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Effective Lagrangian, Casimir Energy and Conformal Field Theory description of the Quantum Hall Effect¹

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Abstract

By using the Girvin-MacDonald effective Lagrangian for the Hall fluid system at filling $\nu = 1$ we are able to evaluate the ground-state energy for a cylinder of infinite length. That is done by using the topological properties of the edge vacuum in the thermodynamic limit, $N_e \rightarrow \infty$, $\nu = 1$ fixed.

From the vacuum energy we extract the value of the Casimir energy for both filling $\nu = 1$ and $\nu = 1/m$. The results obtained fix the central charge c of the underlying 2D Conformal Field Theory at $c = 1$.

This correspondence is further analyzed by using the bosonic realization of the Kac-Moody algebra on the edge vacuum. Then the off-diagonal long-range order of the effective theory can be seen as an equivalent description of the topological order typical of the 2D CFT.

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1 Introduction

Recently in several papers ([1,2]) many matching arguments have been presented which strongly advocate that the Coulomb Gas Vertex Operators (CGVO) realization of the two dimensional conformal field theory (2D CFT) ([3,4]) is an appropriate field theoretic (=many-body) description of the Off-Diagonal Long-Range Order (ODLRO) present in the fractional quantum Hall effect (FQHE). Moreover many results of the one-component 2D plasma analogy of Laughlin [5] have been easily derived and generalized to a torus topology.

In the context of the CGVO approach the edge states for a cylinder geometry are described in terms of a massless chiral scalar field, $\phi(z)$, i.e. by a boson with hard core, which interacts with the 2D Green function: $\ln(z_i - z_j)$, proper of the 2D Coulomb Gas, and a clear connection between the degeneracy of the ground state on the torus and the number of independent edge states is shown.

We also stress that the topological properties implicit in the finite dimensional representations of the magnetic translations operators algebra on the torus generated by $\mathcal{S}, \mathcal{T}$ ([1]) (by acting on the center of mass wave function on the torus) determine the ground state relative to the edges of the (infinite) cylinder for filling $\nu = 1$, where the assumption of non-interacting electrons is justified. Infact our definition of the edge vacuum (ground state in the thermodynamic limit, $N_e \rightarrow \infty$, ν fixed) accounts for the topological properties of the bulk.

Further, from a more formal point of view we get naturally that the CFT current generates a $U(1)$ Kac-Moody algebra in a bosonized form equivalent to the one described in the literature by the use of massless 2D fermions, that is derived by a well known (in 2D CFT) procedure of cutting (or sewing) from a torus (cylinder) to a cylinder (torus).

In the following we shall analyze and clarify in detail some problems left open in [1,2] concerning the relation between the CGVO description of the vacuum states relative to the “edges” of the cylinder and its excitations. Then we shall find in a simple way most of the results known for $\nu = 1$ using the effective Lagrangian of Girvin and MacDonald ([6,7]).

The paper is organized as follows. In sections 2-3 we recall the general definition of the O.D.L.R.O. and the Girvin-MacDonald effective Lagrangian, presented in particular for the cylinder case. The physical role of the magnetic flux neutrality condition is explicitly shown.

In section 4 the “collective” wave function relative to the Hall fluid ground state is given for the torus as well as for the cylinder in the thermodynamic limit. Also its invariance under finite magnetic translations is discussed.

In section 5 we evaluate the energy relative to the edges of the cylinder for $\nu = 1$. Then, by using the above mentioned topological properties, we are able to extend this calculation to the case of filling $\nu = 1/m$. We find that there are m possible solutions, labelled by $l = 1 \dots m$, for the ground-state wave function. The wave function for $l = m$ is identical to the one for the electron at filling $\nu = 1$; it can be interpreted as

the wave function of a superfermion which carries charges $q_e = e$ and $q_m = m\Phi_0$. Then we notice that the first term in the expansion in λ/L_x (L_x is the cylinder length and λ is the magnetic length) of the edge ground-state energy allows us to determine the Casimir Energy.

In section 6 we confront these results with the ones derived in the CGVO approach by using general results on the Casimir energy. The central charge of the underlying CFT is then found to be $c = 1$ (Gaussian chiral model, described by a massless scalar field $\phi(z)$). The role of the $U(1)$ Kac-Moody algebra in the bosonic form is then confronted with the previous analysis by Wen and Stone for $\nu = 1$ (see refs.[9,10]). The extension to the case $\nu = 1/m$ is finally seen, in the CFT approach, as a rescaling of the zero modes of $\phi(z)$ which corresponds to a compactification of such a field on a circle of radius $R = \sqrt{1/\nu}$, i.e., physically, to a rescaling of the magnetic charge associated to the superfermion, $q_m \rightarrow R^2 q_m$.

Finally we stress the strict relation between the ODLRO and the topological order advocated in ref.[2] in the context of 2D CFT.

2 O.D.L.R.O. and neutrality of the magnetic flux

It is well known that a very important property of the Hall fluid is the O.D.L.R.O. which can be related to the existence of delocalized single particle states [6].

Such a property can be expressed by the condition:

$$\rho(\vec{r}, \vec{r}') \sim |\vec{r} - \vec{r}'|^\alpha \quad \text{for } |\vec{r} - \vec{r}'| \rightarrow \infty \quad (1)$$

where ρ is the “correlation density” matrix and α is a negative rational number:

$$\rho(\vec{r}, \vec{r}') \equiv \int d^2 r_2 \dots d^2 r_N \psi^*(\vec{r}, \vec{r}_2 \dots \vec{r}_N) \psi(\vec{r}', \vec{r}_2 \dots \vec{r}_N) \quad (2)$$

ψ being the N_e -electrons wave function.

By using for ψ the Laughlin wave function ([5]):

$$f_L(z_1 \dots z_N) = \prod_{i < j} (z_i - z_j)^m e^{-\frac{1}{4} \sum_{k=1}^N |z_k|^2} \quad (z \equiv x + iy) \quad (3)$$

one finds, from eq.(2), a density ρ constant ($1/m$ times the density for $\nu = 1$). But in [6] it was noted that, by performing a “gauge transformation” induced by

$$a_k(z_j) \equiv \frac{m\Phi_0}{2\pi} \sum_{i \neq j} \partial_i \text{Im} \ln(z_i - z_j) \quad (4)$$

which is the so-called “statistical potential”, one does get in the “bosonic gauge” for the wave function the new expression $\tilde{f}_L$ given by:

$$\tilde{f}_L = \prod_{i < j} |z_i - z_j|^m e^{-\frac{1}{4} \sum_k |z_k|^2} \quad (5)$$

Then the corresponding density matrix $\bar{\rho}$ behaves as $|z - z'|^{-m}$ for $|z - z'| \rightarrow \infty$ showing the long range order just mentioned.

Notice that for such a result it is necessary that the previous gauge transformation gets rid of all the phases in f_L (eq.(3)), thus implying that the O.D.L.R.O. requires the “magnetic charge”, $q_m = m\Phi_0$, to be exactly attached to the electrically charged particles. Then the filling relation $\nu = N_e/N_s = 1/m$ can be read as a complete screening of external magnetic flux, i.e. $\Phi \equiv N_s\Phi_0 = mN_e\Phi_0$.

3 Effective theories and neutrality

A. The general theory

The O.D.L.R.O. in the Hall fluid can be described in terms of an effective field theory which basically has the following properties:

- A) The presence of a gap for phonon excitations;
- B) The existence of localized, finite-energy excitations, to be identified with Laughlin anyons.

Further the theory has to reproduce the stability at a given Hall plateau.

In refs.[7,8] the correct formulation of the effective theory has been proposed for the “collective” wave function $\psi(\vec{r})$ with an action given by:

$$\mathcal{S}[\psi] = \frac{1}{2M} \int d^2r | -i\hbar \vec{\nabla} \psi + \frac{e}{c} \vec{\mathcal{A}} \psi |^2 \quad (6)$$

where $\vec{\mathcal{A}}$ describes the effect of the external magnetic field $\vec{A}_{ext}$, as well as the “statistical field” $\vec{a}$, that is, $\vec{\mathcal{A}} = \vec{A}_{ext} + \vec{a}$.

We must notice that the complete form of Girvin action presents a term describing the constraint of frustration, but which does not contribute to the energy of a physical state of the system.

In fact he introduces a frustration in the system for solutions different than the Hall fluid with constant density $|\psi|^2$ given by

$$|\psi|^2 = n_0 \equiv \frac{\nu}{\lambda^2} \quad (7)$$

Such a frustration is described by the condition:

$$\vec{\nabla} \times \vec{\mathcal{A}} \sim -|\psi|^2 + n_0 \quad (8)$$

For $|\psi|^2 \neq n_0$ one has $\vec{\nabla} \times \vec{\mathcal{A}} \neq 0$ and the action in eq.(6) cannot be minimized by:

$$\psi(\vec{r}) = \psi(0) e^{-i \int d\vec{R} \cdot \vec{\mathcal{A}}(R)/\Phi_0} \quad (9)$$

One can rewrite eq.(8) in standard units as:

$$\vec{\nabla} \times \vec{\mathcal{A}} = B - \Phi_0 2\theta |\psi|^2 \quad (10)$$

where $\Phi_0 \equiv hc/e$ is the fundamental flux quantum and the parameter θ is related to the filling.

For $\nu = 1/m$ a reasonable choice is to take $\oint (\vec{\nabla} \times \vec{A}) \cdot d\vec{r}$ proportional to the external flux Φ . Then using:

$$\int_{\text{sample}} d^2r |\psi|^2 = N_e \quad (11)$$

one finds:

$$\theta = \frac{\pi}{\nu} \quad (12)$$

and the frustration condition is equivalent to $N_s = mN_e$ which is just the neutrality condition of the Hall fluid. The neutrality condition, then, plays a crucial role for the effective theory to be consistent.

It turns out that neutrality is relevant also when we consider the excitations of the theory.

Let us imagine to perturb the density from the equilibrium value n_0 to $n_0 + \delta n_0$. The “statistical field” $b(\vec{r})$ will change correspondingly from b to $b + \delta b$, with: $\delta b = 2\theta\delta n\Phi_0$.

Assume also that, as a consequence of such a perturbation a vortex will be created in the system with the asymptotic behaviour:

$$\psi(\vec{r}) \sim \sqrt{n}e^{i\gamma} ; \quad \vec{a} \sim \frac{l\Phi_0}{r}\hat{\gamma} \quad (13)$$

where the “statistical potential” $\vec{a}$ is defined by:

$$\vec{\nabla} \times \vec{a} = -\Phi_0 |\psi|^2 2\theta \quad (14)$$

as shown in [8] and γ is the polar angle on the plane.

Considering a closed path Γ containing the vortex one has:

$$\int \int_A b(\vec{r}) d^2r = \oint_{\Gamma} \vec{a} d\vec{r} = 2\pi l \Phi_0 \quad (15)$$

where A is the surface enclosed by Γ . Eq.(15) tells us what is the “magnetic charge” of the vortex, while its electric charge $q_e = q_m/m$ has been evaluated in detail in ref.[8]. For $l = 1 \dots m$ eq.(13) describes Laughlin anyon solutions for $\nu = 1/m$.

Then we can conclude that by changing the external magnetic flux from Φ to $\Phi + \delta\Phi$ the Hall system reacts with the creation of localized density changes, given by eq.(13), which describe vortices, whose statistical field “neutralizes” the external δB :

$$\Phi_{ext} + \Phi_{stat} = 0 . \quad (16)$$

Above Φ_{ext} is the flux of the external magnetic field through the system and Φ_{stat} the corresponding one for the statistical field $b(\vec{r})$. Again eq.(16) is in agreement with the neutrality condition.

B. The theory on the cylinder

Let us now apply such an effective theory to a cylindrical sheet of radius $r = L_x/2\pi$ and length $2S$ ($S \gg L_x$).

We start noticing that, by making a gauge transformation from the anyonic gauge to the bosonic one, as in ref.[6], the field $\vec{A}$ can be made equal to $\vec{A}_{ext}$. In this gauge is the phase of ψ to account for the association of the statistical magnetic flux to physical particles; then the action $S[\psi]$ becomes:

$$S[\psi] = \frac{1}{2M} \int_{sample} d^2r \left| -i\hbar \vec{\nabla} \psi + \frac{e}{c} \vec{A}_{ext} \psi \right|^2 \quad (17)$$

where the “collective” wave function ψ satisfies the “boundary condition”:

$$\psi(x + L_x, y) = \psi(x, y) \quad (18)$$

In the Landau gauge, $\vec{A}_{ext} = -By\hat{x}$, the Hamiltonian density looks like:

$$\mathcal{H}[\psi] = \frac{1}{2M} |\pi_- \psi|^2 + \frac{\hbar\omega_c}{2} |\psi|^2 + \frac{i\hbar^2}{2M} \left[\frac{\partial}{\partial y} \left(\psi^\dagger \frac{\partial}{\partial x} \psi \right) - \frac{\partial}{\partial x} \left(\psi^\dagger \frac{\partial}{\partial y} \psi \right) \right] - \frac{\hbar\omega_c}{2} \frac{\partial}{\partial y} (y |\psi|^2) \quad (19)$$

where the operator π_- is given by:

$$\pi_- \equiv i\hbar \frac{\partial}{\partial x} + \hbar \frac{\partial}{\partial y} + \frac{eB}{c} y \quad (20)$$

and ω_c is the cyclotron frequency: $\omega_c \equiv eB/Mc$.

Further

$$\int_{-S}^S dy \int_0^{L_x} \left[\frac{i\hbar^2}{2M} \left(\frac{\partial}{\partial y} \left(\psi^\dagger \frac{\partial}{\partial x} \psi \right) - \frac{\partial}{\partial x} \left(\psi^\dagger \frac{\partial}{\partial y} \psi \right) \right) - \frac{\hbar\omega_c}{2} \frac{\partial}{\partial y} (y |\psi|^2) \right] = H_L^E + H_R^E \quad (21)$$

with

$$H_{(L,R)}^E \equiv \int_0^{2\pi r} dx \left(\pm i \frac{\hbar^2}{2M} \psi^\dagger \frac{\partial}{\partial x} \psi - \frac{\hbar\omega_c}{2} (y \psi^\dagger \psi) \right)_{y=\pm S} \quad (22)$$

where with $H_{L(R)}^E$ is indicated the “edge Hamiltonian” left(right).

As we will see, this edge term will describe the edge excitations of our system, while the “bulk” is described by the remaining term:

$$H_B = \int d^2r \left[\frac{1}{2M} |\pi_- \psi|^2 + \frac{\hbar\omega_c}{2} |\psi|^2 \right] \quad (23)$$

4 Invariance under finite magnetic translations and the thermodynamic limit

In our approach the (physical) properties of the edge vacuum, relative to the geometry of the cylinder, will be derived explicitly by using the invariance under finite magnetic translations. Such a procedure will show that the requirement of periodicity for

the ground-state wave-function along the axes (and, naturally, the implied topological properties of the Hall fluid) is uniquely related to the incompressibility of the ground-state (i.e. the “vacuum” in the thermodynamic limit, $N_e \rightarrow \infty$, ν fixed) and, then, to the nature of the anyonic excitations.

To clarify this statement let us briefly summarize some results reached in refs.[1,2].

For simplicity we consider the case of filling $\nu = 1$ with a torus of size L_x, L_y , with $L_x L_y = 2\pi N_s \lambda^2$, where N_s is an integer number and $\lambda = \sqrt{\frac{\hbar c}{eB}}$ is the magnetic length.

By choosing the Landau gauge $\vec{A} = -By\hat{x}$ one finds that the ground-state wave function for N_e electrons on the torus is given by ([12])

$$\psi(\vec{r}_1 \dots \vec{r}_{N_e}) = \prod_{i < j=1}^{N_e} F(\vec{r}_i - \vec{r}_j) \psi_{cm}(\vec{R}) \quad (24)$$

where the explicit form of $\psi_{cm}(\vec{R})$ is:

$$\psi_{cm}(\vec{R}) = \sum_{n=-\infty}^{\infty} e^{2\pi i n \frac{\vec{R}}{L_x}} \exp \left[-\frac{1}{2\lambda^2} \left(Y - 2\pi n \frac{\lambda^2}{L_x} \right)^2 \right] \quad (25)$$

with the c.m. coordinate $\vec{R}$ given by $\vec{R} = \sum_i \vec{r}_i$.

In refs.[1,2] it has been shown that $\psi_{cm}(\vec{R})$ is a basis for the representation of the finite subgroup of the magnetic translation $\mathcal{S}_{L_x/\lambda N_s}, \mathcal{T}_{L_y/\lambda N_s}$, satisfying the non-commutative algebra:

$$\mathcal{S}_{L_x/\lambda N_s} \mathcal{T}_{L_y/\lambda N_s} = e^{2\pi i \frac{N_s}{N_e}} \mathcal{T}_{L_y/\lambda N_s} \mathcal{S}_{L_x/\lambda N_s} \quad (26)$$

On the other hand $\psi_{cm}(\vec{R})$ can be thought as a “collective” wave function satisfying the equation:

$$\pi_- \psi(\vec{R}) = 0 \quad (27)$$

with periodic boundary conditions along the two directions x, y .

First we notice that ψ_{cm} does not depend on N_s , but only depends on $\nu = N_e/N_s$. Therefore we can perform the thermodynamic limit, $N_e \rightarrow \infty$, with ν fixed to the value $\nu = 1$.

By recalling that the (total) magnetic translations on a torus are defined as a product of magnetic translations, $\mathcal{S}_{L_x/\lambda N_s}^{(i)}, \mathcal{T}_{L_y/\lambda N_s}^{(i)}$, acting on the i th electron, we can imagine the torus with N_s fluxes, as formed by an array of N_s vortices of magnetic charge $q_m = 1 \cdot \Phi_0$ and $q_e = 1 \cdot e$.

In other words the topological properties implicit in the requirement of double periodicity are preserved by the thermodynamic limit so defined (i.e. the field theory is defined on $S^1 \times S^1$).

The relevant topological properties in connection with the representation of the magnetic translation operators have been analyzed in ref.[11].

Let us now go back to the cylinder with $L_x = 2\pi r$ and $S \rightarrow \infty$. For such a case the “collective” wave function can be written as:

$$\psi(\vec{r}) = \sum_{n=-\infty}^{\infty} c_n \psi_n(\vec{r}) \quad (28)$$

where

$$\psi_n(\vec{r}) \equiv e^{2\pi i n \frac{y}{L_x}} \exp \left[-\frac{1}{2} \left(\frac{y - 2\pi n \lambda^2 / L_x}{\lambda} \right)^2 \right] \quad (29)$$

have the same form of the one electron wave function on a cylinder. Moreover the coefficients c_n can be fixed by requiring that the Hall density $|\psi_{cm}|^2$ is uniform on any “elementary cell” of area $2\pi\lambda^2$, i.e.:

$$\int_y^{y+a} dy \sum_n (|c_n|^2 - |c_{n+k}|^2) |\psi_n|^2 = 0 \quad \forall k; \quad (30)$$

where $a = 2\pi\lambda^2/L_x$, implying $|c_n| = |c_{n+k}|$ and, if we normalize to 1 the density, then $|c_n| = 1 \quad \forall n \in \mathbb{Z}$.

The relative phase of c_n is determined by the condition that ψ acquires the correct phase factor, $\exp(i2\pi\Phi/\Phi_0)$, once one goes around the plaquette $L_x \times a$, where $\Phi = \Phi_0$ for $\nu = 1$, implying:

$$c_n/c_{n+1} = 1. \quad (31)$$

So we find as for $\psi(\vec{r})$ the same function as for the torus (see eq.(25)).

Notice that the requirement of periodicity along y of step a implies that, $2S = N_e a$, then S goes to infinity in the thermodynamic limit ($N_e \rightarrow \infty$).

In conclusion, we have a formulation on the cylinder in the thermodynamic limit which preserves all the topological properties of the center of mass wave function on a torus.

These properties allow us to derive the finite size effects as discussed in the next section.

5 Finite-size effects on the cylinder

A. The case $\nu = 1$

The properly normalized “collective” wave function defined in eq.(28) can be then explicitly written as:

$$\psi^{(1)}(\vec{r}) = \sum_{n=-\infty}^{\infty} \frac{e^{2\pi i n \frac{y}{L_x}}}{\sqrt{2\pi L_x \lambda}} \exp \left[-\frac{1}{2\lambda^2} \left(y - 2\pi n \frac{\lambda^2}{L_x} \right)^2 \right]. \quad (32)$$

which can be seen as a sequence of an “infinity of states” carrying charge $q_e = e$ and flux $q_m = \Phi_0$, delocalized in x and “centered” at $y_n = 2\pi n \lambda^2 / L_x$, in perfect analogy with the torus case.

We are now ready to evaluate the energy of the edge vacuum for the cylinder, from which the Casimir energy can be derived.

In fact, let us consider the Hamiltonian of the (right) edge located at $y = 0$, eq.(22).

$$H_{edge}^R = i \frac{\hbar^2}{2M} \int_0^{2\pi R} dx \psi^\dagger(x, y=0) \frac{\partial}{\partial x} \psi(x, y=0) \quad (33)$$

By using eqs.(32,33) one gets

$$\begin{aligned} E_{edge} &= \frac{\hbar\omega_c}{2L_x} \lambda \sum_{n=-\infty}^0 (-n) \exp \left[-\frac{4\pi^2 \lambda^2 n^2}{L_x^2} \right] = \\ &= \sum_{n=0}^{\infty} \frac{\hbar\omega_c}{2} \frac{n}{\alpha_x} e^{-\frac{4\pi^2 \lambda^2 n^2}{L_x^2}} \end{aligned} \quad (34)$$

where $\alpha_x = L_x/\lambda$.

In evaluating the sum in eq.(34) we neglect the exponential term. In Stone's language ([9]) this is equivalent to consider only the contribution of the edge modes with wavelength much bigger than the magnetic length λ .

Formally this means that one first restricts the sum in eq.(34) to a finite number of terms (say N), then one takes the limit $L_x \rightarrow \infty$, obtaining:

$$E_1^N = \frac{\hbar\omega_c}{2} \sum_{n=1}^N \frac{n}{\alpha_x} \quad (35)$$

Then in the $N \rightarrow \infty$ limit, by using the ζ -function regularization, one finds:

$$E_1 = -\frac{\hbar\omega_c}{2\alpha_x} \zeta(-1) = -\frac{v}{24} p_x \quad (36)$$

where one recognizes that v is the edge velocity, $v = \sqrt{eB\hbar/cM^2}$, and where $p_x = \frac{\hbar}{L_x}$ is the compactified momentum unit in the x direction.

The corresponding pressure is negative and is given by:

$$p_C = -\frac{1}{L_x} \frac{\partial E_1}{\partial L_x} = +\frac{1}{L_x^2} E_1 \quad (37)$$

In the next subsection we will make a comment on this.

We notice that the result contained in eq.(36) does not depend on the choice of the position of the edge (i.e., $y = 0$ or $y = na$, n integer) due to the discreteness of the states (fluxes) described by eq.(32).

B. The case $\nu = 1/m$

For $\nu = 1/m$ we define the ground-state for the cylinder topology as in the previous case, but the flux balance condition now associates to each electron, with electric charge $q_e = e$, a magnetic charge $q_m = m\Phi_0$ ([1,2]). We will refer to such an object as to a "superfermion".

In fact the properly normalized wave functions on the infinite cylinder is given (in analogy with the torus case [1,2]) by:

$$\psi_l^{(m)}(\vec{r}) = \sum_{n=-\infty}^{\infty} \frac{\exp \left[2\pi i m \left(n + \frac{l}{m} \right) \frac{x}{L_x} \right]}{\sqrt{2\pi L_x \lambda}} \exp \left[-\frac{m}{2\lambda^2} \left(y - 2\pi(nm + l) \frac{\lambda^2}{mL_x} \right)^2 \right] \quad (38)$$

with $l = 1 \dots m$. For $l = m$ eq.(38) is “identical” to eq.(32) for $\nu = 1$ up to the rescaling $n \rightarrow mn \equiv n \frac{1}{\nu}$.

Therefore $\psi_m^{(m)}$ can be interpreted as describing a sequence of infinite super-fermions of “magnetic” charge (flux) m -times the corresponding one for filling $\nu = 1$.

As for the case $\nu = 1$ one finds that the value of the Casimir energy is given by:

$$E_1 = -\frac{v^{(m)}}{24L_x} p_x^{(m)} \quad (39)$$

where $v^{(m)} = \sqrt{m}v$ is the “new” edge velocity and $p_x^{(m)} = m\hbar/L_x$ is the new momentum unit in the x direction.

As for the case $\nu = 1$ (eq.(37)) the pressure is still negative and the system (cylinder or torus) would collapse if it were not counterbalanced by the effect of the neutrality condition, $N_s = mN_e$.

Then the “stability” of the ground-state (in the thermodynamic limit, at least) is strictly related to the Hall fluid incompressibility; we shall see that this fact can be expressed as an algebraic constraint on the edge vacuum indicated by $|0\rangle$, as first pointed out by M. Stone in ref.[9].

From a physical point of view we notice that the droplet of the Hall fluid for a cylinder (equivalent to a Corbino disk) is seen, in our analysis, as a sequence of states (vortices of the statistical field) carrying magnetic charge $\nu^{-1}\Phi_0$.

6 Finite size effects and CFT

In the previous section we have evaluated the Casimir energy (eq.(36)) for a Hall system of cylindrical or toroidal shape by using Girvin-MacDonald effective Lagrangian for $\nu = 1$ and, then, extended to $\nu = 1/m$ (see eq.(40)).

We now show how such a result is relevant in determining the central charge of the 2D CFT, used in order to describe the Hall system with doubly periodic boundary conditions at any rational value of the filling ν [1,2].

It has been proved that the physical states are just the highest weight states of the underlying CFT and are defined in terms of a massless scalar chiral field $\phi(z)$ compactified on a circle with radius $R = \sqrt{(1/\nu)}$ with mode expansion:

$$\phi(z) = x_0 + p_0 \ln z + i \sum_{n \neq 0} \frac{a_n}{n} z^{-n} \quad (40)$$

with the oscillators a_n obeying the commutation relations:

$$[a_n, a_m] = n\delta_{n+m,0} \quad (41)$$

and the “zero modes” x_0, p_0 , obeying:

$$[x_0, p_0] = 1 \quad (42)$$

The field $\phi(z)$ above describes the 2D massless Green function (i.e. the vortex interaction), being:

$$\langle \phi(z_1)\phi(z_2) \rangle = \ln(z_1 - z_2) \quad (43)$$

In refs.[1,2] the cylindrical configuration for the Hall system is viewed as a “closed string state” propagating in “time” from minus infinity to plus infinity, where for a toroidal configuration the initial and final states are identified and a trace is taken over the Hilbert space relative to them, which is the sewing procedure of string or conformal field theory (see the third of refs.[1]).

As a result the Laughlin wave function on the torus (24) has been evaluated for the filling $\nu = 1/m$ for any odd m , with the contribution to the center of mass, $\psi_{cm}(\vec{R})$, given by eq.(25), coming from the sum over the zero modes i.e. the eigenvalues of the momentum p_0 ; as well known (see ref.[1c]) they are fixed by the global (topological) properties. At this stage it is appropriate to remark that the sum over the oscillator modes in the evaluation of the ψ_{cm} gives the well known Dedekind function $\eta(\tau)$ which describes the contribution of the so-called neutral excitations to the partition function ([9]); this fact has not been discussed in ref. [1], neither it will be analyzed here, because we are interested only in its topological properties.

Let us go back to the zero-modes; they characterize the finite magnetic translations operators on the torus, $S_{L_x/\lambda N_x}$ and $T_{L_y/\lambda N_y}$, whose algebra is given by:

$$S_{L_x/\lambda N_x} T_{L_y/\lambda N_y} = e^{\frac{2\pi i}{m}} T_{L_y/\lambda N_y} S_{L_x/\lambda N_x} . \quad (44)$$

Their m -dimensional representation fixes the degeneracy of the ground state on the torus (see ref.[1]). Moreover, as shown in ref. [1] $S_{L_x/\lambda N_x}$ and $T_{L_y/\lambda N_y}$ are explicitly realized in the 2D CFT as the Verlinde operators [14]:

$$S_{L_x/\lambda N_x} \equiv \exp \left[\frac{i}{R} \oint_A \partial_z \phi \right] \quad (45)$$

$$T_{L_y/\lambda N_y} \equiv \exp \left[\frac{i}{R} \oint_B \partial_z \phi \right]$$

defined on the two homology cycles A, B of the torus, which implement the modular transformations $SL(2, Z)$ or physically topological non trivial gauge transformations.

Now we relate the edge vacuum energy in the thermodynamic limit as defined in sect.4, i.e. the Casimir energy given by eqs.(36,39) for $\nu = 1, 1/m$, to the Fock-vacuum energy of 2D CFT on the cylinder properly defined in ref. [1c].

As for the Casimir energy we will relate it to the “central charge” c (the vacuum energy of the underlying 2D CFT). In fact in ref. [15] it has been derived by using general techniques of 2D CFT the vacuum energy for a cylinder topology, taking into account also finite size effect. The result obtained there for the Casimir Energy is (in our notations) given by:

$$E_{Cas} = -vp_x \frac{c}{24} + O(\lambda/L_x) \quad (46)$$

which is identical to eqs.(36,39) iff $c = 1$.

Such a result is then valid both for the filling $\nu = 1$ and $\nu = 1/m$.

Also we get naturally the bosonic realization of the Kac-Moody algebra by identifying the current j_z as $j_z \equiv R\partial_z\phi$, where the scalar field ϕ is compactified on a circle of radius $R = \sqrt{1/\nu}$. By defining the moments j_n of the current as

$$j_n = R \oint \frac{dz}{2\pi i} z^n \partial_z \phi \quad (47)$$

we get the conditions:

$$j_n|0\rangle = Ra_n|0\rangle = 0 \quad (n > 0) \quad (48a)$$

$$j_0|0\rangle = Rp_0|0\rangle = nR^2|0\rangle \quad (n \in \mathbb{Z}) \quad (48b)$$

The first of eqs.(48) is naturally interpreted as incompressibility condition of the Hall fluid once the stability of the vacuum $|0\rangle$ is assumed, while the second one can be seen as a consequence of the discreteness of the “spectrum” in eqs.(32, 38) (naturally the minimum value for j_0 is $n = 0$ for the choice of the edge at $y = 0$).

We remind that eq.(48a) was first derived in ref.[9] using a fermionic realization of the $U(1)$ K.M. algebra in the case $\nu = 1$. The above equation for the ground state, which describes deformations of the edge, is naturally interpreted as an incompressibility condition of the Hall fluid and the vacuum viewed as a filled fermionic Dirac (Weyl) sea relative to the 2D chiral fermion (i.e. the filled Landau level is equivalent to the filled Dirac sea).

In a recent work a detailed analysis has been performed within the context of second quantized non-interacting many fermion theory which has confirmed the results of Stone and evaluated the Casimir effect at $\nu = 1$ [16].

Here we emphasize that our derivation is quite different in spirit because we use the GMD effective theory for $\nu = 1$ as starting point; then the topological properties of the system determine the edge vacuum.

Moreover the extension to the filling $\nu = 1/m$ is quite natural (see Sect.5) and the interpretation of eqs.(48) given above is also valid for the superfermion, i.e. an object of electric charge $q_e = e$ and magnetic charge $q_m = m\Phi_0$ (i.e. the vacuum can be interpreted as a filled Dirac sea for a “superfermion”).

The difference between the two cases is basically in the rescaling of the momentum $n \rightarrow mn$ (R^2n in the CGVO approach). This rescaling of the zero-modes (momentum p_0) only changes the topological properties, as, for example, the m -fold degeneracy of the vacuum on the torus² and the anyonic charges, $q_e = l/m$ and $q_m = l$, carried by the m independent vertices $V_l(z) =: \exp(i\alpha_l\phi(z))$;, where $\alpha_l = l/m$ and $l = 1 \dots m$.

These vertices are the highest weight states of 2D CFT which propagate in “time” along the cylinder as described in [1].

In conclusion our analysis relates in a deep way the topological properties of the vacuum on the infinite cylinder to the conformal anomaly, $c = 1$, of the underlying 2D CFT for $\nu = 1/m$.

More precisely, an exact relationship between the topological order realized in

²The role of the degeneracy as a signal of the topological order, we are addressing in the CFT approach, has been also noticed in refs.[17,18] within the context of the GMD effective Lagrangian with a Chern-Simons term (i.e. a 2 + 1 field theory).

the CGVO approach by the finite magnetic translations operators and the off-diagonal long-range order of the GMD effective theory has been proved.

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