3)}(u) : [\frac{1}{4}, -\frac{1}{4}],$$

$$W_1^{(4)}(u) : [0], \quad \tilde{W}_1^{(4)}(u) : [\frac{3}{2}, -\frac{3}{2}]. \hspace{1cm} (4.32c)$$

$$g_m^{(1)}(u) = \prod_{k=1}^{m} g(u - \frac{m - 2k + 1}{2}) , \quad \tilde{T}_1^{(4)}(u) = g(u) \tilde{T}_1^{(4)}(u). \hspace{1cm} (4.32d)$$

$X_r = E_6$: Like the $D_r$ case, we use the conjugate pair $W_1^{(1)}(u)$ and $W_1^{(5)}(u)$ as building blocks. The spectral decomposition of $\tilde{R}_{\Lambda_5 \Lambda_1}(u)$ coincides with that of $\tilde{R}_{\Lambda_1 \Lambda_5}(u)$. The
spectral decomposition of $\tilde{R}_{\Lambda_5,\Lambda_5}(u)$ is obtained from that of $\tilde{R}_{\Lambda_1,\Lambda_1}(u)$ by the simultaneous replacements $\Lambda_a \leftrightarrow \Lambda_{6-a}, 1 \leq a \leq 5$. (There is a discrepancy between the expressions of the $R$-matrix $\tilde{R}_{\Lambda_5,\Lambda_1}(u)$ in refs. [56] and [43] and the latter seems erroneous.) The fusion content of $W_1^{(3)}(u)$ is deduced in a similar way as in $F_4$.

$$
\tilde{R}_{\Lambda_1,\Lambda_1}(u) = (1 + u)(4 + u)P_{2\Lambda_1} \oplus (1 - u)(4 + u)P_{\Lambda_2} \oplus (1 - u)(4 - u)P_{\Lambda_5},
$$

$$
\tilde{R}_{\Lambda_5,\Lambda_1}(u) = (3 + u)(6 + u)P_{\Lambda_1 + \Lambda_5} \oplus (3 - u)(6 - u)P_0 \oplus (3 - u)(6 + u)P_{\Lambda_5}.
$$

(4.33a)

$$
W_1^{(1)} = V_{\Lambda_1}, \ W_1^{(2)} = V_{\Lambda_2} \oplus V_{\Lambda_5}, \ W_1^{(3)} = V_{\Lambda_2}, \ W_1^{(6)} = V_{\Lambda_5} \oplus V_0.
$$

(4.33b)

$$
W_1^{(1)}(u) : [0], \ W_1^{(2)}(u) : [\frac{1}{2}, -\frac{1}{2}], \ W_1^{(3)}(u) : [[1, 0, -1]] \simeq [[1, 0, -1]],
$$

$$
W_1^{(4)}(u) : [\frac{1}{2}, -\frac{1}{2}], \ W_1^{(5)}(u) : [0], \ W_1^{(6)}(u) : [\frac{3}{2}, -\frac{3}{2}] \simeq [\frac{3}{2}, -\frac{3}{2}],
$$

$$
\bar{W}_1^{(5)}(u) : [2, -2].
$$

(4.33c)

$$
g_m^{(6)}(u) = \prod_{k=1}^{m} g(u - m - 2k + 1), \ \bar{T}_1^{(5)}(u) = g(u)\bar{T}_1^{(5)}(u).
$$

(4.33d)

5. Determinant formulas for $B_r, C_r$ and $D_r$

In this section we consider the classical series $X_r = B_r, C_r, D_r$ and observe that (3.20) leads to remarkable conjectures of determinant formulas analogous to (2.20). We regard these as further supports of our $T$-system from an algebraic viewpoint.

As is well known, by the tensor product the FDRs of $X_r$ generate a ring $R(X_r)$ called the representation ring [60]. The character (3.6b) is a one-dimensional representation of $R(X_r)$. From the definition of $Q_m^{(a)}$ in (3.6a), the $Q$-system is a set of relations of the ring $RQ(X_r) \subset R(X_r)$ generated by $Q_m^{(a)}$‘s. In the same way, the restricted $Q$-system is that of the ring $RQ_{\ell}(X_r) \subset R_{\ell}(X_r) = R(X_r)/I_\ell$ generated by the images of $Q_m^{(a)}$‘s. Here $I_\ell$ is the ideal generated by the FDRs of $X_r$-modules whose quantum dimensions at $q = \exp(2\pi i/(\ell + g))$ are zero. The matrices $\overline{M}$ and $M$ in (3.13) are representations of $RQ(X_r)$ and $RQ_{\ell}(X_r)$, respectively. It is now natural to regard the rings $RT(X_r)$ and $RT_{\ell}(X_r)$ of the transfer matrices for the vertex and level $\ell$ RSOS models as quantum analogues of
the $RQ(X_r)$ and $RQ_e(X_r)$, respectively. Though the tensor products of quantum group modules here are indecomposable, the transfer matrices have a decomposability property and play a role of the “character” in $RT(X_r)$.

For our $T$-system (3.20), one can in principle solve it successively for $T_m^{(a)}(u)$ in terms of the $T_1^{(a)}(u)$’s. Based on studies of several examples, we conjecture that $T_m^{(a)}(u)$ is expressed as a polynomial in $T_1^{(a)}(u)$’s, i.e., the ring $RT(X_r)$ is generated by $T_1^{(a)}(u)$’s for any $X_r$. This is a natural quantum analogue of the fact that $R(X_r)$ is generated by the fundamental representations [60].

Furthermore, in the $A_r$ case, all the $T_m^{(a)}(u)$ are expressed through a beautiful determinant formula (2.20b). This was a consequence of the resolutions of quantum group modules [41], whose classical counterpart has been also studied in [61, 62]. Below we present the conjectures on determinant formulas for $B_r, C_r, D_r$ analogous to (2.20b). They seem novel even in the classical context to authors’ knowledge. The following notations are adopted below.

1. $T_1^{(a)}(u + k) = x_k^a$ for $0 \leq a \leq r$.

2. For a semi-infinite dimensional matrix $M = (M_{ij})_{1 \leq i, j < \infty}$, we denote by $M(a, b, m)$ the $m \times m$ submatrix $(M_{ij})_{a \leq i \leq a+m-1, b \leq j \leq b+m-1}$.

$X_r = B_r$:

We find it convenient to deal with the following $T$-system which is equivalent to (3.20b) through some elementary redefinitions.

$$T_m^{(a)}(u)T_m^{(a)}(u + 2) = T_{m+1}^{(a)}(u)T_{m-1}^{(a)}(u + 2) + T_m^{(a+1)}(u)T_m^{(a-1)}(u + 2),$$

$$(1 \leq a \leq r - 2),$$

$$T_m^{(r-1)}(u)T_m^{(r-1)}(u + 2) = T_{m+1}^{(r-1)}(u)T_{m-1}^{(r-1)}(u + 2) + T_m^{(r-2)}(u + 2)T_{2m}^{(r)}(u + 1),$$

$$T_{2m}^{(r)}(u)T_{2m}^{(r)}(u + 1) = T_{2m+1}^{(r)}(u)T_{2m-1}^{(r)}(u + 1) + T_{m-1}^{(r-1)}(u)T_{m}^{(r-1)}(u + 1),$$

$$T_{2m+1}^{(r)}(u)T_{2m+1}^{(r)}(u + 1) = T_{2m+2}^{(r)}(u)T_{2m}^{(r)}(u + 1) + T_{m-1}^{(r-1)}(u + 1)T_{m+1}^{(r-1)}(u).$$

In the above, $T_m^{(0)}(u)$ is to be understood as the scalar function obeying $T_m^{(0)}(u) = T_1^{(0)}(u)T_1^{(0)}(u + 2)\cdots T_1^{(0)}(u + 2m - 2)$ corresponding to (4.29d). Solving (5.1) recursively, one finds, for example in $B_3$ case, that

$$T_2^{(1)}(u) = x_0^1x_1^1 - x_2^0x_0^2,$$

$$T_2^{(2)}(u) = x_0^2x_2^1 - x_2^1x_1^3 + x_2^1x_1^2.$$
Then, we find the following determinant expressions.

\[ T_2^{(3)}(u) = x_0^3 x_1^3 - x_0^2, \]
\[ T_3^{(1)}(u) = x_0^1 x_2^1 x_4^1 - x_0^0 x_1^0 x_2^0 - x_2^0 x_4^0 x_0^0 - x_2^0 x_0^0 x_4^0 - x_0^0 x_4^0 x_2^0, \]
\[ T_3^{(2)}(u) = -x_0^1 x_3^1 x_4^1 + x_4^0 x_2^0 x_3^0 + x_2^0 x_1^0 x_4^0 - x_1^0 x_3^0 x_4^0 + x_2^0 x_0^0 x_3^0 + x_0^0 x_1^0 x_3^0 - x_4^0 x_1^0 x_3^0 + x_4^0 x_0^0 x_2^0 - x_4^0 x_2^0 x_0^0 - x_2^0 x_0^0 x_4^0, \]
\[ T_3^{(3)}(u) = -x_1^2 x_0^3 - x_0^2 x_2^3 + x_0^2 x_1 x_3. \] (5.2)

In (5.3) we separate the matrices into two parts by intention. As seen immediately, two matrices in (5.3a,b) are submatrices of the corresponding ones in (5.3c,d). Furthermore, the second matrices in (5.3c) and (5.3d) are identical while the first ones seem to be submatrices of a common bigger size matrix. These observations are extended to a general conjecture for \( B_r \) as follows. Let us define the semi-infinite dimensional matrices \( T^{B_r} \) and \( \varepsilon^{B_r} \) by

\[
T_{ij}^{B_r} = \begin{cases} 
\frac{i-j+1}{2} & \text{if } i \in 2\mathbb{Z} + 1 \text{ and } \frac{i-j}{2} \in \{1, 0, \ldots, 2 - r\}, \\
-x_{i-j+2r-2} & \text{if } i \in 2\mathbb{Z} + 1 \text{ and } \frac{i-j}{2} \in \{1 - r, -r, \ldots, 2 - 2r\}, \\
x_{i-j+2r} & \text{if } i \in 2\mathbb{Z} \text{ and } j = i + 2r - 3, \\
0 & \text{otherwise},
\end{cases}
\]

\[
\varepsilon_{ij}^{B_r} = \begin{cases} 
\pm 1 & \text{if } i = j - 1 \pm 1 \text{ and } i \in 2\mathbb{Z}, \\
x_i^r & \text{if } i = j - 1 \text{ and } i \in 2\mathbb{Z} + 1, \\
0 & \text{otherwise}.
\end{cases}
\]
For example, for $B_4$, they look as

$$T^{B_4} = \begin{pmatrix}
x_1^4 & 0 & x_0^2 & 0 & x_3^4 & 0 & -x_1^3 & 0 & -x_3^2 & 0 & -x_1^1 & 0 & -x_7^0 \\
0 & 0 & 0 & 0 & 0 & -x_4^4 & 0 & 0 & 0 & 0 & 0 & 0 \\
x_2^0 & 0 & x_2^1 & 0 & x_3^2 & 0 & -x_3^3 & 0 & -x_2^2 & 0 & -x_1^1 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & -x_4^4 & 0 & 0 & 0 & 0 & 0 
\end{pmatrix}, \quad (5.5a)$$

$$\varepsilon^{B_4} = \begin{pmatrix}
0 & x_1^4 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & x_3^4 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & x_5^4 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 1 & 0 & -1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & x_7^4 & 0 & \cdots & \cdots 
\end{pmatrix}. \quad (5.5b)$$

Then we have a conjecture

$$T_m^{(a)}(u) = \det(T^{B_r}(1, 2a - 1, 2m - 1) + \varepsilon^{B_r}(1, 1, 2m - 1)), \quad (5.6)$$

for $1 \leq a \leq r - 1, m \geq 1$. This has been verified up to $r = 5, m = 4$.

$X_r = C_r$:

We consider the following $T$-system which is equivalent to (3.20c) through some elementary redefinitions.

$$T_m^{(a)}(u)T_m^{(a)}(u + 1) = T_{m+1}^{(a)}(u)T_{m-1}^{(a)}(u + 1) + T_m^{(a+1)}(u)T_m^{(a-1)}(u + 1), \quad (1 \leq a \leq r - 2),$$

$$T_{2m}^{(r-1)}(u)T_{2m}^{(r-1)}(u + 1) = T_{2m+1}^{(r-1)}(u)T_{2m-1}^{(r-1)}(u + 1) \quad + T_{2m-1}^{(r-2)}(u + 1)T_m^{(r)}(u)T_m^{(r)}(u + 1), \quad (5.7)$$

Here we take $T_m^{(0)}(u) \equiv 1$ in the first equation and the scalar in the last equation satisfies $g_m(u) = g_1(u)g_1(u + 2)\cdots g_1(u + 2m - 2)$ corresponding to (4.30d). Define the semi-infinite dimensional matrix $T^{C_r}$ by

$$T_{ij}^{C_r} = \begin{cases}
x_{i-1}^{j-i+1} & \text{if } i - j \in \{-1, 0, \ldots, 1 - r\}, \\
y_{i,j}x_{i-j-1}^{2r+1} & \text{if } i - j \in \{-1 - r, -2 - r, \ldots, -1 - 2r\}, \\
0 & \text{otherwise},
\end{cases} \quad (5.8)$$

$$y_{ij} = g_1(u + i - 1)g_1(u + i)\cdots g_1(u + j - r - 2),$$

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where we assume $\forall x_k^0 = 1$. For example, for $C_4$, it looks as

$$T^{C_4} = \begin{pmatrix}
  x_1^4 & x_0^4 & 0 & -y_16 x_1^4 & -y_17 x_0^4 & -y_18 x_1^3 & -y_19 x_0^3 \\
  1 & x_1^3 & x_0^3 & x_0^3 & 0 & -y_27 x_1^2 & -y_28 x_0^2 & -y_29 x_1 \\
  0 & 1 & x_2^2 & x_1^2 & x_0^2 & x_0^2 & 0 & -y_38 x_1 & -y_39 x_0 \\
  \vdots & & & & & & & & \vdots
\end{pmatrix}. \quad (5.9)$$

Then we have a conjecture

$$T_m^{(a)}(u) = \det T^{C_r}(1, a, m), \quad (5.10)$$

for $1 \leq a \leq r - 1, m \geq 1$, which can be shown to actually satisfy the first equation in (5.7) as in section 2.3. From (5.10), $T_m^{(r)}(u)$ can be solved recursively as

$$T_m^{(r)}(u) = T_{m-1}^{(r)}(u) \det T^{C_r}(1, r, 2m - 1) \det T^{C_r}(1, r, 2m - 2). \quad (5.11)$$

We have verified that (5.11) actually yields a polynomial in $x_k^a$'s up to $r = 5, m = 4$.

$X_r = D_r$:

We consider the following $T$-system which is equivalent to (3.20a) through some elementary redefinitions.

$$T_m^{(a)}(u)T_m^{(a)}(u + 1) = T_{m+1}^{(a)}(u)T_{m-1}^{(a)}(u + 1) + T_{m+1}^{(a+1)}(u)T_{m-1}^{(a-1)}(u + 1), \quad (1 \leq a \leq r - 3), \quad (5.12)$$

$$T_m^{(r-2)}(u)T_m^{(r-2)}(u + 1) = T_{m+1}^{(r-2)}(u)T_{m-1}^{(r-2)}(u + 1) + T_{m+1}^{(r-3)}(u + 1)T_{m-1}^{(r)}(u + 1), \quad (5.12)$$

$$T_m^{(b)}(u)T_m^{(b)}(u + 1) = T_{m+1}^{(b)}(u)T_{m-1}^{(b)}(u + 1) + T_{m+1}^{(r-2)}(u), \quad b = r - 1, r. \quad (5.12)$$

As in $B_r$ case, we here interpret $T_m^{(0)}(u)$ as the scalar function obeying $T_m^{(0)}(u) = T_1^{(0)}(u)T_1^{(0)}(u + 1) \cdots T_r^{(0)}(u + m - 1)$ corresponding to (4.31d). Define the semi-infinite dimensional matrices $T^{D_r}$ and $\varepsilon^{D_r}$ by

$$T_{ij}^{D_r} = \begin{cases}
  \pm 1 & \text{if } i = j - 2 \pm 2 \text{ and } i \in 2\mathbb{Z}, \\
  x_{i-2}^{r+1} & \text{if } i = j - 3 \text{ and } i \in 2\mathbb{Z}, \\
  -x_{i-2}^{r-1} & \text{if } i = j - 2 \pm 2 \text{ and } i \in 2\mathbb{Z}, \\
  x_{i-2}^{r} & \text{if } i = j - 2 \pm 2 \text{ and } i \in 2\mathbb{Z},
\end{cases} \quad (5.13a)$$

$$\varepsilon_{ij}^{D_r} = \begin{cases}
  \pm 1 & \text{if } i = j - 2 \pm 2 \text{ and } i \in 2\mathbb{Z}, \\
  x_{i-2}^{r+1} & \text{if } i = j - 3 \text{ and } i \in 2\mathbb{Z}, \\
  -x_{i-2}^{r-1} & \text{if } i = j - 2 \pm 2 \text{ and } i \in 2\mathbb{Z}, \\
  x_{i-2}^{r} & \text{if } i = j - 2 \pm 2 \text{ and } i \in 2\mathbb{Z},
\end{cases} \quad (5.13b)$$

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For example, for $D_4$, they look as

\[
T^{D_4} = \begin{pmatrix}
  x_0^4 & 0 & x_0^3 & -x^3_1 & 0 & -x^4_1 & -x_1^2 & 0 & -x_1^3 & 0 \\
  0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
  x_1^0 & 0 & x_1^1 & 0 & x_1^2 & -x^3_2 & 0 & -x^4_2 & -x_2^2 & 0 & -x_2^3 \\
  0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
  & & & & & & & & & & \\
  & & & & & & & & & & \\
  & & & & & & & & & & \\
  & & & & & & & & & & \\
  & & & & & & & & & & \\
  & & & & & & & & & & \\
\end{pmatrix}, \quad (5.14a)
\]

\[
\varepsilon^{D_4} = \begin{pmatrix}
  0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
  0 & 1 & x_0^4 & 0 & x_1^3 & -1 & 0 & 0 & 0 & 0 \\
  0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
  0 & 0 & 0 & 1 & x_1^4 & 0 & x_2^3 & -1 & 0 \\
  & & & & & & & & & & \\
  & & & & & & & & & & \\
  & & & & & & & & & & \\
  & & & & & & & & & & \\
  & & & & & & & & & & \\
  & & & & & & & & & & \\
\end{pmatrix}. \quad (5.14b)
\]

Then we have a conjecture

\[
T_m^{(a)}(u) = \det(T^{D_r}(1, 2a - 1, 2m - 1) + \varepsilon^{D_r}(2a - 1, 2a - 1, 2m - 1)), \quad (5.15)
\]

for $1 \leq a \leq r - 2, \; m \geq 1$.

We have not found simple expressions for $T_m^{(r)}(u)$ for $B_r$ and $T_m^{(r-1)}(u)$ and $T_m^{(r)}(u)$ for $D_r$ but checked that they are expressed as polynomials in $T_1^{(a)}(u)$’s at least for small $r$ and $m$.

6. Summary

In this paper we have proposed the $T$-system (3.20), a new set of functional relations among the commuting family of the row-to-row transfer matrices, for a class of solvable lattice models. They are the vertex and RSOS type models associated with the simple Lie algebras $X_r$ as sketched in section 3.3. For $X_r = sl(2)$, we have shown in section 2.2 that the $T$-system is governed by exact sequences of the quantum group modules, which provide a representation theoretical background for a general $X_r$ case as well. An intriguing connection has thereby been proved in section 2.3 among the $T$, $Q$ and $Y$-systems for $X_r = A_r$, indicating deep relations to the TBA and dilogarithm identities, etc. Though the meaning of the connection is yet to be understood, postulating it for the other $X_r$’s has led to our main proposal (3.20). In fact, (3.20) is almost the unique possible system of the FRs once one admits the underlying exact sequences as argued in section 4. We have also observed that our $T$-system leads to remarkable determinant formulas for
$B_r, C_r$ and $D_r$ in section 5, which may be viewed as a further support for our proposal. The $T$-system is to hold either for critical or off-critical models and for finite or infinite system sizes. Its application to the calculation of various thermodynamic quantities will be presented in Part II.

An important open problem is to understand the meaning of the $Y$-system (B.6) and its connection to the $T$-system (3.19). While the $T$-system involves the scalar functions (3.18), the $Y$-system does not. This indicates that the latter is an object related to $Y(X_r)$ or $U_q(X_r^{(1)})$ and not their central extensions. We believe that the connection (3.19) is deep and its explanation will shed a new light to the theories of solvable lattice models and related subjects.

There are other kinds of FRs known for many models described in section 3.3. For instance, those in eqs.(9.8.40) and (10.5.32) of [1] are typical such examples, which may be viewed more broadly as the FRs from the analytic Bethe ansatz [63]. It would be interesting to investigate the implication of our approach in the light of the analytic Bethe ansatz. Finally, we remark that the FRs of the transfer matrices have also been studied [64–66] for the chiral Potts model [67]. The relevant representations there are so-called cyclic representations, which only exist at $q$ a root of unity. To widen our scope to such “genuine” quantum representations would be an important subject.

Acknowledgements

The authors would like to thank V.V. Bazhanov, I. Cherednik, G. Delius, E. Frenkel, T. Hayashi, M. Hashimoto, S. Hosono, M. Jimbo, A. Matsuo, T. Miwa, Y-H. Quano and V.O. Tarasov for their helpful discussions. This work is supported in part by JSPS fellowship, NSF grants PHY-87-14654, PHY-89-57162 and Packard fellowship.

Appendix A. Remarks on $Q$-system

A.1. Origin of eq.(3.5)

Let us sketch a derivation of (3.5) (eq.(21) in [26]). Let $W_{m_j}^{(a_j)}(u_j), j = 1, 2, \ldots, N$ be the IFDRs of $Y(X_r)$ as described in section 3.2 for some integers $m_j \geq 0, 1 \leq a_j \leq r$ and complex parameters $u_j$. For those $u_j$’s in a general position, the $N$-fold tensor product
\( \bigotimes_{j=1}^{N} W_{m_{ij}}^{(a_j)}(u_j) \) is known to be independent of the order and irreducible as a \( Y(X_r) \)-module. Decomposing it into IFDRs of \( X_r \subset Y(X_r) \), one has

\[
\bigotimes_{j=1}^{N} W_{m_{ij}}^{(a_j)}(u_j) = \bigoplus_{\Lambda} Z(\{a_j\}, \{m_j\}, \sum_{j=1}^{N} m_j a_j - \Lambda) V_{\Lambda}, \tag{A.1}
\]

where \( V_{\Lambda} \) is the IFDR of \( X_r \) with highest weight \( \Lambda \) and \( Z(\cdots) \) denotes its multiplicity which is independent of \( u_j \)'s in a general position. By the definition, the sum in (A.1) extends over those \( \Lambda \) having the form

\[
\Lambda = \sum_{j=1}^{N} m_j a_j - \sum_{b=1}^{r} n_b a_b \tag{A.2}
\]

for some non-negative integers \( n_b \). The \( Z \) in (3.5) is the special case of the above multiplicity corresponding to the simplest situation \( N = 1, a_1 = a, m_1 = m \). The function \( Z(\{a_j\}, \{m_j\}, \sum_{b=1}^{r} n_b a_b) \) for general \( \{a_j\}, \{m_j\} \) has been computed in [26,68] from rational Bethe ansatz equations. Their calculation essentially reduces to counting available ranges for “branch cut integers” of the Bethe ansatz equation (BAE) (B.2).

Here we provide an alternative prescription to recover their formula at least formally based on the observation in [69]. The argument is essentially equivalent to [26] but closer to the TBA context in appendix B. Consider the “inhomogeneous version” of the BAE (B.2) that corresponds to the quantum space choice (A.1) in the rational limit. The equation (B.3) may then formally be replaced by

\[
\frac{1}{N} \sum_{j=1}^{N} \delta_{a a_j} \hat{A}_{a_j a}^{m_j m} = \sigma_m(a) + \sum_{(b,k) \in G} K_{ab} m_k N_k(b) \tag{A.3}
\]

for \( N \) tending to infinity. Taking the \( x = 0 \)-th Fourier component (B.8) of this, we have

\[
\sum_{j=1}^{N} \delta_{a a_j} \hat{A}_{a_j a}^{m_j m}(0) = M_m(a) + \sum_{(b,k) \in G} K_{ab} m_k N_k(b) \tag{A.4}
\]

by using (B.21a). Here, \( N_m(a) \), \( M_m(a) \) are the total number (infinite actually) of color \( a \) \( m \)-strings and holes by which we have put \( \hat{\rho}_m(a)(0) = N_m(a)/N \), \( \sigma_m(a)(0) = M_m(a)/N \). By considering the rational limit \( \ell \to \infty \) further, (A.4) can be solved for \( M_m(a) \) as

\[
M_m(a) = \sum_{j=1}^{N} \delta_{a a_j} \min(m, m_j) - \sum_{b=1}^{r} \sum_{k=1}^{\infty} (\alpha_a | \alpha_b)(a, b) \min(t_k m, t_a k) N_k(b) \tag{A.5}
\]
by using (B.11) and (B.21b). Now regard (A.5) as a formal relation among the infinitely many integers \( M_m^{(a)}, N_m^{(a)} (1 \leq a \leq r, m \in \mathbb{Z}_{\geq 1}) \) for any finite \( N \). This makes sense since \( N \) now only appears as a parameter. Then the multiplicity formula in [26] is given as follows:

\[
Z(\{a_j\}, \{m_j\}, \sum_{b=1}^{r} n_b a_b) = \sum_{\{N^{(b)}_k\}} \prod_{a=1}^{r} \prod_{m=1}^{\infty} \left( \frac{N_m^{(a)} + M_m^{(a)}}{N_m^{(a)}} \right),
\]

(A.6a)

where the sum extends over all the non-negative integers \( \{N^{(b)}_k\} \) such that

\[
n_b = \sum_{k=1}^{\infty} k N_k^{(b)} \quad \text{for all } 1 \leq b \leq r, \quad \text{(A.6b)}
\]

\[
M_m^{(a)} \text{ in (A.5) } \geq 0 \quad \text{for all } 1 \leq a \leq r, m \geq 1. \quad \text{(A.6c)}
\]

The formula is just counting the total number of those string-hole arrangements that obey the BAE constraint (A.5).

A.2. General specializations, charge function and congruence

It is possible to generalize the restricted \( Q \)-system (3.10) by considering the specializations \( z = \Lambda \in P_\ell \). Supported by numerical experiments we assume that

\[
Q_m^{(a)}(\Lambda) = Q_{\ell_a}^{(a)}(\Lambda)Q_{\ell_a-m}^{(a)}(\Lambda)^* \quad \text{for any } -1 \leq m \leq \ell_a + 1 \text{ and } \Lambda \in P_\ell, \quad \text{(A.7)}
\]

where * denotes complex conjugation. Note in particular that it implies

\[
Q_{\ell_a+1}^{(a)}(\Lambda) = 0, \quad \text{(A.8)}
\]

from which one finds that the \( Q \)-system (3.7) closes within those \( Q_m^{(a)}(\Lambda) \) with \( 0 \leq m \leq \ell_a \).

Due to (A.7) and (B.23b), we have

\( \Lambda \)-restricted \( Q \)-system

\[
Q_m^{(a)}(\Lambda)^2 = Q_{m-1}^{(a)}(\Lambda)Q_{m+1}^{(a)}(\Lambda) + \prod_{b=1}^{r} Q_{b}^{(b)}(\Lambda)^2 \prod_{k=1}^{\ell_b} Q_{k}^{(b)}(\Lambda)^{-2} \prod_{m}^{\infty} \left( \frac{N_m^{(a)} + M_m^{(a)}}{N_m^{(a)}} \right)^m \quad \text{for } (a, m) \in G, \quad \text{(A.9a)}
\]

\[
\prod_{b=1}^{r} Q_{b}^{(b)}(\Lambda)^{C_{ab}} = 1 \quad \text{for } 1 \leq a \leq r. \quad \text{(A.9b)}
\]
It is possible to determine the values $Q^{(a)}_{\ell_a}(\Lambda)$ as we shall describe below. Thus, (A.9a) can be also viewed as relations among $\{Q^{(a)}_m(\Lambda) \mid (a,m) \in G\}$ with $Q^{(a)}_{\ell_a}(\Lambda)$ as an external input.

The eq.(A.9b) says that $Q^{(a)}_{\ell_a}(\Lambda)$'s are $\kappa$-th roots of unity with $\kappa = \det(C_{ab})_{1 \leq a,b \leq r}$ = number of the $X^{(1)}_r$. Kac labels $a_i (0 \leq i \leq r)$ equal to 1. Explicitly, $\kappa$ is given by

\[
X_r \quad A_r \quad B_r \quad C_r \quad D_r \quad E_6 \quad E_7 \quad E_8 \quad F_4 \quad G_2 \\
\kappa \quad r + 1 \quad 2 \quad 2 \quad 4 \quad 3 \quad 2 \quad 1 \quad 1 \quad 1 .
\]

(A.10)

Being $\kappa$-th roots of unity, their actual values can easily be fixed from a numerical analysis. The following is the so obtained result and was announced earlier in [44]. (Eqs. (6) and (7) therein contain misprints and should read as (A.11,12) below.)

\[
Q^{(a)}_{\ell_a}(\Lambda) = \exp(-2\pi i \Gamma(\Lambda)\tau_a/\kappa) \quad \text{for} \quad X_r \neq D_r, \\
= \exp(2\pi i((r \Gamma(\Lambda) + \Gamma_2(\Lambda))\gamma^{(1)}_a + \Gamma_1(\Lambda)\gamma^{(2)}_a)/\kappa) \quad \text{for} \quad X_r = D_r, \quad r : \text{even}, \\
= \exp(2\pi i \Gamma_2(\Lambda)((r + 1)\gamma^{(1)}_a + \gamma^{(2)}_a)/\kappa) \quad \text{for} \quad X_r = D_r, \quad r : \text{odd}.
\]

(A.11)

Here, for $\mu = \sum_{a=1}^r \mu_a \Lambda_a \in \mathcal{H}^*$, we have set

\[
\Gamma(\mu) = \sum_{a=1}^r \gamma_a \mu_a \quad \text{for} \quad X_r \neq D_r, \\
\Gamma_i(\Lambda) = \sum_{a=1}^r \gamma^{(i)}_a \mu_a \quad (i = 1, 2) \quad \text{for} \quad X_r = D_r,
\]

(A.12a)

and $\gamma = (\gamma_1, \ldots, \gamma_r)$ is the rank-dimensional integer vector given by

- $A_r, C_r, E_6 : \gamma_a = a,$
- $B_r : \gamma = (0, \ldots, 0, 1),$
- $D_r : \gamma^{(1)} = (0, \ldots, 0, 1, 1), \quad \gamma^{(2)} = (2, 4, 6, \ldots, 2(r - 2), r - 2, r),$
- $E_7 : \gamma = (0, 0, 0, 1, 0, 1, 1),$
- $E_8, F_4, G_2 : \gamma = (0, \ldots, 0).$

(A.12b)

The $\tau$ in the first line of (A.11) is specified from the above $\gamma$ by the rule $\tau = -\gamma$ if $X_r = A_r$ and $\tau = \gamma$ of the dual algebra of $X_r$ if $X_r \neq A_r, D_r$. We note that when $X_r = D_r$, there is an obvious constraint $1 - \frac{(-1)^r}{2} \Gamma_1(\Lambda) + \Gamma_2(\Lambda) \in 2\mathbb{Z}$ for $\Lambda \in P$. It is zero for $\Lambda = 0$ hence
\[ (3.9b) \text{ is just the special case of } (A.11). \text{ We call } \Gamma(\mu) \text{ the charge function. Below we summarize its properties that will be of use in Part II.} \]

The charge function dictates the congruence property of the weight lattice element
\[ \mu = \sum_{a=1}^{r} \mu_a \Lambda_a \in P \text{ mod root lattice } Q \text{ as follows,} \]
\[
\mu \equiv 0 \mod Q \iff \Gamma(\mu) \equiv 0 \mod \kappa \text{ for } X_r \neq D_r,
\]
\[
\iff \Gamma_i(\mu) \equiv 0 \mod 2i \ (i = 1, 2) \text{ for } X_r = D_r. \tag{A.13}
\]

This originates in the relation \( \alpha_a = \sum_{b=1}^{r} C_{ba} \Lambda_b \) and the property of the \( \gamma \)-vectors in \( (A.12) \) as follows:
\[
\sum_{a=1}^{r} \gamma_a C_{ab} \equiv \sum_{b=1}^{r} C_{ab} \gamma_b \equiv 0 \mod \kappa \text{ for } X_r \neq D_r,
\]
\[
\sum_{a=1}^{r} \gamma_a^{(i)} C_{ab} \equiv 0 \mod 2i \ (i = 1, 2) \text{ for } X_r = D_r. \tag{A.14}
\]

As is well known, the weight lattice is divided into \( \kappa \) distinct classes each of which consists of the elements congruent under the root lattice \( Q \). In particular for \( X_r = E_8, F_4 \) and \( G_2 \), one has \( \kappa = 1 \) hence every integral weight is congruent under \( Q \). The following formula will also be useful as well as \( (A.13) \),
\[
\frac{1}{\kappa} \sum_{1 \leq a, b \leq r} \gamma_a C_{ab} \gamma_b \equiv 1 \mod \kappa \text{ for } X_r \neq D_r, \tag{A.15a}
\]
\[
\frac{1}{\kappa} \sum_{1 \leq a, b \leq r} \gamma_a^{(i)} C_{ab} \gamma_b^{(j)} = \begin{cases} r & \text{if } i = j = 2 \\ 1 & \text{otherwise} \end{cases} \text{ for } X_r = D_r. \tag{A.15b}
\]

**Appendix B. \( Y \)-system**

**B.1. \( Y \)-system from TBA**

Let us recapitulate the \( U_q(X_r^{(1)}) \) functional relation introduced in [22] based on the TBA analyses [9,13,17]. Formally the same relation has been noticed for some \( X_r \) in the context of the TBA for the perturbed CFTs [18–23]. Here we call the functional relation simply the \( Y \)-system borrowing the naming in [23]. See section 3.1 for the notations.
Take any simple Lie algebra $X_r$ and fix the integers $\ell \geq 1$, $p$ and $s$ so that $(p, s) \in G$. Choose a positive integer $N_a$ such that

\[ N_a \stackrel{\text{def}}{=} N s(C^{-1})_{a p} \in \mathbb{Z} \quad \text{for all} \quad 1 \leq a \leq r. \quad (B.1) \]

Then the following Bethe ansatz equation (BAE) for $\{ u_{j}^{(a)} | 1 \leq a \leq r, 1 \leq j \leq N_a \}$ was considered in [9,13,17]:

\[ \left( \frac{\sinh \left( \frac{\varpi}{2 L} (u_{j}^{(a)} + i \frac{s}{L} \delta_{a p}) \right)}{\sinh \left( \frac{\varpi}{2 L} (u_{j}^{(a)} - i \frac{s}{L} \delta_{a p}) \right)} \right)^N = \Omega_{j}^{(a)} \prod_{b=1}^{r} \prod_{k=1}^{N_b} \frac{\sinh \left( \frac{\varpi}{2 L} (u_{j}^{(a)} - u_{k}^{(b)} + i (a | \alpha_b)) \right)}{\sinh \left( \frac{\varpi}{2 L} (u_{j}^{(a)} - u_{k}^{(b)} - i (a | \alpha_b)) \right)}, \quad (B.2) \]

where $L = \ell + g$ and $\Omega_{j}^{(a)}$ is some phase factor without which (B.2) is essentially the BAE proposed in [63]. The above BAE is indeed valid [9,13] for the $A_{p}^{(1)}$ RSOS model [35] and is a candidate describing the transfer matrix eigenvalues of the level $\ell$ critical $X_{p}^{(1)}$ RSOS model with fusion type $W_s^{(p)}$ in general. In Part II, we will investigate such RSOS models with the aid of the $T$-system (3.20). However there is yet an alternative approach, namely, the TBA analysis of (B.2) upon the special string hypothesis employed in [9,13,17]. The analysis was the source of our $Y$-system and the rest of this section is a quick digest of it following section 2 in [17].

Denote by $\mathcal{A}_{m}^{(a)}$ the number of $u_{j}^{(a)}$'s that form the color $a$ $m$-string $\{ u + it^{a}^{-1}(m + 1 - 2n) | 1 \leq n \leq m \}$ $(u \in \mathbb{R})$ in the thermodynamic limit $N \to \infty$. Then the hypothesis is to assume $\lim_{N \to \infty} \sum_{m=1}^{\ell_a} \mathcal{A}_{m}^{(a)} / N_a = 1$ for all $1 \leq a \leq r$. It means that for color $a$, only those strings with length $\leq \ell_a$ contribute to the thermodynamic quantities. Then one introduces the string and hole densities $\rho_{m}^{(a)}(u)$ and $\sigma_{m}^{(a)}(u)$, respectively $(1 \leq a \leq r, 1 \leq m \leq \ell_a)$ and transforms (B.2) into an integral equation. This is a fairly standard calculation going back to [16]. A peculiar feature here is that $\sigma_{a}^{(a)}(u) \equiv 0$ follows automatically from the resulting equation, string hypothesis and (B.1). After some reduction owing to this, the integral equation becomes

\[ \delta_{pa} \mathcal{A}_{pa}^{m} = \sigma_{m}^{(a)} + \sum_{(b,k) \in G} \mathcal{K}_{ab}^{mk} * \rho_{k}^{(b)} \quad \text{for} \quad (a, m) \in G. \quad (B.3) \]

See section B.2 for the definitions of $\mathcal{A}_{ab}^{mk}, \mathcal{K}_{ab}^{mk}$ and the symbol $*$. Under the constraint (B.3), one demands the free energy be extremum with respect to $\rho_{m}^{(a)}$ and deduces the
equilibrium condition

\[
\frac{e^{\ell_a \delta_{pa}} \delta_{sm}}{4T \cosh(t_a \pi u/2)} = \sum_{n=1}^{\ell_a-1} \int_{-\infty}^{\infty} dv K_a^{mn}(u - v) \log(1 + \exp(\epsilon_n^{(a)}(v))) - \sum_{(b,k) \in G} \int_{-\infty}^{\infty} dv J_{ab}^{mk}(u - v) \log(1 + \exp(-\epsilon_k^{(b)}(v))),
\]  

(B.4a)

\[\sigma_m^{(a)}(u)/\rho_m^{(a)}(u) = \exp(\epsilon_m^{(a)}(u)),\]  

(B.4b)

for \((a, m) \in G\). Here \(T\) denotes the temperature and \(\epsilon = \pm 1\) on the lhs, which specifies the two regimes as in [13,17]. \(K_a^{mn}\) and \(J_{ab}^{mk}\) are available in (B.16) and (B.19). Eq.(B.4a) is called the TBA equation and played a central role in the study of thermodynamics of the RSOS models [9,13,17]. The rhs is universal in that it is only governed by the data \(X_r, \ell\) and reflects the structure of the rhs in BAE (B.2) under the string hypothesis. On the other hand, the lhs of (B.4a) depends on the model details like fusion type \(W_s^{(p)}\) as well as regimes. Now consider the high temperature limit \(T \to \infty\), where one formally drops the lhs and is left with the equation

\[\sum_{n=1}^{\ell_a-1} K_a^{mn} \star \log(1 + Y_n^{(a)-1}) = \sum_{(b,k) \in G} J_{ab}^{mk} \star \log(1 + Y_k^{(b)}),\]  

(B.5a)

\[Y_m^{(a)}(u) = \exp(-\epsilon_m^{(a)}(u)),\]  

(B.5b)

which is universal in the above sense. Formally passing to the Fourier components by using (B.16)-(B.19) and transforming back after some rearrangements, one can also rewrite (B.5) in the form

\[Y_m^{(a)}(u + \frac{i}{\ell_a})Y_m^{(a)}(u - \frac{i}{\ell_a}) = \prod_{b=1}^{k-1} \prod_{k=1}^{3} F_k(a, m, b; u) L_a^{(k_a)} L_s^{(k_s)} L_a^{(k_h)},\]  

(B.6a)

\[F_k(a, m, b; u) = \prod_{j=1}^{-k+1} \prod_{n=0}^{k-1} \left(1 + Y_{i_m/m + j}^{(a)}(u + i(k - 1 - |j| - 2n)/k_b)\right),\]  

(B.6b)

where by convention \(Y_0^{(a)}(u)-1 = Y_{\ell_a}^{(a)}(u)-1 = 0\) in (B.6a) and \(Y_m^{(a)}(u) = 0\) if \(m \notin \mathbb{Z}\) in (B.6b). Eq.(B.6) closes among \((a, m) \in G\). However, it extends to an infinite system of simultaneous equations for all \(m \geq 1\) if we disregard \(Y_{\ell_a}^{(a)}(u)-1 = 0\). We call them the (level \(\ell\)) restricted and unrestricted \(Y\)-system, respectively. The former refers to the data

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\( X_r \) and \( \ell \) while the latter is specified only by \( X_r \). They were firstly introduced in the above generality in [22]. The level \( \ell \) restricted \( Y \)-system is a closed and finite set of functional relations. Repeated use of it seems to yield the following periodicity in general:

\[
Y^{(a)}_m(u) = Y^{(a)}_m(u + 2i(\ell + g)) \quad \text{for} \quad (a, m) \in G. \tag{B.7}
\]

In view of the identification \( Y^{(a)}_m(u) = \rho^{(a)}_m(u) / \sigma^{(a)}_m(u) \), this is consistent with the invariance of the BAE (B.2) under \( u^{(a)}_j \rightarrow u^{(a)}_j + 2i(\ell + g) \). Note however that the period can be a divisor of the above. For example in the simplest \( X_r = A_1 \), \( \ell = 2 \) case, one has \( Y^{(1)}_1(u - i)Y^{(1)}_1(u + i) = 1 \) hence the period is \( 4i \), which is the half of (B.7).

**B.2. Summary of relevant functions**

Let us summarize the definitions of the functions \( A^{mk}_{ab} \) and \( K^{mk}_{ab} \) etc and their properties relevant to the TBA analysis. They will be also of use in many places in Part II. Given a function \( f(u) \) we shall frequently work with its Fourier component \( \hat{f}(x) \) defined as

\[
f(u) = \hat{f}(u) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \hat{f}(x)e^{iu x} dx, \quad \hat{f}(x) = \int_{-\infty}^{\infty} f(u)e^{-iu x} du. \tag{B.8}
\]

The convolution of the two functions will be denoted by

\[
(f_1 * f_2)(u) = \int_{-\infty}^{\infty} dv f_1(u - v)f_2(v). \tag{B.9}
\]

We shall often suppress the arguments as \( f = f(u) \), \( \hat{f} = \hat{f}(x) \). Below we list the definitions and the useful properties. The indices \( a \) and \( b \) are always taken in \( \{1, 2, \ldots, r\} \).

\[
\hat{M}_{ab} = \hat{M}_{ba} = B_{ab} + 2\delta_{ab}(\cosh(\frac{x}{t_a}) - 1) = 2\cosh(\frac{x}{t_a})(\delta_{ab} - \frac{I_{ab}}{2\cosh(\frac{x}{t_a})}),
\]

\[
\hat{A}^{mk}_{ab} = \frac{\sinh(\min(\frac{m}{t_a}, \frac{k}{t_b})x) \sinh((\ell - \max(\frac{m}{t_a}, \frac{k}{t_b}))x)}{\sinh(\frac{x}{t_a})\sinh(\ell x)}, \tag{B.10}
\]

\[
\hat{K}^{mk}_{ab} = \hat{A}^{mk}_{ab} \hat{M}_{ab}, \tag{B.11}
\]

\[
\hat{\Psi}^{mk}_{ab} = \delta_{ab}\delta_{mk} - \hat{K}^{mk}_{ab}, \tag{B.12}
\]

where \( (a, m), (b, k) \in G \) in (B.11-13) and \( B_{ab}, I_{ab} \) and \( t_{ab} \) are given in (3.1). From (B.10-13) we see that

\[
\hat{\Psi}^{mk}_{ab}(u) = \hat{\Psi}^{mk}_{ab}(-u) = \hat{\Psi}^{km}_{ba}(u), \tag{B.14}
\]

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and \( \Psi_{ab}^{mk}(u) \) decays rapidly when \( u \rightarrow \pm \infty \). The same property holds also for \( \mathcal{A}_{ab}^{mk}(u) \) and \( \mathcal{K}_{ab}^{mk}(u) \). In particular, we have the asymptotics

\[
\mathcal{A}_{aa}^{mk}(u) \xrightarrow{u \rightarrow \pm \infty} \frac{\sin \frac{\pi}{\ell} \sin \frac{\pi}{\ell} e^{-\frac{|x|}{\ell}}}{\ell \sin \frac{\pi}{\ell}}. \tag{B.15}
\]

Definitions continue.

\[
\hat{K}^{mn}_a = \hat{K}^{nm}_a = \delta_{mn} + \frac{1}{2 \cosh(\frac{x}{\ell})} (\bar{C}^{a}_{mn} - 2 \delta_{mn}) = \delta_{mn} - \frac{\bar{I}^{a}_{mn}}{2 \cosh(\frac{x}{\ell})} \quad 1 \leq m, n \leq \ell - 1,
\]

where \( \bar{C}^a \) and \( \bar{I}^a \) denote the Cartan and the incidence matrices of \( A_{\ell_a-1} \), respectively, i.e.,

\[
\bar{C}^a_{mn} = 2 \delta_{mn} - \bar{I}^a_{mn} = 2 \delta_{mn} - \delta_{mn-1} - \delta_{mn+1}, \quad 1 \leq m, n \leq \ell - 1. \tag{B.16b}
\]

\( \hat{K}^{mn}_a \) above should not be confused with \( \hat{K}^{mk}_a \) in (B.12). By the definition we have

\[
\hat{K}^{mn}_a = \frac{\hat{M}_{mn}}{2 \cosh(\frac{x}{\ell})} \quad \text{for } X_r = A_{\ell_a-1},
\]

\[
2 \hat{K}^{mn}_a(0) = \bar{C}^a_{mn}. \tag{B.17b}
\]

The following inversion property is valid:

\[
2 \cosh(\frac{x}{\ell}) \sum_{n=1}^{\ell-1} \hat{A}_{aa}^{mn} \hat{K}^{nk}_a = \delta_{mk} \quad \text{for } 1 \leq m, k \leq \ell - 1. \tag{B.18}
\]

We further introduce

\[
\hat{J}^{mk}_{ab} = \hat{M}_{ab} \sum_{n=1}^{\ell_a-1} \hat{K}^{mn}_a \hat{K}^{nk}_b = \frac{\hat{M}_{ab}}{2 \cosh(x/\ell)} \left( \sinh(\frac{x}{\ell_a}) \sinh(\frac{x}{\ell_b}) \delta_{\ell_m, \ell_k} \right.
\]

\[
+ \left. \sum_{j=1}^{\ell_a-\ell_k} \frac{\sinh(\frac{j}{\ell_b})}{\sinh(\frac{x}{\ell_b})} \left( \delta_{\ell_m(m+1)-\ell_j, t_a}, \ell_k + \delta_{\ell_m(m+1)+\ell_j, t_a}, \ell_k \right) \right), \tag{B.19b}
\]

for \((a, m), (b, k) \in G\). The sum \( \sum_{j=1}^{\ell_a-\ell_k} \) in (B.19b) is to be understood as zero if \( t_a \geq t_b \).

Since the expression (B.19b) does not contain \( \ell \), we extend the definition of \( \hat{J}^{mk}_{ab} \) to all the
integers $m, k \geq 0$. $\mathcal{J}_{ab}^{mk}(u)$ is an even function of $u$ but $\mathcal{J}_{ab}^{mk}(u) \neq \mathcal{J}_{ba}^{km}(u)$ in general as opposed to the latter property of (B.14). Combining (B.18) and (B.19a) we have

$$2 \cosh\left(\frac{x}{t_a}\right) \sum_{m=1}^{t_a-1} \hat{A}_{aa}^{jm} \hat{J}_{ab}^{mk} = \hat{M}_{ab} \hat{J}_{ab}^{jk} = \hat{K}_{ab}^{jk} \quad \text{for } (a, j), (b, k) \in G,$$

(B.20)

by noting (B.12).

Special values of these functions also play an important role and we prepare the notation

$$K_{ab}^{mk} = \hat{K}_{ab}^{mk}(0)$$

(B.21a)

$$= \left(\min(t_{ha}, t_{kb}) - \frac{mk}{\ell}\right)(\alpha_a | \alpha_b),$$

(B.21b)

for $(a, m), (b, k) \in G$. We also define

$$J_{ab}^{mk} = \hat{J}_{ab}^{mk}(0)$$

(B.22a)

$$= \frac{B_{ab}}{2} \left(\frac{t_{ab}}{t_a} \delta_{t_a k, t_b m} + \sum_{j=1}^{t_b - t_a} j\left(\delta_{t_b(m+1)-t_a j, t_a k} + \delta_{t_b(m-1)-t_a j, t_a k}\right)\right),$$

(B.22b)

for all integers $m, k \geq 0$. Then one has

$$2J_{ba}^{km} = \sum_{n=1}^{t_b-1} \tilde{C}_{nk}^{b} K_{ab}^{mn} \quad \text{for } (a, m), (b, k) \in G,$$

(B.23a)

$$2J_{ba}^{k0} = C_{ab} \delta_{t_b 0}, \quad 2J_{ba}^{k\ell_a} = C_{ab} \delta_{t_b \ell_a} \quad \text{for } k \geq 0,$$

(B.23b)

$$2J_{ba}^{\ell_a m} + K_{ab}^{m \ell_a - 1} = \frac{m}{\ell^a} C_{ab} \quad \text{for } (a, m) \in G.$$

(B.23c)

Eq. (B.23a) is a direct consequence of (B.19a), and (B.23b,c) can be checked for all the possibilities $(t_a, t_b) = (1, 1), (1, 2), \ldots, (3, 3)$ case by case.

### B.3. Alternative forms of Y-system

Let us formally rewrite the Y-system in alternative forms. Consider the Fourier component of (B.5a) and take the sum $\sum_{m=1}^{t_a-1} \hat{A}_{aa}^{jm}$ of both sides by means of (B.18) and (B.20). After the inverse Fourier transformation the result reads

$$\log(1 + Y^{(a) - 1}_j) = \sum_{(b, k) \in G} \mathcal{K}_{ab}^{jk} * \log(1 + Y^{(b)}_k) \quad \text{for } (a, j) \in G.$$

(B.24)
By (B.13) this can be further rewritten as

$$\log Y_j^{(a)} = \sum_{(b,k) \in G} \Psi_{ab}^{jk} * \log (1 + Y_k^{(b)}).$$  \hspace{1cm} (B.25)

Thus we have seen that the level $\ell$ restricted $Y$-system (B.6) can formally be transformed into three equivalent logarithmic forms (B.5a), (B.24) and (B.25).

**B.4. Constant $Y$-system as $Q$-system**

Here we show how the restricted $Q$-system (3.10) is recovered from the restricted $Y$-system by dropping the spectral parameter dependence. This is in a sense the inverse procedure of the Yang-Baxterization exploited in section 3.

Let $Y_m^{(a)} > 0$ be a constant solution of (B.6). Then, from (B.24) and (B.21a), the resulting algebraic equation is equivalent to

$$\log(1 + Y_j^{(a)} - 1) = \sum_{(b,k) \in G} K_{ab}^{jk} \log(1 + Y_k^{(b)}).$$  \hspace{1cm} (B.26)

Define

$$f_m^{(a)} = 1 - \frac{Q_{m-1}^{(a)}(0)Q_{m+1}^{(a)}(0)}{Q_m^{(a)}(0)^2} \quad \text{for } (a, m) \in G,$$  \hspace{1cm} (B.27a)

or equivalently,

$$\log(1 - f_n^{(b)}) = -\sum_{k=1}^{\ell_k - 1} \bar{C}_{nk}^b \log Q_k^{(b)}(0),$$  \hspace{1cm} (B.27b)

due to (3.9b) and $\forall Q_k^{(b)}(0) > 0$ for $(b, k) \in G$. From (3.10), we have

$$\log f_m^{(a)} = -2 \sum_{(b,k) \in G} J_{ba}^{km} \log Q_k^{(b)}(0)$$

$$= -\sum_{(b,k) \in G} \bar{C}_{nk}^b K_{ab}^{mn} \log Q_k^{(b)}(0)$$

$$= \sum_{(b,n) \in G} K_{ab}^{mn} \log(1 - f_n^{(b)}),$$  \hspace{1cm} (B.28)

by using (B.23a) and (B.27). Comparing this with (B.26) we find a constant solution

$$\frac{Y_m^{(a)}}{1 + Y_m^{(a)}} = f_m^{(a)}, \quad Y_m^{(a)} = \frac{Q_m^{(a)}(0)^2 \prod_{(b,k) \in G} Q_k^{(b)}(0)^{-2 J_{ba}^{km}}}{Q_{m-1}^{(a)}(0)Q_{m+1}^{(a)}(0)}. \hspace{1cm} (B.29)$$
An analogous identification to (B.29) also holds between the unrestricted $Y$-system and the unrestricted $Q$-system as eq.(6) of [22].

**B.5. Simply laced algebra case**

Here we shall exclusively consider the simply laced algebras $X_r = A_r, D_r$ and $E_{6,7,8}$, where many formulas in sections B.1–B.4 simplify considerably.

By virtue of $\forall t_a = 1$, we shall write $A_{mk}^{(\ell)}$ (B.11), $\tilde{\kappa}_{mn}^{(\ell)}$ (B.16a), $\tilde{C}^a$ and $\tilde{I}^a$ (B.16b) just as $A_{mk}, \kappa_{mn}, \tilde{C}$ and $\tilde{I}$, respectively. Then the restricted $Y$-system (B.6) takes the form

$$Y^{(a)}_m(u + i)Y^{(a)}_m(u - i) = \frac{\prod_{b=1}^{r}(1 + Y^{(b)}_m(u))I_{ab}}{\prod_{k=1}^{r-1}(1 + Y^{(a)}_k(u-1))\tilde{I}_{mk}},$$

where $\tilde{I}$ is the incidence matrix for $A_{\ell-1}$ as specified just above. See (3.8) for the corresponding $Q$-system. Set

$$\hat{J}_{mk}^{ab} = \delta_{ab}\delta_{mk} - 2\cosh x(\hat{\kappa}^{-1})_{ab}\hat{C}_{mk}$$

for $(a, m), (b, k) \in G$, (B.31)

which enjoys the same property as (B.14). Now we list the useful simplifications.

$$J_{ab}^{mk} = \frac{\delta_{mk}\kappa_{ab}}{2\cosh x},$$

(B.32a)

$$J_{ab}^{mk} = 2\delta_{mk}C_{ab} = 2\delta_{mk}\delta_{ab} - \delta_{mk}I_{ab},$$

(B.32b)

$$K_{ab}^{mk} = 2\delta_{ab}\delta_{mk} - \hat{J}_{ab}^{mk}(0) = C_{ab}(\tilde{C}^{-1})_{mk},$$

(B.32c)

$$\hat{J}_{ab}^{mk}(0) = 2\delta_{ab}\delta_{mk} - (C^{-1})_{ab}\tilde{C}_{mk},$$

(B.32d)

$$A_{mk}^{(\ell)} = \frac{\sinh((m, k)x)\sinh((\ell - \max(m, k))x)}{\sinh x\sinh(\ell x)}.$$

(B.32e)

Due to (B.32b) above, the latter of (B.29) becomes

$$Y^{(a)}_m = \frac{\prod_{b=1}^{r}Q^{(b)}_{m}(0)^{I_{ab}}}{Q^{(a)}_{m-1}(0)Q^{(a)}_{m+1}(0)},$$

(B.33)

It is elementary to check that (B.33) is a constant solution of (B.30) by using (3.8) and (3.9b). Finally we note the property

$$\chi^{(a)}(\hat{\kappa}^{-1})_{ab}(u) \xrightarrow{u \to \pm\infty} \frac{\lambda^{(a)}\lambda^{(b)}}{2\sin \frac{\pi}{g}} e^{-\frac{|n-1|}{g}},$$

(B.34)
where $g$ is the (dual) Coxeter number and $\chi = (\chi^{(a)})_{1 \leq a \leq r}$ is the normalized Perron-Frobenius eigenvector of the incidence matrix $I_{ab}$, i.e.,

$$\sum_{b=1}^{r} I_{ab} \chi^{(b)} = (2 \cos \frac{\pi}{g}) \chi^{(a)} \quad \text{for } 1 \leq a \leq r, \quad (B.35a)$$

$$\sum_{a=1}^{r} \chi^{(a)^2} = 1. \quad (B.35b)$$

In view of (B.17a) and (B.18), the asymptotics (B.34) above actually includes (B.15) as the special case $X_r = A_{\ell_n-1}$. 
References
The classical simple Lie algebra $X_r$, the Dynkin diagram, the dimension of $X_r$ and the dual Coxeter number $g$. In each Dynkin diagram, the nodes are numerated from 1 to $r$. The parameter $t_a$ (3.1a) has been given above the node $a$ only when $t_a \neq 1$. 

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