Bifurcation Phenomena Of The Non-Associative Octonionic Quadratic.

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In this work we have performed an investigation into the limiting dynamics and bifurcation phenomena of the non-associative Octonionic quadratic map. It displayed a wealth of non-linear structure including fixed points, Hopf bifurcations, phase locking, periodic cycles, tori, non-trivial knots, loop doubling and tripling, infinite period doubling cascades and hyperchaos. The evolution of the limiting structure was characterised by the recursive interplay of these various bifurcation mechanisms, which led to the appearance of complex attracting structures. Connections from this behaviour to the theory of the Mandelbrot set were established.

1. The map.

In this paper, we further extend our investigations of the Octonionic generalisation of Mandelbrot's complex quadratic map [1-4],

$$Z \rightarrow Z^2 + c + (Zc)\alpha - Z(c\alpha) \quad c, Z, \alpha \in \mathcal{O}. \quad (1)$$

In the associative (quaternionic) subspace, this map reduces to the more familiar complex quadratic,

$$Z \rightarrow Z^2 + c \quad Z, c \in \mathbb{C}. \quad (2)$$

Mandelbrot's map has a beautiful, well-defined limit structure, comprehensively categorised by the Mandelbrot set, $\mathcal{M}$, [5,6]. The spirit of this paper is to relate the new dynamical possibilities to the understood associative structure, and to present entirely novel phenomenology.

In a reference, [1] investigates the associated Julia sets, for particular choices of c , and reports a structural phase transition, from a disconnected Cantor dust to a smooth, locally connected set as non-associativity was increased. This corresponds with the sudden emergence of an attracting limit structure. Here we focus on fixed c values with no attractor in the associative limit, which as non-associativity is increased display the sudden emergence of an attractor (i.e. those c 's that display a transition in the Julia set).

Acknowledging that we are seeking the limiting dynamics we can eliminate redundancy in the description and employ a reduced map, [2]. Call the map P^α , $\alpha \in \{z_0, z_1, x\}$,

$$\begin{aligned} z_0 &\rightarrow z_0^2 - z_1^2 + c_0 - x, \\ z_1 &\rightarrow 2z_0z_1 + c_1, \\ x &\rightarrow 4(z_0^2 + \zeta^2)x. \end{aligned} \quad (3)$$

Note that x plays $x = \sum_{i=4}^7 z_i^2$ and $\zeta = c_1\alpha$. Non-associativity is linearly related to the bifurcation parameter, ζ .

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2. Parameter regions with attractive structure.

As it happens, the Mandelbrot set is also useful in categorizing the limiting dynamics of the non-associative map. Firstly we seek to characterise those values of c , which with increasing non-associativity, display this generic transition. To do this we take constant ζ slices of parameter space, (c_0, c_1, ζ) , and plot those c 's which admit stable non-diverging trajectories. Define the set

$$A_\zeta = \{c \in \mathbb{C} \text{ s.t. non-diverging trajectories of } P_{c,\zeta} \text{ are dense in some neighbourhood of } \mathbb{R}^3\}$$

In the associative limit, this is by definition the interior of the Mandelbrot set, see [6]. The sets A_ζ for representative ζ 's are displayed in figure 1. These sets were generated by evaluating the trajectory from one thousand random initial points in phase space; a map fell in the set if any of these trajectories were non-diverging (with a cutoff of about 200000 iterations). Furthermore, we have coloured these maps according to the periodicity of the limit cycle, if that cycle proved to be periodic. Fig 1(b), by indexing the periodicity of 1(a), provides a key.

Note that P^α implies that all trajectories diverge for $\zeta > \frac{1}{2}$. In the interval $0 < \zeta < \frac{1}{2}$, these plots demonstrate that for a map to show the sudden emergence of non-diverging trajectories, it's c must be located at such a point that a leftwards line at constant imaginary part intersects some part of the Mandelbrot set. This is a first connection between properties of Mandelbrot's quadratic and inherently non-associative phenomena.

The evolution of limiting structure is most naturally observed as ζ is decreased from $\frac{1}{2}$ to the transition point, where the finite attracting structure disappears. In this direction the interplay of several mechanisms readily explains some otherwise mysterious structure.

We shall introduce the mechanisms in order of appearance as ζ is reduced, illustrating with examples. The full menagerie is seen for small $\text{Im}\{c\}$, however for more distant choices only an interval of the full phenomenology is observed.

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3. Fixed points.

The first structure ($\zeta \simeq \frac{1}{2}$) is simply an attractive fixed point. This exists, intuitively, because for fixed x , the (z_0, z_1) mapping simply forms Mandelbrot's mapping with an effective value of c , $c_{eff} = c - x$,

$$\begin{aligned} Z &\rightarrow Z^2 + (c - x), \\ x &\rightarrow x(4(\operatorname{Re}\{Z\}^2 + \zeta^2)), \\ \zeta, x \in \mathbb{R} : Z, c \in \mathbb{C}, \end{aligned} \quad (4)$$

Recall the observed criteria for a mapping to display phase transitions as γ is increased. The complex constant c must be located directly to the right of the Mandelbrot set as discovered in figure 1. Note also that the cardioid, or main body, of the Mandelbrot set classifies those maps which in the associative limit admit attractive fixed points, [6]. It is reasonable to suggest that these maps whilst located outside the Mandelbrot set, might in the complex subspace form the complex quadratic with an effective c within the cardioid.

For this identification to imply stability requires that a planar eigenspace of the Jacobian evaluated at the fixed point, should approximate the complex plane; and also that the third eigenvector be stable. This understanding proved largely valid, and the nature of it's breakdown led to novel phenomenology, detailed below.

To keep x fixed, reducing ζ implies increasing z_0 . Given a particular c and fixed z_{0f} , there is a unique consistent c_{eff} ,

$$(c_0 - x) = z_0 - z_0^2 - \frac{c_1^2}{(1 - 2z_0)^2}. \quad (5)$$

$$\frac{\partial c_{eff}}{\partial z_0} = 1 - 2z_0 + \frac{4c_1^2}{(1 - 2z_0)^3}. \quad (6)$$

P^∞ indicates that having $|Z| > \frac{1}{2}$ leads to diverging trajectories. Thus decreasing ζ implies c_{eff} travels rightwards towards the boundary of the cardioid (main body of $\mathcal{M}$). Complex analytic theory completely describes the boundary of $\mathcal{M}$, and the limiting behaviour of the maps there chosen, ([6], or more comprehensively [7]). The eigenvalues of the Jacobian evaluated at the fixed point of a map on the boundary, map a homeomorphism between the cardioid's boundary and the unit circle. Mappings chosen from the boundary will either produce the germination point for an n -cycle bud (Parabolic point), or a region of irrational rotation about an indifferent fixed point (Siegel disc). This depends on whether the phase of the complex conjugate eigenvalue pair is rational or irrational. Note that this implies that buds of every conceivable periodicity densely populate the the cardioid's boundary.

As c_{eff} approaches the proximity of the Mandelbrot set boundary, we observe the fixed point destabilise in the planar eigenspace, although not in a parabolic point or a Siegel disc. The perturbation of the third dimension introduces a uniquely non-linear scenario, the Hopf bifurcation.

4: A Hopf bifurcation and the Locking mechanism.

This occurs when a pair of complex conjugate eigenvalues of the Jacobian at the fixed point, exceed unit magnitude. It is characterised by the birth of an attracting invariant loop in the eigenspace of the conjugate pair. The loop is born out of the fixed point, and as ζ is further decreased the loop grows in size.

At parabolic points, the purely complex map spontaneously displays an n -furcation. N -cycles are observed in the non-associative map, yet they are always obtained by phase-locking on a limit loop born from a Hopf bifurcation, never spontaneously from a fixed point.

Figure 1 demonstrates the distribution of periodic cycle regions in parameter space. In the large, they only occur in linear regions at constant $\operatorname{Im}\{c\}$ to the right of a Mandelbrot bud of corresponding periodicity. There also appear peculiar angled rays of defined periodicity on the right edge of A_ζ for high ζ , and as we have only mapped low periodicities the extent of this sequence is not clear.

With some analysis of the locking mechanism, this first observation can be understood. Investigation shows the one-dimensional angular return map to be qualitatively similar to the generic non-linear circle map, [8] Ch 2. In the spirit of universality, we can identify the emergence of periodic cycles with the locking behaviour of this map. Given below, K and Ω are arbitrary constants, and all angles are normalised by 2π ,

$$\theta^{(n+1)} = \theta^{(n)} + \Omega + \frac{K}{2\pi} \sin(2\pi\theta) \pmod{1}. \quad (7)$$

Winding numbers characterise limiting behaviour around the loop. They are defined as $W = \lim_{n \rightarrow \infty} \frac{\theta^{(n)} - \theta^{(0)}}{n}$. This map has the characteristic behaviour that winding numbers, and hence periodic cycles, located at every rational point in the interval $[0, 1]$ have broad locking regions as a function of the rotation rate, Ω , densely filling the Ω axis, see figure 2.

We take a map which is just short of a broad locking to a 5 cycle, located for $c = 0.456 + 0.34i, \zeta = 0.44$. We display the functional form of this effectively one-dimensional map by generating $f(\theta)$, defined as,

$$\theta^{(n+1)} = \theta^{(n)} + f(\theta), \quad (8)$$

parameterising the points on our loop by their angle in the complex projection from the z_0 plane, evaluating from the mean of the set as the origin. This is displayed in figure 3 where one can see the qualitative identification of this angular return map with 7. The kinks in the functional form of the map are a result of our ill-defined parameterisation of points on the loop; corresponding kinks around the cycle prevent an angle labelling from being genuinely monotonic. Seeking the demonstrated universality of this class [9], we look for and expect similar locking properties.

In figure 4(a) we isolate a period adding sequence of lockings which pre-empt the broad 5-cycle locking in our chosen example as γ is decreased. That is, we expect in evolving continuously from winding number $\frac{p}{q}$ to $\frac{r}{s}$ that all members of the sequence $\frac{p+n\tau}{q+n\tau}$ must be visited. Kaneko has identified an important scaling law for this sequence, as displayed by 7, [10]. Denoting by $\Omega_{\frac{p}{q}}$ the omega where first the winding number $\frac{p}{q}$ locks, he showed that

$$(\Omega_{\frac{p+n\tau}{q+n\tau}} - \Omega_{\frac{p}{q}}) \propto n^{-2}. \quad (9)$$

That we observed this relation (figure 4(b), agreement improving with n) supports that locking in our effective one dimensional return map is universal with the generic non-linear circle map. The rotation rate is tangential to the phase of the eigenvalue pair at the Hopf bifurcation, see figure 5, and we expect from eigenvalue phase calculations that the rotation rate will evolve linearly with ζ .

We can understand now the observation from the plots, that locking regions largely occur when c 's are located to the right of Mandelbrot buds of corresponding periodicity.

The rotation rate around the loop born from a Hopf bifurcation is, locally, the phase of the eigenvalue pair at the bifurcation. Recalling that the associated eigenspace approximates the complex subspace, this phase is approximately determined by where c_{eff} for that c , will meet the boundary of the Mandelbrot set. On the Mandelbrot boundary, buds of defined periodicity germinate at points where the eigenvalue pair has rational phase, with the period of the cycle dictated by the minimal denominator of the phase. So, having a c at the height of a bud, implies it will Hopf bifurcate with a rotation rate local to the relevant rational rate, so that as non-linearity in the angular return map develops, the desired locking will ensue in the broad locking regions.

We are in a position now to appreciate how the interplay of these mechanisms can bring surprising limit structure. Figure 6 is an example of a limit cycle achieved via the sequence -

Fixed Point $\rightarrow$ Hopf Bifurcation. $\rightarrow$ Locking to a 51 cycle $\rightarrow$ Hopf bifurcation in the 51st composed map.

A 51 cycle, is seen as a fixed point of the 51st composed map. We observe the hopf bifurcation of this fixed point, which implies 51 loops in phase space. Eigenvalue calculations showed a pair of complex conjugate eigenvalues of the 51st composed map crossing the unit circle at this point.

5: Knots, Tori and Loop Doubling.

A non-trivial observed mechanism, was when a limit loop *knotted* up. Such a progression as ζ is reduced is presented in figure 7, where we show the complex projection of the full structure. The non-triviality of this mechanism is highlighted by figure 8, resolved in three dimensions.

The provided explanation is that this knotting is an example of the emergence of a 2-Torus in the limit structure. The observed 1-dimensional knot is a consequence again of phase locking, into a rational ratio of rotation rates around the two loops which form the torus.

An abstracted analysis of non-linear torus mappings may be found in [8] ch 6, who investigated coupled generic non-linear circle map pairs. It is consistent with his analysis that 1-dimensional and pointwise lockings should result from a non-linear evolution on a 2-torus. No explicit mechanism is provided in the literature for the emergence of a 2-torus in a three dimensional map, though tori are occasionally observed, [11]. This is the subject of further study and there are indications that it is the result of some style of Hopf bifurcation in a dimensionally reduced return map for radial and vertical perturbations about the loop.

Our main confidence in this explanation, is that for certain parameter values, the actual tori appear intermittently with the associated knots. One superlative example is here offered in fig 9, which also illustrates the map's capacity for recursive mechanism. The observed limit cycle is obtained via the sequence-

Fixed point $\rightarrow$ Hopf bifurcation $\rightarrow$ Locking to a three cycle $\rightarrow$ Hopf Bifurcation $\rightarrow$ Torus/Knot bifurcation.

We see for various ζ 's, the tori, and several non-similar knots representing various ratios of rotation rates.

Kaneko reported also the phenomena of loop doubling [12], whereby the loop can destabilise by splitting spontaneously into a pair of detached loops. This phenomena is obviously distinct from toroidal transitions, and generally can only support a finite number of doublings.

In this system, loop doubling arose as a natural development, and often appeared in tandem with knotting transitions as figures 10 demonstrate. One choice example which highlights the important recursive nature of these mechanisms is figure 11. The sequence which this example follows is -

Fixed Point $\rightarrow$ Hopf bifurcation $\rightarrow$ Locking to a 55 cycle. $\rightarrow$ A Hopf bifurcation in the 55th-composed map $\rightarrow$ A Torus/Knot transition. $\rightarrow$ These loops double.

This leads to the somewhat counterintuitive limit cycle of 55 doubled knots distributed throughout phase space which can be explained by these recursive mechanisms but despairs of explanation on it's own.

Indeed, Kaneko's loop doubling has found generalisation in this system. In figure 12 we present a single instance where a tripling on a highly non-trivial knot occurs.

Knots appear to behave similarly to limit loops - in that they admit lockings. This is consistent with Kaneko's result on his abstract torus evolution study. Figure 13 illustrates the product of the sequence -

Fixed point $\rightarrow$ Hopf bifurcation $\rightarrow$ Knot/Torus transition $\rightarrow$ Locking to a 99 cycle $\rightarrow$ Hopf bifurcation in the 99th composed map.

6: To Chaos

Generically, the last attractive structures as ζ is reduced are strange chaotic attractors. Note that the associative limit fails to display strange or chaotic attractors (although the repelling Julia sets show chaotic dynamics thereon and are definitely strange!).

Chaos is achieved here in two stages. First a periodic cycle (achieved via a locking) undergoes an infinite sequence of period doubling in the style of Feigenbaum, based on each point of the periodic cycle. At this point, iterates populate one dimensional line segments based on each of the original points of the periodic cycle. Motion is chaotic in one effective dimension. We observed the period doublings to scale in the precise fashion one would expect for higher dimensional period doublings in dissipative maps, [13]. In figure 14 we show two stages in the sequence - Fixed Point $\rightarrow$ Hopf Bifurcation $\rightarrow$ Locking to an 8 cycle $\rightarrow$ Hopf Bifurcation in the 8th composed map (shown (a)) $\rightarrow$ Locking to a 10 cycle in the 8th composed map (in essence an eighty cycle) $\rightarrow$ An infinite cascade of period doublings $\rightarrow$ Iterates chaotically populate one dimensional line segments on each of the original eighty points (also shown, (b)).

Generically, as ζ is reduced further, this situation undergoes some dimensionally expansive mechanism, to a generic hyperchaotic attractor as displayed in figure 15. It appears to have a well defined twisting structure, although it does have fractal dimension greater than two. Analysis shows this affords two positive Liapunov exponents. The chaotic regime in this system has been largely ignored in this work and is the subject of further study.

7: Concluding remarks.

We have found that this system is wealthy with non-linear phenomena. All classic non-linear mechanisms are represented and more besides. Yet the importance of this system is that these mechanisms appear recursively, and build constructively. It is probable that this recursion, whilst being always of finite order, has it's origins in the infinitely nested n-furcations of the Mandelbrot quadratic.

It is an interesting question as to whether the observed behaviour can be unified in a simple mathematical description, or whether it must always be treated as a perturbation of it's beautiful associative limit.

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Captions.

Note: Figures are numbered on the back of the printouts.

Figure 1

- (a) $A_0 - 1.5 < c_0 < 1.5 - 1.0 < c_1 < +1.0$ as for (c), (d) and (e).
- (b) Key to periodicity.
- (c) $A_{0.25}$
- (d) $A_{0.41}$
- (e) $A_{0.45}$
- (f) $A_{0.491} - 0.2 < c_0 < 2.7 - 1.0 < c_1 < +1.0$.

Figure 2

Rational phase lockings from the critical ($K=1$) circle map.

Figure 3

Pre-emptory lockings display a period-adding sequence.

Figure 4

Representing the functional form of the angular return map.

Figure 5

Rotation is tangential to the phase of the eigenvalue pair at a Hopf bifurcation.

Figure 6

A Hopf bifurcation in the 51st composed map.

Figure 7

As ζ decreases, the knot emerges from the limit loop.

Figure 8

$$c = 0.366 + 0.4i\zeta = 0.4718614$$

Figure 9

- (a) Three knots. (b) Close up of a knot. (c) The tori on which these knots lock. (d) Close up of a tori. (e) Higher order locking. (f) Lower order locking.

Figure 10

Loop doubling.

Figure 11

- (a) 55 knots in phase space. (b) Close up. (c) Same knot after doubling.

Figure 12

Observation of a highly non-trivial knot, tripling.

Figure 13

99 loops distributed around a knot.

Figure 14

Before and after a cascade of period doublings.

Figure 15

A Hyperchaotic attractor.

$c = 0.366 + 0.4i$ $\text{zeta} = 0.4718614$

$c = 0.38 + 0.3\,i$ $\text{zeta} = 0.4381$