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Translational Gauge Invariance and Classical Gravitodynamics

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Abstract

A generalization of gauge theories based on non-abelian matrix groups to the operator group of translations is given. It leads to the concept of local translation invariance and to the existence of an operator-valued gauge field in Minkowski space. Its dynamics is uniquely determined using the RG and new heat kernel techniques developed in the Appendix. The resulting theory is shown to be a gauge theory of gravitation which is physically complementary and mathematically equivalent to the general theory of relativity.

ZU-TH-12/94

1 Introduction

In particle physics the concept of gauge field theories has led to a successful and detailed description of elementary particle dynamics. Two features are essential thereby.

The first of them is the validity of special relativity, i.e. space-time is described as a four-dimensional Minkowski space-time $(\mathbf{R}^4, \eta)$ endowed with an indefinite metric η . Restriction to Cartesian coordinates leads to the usual form of $\eta = \text{diag}(1, -1, -1, -1)$. The corresponding maximal symmetry group is the Poincaré group $\mathbf{P}$ consisting of translations $a \in \mathbf{T}$ and Lorentz rotations $\Lambda \in \mathbf{L}$.

The second essential feature is the description of matter by local, causal fields defined on $(\mathbf{R}^4, \eta)$. These fields are classified according to their behaviour under irreducible representations of the Poincaré group $\mathbf{P}$. Furthermore the field theoretical approach allows a very elegant incorporation of experimentally checked conservation laws into the theory by means of the Noether theorem. We persist in both of these features unless stated explicitly throughout the paper.

Our considerations are based on energy-momentum conservation and its consequences. Experimentally the conservation of energy-momentum is tested to a high accuracy [1]. On the theoretical side this is taken into account by the assumption of global translation invariance made for all theories describing fundamental interactions and the subsequent conservation laws.

It is now important for us to note that the usual treatment of the connection between global continuous symmetries and the corresponding conserved currents and charges in a Lagrangian field theory given in the Noether theorem makes a clear distinction between the so-called inner symmetries and the space-time symmetries. The inner symmetries are built in a theory by the requirement that the Lagrangian density $\mathcal{L}$ be invariant under the action of some Lie group which is represented in the space of fields only and does not act on space-time. Invariance under $U(1)$ is an important example here. Space-time symmetries on the other hand require the invariance of $\mathcal{L}$ under a combined transformation of coordinates and fields. Invariance under $\mathbf{T}$ or $\mathbf{L}$ are here the important examples.

We concentrate on $\mathbf{T}$ in the following. Our first step is to recast $\mathbf{T}$ symmetry and its consequences in the form of an inner symmetry (Section 2). This leads to a complementary description of the global action of $\mathbf{T}$ which is in complete analogy to the description of the action of inner sym-

metry groups as groups of generalized 'rotations' in field space. In particular the coordinate system used to specify the space-time events is not affected anymore by $\mathbf{T}$ transformations.

As global translations correspond in this way to global generalized 'phase rotations' the same locality arguments that lead to the concept of local generalized 'phase rotations' [2] and all its consequences should again apply. We thus introduce local translations as local $\mathbf{T}$ gauge transformations and demand the invariance of physical processes under those (Sections 3 and 4). This necessarily leads to the existence of gauge fields with definite behaviour under local $\mathbf{T}$ gauge transformations [3]. Their coupling to any other field is essentially fixed as in the case of other gauge field theories (Section 5), whereas the usual gauge symmetry and renormalizability arguments do not allow to determine their dynamics uniquely.

We use instead the renormalization group to fix the gauge field dynamics in the following way [4] (Section 7). Perturbative renormalizability of a scalar matter field theory minimally coupled to the $\mathbf{T}$ gauge fields implies that the change of the one-loop partition function of the whole system under rescaling may be absorbed in a nontrivial scale dependence of the different couplings, masses and wavefunction normalizations occurring in the classical action [5]. Using new heat kernel techniques developed in Appendix A we may on the other hand explicitly compute the part of the change of the one-loop partition function coming from the matter sector. Renormalizability then requires that this part may be reabsorbed in the classical action of the theory. Thus the classical action must contain at least all the terms controlling the scaling behaviour of the one-loop partition function. Restriction to just those fixes thus the minimal classical gauge field dynamics in an essentially unique way. This reasoning leads in the case of non-abelian matrix groups indeed to the usual Yang-Mills action.

In order to get the interpretation of the resulting theory as a gauge theory of gravitation we show that it may be recast in usual geometrical terms [6] replacing local $\mathbf{T}$ gauge invariance by general coordinate invariance and is then mathematically equivalent to the general theory of relativity (Sections 6 and 8). The $\mathbf{T}$ gauge theory of gravitation allows a complementary description of gravitational effects in which the mathematical structure of the underlying space-time is not directly affected by physical events.

Throughout this paper indices $\alpha, \beta, \gamma, \dots$ from the first half of the Greek alphabet denote quantities defined on Minkowski space-time ($\mathbf{R}^4, \eta$) which transform covariantly w.r.t. the Lorentz group. They are correspondingly raised and lowered with η . Indices $\mu, \nu, \rho, \dots$ from the second half of the

Greek alphabet denote coordinate covariant quantities defined on a general Riemannian manifold $(\mathbf{R}^4, g)$ and are raised and lowered with the metric g .

2 Global translation invariance as an inner symmetry

In this Section we review the two complementary conceptions of global translation invariance in field theory. We construct the corresponding conserved currents $\Theta^\beta{}_\alpha$, i.e. the canonical energy-momentum tensor and the charges P_α , i.e. the energy-momentum four-vector.

Let us begin stating the Noether theorem in a quite general form [7]. Consider a set of fields $\varphi_j(x)$ with $j = 1, \dots, n$. Their dynamics shall be specified by the action $S_M = \int d^4x \mathcal{L}_M(x, \varphi_j, \partial_\alpha \varphi_j)$. $\delta S_M = 0$ yields then the equations of motion. Consider further the infinitesimal transformations

$$x^\alpha \longrightarrow x'^\alpha = x^\alpha + \delta x^\alpha \quad (1)$$

of the coordinates and

$$\varphi_j(x) \longrightarrow \varphi'_j(x') = \varphi_j(x) + \delta \varphi_j(x) \quad (2)$$

of the fields. If there are functions $\delta f^\beta(x)$ for which

$$\begin{aligned} d^4x' \mathcal{L}_M(x', \varphi'_j(x'), \partial'_\alpha \varphi'_j(x')) &= \\ d^4x \{ \mathcal{L}_M(x, \varphi_j(x), \partial_\alpha \varphi_j(x)) &+ \partial_\beta \delta f^\beta(x) \} \end{aligned} \quad (3)$$

holds, then there is a conserved current J^β found to be

$$J^\beta = - \frac{\partial \mathcal{L}_M}{\partial(\partial_\beta \varphi_j)} \cdot \delta \varphi_j + \delta f^\beta + \Theta^\beta{}_\alpha \cdot \delta x^\alpha \quad (4)$$

as well as a conserved charge Q given by

$$Q = \int_{x^0 = \text{const}} d^3x J^0. \quad (5)$$

The fields φ_j above must obey the equations of motion. $\Theta^\beta{}_\alpha$ is the canonical energy-momentum tensor

$$\Theta^\beta{}_\alpha = \frac{\partial \mathcal{L}_M}{\partial(\partial_\beta \varphi_j)} \cdot \partial_\alpha \varphi_j - \delta^\beta{}_\alpha \cdot \mathcal{L}_M. \quad (6)$$

We shall apply this theorem now in two different though equivalent ways to a globally translation invariant theory of the fields φ_j .

The usual conception of translation symmetry as a space-time symmetry is expressed in the transformation formulae

$$\begin{aligned} x^\alpha &\longrightarrow x'^\alpha &= x^\alpha + \varepsilon^\alpha \\ \varphi_j(x) &\longrightarrow \varphi'_j(x') &= \varphi_j(x). \end{aligned} \quad (7)$$

$\delta x^\alpha = \varepsilon^\alpha$ is a constant infinitesimal space-time translation and $\delta\varphi_j = 0$. One easily convinces oneself that (3) then holds for $\delta f^\beta = 0$ and gets with (4) the conserved current

$$J_1^\beta = \Theta^\beta{}_\alpha \cdot \varepsilon^\alpha.$$

Now we recast the transformations above in the equivalent form of a global **T** gauge transformation

$$\begin{aligned} x^\alpha &\longrightarrow x'^\alpha &= x^\alpha \\ \varphi_j(x) &\longrightarrow \varphi'_j(x) &= \varphi_j(x) - \varepsilon^\alpha \cdot \partial_\alpha \varphi_j(x). \end{aligned} \quad (8)$$

T acts thus as a group of generalized 'phase rotations' in field space only and leaves the space-time coordinates x unchanged. This leads to the complementary conception of translation symmetry as an inner symmetry. We now have $\delta x^\alpha = 0$, $\delta\varphi_j = -\varepsilon^\alpha \cdot \partial_\alpha \varphi_j$ and

$$\begin{aligned} d^4x \mathcal{L}_M(x, \varphi'_j(x), \partial_\alpha \varphi'_j(x)) &= \\ d^4x \{ \mathcal{L}_M(x, \varphi_j(x), \partial_\alpha \varphi_j(x)) - \varepsilon^\beta \cdot \partial_\beta \mathcal{L}_M(x, \varphi_j(x), \partial_\alpha \varphi_j(x)) \} \end{aligned} \quad (9)$$

so that (3) holds with $\delta f^\beta = -\delta^\beta{}_\alpha \cdot \varepsilon^\alpha \mathcal{L}_M$. The conserved current reads therefore

$$\begin{aligned} J_2^\beta &= -\frac{\partial \mathcal{L}_M}{\partial(\partial_\beta \varphi_j)} \cdot \delta\varphi_j + \delta f^\beta \\ &= \left\{ \frac{\partial \mathcal{L}_M}{\partial(\partial_\beta \varphi_j)} \cdot \partial_\alpha \varphi_j - \delta^\beta{}_\alpha \mathcal{L}_M \right\} \varepsilon^\alpha \\ &= \Theta^\beta{}_\alpha \cdot \varepsilon^\alpha \end{aligned}$$

where $\Theta^\beta{}_\alpha$ is again the canonical energy-momentum tensor (6). As the conserved charges one gets in both cases the energy-momentum four-vector

$$P_\alpha = \int d^3x \Theta^0{}_\alpha. \quad (10)$$

We have obtained a description of the global action of $\mathbf{T}$ which resembles very much the manner the action of well-known inner symmetries is usually described in field theory (see e.g. [8]).

Let us now go one step farther and let us gauge the translation group $\mathbf{T}$ in a way which follows as much as possible the steps one usually performs gauging an inner symmetry group [2,9].

3 Local $\mathbf{T}$ translation invariance, the covariant derivative D_α and the gauge field B_α

In this Section we introduce local $\mathbf{T}$ gauge transformations. We construct the covariant derivative $D_\alpha = \partial_\alpha + B_\alpha$ respecting the local $\mathbf{T}$ gauge symmetry and discuss the behaviour of the compensating field B_α under $\mathbf{T}$ gauge transformations. Throughout this Section we emphasize the structural similarity to non-abelian gauge field theories introducing analogous notions and notations wherever possible.

In the previous Section we recast $\mathbf{T}$ symmetry in the form of an inner symmetry. Let us rewrite the formulae (8) for global infinitesimal $\mathbf{T}$ gauge transformations as

$$\begin{aligned} x^\alpha &\longrightarrow x'^\alpha = x^\alpha \\ \varphi_j(x) &\longrightarrow \varphi'_j(x) = (U(\Theta)\varphi_j)(x) \end{aligned} \tag{11}$$

where we introduce in complete analogy to notions used in non-abelian gauge field theory the unitary infinitesimal representations of $\mathbf{T}$ transformations in field space

$$U(\Theta) = 1 + \Theta$$

forming a Lie group, the infinitesimal hermitean gauge parameters

$$\Theta = i\varepsilon^\alpha p_\alpha$$

and the generators of the Lie algebra $\mathfrak{t}$ in field space

$$p_\alpha = i\partial_\alpha.$$

Above hermiteicity and unitarity are understood w.r.t. the usual scalar product in field space. Let us extend now $\mathbf{T}$ to a Lie group of local infinitesimal gauge transformations

$$\begin{aligned} U(\Theta) &\longrightarrow U(\Theta(x)) \\ \Theta = i\varepsilon^\alpha p_\alpha &\longrightarrow \Theta(x) = i\varepsilon^\alpha(x)p_\alpha. \end{aligned}$$

It is understood that $\Theta(x)$ in its above form only acts to the right so that no ordering problems may occur. The formulae (11) still define the representation of $\mathbf{T}$ in the space of fields.

In order to recast a given matter theory in a locally $\mathbf{T}$ gauge invariant form we must introduce a covariant derivative D_α . To be more precise we demand that the theory with covariant derivatives D_α replacing the usual ones behave under local infinitesimal $\mathbf{T}$ transformations the same way it behaves with the usual derivatives ∂_α under global infinitesimal $\mathbf{T}$ transformations. Instead of

$$\begin{aligned}\mathcal{L}_M(x, \varphi'_j(x), \partial_\alpha \varphi'_j(x)) &= \\ \mathcal{L}_M(x, \varphi_j(x), \partial_\alpha \varphi_j(x)) &- \varepsilon^\beta \cdot \partial_\beta \mathcal{L}_M(x, \varphi_j(x), \partial_\alpha \varphi_j(x))\end{aligned}$$

resulting from a constant infinitesimal change of gauge we demand therefore

$$\begin{aligned}\mathcal{L}_M(x, \varphi'_j(x), D'_\alpha \varphi'_j(x)) &= \\ \mathcal{L}_M(x, \varphi_j(x), D_\alpha \varphi_j(x)) &- \varepsilon^\beta(x) \cdot \partial_\beta \mathcal{L}_M(x, \varphi_j(x), D_\alpha \varphi_j(x))\end{aligned}\tag{12}$$

where $'$ denotes the infinitesimally $\mathbf{T}$ transformed fields (11) and D'_α the covariant derivative in the new gauge. Note that (12) alone does not ensure the local $\mathbf{T}$ invariance of the original action $S_M = \int \mathcal{L}_M$ because the second term in (12) is not any longer a pure divergence as it was in the case of global infinitesimal transformations.

As usual it is sufficient to construct a covariant derivative which fulfills

$$D'_\alpha U(\Theta(x)) = U(\Theta(x)) D_\alpha.\tag{13}$$

For the infinitesimal transformation $U(\Theta) = \mathbf{1} + \Theta$ considered eqn. (13) ensures indeed the proper transformation behaviour (12). We try the usual ansatz

$$\partial_\alpha \longrightarrow D_\alpha = \partial_\alpha + B_\alpha$$

together with the decomposition of B_α w.r.t. the generators p_α of $\mathfrak{t}$

$$B_\alpha \equiv -i B_\alpha{}^\gamma \cdot p_\gamma,$$

introducing thus the compensating fields $B_\alpha{}^\gamma$. Note that B_α acts here as a differential operator in field space and not as usual as a matrix. Note also that the Leibniz rule still holds for D_α acting on products of fields.

We are ready now to discuss the behaviour of B_α under local gauge transformations demanding that eqn. (13) holds.

$$(\partial_\alpha + B'_\alpha) U(\Theta) = U(\Theta)(\partial_\alpha + B_\alpha)$$

yields

$$B'_\alpha U(\Theta) = U(\Theta)B_\alpha - \partial_\alpha U(\Theta).$$

Acting with this equation on $U(-\Theta)$ from the right leads then together with $U(\Theta) \cdot U(-\Theta) = \mathbf{1}$ to

$$B'_\alpha = U(\Theta)(B_\alpha + \partial_\alpha)U(-\Theta).$$

With the infinitesimal $U(\Theta) = 1 + \Theta$ considered we get

$$\begin{aligned} \delta B_\alpha &= B'_\alpha - B_\alpha \\ &= -\partial_\alpha \Theta + [\Theta, B_\alpha] \end{aligned}$$

defining the representation of the translation group $\mathbf{T}$ in the gauge field space. Note the structural similarity of all the results obtained up to now to similar ones in the discussion of non-abelian gauge symmetry [8]. Next we decompose δB_α w.r.t. the generators p_α of $\mathbf{t}$ and obtain

$$\begin{aligned} \delta B_\alpha &= -\partial_\alpha \Theta + [\Theta, B_\alpha] \\ &= -i \partial_\alpha \varepsilon^\gamma \cdot p_\gamma + [i \varepsilon^\delta p_\delta, -i B_\alpha{}^\gamma p_\gamma] \\ &= -i \left\{ \partial_\alpha \varepsilon^\gamma + B_\alpha{}^\delta \cdot \partial_\delta \varepsilon^\gamma - \varepsilon^\delta \cdot \partial_\delta B_\alpha{}^\gamma \right\} p_\gamma \\ &\equiv -i \delta B_\alpha{}^\gamma \cdot p_\gamma \end{aligned}$$

or

$$\delta B_\alpha{}^\gamma = \partial_\alpha \varepsilon^\gamma + B_\alpha{}^\delta \cdot \partial_\delta \varepsilon^\gamma - \varepsilon^\delta \cdot \partial_\delta B_\alpha{}^\gamma. \quad (14)$$

Of course this transformation formula strongly differs from analogous ones in non-abelian gauge field theory.

Let us finally introduce the field strenght tensor

$$\begin{aligned} H_{\alpha\beta} &= [D_\alpha, D_\beta] \\ &= \partial_\alpha B_\beta - \partial_\beta B_\alpha + [B_\alpha, B_\beta] \end{aligned} \quad (15)$$

together with its decomposition w.r.t. p_α

$$H_{\alpha\beta} \equiv -i G_{\alpha\beta}{}^\gamma \cdot p_\gamma.$$

Obviously it transforms homogeneously under local **T** gauge transformations

$$\begin{aligned}
H'_{\alpha\beta} &= [D'_\alpha, D'_\beta] \\
&= [U(\Theta)D_\alpha U(-\Theta), U(\Theta)D_\beta U(-\Theta)] \\
&= U(\Theta)H_{\alpha\beta}U(-\Theta)
\end{aligned}$$

where we used (13). With $U(\Theta) = 1 + \Theta$ we get [8]

$$\begin{aligned}
\delta H_{\alpha\beta} &= H'_{\alpha\beta} - H_{\alpha\beta} \\
&= [\Theta, H_{\alpha\beta}].
\end{aligned}$$

We also decompose $\delta H_{\alpha\beta}$ w.r.t. p_α and obtain

$$\begin{aligned}
\delta H_{\alpha\beta} &= [\Theta, H_{\alpha\beta}] \\
&= [i\varepsilon^\delta p_\delta, -iG_{\alpha\beta}{}^\gamma p_\gamma] \\
&= -i\{G_{\alpha\beta}{}^\delta \cdot \partial_\delta \varepsilon^\gamma - \varepsilon^\delta \cdot \partial_\delta G_{\alpha\beta}{}^\gamma\} p_\gamma \\
&\equiv -i\delta G_{\alpha\beta}{}^\gamma p_\gamma,
\end{aligned}$$

or

$$\delta G_{\alpha\beta}{}^\gamma = G_{\alpha\beta}{}^\delta \cdot \partial_\delta \varepsilon^\gamma - \varepsilon^\delta \cdot \partial_\delta G_{\alpha\beta}{}^\gamma. \quad (16)$$

We finally remark that the first term in the transformation law (16) for $G_{\alpha\beta}{}^\gamma$ prevents us from building up a simple **T** invariant action for the gauge fields $B_\alpha{}^\gamma$ in terms of $G_{\alpha\beta}{}^\gamma$. The necessary covariant decomposition of the field strength will be given in the next Section.

4 The gauge fields $e_\alpha{}^\gamma$ and the covariant components $H_{\alpha\beta}{}^\gamma$ of the field strenght operator

In this Section we recast the previous results in terms of the gauge fields $e_\alpha{}^\gamma = \delta_\alpha{}^\gamma + B_\alpha{}^\gamma$. We discuss the **T** transformation properties of the covariant decomposition of the field strenght operator $H_{\alpha\beta} = H_{\alpha\beta}{}^\gamma D_\gamma$. Finally we determine the Bianchi constraints on $H_{\alpha\beta}{}^\gamma$.

In the previous Section we introduced the covariant derivative D_α . We recast it now introducing the matrix field $e_\alpha{}^\gamma$ [3,6]

$$\begin{aligned}
D_\alpha &= \partial_\alpha + B_\alpha \\
&= (\delta_\alpha{}^\gamma + B_\alpha{}^\gamma)\partial_\gamma \\
&= e_\alpha{}^\gamma \partial_\gamma
\end{aligned}$$

where

$$e_\alpha{}^\gamma = \delta_\alpha{}^\gamma + B_\alpha{}^\gamma. \quad (17)$$

At this stage we recall that it was the extension of global to local $\mathbf{T}$ symmetry transformations conceived as inner symmetry transformations which has led us to the introduction of both matrix fields $B_\alpha{}^\gamma$ and $e_\alpha{}^\gamma$. They serve as compensating fields in exactly the same manner as e.g. the Maxwell field serves as compensating field which has to be introduced gauging $U(1)$. In this sense they may indeed be understood as gauge fields.

As in our conception coordinate and $\mathbf{T}$ gauge transformations are strictly separated we emphasize that the introduction of $B_\alpha{}^\gamma$ and $e_\alpha{}^\gamma$ has neither implications on the structure of the underlying space-time which we assumed to be $(\mathbf{R}^4, \eta)$ endowed with the Minkowski metric η . Nor has it implications on the maximal symmetry group $\mathbf{P}$ of $(\mathbf{R}^4, \eta)$ if we still restrict ourselves to the use of Cartesian coordinates only. This fact will allow us later to obtain the energy-momentum of $e_\alpha{}^\gamma$ in the usual way.

Let us discuss now some of the properties of the gauge fields $e_\alpha{}^\gamma$. The variation of $e_\alpha{}^\gamma$ under local $\mathbf{T}$ gauge transformations reads with the use of eqn. (14)

$$\begin{aligned} \delta e_\alpha{}^\gamma &= e'_\alpha{}^\gamma - e_\alpha{}^\gamma \\ &= e_\alpha{}^\delta \cdot \partial_\delta \varepsilon^\gamma - \varepsilon^\delta \cdot \partial_\delta e_\alpha{}^\gamma. \end{aligned} \quad (18)$$

As the determinant $\det e^{-1}$ will enter the locally $\mathbf{T}$ invariant actions we determine its transformation behaviour next.

$$\begin{aligned} \delta \det e^{-1} &= -\det e^{-1} \operatorname{tr} e^{-1} \cdot \delta e \\ &= -\det e^{-1} (\partial_\alpha \varepsilon^\alpha + \varepsilon^\alpha \cdot \partial_\alpha \operatorname{tr} \log e^{-1}) \\ &= -\det e^{-1} \cdot \partial_\alpha \varepsilon^\alpha - \varepsilon^\alpha \cdot \partial_\alpha \det e^{-1}. \end{aligned} \quad (19)$$

Finally we turn to the field strenght operator. We already mentioned that the decomposition of $H_{\alpha\beta}$ w.r.t. the p_α shows not yet the desired behaviour under gauge transformations. Let us give therefore its decomposition w.r.t. D_α now

$$\begin{aligned} [D_\alpha, D_\beta] &= H_{\alpha\beta} \\ &= G_{\alpha\beta}{}^\gamma \partial_\gamma \\ &= G_{\alpha\beta}{}^\gamma e^{-1}{}^\delta{}_\gamma \cdot e_\delta{}^\eta \partial_\eta \\ &= H_{\alpha\beta}{}^\delta D_\delta \end{aligned}$$

where

$$H_{\alpha\beta}{}^\delta = e^{-1\delta}{}_\gamma G_{\alpha\beta}{}^\gamma$$

and $e^{-1\delta}{}_\gamma$ is the matrix inverse to $e_\alpha{}^\gamma$, i.e.

$$e_\alpha{}^\gamma \cdot e^{-1\beta}{}_\gamma = \delta_\alpha{}^\beta.$$

For $H_{\alpha\beta}{}^\gamma$ we obtain [3]

$$\begin{aligned} H_{\alpha\beta}{}^\gamma &= e^{-1\gamma}{}_\delta (e_\alpha{}^\eta \cdot \partial_\eta e_\beta{}^\delta - e_\beta{}^\eta \cdot \partial_\eta e_\alpha{}^\delta) \\ &= (e_\alpha{}^\delta e_\beta{}^\eta - e_\beta{}^\delta e_\alpha{}^\eta) \partial_\eta e^{-1\gamma}{}_\delta. \end{aligned} \quad (20)$$

To find the behaviour of $H_{\alpha\beta}{}^\gamma$ under infinitesimal **T** gauge transformations we note that

$$H'_{\alpha\beta} = U(\Theta) H_{\alpha\beta} U(-\Theta)$$

may be rewritten as

$$\begin{aligned} H'_{\alpha\beta}{}^\gamma D'_\gamma &= H'_{\alpha\beta}{}^\gamma U(\Theta) D_\gamma U(-\Theta) \\ &= U(\Theta) H_{\alpha\beta}{}^\gamma D_\gamma U(-\Theta) \end{aligned}$$

leading to the homogeneous transformation law

$$H'_{\alpha\beta}{}^\gamma = U(\Theta) H_{\alpha\beta}{}^\gamma U(-\Theta).$$

With $U(\Theta) = 1 + \Theta$ we thus get

$$\begin{aligned} \delta H_{\alpha\beta}{}^\gamma &= H'_{\alpha\beta}{}^\gamma - H_{\alpha\beta}{}^\gamma \\ &= [\Theta, H_{\alpha\beta}{}^\gamma] \\ &= -\varepsilon^\delta \cdot \partial_\delta H_{\alpha\beta}{}^\gamma, \end{aligned} \quad (21)$$

i.e. $H_{\alpha\beta}{}^\gamma$ transforms as the matter fields in eqn. (11). This is the desired covariant behaviour. Any locally **T** invariant action for the gauge fields $e_\alpha{}^\gamma$ may therefore be given most simply in terms of $H_{\alpha\beta}{}^\gamma$ and its covariant derivatives, a task we postpone to Section 7.

Next we turn to the Bianchi identities following from the Jacobi identity for covariant derivatives

$$[D_\alpha, [D_\beta, D_\gamma]] + \text{cycl.perm.} = 0. \quad (22)$$

We rewrite (22) in terms of H and its first derivatives

$$\begin{aligned} [D_\alpha, H_{\beta\gamma}] + \text{cycl.perm.} &= [D_\alpha, H_{\beta\gamma}{}^\delta D_\delta] + \text{cycl.perm.} \\ &= \left(D_\alpha H_{\beta\gamma}{}^\eta + H_{\beta\gamma}{}^\delta H_{\alpha\delta}{}^\eta \right) D_\eta + \text{cycl.perm.} \\ &= 0. \end{aligned}$$

Thereby we get the four identities

$$D_\alpha H_{\beta\gamma}{}^\eta + H_{\beta\gamma}{}^\delta H_{\alpha\delta}{}^\eta + \text{cycl.perm.} = 0 \quad \eta = 0, \dots, 3 \quad (23)$$

expressing constraints on the field strength related to the four degrees of **T** gauge freedom.

5 **T** gauge invariant matter actions

In this Section we discuss the extension of globally **T** invariant matter actions on $(\mathbf{R}^4, \eta)$ to locally **T** gauge invariant ones. We then derive the quasi conservation law of energy-momentum valid for the locally **T** invariant theory.

Let us consider a globally **T** invariant theory for n fields φ_j specified by the Lagrangian $\mathcal{L}_M(\varphi_j, \partial_\alpha \varphi_j)$ assumed to be real $\mathcal{L}_M^* = \mathcal{L}_M$. In Section 3 we have constructed the covariant derivative D_α respecting the behaviour of $\mathcal{L}_M$ under global translations extended to local ones, i.e. for

$$\varphi_j(x) \longrightarrow \varphi'_j(x) = \varphi_j(x) - \varepsilon^\alpha(x) \cdot \partial_\alpha \varphi_j(x)$$

we have eqn. (12)

$$\mathcal{L}_M(\varphi'_j, D'_\alpha \varphi'_j) = \mathcal{L}_M(\varphi_j, D_\alpha \varphi_j) - \varepsilon^\beta \cdot \partial_\beta \mathcal{L}_M(\varphi_j, D_\alpha \varphi_j).$$

But as we already mentioned (12) is not sufficient for the original action $S_M = \int \mathcal{L}_M$ to be locally **T** gauge invariant.

We must complete the Lagrangian density with another term ensuring that the change of both terms together under a local **T** transformation will yield a pure divergence term only [3]. Now remember the transformation law (19) for $\det e^{-1}$

$$\det e'^{-1} = \det e^{-1} - \det e^{-1} \cdot \partial_\alpha \varepsilon^\alpha - \varepsilon^\alpha \cdot \partial_\alpha \det e^{-1}.$$

With this result we get for the behaviour of the combination

$$\det e^{-1} \cdot \mathcal{L}_M(\varphi_j, D_\alpha \varphi_j) \quad (24)$$

under local **T** gauge transformations

$$\begin{aligned} \det e'^{-1} \cdot \mathcal{L}_M(\varphi'_j, D'_\alpha \varphi'_j) &= \\ \det e^{-1} \cdot \mathcal{L}_M(\varphi_j, D_\alpha \varphi_j) &- \partial_\beta \left(\varepsilon^\beta \cdot \det e^{-1} \cdot \mathcal{L}_M(\varphi_j, D_\alpha \varphi_j) \right), \end{aligned}$$

i.e. the change of the combination (24) is indeed a pure divergence.

Therefore the minimally extended locally $\mathbf{T}$ invariant matter action becomes [3,6]

$$S_M = \int d^4x \det e^{-1}(x) \cdot \mathcal{L}_M(\varphi_j(x), D_\alpha \varphi_j(x)). \quad (25)$$

S_M remains of course globally $\mathbf{P}$ invariant if we require that $B_\alpha{}^\gamma$ and $e_\alpha{}^\gamma$ transform under $\mathbf{L}$ as does $\delta_\alpha{}^\gamma$.

It is the conception of $\mathbf{T}$ symmetry as an inner symmetry together with the gauge principle which has lead us to this minimal coupling prescription. As in this conception the gauge fields and their transformation behaviour do not interfere with the space-time structure $(\mathbf{R}^4, \eta)$ fixed by an apriori convention the underlying geometry remains separated from the physics described by the $\mathbf{T}$ gauge fields in the same manner it remains separated from the physics described by any matrix gauge field.

We postpone the difficulties arising with matter fields carrying spin [10,11] which demand an extension of the actual gauge group $\mathbf{T}$ to the full Poincaré to a separate discussion and turn to apply the general framework developed so far to the globally $\mathbf{T}$ invariant action for one real scalar field of mass m

$$S_M = \int d^4x \left\{ \frac{1}{2} \partial_\alpha \varphi \cdot \partial^\alpha \varphi - \frac{1}{2} m^2 \varphi^2 \right\}. \quad (26)$$

For this example the minimal extension (25) yields the locally $\mathbf{T}$ gauge invariant action

$$S_M = \int d^4x \det e^{-1} \left\{ \frac{1}{2} D_\alpha \varphi \cdot D^\alpha \varphi - \frac{1}{2} m^2 \varphi^2 \right\}. \quad (27)$$

As φ is a scalar S_M is in fact even locally $\mathbf{P}$ invariant. Note that in the presence of $e_\alpha{}^\gamma$ the scalar product in real scalar field space becomes

$$(\chi, \varphi)_e = \int d^4x \det e^{-1} \chi \cdot \varphi$$

and that the infinitesimal local representation of $\mathbf{T}$ is unitary w.r.t. it.

Let us finally investigate the current to which $e_\alpha{}^\gamma$ couples, i.e. for the general theory (25) $\frac{\delta S_M}{\delta e_\alpha{}^\gamma}$ and derive the quasi conservation law it fulfills. By construction the action S_M is invariant under local $\mathbf{T}$ gauge transformations. Demanding that the fields φ_j obey the equations of motion $\frac{\delta S_M}{\delta \varphi_j} = 0$ we obtain for the special variations (18)

$$\delta e_\alpha{}^\gamma = e_\alpha{}^\delta \cdot \partial_\delta \varepsilon^\gamma - \varepsilon^\delta \cdot \partial_\delta e_\alpha{}^\gamma$$

expressing the effect of such a local **T** gauge transformation

$$\begin{aligned}\delta S_M &= \int \frac{\delta S_M}{\delta e_\alpha^\gamma} \left\{ e_\alpha^\delta \cdot \partial_\delta \varepsilon^\gamma - \varepsilon^\delta \cdot \partial_\delta e_\alpha^\gamma \right\} \\ &= - \int \left\{ \partial_\delta \cdot (e_\alpha^\delta \frac{\delta S_M}{\delta e_\alpha^\gamma}) + \frac{\delta S_M}{\delta e_\alpha^\delta} \cdot \partial_\gamma e_\alpha^\delta \right\} \varepsilon^\gamma \\ &= 0.\end{aligned}$$

Here we performed a partial integration in the first term.

We therefore get the quasi conservation of energy-momentum in the form

$$\partial_\delta \Theta^\delta_\gamma + \frac{\delta S_M}{\delta e_\alpha^\delta} \cdot \partial_\gamma e_\alpha^\delta = 0 \quad (28)$$

introducing the energy-momentum tensor of matter in the presence of e_α^γ

$$\Theta^\delta_\gamma \equiv e_\alpha^\delta \frac{\delta S_M}{\delta e_\alpha^\gamma}. \quad (29)$$

Note that Θ^δ_γ does not transform covariantly w.r.t. **T** gauge transformations.

For comparison with later results we rewrite Θ^δ_γ now making use of the explicit dependence of S_M on e_α^γ occuring only in $\det e^{-1}$ and in covariant derivatives $D_\alpha = e_\alpha^\gamma \partial_\gamma$. We get

$$\frac{\delta S_M}{\delta e_\alpha^\gamma} = \det e^{-1} \left\{ \frac{\partial \mathcal{L}_M}{\partial (D_\alpha \varphi_j)} \cdot \partial_\gamma \varphi_j - e^{-1\alpha}_\gamma \mathcal{L}_M \right\}.$$

Together with

$$\frac{\partial \mathcal{L}_M}{\partial (D_\alpha \varphi_j)} = e^{-1\alpha}_\gamma \frac{\partial \mathcal{L}_M}{\partial (\partial_\gamma \varphi_j)}$$

we finally obtain

$$\Theta^\delta_\gamma = \det e^{-1} \left\{ \frac{\partial \mathcal{L}_M}{\partial (\partial_\delta \varphi_j)} \cdot \partial_\gamma \varphi_j - \delta^\delta_\gamma \mathcal{L}_M \right\}, \quad (30)$$

a result we may later derive directly from the Noether theorem applied to the action S_M combined with the action S_G for e_α^γ . Θ^β_α reduces in the weak-field limit to the energy-momentum tensor (6) as is easily seen by setting e_α^γ equal to δ_α^γ and (28) reduces to the ordinary conservation of Θ^β_α . In the general case we will have a real conservation law only after adding the contribution coming from the dynamics of the gauge field itself as expected.

6 Coordinate invariance as complementary conception

In this Section we recast the locally **T** gauge invariant scalar field theory (27) in terms of the field $g^{\gamma\delta} = e_\alpha{}^\gamma e^{\alpha\delta}$. We discuss the transformation properties of $g^{\gamma\delta}$. Its introduction allows us to reinterpret the formalism in geometrical terms.

In the previous Section we gave the minimal locally **T** gauge invariant action for a real scalar field eqn. (27). We recast it now

$$\begin{aligned} S_M &= \int d^4x \det e^{-1} \left\{ \frac{1}{2} e_\alpha{}^\gamma \partial_\gamma \varphi \cdot e^{\alpha\delta} \partial_\delta \varphi - \frac{1}{2} m^2 \varphi^2 \right\} \\ &= \int d^4x \det^{\frac{1}{2}}(-g) \left\{ \frac{1}{2} g^{\gamma\delta} \partial_\gamma \varphi \cdot \partial_\delta \varphi - \frac{1}{2} m^2 \varphi^2 \right\} \end{aligned} \quad (31)$$

introducing thus the symmetric combination

$$g^{\gamma\delta} = e_\alpha{}^\gamma e^{\alpha\delta}. \quad (32)$$

of the gauge fields $e_\alpha{}^\gamma$. $\det(-g)$ denotes the determinant of the matrix $g_{\beta\gamma}$ inverse to $g^{\gamma\delta}$, i.e.

$$g_{\beta\gamma} \cdot g^{\gamma\delta} = \delta_\beta{}^\delta.$$

Note that it is the quadratic occurrence of the covariant derivatives in (27) which permits this recasting in terms of $g^{\gamma\delta}$ only.

The variation of $g^{\gamma\delta}$ under local **T** gauge transformations becomes with the use of eqn. (18)

$$\begin{aligned} \delta g^{\gamma\delta} &= g'^{\gamma\delta} - g^{\gamma\delta} \\ &= g^{\gamma\alpha} \cdot \partial_\alpha \varepsilon^\delta + g^{\delta\alpha} \cdot \partial_\alpha \varepsilon^\gamma - \varepsilon^\alpha \cdot \partial_\alpha g^{\gamma\delta}. \end{aligned} \quad (33)$$

Let us now reinterpret the formalism developed so far in geometrical terms. Up to now we relied on the conception of **T** symmetry as an inner symmetry expressed in the transformation behaviour eqn. (11) for matter fields and eqns. (14), (18) for the gauge fields. It allowed us to extend the framework of gauge theories of matrix groups to the operator gauge group **T**. As the group **P** of global space-time transformations relating different observers and the local gauge group **T** were strictly separated the apriori geometry of space-time $(\mathbf{R}^4, \eta)$ chosen to be Minkowskian was not affected by the introduction of the **T** gauge fields $e_\alpha{}^\gamma$.

We turn now to the complementary conception of translation symmetry as a space-time symmetry [3,10]. The infinitesimal transformation formulae involve now both coordinates and fields as in eqn. (7) introducing covariant indices $\mu, \nu, \rho, \dots$

$$\begin{aligned} x^\mu &\longrightarrow x'^\mu &= x^\mu + \varepsilon^\mu(x) \\ \varphi_j(x) &\longrightarrow \varphi'_j(x') &= \varphi_j(x). \end{aligned} \quad (34)$$

Eqn. (18) is equivalent to

$$e_\alpha{}^\mu(x) \longrightarrow e'_\alpha{}^\mu(x') = e_\alpha{}^\nu(x) \cdot (\delta_\nu{}^\mu + \partial_\nu \varepsilon^\mu). \quad (35)$$

For $g^{\mu\nu}$ this leads to the transformation behaviour

$$g^{\mu\nu}(x) \longrightarrow g'^{\mu\nu}(x') = g^{\rho\sigma}(x) \cdot (\delta_\rho{}^\mu + \partial_\rho \varepsilon^\mu)(\delta_\sigma{}^\nu + \partial_\sigma \varepsilon^\nu). \quad (36)$$

Note that the covariant upper indices $\mu, \nu, \rho, \dots$ are transformed with the Jacobi matrix $\frac{\partial x'^\mu}{\partial x^\rho} = \delta_\rho{}^\mu + \partial_\rho \varepsilon^\mu$ resulting from the infinitesimal coordinate change and correspondingly lower covariant indices with its inverse $\frac{\partial x^\sigma}{\partial x'^\nu} = \delta_\nu{}^\sigma - \partial_\nu \varepsilon^\sigma$.

The gauge group $\mathbf{T}$ and the requirement of local $\mathbf{T}$ gauge invariance are thus replaced by the group of general (infinitesimal) coordinate transformations and the requirement of coordinate invariance. $g^{\mu\nu}$ has to be reinterpreted as metric tensor. The geometry of space-time is now necessarily linked with the complementary symmetry requirement of general coordinate covariance.

The natural framework to deal with this point of view is thus Riemannian geometry [12]. ∂_μ and D_α fit in as coordinate and orthonormal non-coordinate basis vectors in the tangential spaces belonging to the Riemannian manifold $(\mathbf{R}^4, g)$ endowed with the indefinite metric $g^{\mu\nu}$. $e_\alpha{}^\mu$ enters as the vierbein relating the two basis systems as $D_\alpha = e_\alpha{}^\mu \partial_\mu$ and $H_{\alpha\beta}{}^\gamma$ as anholonomy coefficients. Finally the equation of motion for φ obtained from varying eqn. (31) are as usual expressed in terms of the coordinate covariant d'Alembertian $\det^{-\frac{1}{2}}(-g) \cdot \partial_\mu \left(\det^{\frac{1}{2}}(-g) g^{\mu\nu} \partial_\nu \right)$ which in turn leads to the introduction of the Christoffel symbols $\Gamma^\mu{}_{\rho\sigma} = g^{\mu\nu} \Gamma_{\nu\rho\sigma}$ with $\Gamma_{\nu\rho\sigma} = \frac{1}{2} (\partial_\sigma g_{\rho\nu} + \partial_\rho g_{\sigma\nu} - \partial_\nu g_{\rho\sigma})$. The connection naturally involved is thus the Levi-Civita connection $\nabla_{L.C.}$. As its torsion tensor vanishes we may express any coordinate invariant action for $g^{\mu\nu}$ in terms of the curvature tensor and its covariant derivatives.

For T gauge invariant matter actions which may be reexpressed in terms of $g^{\mu\nu}$ only we determine the current $\frac{\delta S_M}{\delta g^{\mu\nu}}$ to which the metric couples. As

$$\frac{\delta S_M}{\delta e_\alpha{}^\mu} = \frac{\delta S_M}{\delta g^{\mu\rho}} e^{\alpha\rho} + \frac{\delta S_M}{\delta g^{\sigma\mu}} e^{\alpha\sigma}$$

we obtain

$$e_\alpha{}^\nu \frac{\delta S_M}{\delta e_\alpha{}^\mu} = \Theta^\nu{}_\mu = 2g^{\nu\rho} \frac{\delta S_M}{\delta g^{\mu\rho}}$$

or finally

$$T_{\mu\nu} \equiv \frac{\delta S_M}{\delta g^{\mu\nu}} = \frac{1}{2} g_{\mu\rho} \Theta^\rho{}_\nu \quad (37)$$

which is the usual symmetric coordinate covariant energy-momentum tensor [13]. The quasi conservation law (28) for $\Theta^\beta{}_\alpha$ finally yields the coordinate covariant conservation of $T_{\mu\nu}$

$$\nabla_{L.C.}^\mu T_{\mu\nu} = 0. \quad (38)$$

7 Renormalizability and the dynamics of the gauge fields

In this Section we determine the classical gauge field dynamics using the scaling behaviour of the one-loop partition function for matter in the presence of $e_\alpha{}^\gamma$ and the requirement of renormalizability.

In the previous Sections symmetry requirements in classical field theory have led us to introduce the gauge fields $e_\alpha{}^\gamma$ with definite coupling to given matter fields. Symmetry requirements in quantum field theory will lead us to an essentially unique dynamics for the gauge fields. For let us suppose that a given theory for the matter field φ and the gauge field $e_\alpha{}^\gamma$ is perturbatively renormalizable. We thus know that the change of the partition function of the whole system under rescaling may be absorbed in a non-trivial scale dependence of the different couplings, masses and wavefunction normalizations. In other words we can restore the scale invariance of the partition function order by order in a perturbative loop expansion up to 'running couplings, masses and wavefunction normalizations'.

Using heat kernel techniques we may now explicitly compute the part of the change of the one-loop partition function coming from the matter sector. Renormalizability and the subsequent scale invariance of the one-loop partition function then require that this part may be reabsorbed in

the classical action of the theory. Thus the classical action must contain at least all the terms controlling also the scaling behaviour of the one-loop partition function. Restriction to just those fixes on the other hand the minimal classical gauge field dynamics in an essentially unique way [4].

Note that this reasoning leads in the case of non-abelian matrix groups indeed to the usual Yang-Mills action. Note also that in this way renormalizability is of course ensured only at the one-loop level. As technical subtleties arising in the computation of the change of the one-loop partition function have already been discussed elsewhere [5] we restrict ourselves to the main line of argumentation.

To be definite let us investigate the scalar field theory (27) coupled to the gauge fields $e_\alpha{}^\gamma$. We first recast the action

$$\begin{aligned} S_M &= \int d^4x \det e^{-1} \left\{ \frac{1}{2} D_\alpha \varphi \cdot D^\alpha \varphi - \frac{1}{2} m^2 \varphi^2 \right\} \\ &= \frac{1}{2} \int d^4x \det e^{-1} \varphi \left\{ -(D_\alpha + H_{\gamma\alpha}{}^\gamma) \cdot D^\alpha - m^2 \right\} \varphi \end{aligned}$$

using partial intergration and

$$\partial_\gamma (\det e^{-1} e_\alpha{}^\gamma) = \det e^{-1} \cdot H_{\gamma\alpha}{}^\gamma. \quad (39)$$

We farther rewrite it as

$$S_M = \frac{1}{2} (\varphi, M(e, m) \varphi)_e$$

introducing the hyperbolic fluctuation operator $M(e, m)$

$$\begin{aligned} M(e, m) &= -(D_\alpha + \frac{1}{2} H_{\gamma\alpha}{}^\gamma) \cdot (D^\alpha + \frac{1}{2} H_\delta{}^{\alpha\delta}) \\ &\quad + \frac{1}{2} D_\alpha H_{\gamma}{}^{\alpha\gamma} + \frac{1}{2} H_{\gamma\alpha}{}^\gamma \cdot H_\delta{}^{\alpha\delta} - m^2 \end{aligned} \quad (40)$$

which is hermitean w.r.t. $(\cdot, \cdot)_e$.

The contribution of the scalar field sector to the partition function reads

$$\mathcal{Z}_\varphi[e_\alpha{}^\gamma] = \frac{1}{\mathcal{N}} \int \mathcal{D}\varphi e^{\frac{i}{2} (\varphi, M(e, m) \varphi)_e} \quad (41)$$

with the normalization constant $\mathcal{N} = \mathcal{Z}_\varphi[\delta_\alpha{}^\gamma]$. Performing the Gaussian integration yields formally

$$\mathcal{Z}_\varphi[e_\alpha{}^\gamma] = e^{-\frac{1}{2} \{ \log \det M(e, m) - \log \det M(\delta, m) \}}. \quad (42)$$

As we are only interested in the behaviour of $\mathcal{Z}_\varphi[e_\alpha \gamma]$ under rescaling the most suited regularization of the ultraviolet divergent determinants above is the ζ -function regularization [14,15]. We define for a general hermitean second-order fluctuation operator M

$$\log \det M \equiv - \lim_{u \rightarrow 0} \frac{d}{du} \zeta(u; \mu; M) \quad \zeta(u; \mu; M) \equiv \mu^{2u} \text{Tr } M^{-u} \quad (43)$$

introducing the scale μ at which couplings, masses and wavefunction normalizations have to be adjusted. Note that in the formal expression (42) μ drops out. The above definition of ζ does not allow to take the u -derivative at $u = 0$ since the trace is defined only for $\text{Re } u > \frac{d}{2}$. The necessary analytic continuation may be achieved by recasting ζ as the Mellin transformation of the heat kernel

$$\zeta(u; \mu; M) = \frac{i\mu^{2u}}{\Gamma(u)} \int_0^\infty ds (is)^{u-1} \text{Tr } e^{-isM}. \quad (44)$$

The scalar contribution to the partition function normalized at scale μ thus becomes

$$\mathcal{Z}_\varphi[\mu, e_\alpha \gamma] = e^{\frac{1}{2} \{ \zeta'(0; \mu; M(e, m)) - \zeta'(0; \mu; M(\delta, m)) \}}. \quad (45)$$

Let us next consider this contribution at the new scale $\tilde{\mu} = \lambda\mu$ and determine the corresponding change of $\mathcal{Z}_\varphi$

$$\mathcal{Z}_\varphi[\tilde{\mu}, e_\alpha \gamma] = e^{\frac{1}{2} \{ \zeta'(0; \tilde{\mu}; M(e, m)) - \zeta'(0; \tilde{\mu}; M(\delta, m)) \}}.$$

Using

$$\zeta'(0; \lambda\mu; M) = \zeta'(0; \mu; M) + 2 \log \lambda \cdot \zeta(0; \mu; M)$$

we get

$$\mathcal{Z}_\varphi[\tilde{\mu}, e_\alpha \gamma] = \mathcal{Z}_\varphi[\mu, e_\alpha \gamma] \cdot e^{\log \lambda \cdot \{ \zeta(0; \mu; M(e, m)) - \zeta(0; \mu; M(\delta, m)) \}}. \quad (46)$$

Renormalizability of the full theory now requires that the anomalous contribution $\log \lambda \cdot \{ \zeta(0; \mu; M(e, m)) - \zeta(0; \mu; M(\delta, m)) \}$, which is a local polynomial in $e_\alpha \gamma$ and its derivatives, may be reabsorbed in the classical action for the gauge fields $e_\alpha \gamma$ leading at most to a nontrivial scale dependence of couplings, masses and wavefunction normalizations. As the explicit change in the case of usual Yang-Mills theories is found to be proportional to the

Yang-Mills action we are led by analogy to take some multiple of the anomalous contribution as minimal action for the gauge fields $e_\alpha{}^\gamma$.

It thus remains to determine finally $\zeta(0; \mu; M)$. In the limit $u \rightarrow 0$ it is the singular part of the s integration in the representation (44) which yields a nonvanishing value for $\zeta(0; \mu; M)$. As this singular part comes from the small s region we may use the small s expansion for the trace of the heat kernel [16]

$$\text{Tr } e^{-isM} \underset{s \rightarrow 0}{\sim} \frac{i}{(4\pi is)^{\frac{d}{2}}} \sum_{k=0}^{\infty} (is)^k \int d^d x \det e^{-1} \text{tr } c_k(x). \quad (47)$$

Performing the s integration singles out the contribution for $k = \frac{d}{2}$ from the sum and one obtains [17]

$$\zeta(0; \mu; M) = \frac{i}{(4\pi)^{\frac{d}{2}}} \int d^d x \det e^{-1} \text{tr } c_{\frac{d}{2}}(x). \quad (48)$$

Let us apply this result to $\zeta(0; \mu; M(e, m))$. Separating off m^2 explicitly we get with $d = 4$

$$\begin{aligned} \zeta(0; \mu; M(e, m)) \\ = \frac{i}{(4\pi)^2} \int d^4 x \det e^{-1} \left\{ \frac{1}{2} m^4 c_0(x) + m^2 c_1(x) + c_2(x) \right\} \end{aligned} \quad (49)$$

where the $c_k(x)$ belong to operator $M(e, 0)$. As we shall see only $c_1(x)$ is relevant for the classical dynamics of $e_\alpha{}^\gamma$ at sufficiently low momentum scales. Using the results (83) and (88) from Appendix A we thus find

$$\begin{aligned} \zeta(0; \mu; M(e, m)) - \zeta(0; \mu; M(\delta, m)) \\ = \frac{i}{(4\pi)^2} \int d^4 x \det e^{-1} \left[-\frac{m^2}{24} \{ -4D_\alpha H_\gamma{}^{\alpha\gamma} + 2H_{\alpha\beta\gamma} \cdot H^{\alpha\gamma\beta} \right. \\ \left. + H_{\alpha\beta\gamma} \cdot H^{\alpha\beta\gamma} - 2H_{\gamma\alpha}{}^\gamma \cdot H_\delta{}^{\alpha\delta} \} - m^2 E + O(\partial^4) \right]. \end{aligned}$$

The normalization has canceled the first term. $O(\partial^4)$ denotes the contributions from $c_2(x)$ which all contain four derivatives. E is determined from $M(e, 0)$

$$E = \frac{1}{2} D_\alpha H_\gamma{}^{\alpha\gamma} + \frac{1}{4} H_{\gamma\alpha}{}^\gamma \cdot H_\delta{}^{\alpha\delta}.$$

We thus finally obtain

$$\begin{aligned}
& \zeta(0; \mu; M(e, m)) - \zeta(0; \mu; M(\delta, m)) \\
&= \frac{i}{(4\pi)^2} \int d^4x \det e^{-1} \left[-\frac{m^2}{24} \{ 8D_\alpha H_\gamma{}^{\alpha\gamma} + 2H_{\alpha\beta\gamma} \cdot H^{\alpha\gamma\beta} \right. \\
&\quad \left. + H_{\alpha\beta\gamma} \cdot H^{\alpha\beta\gamma} + 4H_{\gamma\alpha}{}^\gamma \cdot H_\delta{}^{\alpha\delta} \} + O(\partial^4) \right] \\
&= \frac{i}{(4\pi)^2} \int d^4x \det e^{-1} \left[-\frac{m^2}{6} \{ -H_{\gamma\alpha}{}^\gamma \cdot H_\delta{}^{\alpha\delta} \right. \\
&\quad \left. + \frac{1}{2} H_{\alpha\beta\gamma} \cdot H^{\alpha\gamma\beta} + \frac{1}{4} H_{\alpha\beta\gamma} \cdot H^{\alpha\beta\gamma} \} + O(\partial^4) \right]
\end{aligned} \tag{50}$$

performing a partial integration in the first term. As minimal classical action for $e_\alpha{}^\gamma$ we now take

$$\begin{aligned}
S_G &= \frac{1}{\kappa^2} \int d^4x \det e^{-1} \{ -H_{\gamma\alpha}{}^\gamma \cdot H_\delta{}^{\alpha\delta} \\
&\quad + \frac{1}{2} H_{\alpha\beta\gamma} \cdot H^{\alpha\gamma\beta} + \frac{1}{4} H_{\alpha\beta\gamma} \cdot H^{\alpha\beta\gamma} \}
\end{aligned} \tag{51}$$

omitting higher derivative terms and introducing the coupling κ of inverse mass dimension. We emphasize that S_G is the action for a gauge field defined on the Minkowski space-time $(\mathbf{R}^4, \eta)$ and is invariant on one hand under local **T** gauge transformations reflecting the gauge symmetry, on the other hand under global Poincaré transformations reflecting the symmetries of the underlying space-time.

Note that our reasoning directly leads to the only combination of the different locally **T** invariant terms quadratic in the field strength which is also invariant under local Lorentz rotations of D_α [6]. As already $M(e, m)$ was rotation invariant this serves as an independent check of our results in the Appendix A.

8 Relation to the general theory of relativity

In this Section we determine the action for the metric field $g^{\mu\nu}$ to be the Einstein-Hilbert action of general relativity. We accordingly may interpret the gauge fields $e_\alpha{}^\gamma$ as gravitational potentials and $H_{\alpha\beta}{}^\gamma$ as gravitational fields. We finally obtain the conservation law for the total energy-momentum of matter and gauge fields.

Recasting the locally **T** gauge invariant scalar theory in geometrical terms we obtained in Section 6 the coordinate invariant scalar theory (31)

coupled to the metric field $g^{\mu\nu}$. Let us determine now the action for $g^{\mu\nu}$ following the lines of reasoning of the previous Section. We have

$$\begin{aligned} S_M &= \int d^4x \det^{\frac{1}{2}}(-g) \left\{ \frac{1}{2} g^{\mu\nu} \partial_\mu \varphi \cdot \partial_\nu \varphi - \frac{1}{2} m^2 \varphi^2 \right\} \\ &= \frac{1}{2} (\varphi, M(g, m) \varphi)_g \end{aligned} \quad (52)$$

where the hyperbolic fluctuation operator $M(g, m)$ is now the covariant d'Alembertian

$$M(g, m) = -\det^{-\frac{1}{2}}(-g) \partial_\mu \left(\det^{\frac{1}{2}}(-g) g^{\mu\nu} \partial_\nu \right) - m^2. \quad (53)$$

Calculating again the explicit change of the partition function under rescaling leads here to the result

$$\begin{aligned} &\zeta(0; \mu; M(g, m)) - \zeta(0; \mu; M(\eta, m)) \\ &= \frac{i}{(4\pi)^2} \int d^4x \det^{\frac{1}{2}}(-g) \left[-\frac{m^2}{6} R + O(\partial^4) \right]. \end{aligned} \quad (54)$$

As minimal classical action for the metric field we thus take

$$S_G = \frac{1}{\kappa^2} \int d^4x \det^{\frac{1}{2}}(-g) R \quad (55)$$

again omitting the higher derivative terms. Note that S_G is here the action for the metric $g^{\mu\nu}$ occuring in the Riemannian space-time $(\mathbf{R}^4, g)$ and is invariant under general coordinate transformations.

The identification $\kappa^2 = 16\pi G$ makes S_G to the usual Einstein-Hilbert action for the metric field in general relativity [18]. The smallness of κ^2 justifies finally the omission of the higher derivative terms $O(\partial^4)$ in (51) and (55) as long as we are interested in the classical effects of the theory only. These terms become of physical relevance at very small distances or very large momentum scales ensuring the perturbative renormalizability of the full quantum theory [20].

By construction the gauge field action (51) and the Einstein-Hilbert action (55) for the metric are mathematically equivalent. Together with the respective matter actions (27), being locally $\mathbf{T}$ gauge invariant and (52) being coordinate invariant they lead to complementary, but equivalent theories of matter coupled to gravitation.

The concept of coordinate invariance on one hand leads to general relativity and the subsequent description of gravitational effects by the geometrical entities g , Γ and R [18]. The concept of $\mathbf{T}$ gauge invariance on the

other hand leads to a description of the effects of gravitation in terms of the gravitational potentials e and the corresponding gravitational fields H .

Let us conclude this Section stating the conservation law for the total energy-momentum of matter and gauge fields in the $\mathbf{T}$ gauge theory of gravitation. We apply the Noether theorem to $S = S_G + S_M$. Under a global $\mathbf{T}$ gauge transformation the coordinates and fields behave as

$$\begin{aligned} x^\alpha &\longrightarrow x'^\alpha = x^\alpha \\ \varphi_j(x) &\longrightarrow \varphi'_j(x) = \varphi_j(x) - \varepsilon^\beta \cdot \partial_\beta \varphi_j(x) \\ e_\alpha{}^\gamma(x) &\longrightarrow e'_\alpha{}^\gamma(x) = e_\alpha{}^\gamma(x) - \varepsilon^\beta \cdot \partial_\beta e_\alpha{}^\gamma(x). \end{aligned} \quad (56)$$

This yields $\delta x^\alpha = 0$, $\delta \varphi_j = -\varepsilon^\beta \cdot \partial_\beta \varphi_j$, $\delta e_\alpha{}^\gamma = -\varepsilon^\beta \cdot \partial_\beta e_\alpha{}^\gamma$ and

$$\begin{aligned} d^4x \det e'^{-1}(\mathcal{L}'_G + \mathcal{L}'_M) &= \\ d^4x \{ \det e^{-1}(\mathcal{L}_G + \mathcal{L}_M) - \varepsilon^\gamma \cdot \partial_\gamma \det e^{-1}(\mathcal{L}_G + \mathcal{L}_M) \} \end{aligned}$$

so that if φ_j and $e_\alpha{}^\gamma$ obey their equations of motion the assumption (3) of the Noether theorem holds with $\delta f^\beta = -\delta^\beta{}_\gamma \cdot \varepsilon^\gamma \det e^{-1}(\mathcal{L}_G + \mathcal{L}_M)$. The conserved current then reads

$$\begin{aligned} J^\beta &= -\det e^{-1} \frac{\partial \mathcal{L}_G}{\partial(\partial_\beta e_\alpha{}^\gamma)} \cdot \delta e_\alpha{}^\gamma - \det e^{-1} \frac{\partial \mathcal{L}_M}{\partial(\partial_\beta \varphi_j)} \cdot \delta \varphi_j + \delta f^\beta \\ &= \det e^{-1} \left\{ \frac{\partial \mathcal{L}_G}{\partial(\partial_\beta e_\alpha{}^\gamma)} \cdot \partial_\delta e_\alpha{}^\gamma - \delta^\beta{}_\delta \mathcal{L}_G \right. \\ &\quad \left. + \frac{\partial \mathcal{L}_M}{\partial(\partial_\beta \varphi_j)} \cdot \partial_\delta \varphi_j - \delta^\beta{}_\delta \mathcal{L}_M \right\} \varepsilon^\delta \\ &= \{ \Theta^\beta{}_\delta + \Theta^\beta{}_\delta \} \varepsilon^\delta \end{aligned}$$

where

$$\Theta^\beta{}_\delta = \det e^{-1} \left\{ \frac{\partial \mathcal{L}_G}{\partial(\partial_\beta e_\alpha{}^\gamma)} \cdot \partial_\delta e_\alpha{}^\gamma - \delta^\beta{}_\delta \mathcal{L}_G \right\} \quad (57)$$

and

$$\Theta^\beta{}_\delta = \det e^{-1} \left\{ \frac{\partial \mathcal{L}_M}{\partial(\partial_\beta \varphi_j)} \cdot \partial_\delta \varphi_j - \delta^\beta{}_\delta \mathcal{L}_M \right\}$$

are the canonical energy-momentum tensors belonging to $e_\alpha{}^\gamma$ and φ_j respectively. Note that we obtained $\Theta^\beta{}_\delta$ already in eqn. (36). The corresponding time-independent energy-momentum four-vector is then

$$P_\alpha = \int d^3x \{ \Theta^0{}_\alpha + \Theta^0{}_\alpha \} \quad (58)$$

and transforms as a Lorentz vector under global $\mathbf{L}$ rotations.

9 Relativistic motion of point particles. Principle of equivalence and renormalized metric

In this Section we give a Hamilton function and the corresponding equations of motion describing the relativistic classical trajectory of a point particle. We then discuss the incorporation of the principle of equivalence in the **T** gauge theory of gravitation. We finally show that $g_{\beta\gamma}$ controls in the sense of a renormalized metric the behaviour of clocks and rods in the presence of gravitation.

In order to get a relativistic classical Hamilton function we follow the same route that leads back from the Klein-Gordon equation to a Hamiltonian describing the relativistic motion of a charged point particle in the presence of an electromagnetic field. Introducing $\hbar$ and c we thus start with the **T** gauge covariant wave equation

$$\left\{ -(D_\alpha + H_{\gamma\alpha}{}^\gamma) \cdot D^\alpha - \left(\frac{mc}{\hbar} \right)^2 \right\} \varphi = 0.$$

Using the correspondence of ∂_α with p_α , i.e. $\partial_\alpha = -\frac{i}{\hbar}p_\alpha$ or of D_α with P_α , i.e. $D_\alpha = -\frac{i}{\hbar}P_\alpha = -\frac{i}{\hbar}e_\alpha{}^\gamma p_\gamma$ we obtain

$$\frac{2m}{\hbar^2} \left\{ \frac{1}{2m} P^2 - \frac{mc^2}{2} + \frac{i\hbar}{2m} H_{\gamma\alpha}{}^\gamma \cdot P^\alpha \right\} \varphi = 0.$$

Dropping the $O(\hbar)$ term we take

$$F_g = \frac{1}{2m} P^2 - \frac{mc^2}{2} \quad (59)$$

as a Hamilton function describing the motion of a massive point particle in the presence of the gauge fields $e_\alpha{}^\gamma$. With p_α and x^α as independent canonically conjugate variables we then obtain the equations of motion

$$\begin{aligned} \frac{\partial F_g}{\partial p_\gamma} &= \dot{x}^\gamma = \frac{1}{m} e_\alpha{}^\gamma e^{\alpha\beta} p_\beta \\ \frac{\partial F_g}{\partial x^\gamma} &= -\dot{p}_\gamma = \frac{1}{m} e_\alpha{}^\delta p_\delta \cdot \partial_\gamma e^{\alpha\beta} p_\beta. \end{aligned}$$

Differentiating the first equation again we finally obtain

$$\ddot{x}^\gamma = \dot{e}_\alpha{}^\gamma e^{-1\alpha}{}_\beta \dot{x}^\beta$$

$$\begin{aligned}
& - e_\alpha^\gamma e^{\alpha\beta} \dot{e}_{\delta\beta}^{-1} e^{-1\delta}{}_\alpha \dot{x}^\alpha \\
& + e_\alpha^\gamma e^{\alpha\beta} \partial_\beta e_{\delta\eta}^{-1} e^{-1\delta}{}_\alpha \dot{x}^\eta \dot{x}^\alpha \\
& = \dot{e}_\alpha^\gamma e^{-1\alpha}{}_\beta \dot{x}^\beta \\
& + e_\alpha^\gamma e^{\alpha\beta} \left(\partial_\beta e_{\delta\eta}^{-1} - \partial_\eta e_{\delta\beta}^{-1} \right) e^{-1\delta}{}_\alpha \dot{x}^\eta \dot{x}^\alpha.
\end{aligned} \tag{60}$$

Note that the mass parameter has disappeared in the above equation.

Let us discuss next the principle of equivalence in the **T** gauge theory of gravitation. In the sense of 'equality of inertial and passive gravitational mass' we just have obtained the weak principle of equivalence: all bodies in a gravitational field fall with the same acceleration regardless of their masses [1,18] as equation (60) shows.

Now there is another, local version of the principle of equivalence stating that at every space-time point in an arbitrary gravitational field it is possible to choose a 'locally inertial coordinate system' [18]. (There is an interesting historical article about this point in [19].)

In general relativity this principle is geometrically incorporated. Using the covariance under general coordinate transformations one may always choose a Riemannian normal coordinate system around a given point. In these coordinates one then has

$$g'_{\mu\nu} = \eta_{\mu\nu}$$

and

$$\Gamma'^{\nu}{}_{\rho\sigma} = 0, \quad \partial'_\mu g'_{\rho\sigma} = 0$$

at the point considered. The normal coordinates now span the desired locally inertial coordinate system.

Let us finally obtain the incorporation of this principle in the **T** gauge theory of gravitation. Using the gauge covariance we may always choose a gauge in which

$$g'_{\beta\gamma} = e'^{-1\alpha}{}_\beta e'^{-1}{}_{\alpha\gamma} = \eta_{\beta\gamma} \tag{61}$$

and

$$\partial_\delta g'_{\beta\gamma} = \partial_\delta e'^{-1\alpha}{}_\beta \cdot e'^{-1}{}_{\alpha\gamma} + e'^{-1\alpha}{}_\beta \cdot \partial_\delta e'^{-1}{}_{\alpha\gamma} = 0 \tag{62}$$

at a given point simply by recasting the above transformation to normal coordinates as a gauge transformation. The first equation (61) yields

$$e'^{-1\alpha}{}_\beta = \eta^\alpha{}_\beta$$

whereas the second (62) leads to

$$\begin{aligned}\partial_\delta e'^{-1\alpha}{}_\beta &= -e'^{\alpha\eta} e'^{-1\gamma}{}_\beta \cdot \partial_\delta e'^{-1}{}_{\gamma\eta} \\ &= e'^{\alpha\eta} \cdot \partial_\delta e'^{-1\gamma}{}_\beta e'^{-1}{}_{\gamma\eta} \\ &= \partial_\delta e'^{-1\alpha}{}_\beta,\end{aligned}$$

which is trivially satisfied. So we finally obtain after this change of gauge at the point considered

$$e'_\alpha{}^\gamma = \eta_\alpha{}^\gamma \quad (63)$$

and

$$H'_{\alpha\beta}{}^\gamma = \partial_\alpha e'_\beta{}^\gamma - \partial_\beta e'_\alpha{}^\gamma. \quad (64)$$

The **T** gauge field strength becomes quite simple, but does thus not vanish unlike the corresponding Christoffel symbols $\Gamma'^\nu{}_{\rho\sigma}$.

As a result it is also in the gauge approach possible to describe the behaviour of a classical test particle locally as if gravitational potentials would be absent choosing the gauge considered above. But in the new coordinates x' corresponding to that gauge, i.e. in locally $B_\alpha{}^\gamma$ -free space-time the line element describing the measured metric relations between different points becomes at the point considered

$$ds^2 = \eta_{\mu\nu} dx'^\mu dx'^\nu, \quad (65)$$

i.e. the usual Minkowski metric. Transforming back to the Cartesian coordinates we started with we get

$$\begin{aligned}ds^2 &= \eta_{\mu\nu} \frac{\partial x'^\mu}{\partial x^\beta} \frac{\partial x'^\nu}{\partial x^\gamma} dx^\beta dx^\gamma \\ &= g_{\beta\gamma} dx^\beta dx^\gamma\end{aligned} \quad (66)$$

as inferring from (36). Thus in the Cartesian coordinates it is $g_{\beta\gamma} = e^{-1\alpha}{}_\beta e^{-1}{}_{\alpha\gamma}$ which governs the behaviour of clocks and rods in the presence of gravitational potentials $e_\alpha{}^\gamma$. For this reason we call $g_{\beta\gamma}$ a renormalized metric in the **T** gauge theory of gravitation.

10 Conclusions

In this paper we have introduced the complementary conception of **T** symmetry as an inner symmetry. This conception has led us then to a natural generalization of gauge theories based on non-abelian matrix groups

to the operator group of translations in Minkowski space-time $(\mathbf{R}^4, \eta)$. Essential features of usual gauge theories such as the unique transformation behaviour of the gauge potentials under the gauge group or the definite minimal coupling to matter fields are thereby preserved.

In order to determine the classical dynamics of the gauge fields we have used the RG and subsequent heat kernel techniques leading to a unique action as in the case of Yang-Mills theories. The separation of the local gauge group $\mathbf{T}$ from the global symmetry group $\mathbf{P}$ of the underlying Minkowski space-time $(\mathbf{R}^4, \eta)$ has allowed us to define the energy-momentum carried by the gauge potentials in the usual way.

We have finally recast the resulting theory in usual geometrical terms replacing local $\mathbf{T}$ gauge covariance by general coordinate invariance and have obtained its interpretation as a gauge theory of gravitation. It has been shown to be mathematically equivalent and physically complementary to the general theory of relativity .

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A Heat kernel coefficients of $\mathbf{T}$ gauge covariant differential operators

In this Appendix we calculate in a new way the small imaginary time expansion of the heat kernel belonging to a general hermitean $\mathbf{T}$ covariant second order differential operator M in the d -dimensional Minkowski space-time $(\mathbf{R}^d, \eta)$.

Let us consider a hermitean operator of the general form

$$M = -\mathcal{D}_\alpha \mathcal{D}^\alpha + E, \quad \mathcal{D}_\alpha = D_\alpha + C_\alpha. \quad (67)$$

$D_\alpha = e_\alpha{}^\gamma \partial_\gamma$ is the $\mathbf{T}$ covariant derivative, $e_\alpha{}^\gamma$ a real matrix field, C_α is an anti-hermitean matrix-valued four-vector and E a hermitean matrix field. The operator (40) considered in Section 7 is obviously of this form.

The heat kernel $K(is; x, y)$, $s > 0$ belonging to M is then defined as

$$K(is; x, y) = \langle x | e^{-isM} | y \rangle$$

and fulfills

$$\left(\frac{\partial}{\partial(is)} + M_x \right) K(is; x, y) = 0 \quad (68)$$

together with the initial condition

$$\lim_{s \rightarrow 0} K(is; x, y) = \langle x|y \rangle = \frac{1}{|\det e^{-1}|} \delta(x - y). \quad (69)$$

We are interested in the small s expansion of $K(is; x, y)$ at $y = x$. Asymptotically we then have [15,21]

$$K(is; x, y) \stackrel{s \rightarrow 0}{\sim} \frac{i}{(4\pi is)^{\frac{d}{2}}} e^{\frac{r^2(x,y)}{4is}} \sum_{k=0}^{\infty} (is)^k c_k(x, y). \quad (70)$$

The main task is to evaluate the coefficient functions $c_k(x, y)$ together with $r^2(x, y)$.

Inserting now the expansion (70) in eqn. (68) and equating equal powers of s we get the three covariant relations

$$r^2(x, y) = \frac{1}{4} D_\alpha r^2(x, y) \cdot D^\alpha r^2(x, y) \quad (71)$$

$$\begin{aligned} \frac{1}{2} D_\alpha r^2(x, y) &= D^\alpha c_0(x, y) \\ &= \left\{ \frac{d}{2} - \frac{1}{4} D_\alpha D^\alpha r^2(x, y) \right\} \cdot c_0(x, y) \end{aligned} \quad (72)$$

and

$$\begin{aligned} \frac{1}{2} D_\alpha r^2(x, y) &= D^\alpha c_{k+1}(x, y) \\ &= \left\{ \frac{d}{2} - k - 1 - \frac{1}{4} D_\alpha D^\alpha r^2(x, y) \right\} \cdot c_{k+1}(x, y) \\ &+ D_\alpha D^\alpha c_k(x, y) - E \cdot c_k(x, y). \end{aligned} \quad (73)$$

Note that all differential operators above act on x only. The first equation allows to evaluate r^2 and all its covariant derivatives at $y = x$ whereas the two other relations (72) and (73) allow a recursive determination of c_k and all its covariant derivatives also at $y = x$. The initial value of c_0 at $y = x$ depends on r^2 and its derivatives. Let us thus turn first to evaluate them.

Differentiation of eqn. (71) leads to the relations

$$r^2 = \frac{1}{4} D_\eta r^2 \cdot D^\eta r^2 \quad (74)$$

$$D_\alpha r^2 = \frac{1}{2} D_{\alpha\eta} r^2 \cdot D^\eta r^2 \quad (75)$$

$$D_{\beta\alpha} r^2 = \frac{1}{2} \{ D_{\beta\alpha\eta} r^2 \cdot D^\eta r^2 + D_{\alpha\eta} r^2 \cdot D_\beta{}^\eta r^2 \} \quad (76)$$

$$D_{\gamma\beta\alpha} r^2 = \frac{1}{2} \{ D_{\gamma\beta\alpha\eta} r^2 \cdot D^\eta r^2 + D_{\beta\alpha\eta} r^2 \cdot D_\gamma{}^\eta r^2 + D_{\gamma\alpha\eta} r^2 \cdot D_\beta{}^\eta r^2 + D_{\alpha\eta} r^2 \cdot D_{\gamma\beta}{}^\eta r^2 \}. \quad (77)$$

We used the shorthand notations $D_{\beta\alpha} = D_\beta D_\alpha$, $D_{\gamma\beta\alpha} = D_\gamma D_\beta D_\alpha$ and introduce $r^2(x) = r^2(x, x)$ for functions taken at $y = x$. As the initial value for r^2 is

$$r^2(x) = 0 \quad (78)$$

the first relation (74) leads thus to

$$D_\alpha r^2(x) = 0 \quad (79)$$

which is consistent with (75). The third relation (76) yields now

$$D_{\beta\alpha} r^2(x) = \frac{1}{2} D_{\alpha\eta} r^2(x) \cdot D_\beta{}^\eta r^2(x)$$

which is solved by

$$D_{\beta\alpha} r^2(x) = 2\eta_{\beta\alpha}. \quad (80)$$

Using next the results (79) and (80) in the forth relation (77) yields

$$D_{\gamma\beta\alpha} r^2(x) = D_{\beta\alpha\gamma} r^2(x) + D_{\gamma\alpha\beta} r^2(x) + D_{\gamma\beta\alpha} r^2(x). \quad (81)$$

Commuting covariant derivatives according to (15) allows us now to reexpress $D_{\gamma\beta\alpha} r^2(x)$ in terms of the field strenght. Inserting

$$\begin{aligned} D_{\beta\alpha\gamma} r^2(x) &= D_{\gamma\beta\alpha} r^2(x) + 2H_{\beta\gamma\alpha} + 2H_{\alpha\gamma\beta}, \\ D_{\gamma\alpha\beta} r^2(x) &= D_{\gamma\beta\alpha} r^2(x) + 2H_{\alpha\beta\gamma} \end{aligned}$$

in (81) leads to

$$\begin{aligned} 0 &= D_{\beta\alpha\gamma} r^2(x) + D_{\gamma\alpha\beta} r^2(x) \\ &= 2D_{\gamma\beta\alpha} r^2(x) + 2\{H_{\beta\gamma\alpha} + H_{\alpha\gamma\beta} + H_{\alpha\beta\gamma}\} \end{aligned}$$

or finally to

$$D_{\gamma\beta\alpha}r^2(x) = H_{\beta\gamma\alpha} + H_{\alpha\gamma\beta} + H_{\alpha\beta\gamma}. \quad (82)$$

Higher covariant derivatives of r^2 at $y = x$ are evaluated in the same recursive way. But as the algebraic work increases considerably we shall only give the results for those eventually needed.

Let us next determine $c_0(x)$ taking the limit $s \rightarrow 0$ in the asymptotic ansatz (70). We thus have to investigate the Taylor expansion of $r^2(x, y)$. Rewriting eqns. (75) and (76) yields

$$\begin{aligned} D_{\alpha}r^2(x) &= e_{\alpha}^{\gamma} \cdot \partial_{\gamma}r^2(x) = 0 \\ D_{\beta\alpha}r^2(x) &= e_{\beta}^{\delta} \partial_{\delta} \left(e_{\alpha}^{\gamma} \partial_{\gamma}r^2(x) \right) \\ &= e_{\beta}^{\delta} e_{\alpha}^{\gamma} \partial_{\delta} \partial_{\gamma}r^2(x) + e_{\beta}^{\delta} \cdot \partial_{\delta} e_{\alpha}^{\gamma} \cdot \partial_{\gamma}r^2(x) \\ &= 2 \eta_{\beta\alpha} \end{aligned}$$

or

$$\begin{aligned} \partial_{\gamma}r^2(x) &= 0 \\ \frac{1}{2} \partial_{\delta} \partial_{\gamma}r^2(x) &= e^{-1\alpha}{}_{\delta} e^{-1}{}_{\alpha\gamma}. \end{aligned}$$

We obtain now

$$r^2(x, y) = e^{-1\alpha}{}_{\delta} (y - x)^{\delta} \cdot e^{-1}{}_{\alpha\gamma} (y - x)^{\gamma} + O((y - x)^3)$$

and may proceed to let $s \rightarrow 0$ in (70)

$$\begin{aligned} \lim_{s \rightarrow 0} \frac{i}{(4\pi i s)^{\frac{d}{2}}} e^{\frac{r^2(x, y)}{4is}} \sum_{k=0}^{\infty} (is)^k c_k(x, y) \\ = c_0(x, y) \delta \left(e^{-1\alpha}{}_{\gamma} (y - x)^{\gamma} \right) \\ = \frac{c_0(x)}{|\det e^{-1}(x)|} \delta(y - x). \end{aligned}$$

Comparing with the initial condition $\lim_{s \rightarrow 0} K(is; x, y) = \frac{1}{|\det e^{-1}|} \delta(x - y)$ yields finally

$$c_0(x) = 1. \quad (83)$$

We turn to the computation of $c_1(x)$. Taking $k = 0$ in the relation (73) we obtain

$$\begin{aligned} \frac{1}{2} D_{\alpha}r^2 \cdot \mathcal{D}^{\alpha} c_1 &= \left\{ \frac{d}{2} - 1 - \frac{1}{4} D_{\alpha}{}^{\alpha} r^2 \right\} \cdot c_1 \\ &+ \mathcal{D}_{\alpha}{}^{\alpha} c_0 - E \cdot c_0. \end{aligned}$$

Using $\frac{1}{4}D_\alpha^\alpha r^2(x) = \frac{1}{2}\delta_\alpha^\alpha = \frac{d}{2}$ and (79) for $y = x$ yields

$$c_1(x) = \mathcal{D}_\alpha^\alpha c_0(x) - E \cdot c_0(x). \quad (84)$$

It remains to determine the above second derivative of c_0 . Differentiating twice the relation (72) for c_0 leads to

$$\begin{aligned} \frac{1}{2}\{D_\alpha^\alpha r^2 \cdot \mathcal{D}^\eta c_0 + D_\eta r^2 \cdot \mathcal{D}^{\alpha\eta} c_0\} \\ = -\frac{1}{4}D_\alpha^\alpha r^2 \cdot c_0 \\ + \left\{\frac{d}{2} - \frac{1}{4}D_\eta r^2\right\} \cdot \mathcal{D}^\alpha c_0 \end{aligned}$$

and to

$$\begin{aligned} \frac{1}{2}\{D_\alpha^\alpha r^2 \cdot \mathcal{D}^\eta c_0 + 2D_{\alpha\eta} r^2 \cdot \mathcal{D}^{\alpha\eta} c_0 + D_\eta r^2 \cdot \mathcal{D}_\alpha^{\alpha\eta} c_0\} \\ = -\frac{1}{4}D_\alpha^\alpha r^2 \cdot c_0 \\ - \frac{1}{2}D_{\alpha\eta} r^2 \cdot \mathcal{D}^\alpha c_0 \\ + \left\{\frac{d}{2} - \frac{1}{4}D_\eta r^2\right\} \cdot \mathcal{D}_\alpha^\alpha c_0. \end{aligned}$$

We have used here the Leibniz rule for $\mathcal{D}_\alpha$ acting on a product of a function f with a matrix N

$$\mathcal{D}_\alpha(fN) = D_\alpha f \cdot N + f \cdot \mathcal{D}_\alpha N.$$

Evaluating the equations above at $y = x$ yields

$$\mathcal{D}^\alpha c_0(x) = -\frac{1}{4}D_\alpha^\alpha r^2 \cdot c_0(x) \quad (85)$$

and

$$\begin{aligned} \mathcal{D}_\alpha^\alpha c_0(x) &= -\frac{1}{8}D_\alpha^\alpha r^2 \cdot c_0(x) \\ &- \frac{1}{4}\{D_{\alpha\eta} r^2(x) + D_\eta r^2(x)\} \cdot \mathcal{D}^\alpha c_0 \\ &= -\frac{1}{8}D_\alpha^\alpha r^2 \cdot c_0(x) \end{aligned} \quad (86)$$

where we used the relation $D_{\alpha\eta}{}^\eta r^2(x) + D_\eta{}^\alpha r^2(x) = 0$ following from (81). We finally need the forth derivative of r^2 which is computed to be

$$\begin{aligned} D_\alpha{}^\alpha{}_\eta{}^\eta r^2(x) &= \frac{1}{3} \{ -4D_\alpha H_\beta{}^{\alpha\beta} + 2H_{\alpha\beta\eta} \cdot H^{\alpha\eta\beta} \\ &+ H_{\alpha\beta\eta} \cdot H^{\alpha\beta\eta} - 2H_{\alpha\eta}{}^\alpha \cdot H_\beta{}^{\eta\beta} \} \end{aligned} \quad (87)$$

and obtain thus together with $c_0(x) = 1$

$$\begin{aligned} c_1(x) &= -\frac{1}{24} \{ -4D_\alpha H_\beta{}^{\alpha\beta} + 2H_{\alpha\beta\eta} \cdot H^{\alpha\eta\beta} \\ &+ H_{\alpha\beta\eta} \cdot H^{\alpha\beta\eta} - 2H_{\alpha\eta}{}^\alpha \cdot H_\beta{}^{\eta\beta} \} \\ &- E. \end{aligned} \quad (88)$$

B Heat kernel coefficients of coordinate covariant differential operators

In this Appendix we give the first two coefficient occuring in the expansion of the heat kernel belonging to a general hermitean $\mathbf{T}$ covariant second order differential operator M in a d -dimensional Riemannian manifold $(\mathbf{R}^d, g)$ of the form

$$M = -\det^{-\frac{1}{2}}(-g) D_\mu \left(\det^{\frac{1}{2}}(-g) g^{\mu\nu} D_\nu \right) + E, \quad D_\mu = \partial_\mu + C_\mu. \quad (89)$$

C_μ above denotes an anti-hermitean matrix-valued coordinate covariant four-vector and E a hermitean matrix field. The operator (53) considered in Section 8 is obviously of this form.

The corresponding small s expansion of the heat kernel yields then the coefficients [15,17,21]

$$c_0(x) = 1 \quad (90)$$

and

$$c_1(x) = -\frac{1}{6} R - E. \quad (91)$$

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