Electron Cloud Effects in the Circulant Matrix Model

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1 Electron Cloud instabilities

Electron clouds are source of several types of instabilities limiting the performance of accelerators, the understanding of their formation and dynamics is of great importance in the future improvement of accelerator performances. Due to the close spacing between the bunches, electrons can accumulate in the beam pipe. As indicated on Fig.1, each passing bunch can produce electrons through different processes, such as photoemission from the pipe surface due to synchrotron radiation, or ionisation of the residual gas. These primary electrons will be accelerated by the beam electric field up to several electron-volts and can impact on the other side of the pipe. They can then generate secondary electrons if they have a sufficient energy, or can simply bounce back elastically. Those secondary electrons are emitted with an energy up to few tens of electron-volts and can either be absorbed by the pipe wall or bounce back. If they survive until the passage of the next bunch, they will undergo the same process as the primary electrons. If the time of flight of the electron in the pipe is comparable to the time spacing between bunches, an avalanche multiplication effect can establish leading to the exponential growth in the electron density, forming an electron cloud. After this buildup phase, the electron density will saturate due to the balance between the production rate of electrons and their repulsive space charge force. [1]. The dynamics of the electrons is seriously impacted by the strength of the magnetic field present in the pipe as they are constrained to follow the field lines. This leads to striped electron clouds in dipole sections, or more complicated shapes in higher order magnets.

![Figure 1: Electron cloud buildup process in the LHC, taken from [2]](image)

The presence of electron clouds can cause several negative effects such as particle losses, energy loss, vacuum degradation, heating of the pipe’s surface and transverse instabilities [1]. Those tranverse instabilities are the subject of the present report and can be of different natures. Multi-bunch instabilities can occur if a first bunch passes through the cloud slightly offset with respect to the other bunches. This will induce a perturbation of the cloud distribution that can later on cause additional deflection to a following bunch. The picture is somehow different for single bunch instabilities: the interaction of a bunch with the electron cloud will induce a coupling between the head and the tail of the bunch, possibly leading to head-tail instabilities. In most cases, however, the distribution of the electrons is fully reset to the equilibrium distribution between two bunch passages. Therefore, single bunch instabilities are mainly expected.

2 Project Goals and scope

The main objective of this report is to extend a semi-analytical code for the beam transverse oscillation modes by implementing a model based on linearised equation of motion for protons and electrons. Eventually, the parametric dependance of the instabilities for the High-Luminosity upgrade of the Large Hadron Collider (HL-LHC) at injection energy is discussed.
3 Physical Model

Electron Clouds exhibits a highly nonlinear dynamics, whose accurate simulation requires intensive computations. The model we develop in the following section relies on several simplifying assumptions but is intended to simulate the onset of instabilities induced by electron clouds, at the price of a reduced computational effort. This simplification, however, aims at preserving the nature of the electron cloud dynamics, as opposed to models based on an equivalent impedance, improving the predictions accuracy.

3.1 Electron density dynamics

As the proton bunch passes through an electron cloud, the electrons feel an attractive force leading them to focalise around the beam axis, they undergo a so-called pinching. Since the electric field of a relativistic bunch can be considered as transverse, the electron oscillations in the longitudinal direction can be neglected in comparison to the transverse ones. Chapter 6 of [2] develops a model for the transverse motion of the electrons around a non oscillating relativistic 4-dimensional Gaussian bunch (we exclude the transverse dimensions \( y, y' \) in the present report for the simplicity of the explanation). We present here the main steps in the computation of the electron cloud collective dynamics.

The phase space distribution of the relativistic non oscillating bunch containing \( N_b \) protons of energy \( E \) and relativistic factors \( \gamma, \beta \) is given by:

\[
\Psi_b(x, x', z, \delta) = N_bG(x; 0, \sigma_x)G(x'; 0, \sigma'_x)G(z; 0, \sigma_z)G(\delta; 0, \sigma_\delta),
\]

where \( G(u; \mu, \sigma) = \frac{1}{\sqrt{2\pi\sigma^2}} e^{-(u-\mu)^2/2\sigma^2} \). In this model, the bunch is thus only defined by its second order moments, i.e.: \( \sigma_x, \sigma'_x, \sigma_z, \sigma_\delta \). Taking the expression for the field produced by a Gaussian bunch from [3], the equation of motion of an electron in the bunch potential is given by:

\[
\ddot{x}(t) = \frac{1}{m_e} \frac{\lambda_b(t)e^2}{2\pi\epsilon_0} \frac{1}{x} \left[ 1 - \exp\left( -\frac{x^2}{2\sigma_x^2} \right) \right],
\]

where \( \lambda_b(z) = N_bG(z; 0, \sigma_z) \) is the bunch line density, \( e \) is the proton charge, \( \epsilon_0 \) the vacuum permittivity, \( m_e \) the electron mass, and noting that we can speak indifferently in terms of time \( t \) or longitudinal position \( z \) by relating them through: \( t = (n\sigma_z - z)/c \). The parameter \( n \) is defined as:

\[ n \equiv \frac{1}{2\pi} \sqrt{\frac{N_b\sigma_z}{\sigma_x^2}}, \]

\( r_e \) being the classical electron radius. This relation between \( t \) and \( z \) is practical as it is such that \( t = 0 \) when the electron is located at \( z = n\sigma_z \), \( n \) being approximately equal to 1 for short bunches.

Figure 2: Electric field produced by a Gaussian transverse distribution, almost linear for \( |x| << \sigma_x \)

As seen on Fig.2, the field produced by the Gaussian bunch is linear for \( |x| << \sigma_x \). Linearising Eq. (3.2), we obtain the following linear equation of motion for the electrons:

\[
\ddot{x}(t) = - \frac{\lambda_b(t)e^2}{4\pi m_e\epsilon_0\sigma_x^2} x = -\omega_e^2(t)x(t),
\]
where \( \omega_2^2 = \lambda_b(t)r_e c^2/\sigma_z^2 \) is the *electron plasma frequency*, and \( c \) is the speed of light in the vacuum. This frequency varies as the square root of the bunch intensity \( N_b \). This implies that the electrons will undergo faster oscillations with decreasing amplitude as they approach the center of the bunch (as the electron frequency will increase). After making use of the Liouville’s theorem stating that the electron phase space density is locally preserved and assuming that the cloud is initially distributed in the transverse dimension as

\[
\Psi_e(x,\dot{x}) = \lambda_e G(x;0,\sigma_0)G(\dot{x};0,\dot{\sigma}_0),
\]

where \( \lambda_e = 2\pi\sigma_0^2 \rho_e \) is the initial electron line density, \( \rho_e \) being the initial electron density and \( \sigma_0, \dot{\sigma}_0 \) the initial position and velocity spread of the cloud. The resulting oscillating cloud density can be expressed as:

\[
n_e(x,t) = \lambda_e^2 \sigma_z^2 \pi D(t) e^{-\frac{x^2}{2\sigma_z^2}}, \tag{3.4}
\]

where the r.m.s transverse size of the cloud is given by the following expression:

\[
D(t) = (d(t)\sigma_0)^2 + (b(t)\dot{\sigma}_0)^2, \tag{3.5}
\]

where

\[
b(t) = e^{\frac{\omega}{4}} + \frac{\dot{\omega}}{2} - \frac{1}{\Omega} \sin(S(t)) \tag{3.6}
\]

\[
d(t) = e^{\frac{\omega}{4}} + \frac{\dot{\omega}}{2} \cos(S(t)) + e^{-\frac{\omega}{4}} \cdot \frac{n}{4\sigma_x} \Omega \sin(S(t))
\]

\[
\Omega = \sqrt{\frac{r_e N_b c^2}{\sigma_x^2 \sigma_z^2}} \sqrt{2\pi}
\]

\[
S(t) = \Omega \frac{\sigma_x}{\sigma_z} \left\{ \text{erf} \left( \frac{n}{2} \right) + \text{erf} \left[ \frac{1}{2} \left( \frac{ct}{\sigma_x} - n \right) \right] \right\}
\]

\[
\tilde{z} = \frac{ct}{\sigma_x}
\]

Fig. 3 presents a typical profile for the evolution of the electron density along the passage of the bunch. We see that the density can reach absurdly high values, this is the consequence of the linearisation of the electrons equation of motion. In reality, electrons that are located further away from the beam center will feel a reduced force, and will thus take a longer time to reach the beam center as seen on Fig. 2, leading to an reduction of the amplitude of the variation of the cloud density. It is worth noting that we are able to neglect the effect of the electron direct space charge since their interaction with the relativistic proton bunch is much more intense than the space charge forces.
3.2 Electron cloud centroid dynamics

The previous development assumed a non oscillating bunch, we can extend it to an oscillating bunch by considering that its transverse Gaussian distribution will be centered around a section centroid \( \langle x \rangle_b(z) \), that may depend on the longitudinal position \( z \), given by:

\[
\langle x \rangle_b(z) = \int x \Psi_b(x, z) dx \quad (3.7)
\]

The bunch distribution is now given by:

\[
\Psi_b(x, x', z, \delta) = N_b G(x; \langle x \rangle_b(z), \sigma_x) G(x'; 0, \sigma_x') G(z; 0, \sigma_z) G(\delta; 0, \sigma_\delta). \quad (3.8)
\]

This will modify the non linearised equation of motion (3.2) by shifting the electric field around \( \langle x \rangle_b(z) \). By integrating this equation of motion over the whole transverse electron cloud distribution, one can obtain the equation of motion for the electron cloud centroid \( \langle x \rangle_c(t) \) . This coherent equation of motion was obtained in [4] and reads as:

\[
\ddot{\langle x \rangle}_c(t) = -\frac{\lambda_b(t) r_e c^2}{k\sigma_x^2} (\langle x \rangle_c(t) - \langle x \rangle_b(t)) = -\omega_{c,coh}^2(t) (\langle x \rangle_c(t) - \langle x \rangle_b(t)), \quad (3.9)
\]

where \( \omega_{c,coh}^2(t) = \frac{\lambda_b(t) r_e c^2}{k\sigma_x} \), where \( k = 2 \) for round Gaussian clouds. The coherent oscillation frequency of the electron cloud \( \omega_{c,coh} \), i.e the oscillation frequency of its section centroid differs from the single electron frequency only by a factor \( 1/\sqrt{k} = 1/\sqrt{2} \). This follows again from the fact that electrons far away from the bunch centroid will feel a reduced force, making their oscillation slower. More rigorously, one should include the motion of the bunch centroid around the cloud centroid, fortunately its oscillation frequency is more than 3 orders of magnitude below the electron frequency and allow us to consider that its motion is so slow that it will not be directly affected by the electron cloud centroid position.

The equation of motion is a Hill equation, and doesn’t have an analytical solution. We can nevertheless already conclude that the cloud centroid will oscillate more rapidly and with a smaller amplitude at the center of the bunch, where the density of protons is the highest.

3.3 Electron Cloud kick

In order to compute the kick produced by the electron cloud onto the proton bunch, we will make the assumption that the Gaussian density undergoing the pinching (Eq.(3.10)) obtained in the
section 3.1 in the case of a non oscillating bunch will just be shifted around the electron cloud centroid $\langle x \rangle_e(z)$. The cloud density can then be written as:

$$n_e(x, t) = \frac{\lambda_e}{2\pi D(t)} e^{-\frac{(x - \langle x \rangle_e(z))^2}{2D(t)}}. \tag{3.10}$$

Where the size of the cloud $D(t)$ is taken from Eq.(3.5) the cloud centroid follow the dynamics of Eq.(3.9). The kick generated by this Gaussian electron cloud can be computed in a way that is similar to the Beam-beam kick generated by Gaussian beams [5], with the only difference that the electron cloud is almost at rest in the lab frame compared to the proton beam passing through it. Just as we computed the force acting on electrons produced by the bunch protons in Eq.(3.2), we now compute the force felt by the protons originating from the electron cloud distributed as Eq.(3.10). We make use again of the expression of the electric field produced by a Gaussian transverse distribution and neglect the longitudinal electric field produced by the cloud. The force can then be written as:

$$F(x, z) = -\frac{\lambda_e e^2}{2\pi \varepsilon_0 \gamma \beta} \frac{1}{(x - \langle x \rangle_e(z))} [1 - \exp(-\frac{(x - \langle x \rangle_e(z))^2}{2D(z)})]. \tag{3.11}$$

The kick acting on a proton located at longitudinal position $z$ can be deduced by integrating this force over the duration of the passage of the bunch through the cloud length $L_c/c$, leading to the incoherent or single particle kick:

$$\Delta x'(x, z) = \frac{1}{m_p c \beta \gamma} \frac{L_c}{c} F(x, z)$$

$$= -\frac{2\lambda_e L_c e}{\gamma \beta} \frac{1}{(x - \langle x \rangle_e(z))} [1 - \exp(-\frac{(x - \langle x \rangle_e(z))^2}{2D(z)})], \tag{3.12}$$

where $m_p$ is the proton mass, $\gamma, \beta$ the bunch relativistic factors and $L_c$ the cloud length. As expected, the electron cloud kick exhibits a profile that is similar to the beam-beam kick, with an opposite sign due to the attractive force between protons and electrons.

4 Circulant Matrix Model

![Figure 4: Longitudinal phase space decomposition in 5 slices and 3 rings.](image)

We now present the so-called Circulant Matrix Model of the bunch, based on the developments made in [6].
As shown on Fig. 4, the Circulant Matrix Model (CMM) relies on a polar decomposition of the longitudinal phase space distribution $\Psi(R)$ of a particle bunch, in the normalized phase space $R = \sqrt{(s/\sigma_x)}^2 + (\delta/\sigma_z)^2$. The phase space is decomposed into $N_r - 1$ radial boundaries and $N_s$ angular boundaries, defining a decomposition of $N = N_s \cdot N_r$ cells. The radial boundaries $r_{b,i}$ can be chosen arbitrarily (as long as $r_{b,0} = 0$), while the angular boundaries are given by $\theta_{b,i} = 2\pi i/N_s$, $i = 0, ..., N_s - 1$. Each cell is attributed a center, whose coordinates are defined as follows:

- $r_i = \int_{r_{b,i-1}}^{r_{b,i}} Rd\Psi(R)$;
- $\theta_j = \frac{\theta_{b,i} + \theta_{b,i+1}}{2} = \frac{\pi(2j+1)}{N_s}$.

Considering the cell $(i, j)$, the Cartesian coordinates of its center in the non normalized space are given by

- $z_{i,j} = r_i \cos(\theta_j) \sigma_z$;
- $\delta_{i,j} = r_i \sin(\theta_j) \sigma_z$.

Every cell is also given a certain weight accounting for the proportion of the bunch it contains, computed as follows:

$$w_{i,j} = \frac{\theta_{b,i} + 1}{\theta_{b,0}} \int_{r_{b,i-1}}^{r_{b,i}} Rd\Psi(R) = \frac{2\pi}{N_s} \int_{r_{b,i-1}}^{r_{b,i}} Rd\Psi(R). \tag{4.1}$$

In the case of a Gaussian distribution, we can choose the radial boundaries such that all weights equal $1/(N_r N_s)$ and that all cells are *equipopulated*. This is achieved with the following expression:

$$e^{-r_i} - e^{-r_{i+1}} = 1/N_r. \tag{4.2}$$

We could also chose to define *equidistant* radial boundaries. The model relies on the assumption that the transverse distribution of every cell is given by a 2D-Gaussian of mean $(x_{i,j}, x'_{i,j})$ and variances $\sigma_x, \sigma_{x'}$. Those variances are the initial beam variances. The cells distribution is therefore given by:

$$\Psi_j(x) = w_j \frac{1}{\sqrt{2\pi\sigma_x}} \exp\left(-\frac{(x - x_j)^2}{2\sigma_x^2}\right), \tag{4.3}$$

ponderated with the cell weight $w_j$ so that the cell contains a number of particles amounts to $w_j N_b$.

Every cell can thus be thought as a sort of macroparticle modelized by a Gaussian distribution in the transverse phase space, the objective of the model will be to provide information about the dynamics of this collection of macroparticles. We will not consider the $y$ transverse plane, but the results can be easily extended to include it.

We can now derive the one turn transfer matrix $M$ (of size $2 \cdot N$) mapping the mean transverse position of each cells center from one turn to the other. It is formally defined as follows:

$$\begin{pmatrix}
  x_{1,k+1} \\
  x'_{1,k+1} \\
  x_{2,k+1} \\
  x'_{2,k+1} \\
  \vdots \\
  x_{N,k+1} \\
  x'_{N,k+1}
\end{pmatrix} = M \cdot \begin{pmatrix}
  x_{1,k} \\
  x'_{1,k} \\
  x_{2,k} \\
  x'_{2,k} \\
  \vdots \\
  x_{N,k} \\
  x'_{N,k}
\end{pmatrix} \tag{4.4}$$

where $x_{n,k}$ denotes the mean transverse coordinate at turn $k$ for the cell $(i, j)$ with $n = N_s(i-1)+j$, and conversely for $x'_{n,k}$. In the following developments, we will refer to the basis vector as

$$Y_k = \begin{pmatrix}
  x_{1,k} \\
  x'_{1,k} \\
  x_{2,k} \\
  x'_{2,k} \\
  \vdots \\
  x_{N,k} \\
  x'_{N,k}
\end{pmatrix} \tag{4.5}$$
for simplicity of notation.

It is worth noting that the one turn matrix is usually defined for any given number of bunches and can also be defined for the two counter-circulating beams of a collider. We will nevertheless limit the developments to the basic case of one transverse dimension of a single bunch in a single beam.

This matrix needs to take into account any effect that we want to model in the transport of the beam. We will focus on the synchro-betatron transport and electron cloud effects.

### 4.1 Synchro-betatron transport

The synchrotron motion over a whole turn corresponds to a rotation in the normalized longitudinal phase space of angle $2\pi Q_s$. Its transfer matrix can be written as:

$$S_r = P_{N_s}^{N_s}Q_s,$$  \hspace{1cm} (4.6)

where $P_{N_s}$ is the following circulant permutation matrix of size $N_s$ causing a rotation of angle $1/N_s$:

$$P_{N_s} = \begin{pmatrix}
0 & 1 & \ldots & 0 \\
0 & 1 & & \ddots \\
\vdots & \ddots & \ddots & \ddots \\
1 & \ldots & 0 & 0
\end{pmatrix},$$  \hspace{1cm} (4.7)

Accounting for the fact that each cell will stay on the same ring, the synchrotron motion can therefore be modeled by:

$$S_0 = I_{N_r} \otimes S_r.$$  \hspace{1cm} (4.8)

Recalling that the betatron motion can also be expressed as a rotation angle $2\pi Q_x$ on the surface of the betatron ellipse in the transverse phase space, each cell will be affected by the following rotation:

$$B_0 = \begin{pmatrix}
\cos(2\pi(Q_x + \delta Q'_x)) & \beta_x \sin(2\pi(Q_x + \delta Q'_x)) \\
\frac{\beta_x}{\alpha_x} \sin(2\pi(Q_x + \delta Q'_x)) & \cos(2\pi(Q_x + \delta Q'_x))
\end{pmatrix}.$$  \hspace{1cm} (4.9)

where we cleverly chose the starting point such that $\alpha(0) = 0$, and $Q'_x$ is the chromaticity. The one turn matrix is given by the composition of betatron and synchrotron rotations for each cell, which can be simply written as:

$$M_0 = S_0 \otimes B_0.$$  \hspace{1cm} (4.10)

This matrix is called the unperturbed synchro-betatron one turn matrix for a single bunch.

### 4.2 Normal mode analysis

One of the principal use of the CMM is that the stability of the beam can be easily assessed from the analysis of the eigenvalues of the one turn matrix. Let $W$ be the change of basis matrix diagonalizing $M$ into the diagonal matrix $D$ carrying the eigenvalues of $M$ on its diagonal:

$$M = W^{-1}DW.$$  \hspace{1cm} (4.11)

The behaviour of the beam after many turns is described by the n-turn matrix:

$$M^n = W^{-1}D^nW,$$  \hspace{1cm} (4.12)

where $D^n$ carries the n-th power of the eigenvalues on its diagonal. Let us decompose the initial configuration of the system in the basis of the eigenvectors $v(n = 0) = \sum_i a_i V_i$, where $V_i$ is the eigenvector corresponding to the eigenvalue $\lambda_i$. After $n$ turns, the system will be in the configuration:

$$\sum_j a_j \lambda_j^n V_j = \sum_j a_j e^{-in2\pi Q_i} V_j, Q_i = i\log(\lambda_i)/(2\pi).$$  \hspace{1cm} (4.13)

The consequence of this expression is quite straightforward: if the imaginary part of $Q_i$ is strictly positive, any system with a non-zero component of this mode ($a_j \neq 0$) will eventually diverge.
and be called unstable. Conversely, any mode associated with an eigenvalue with negative real value will be spontaneously damped. It is worth noticing that this definition of the growth rate is expressed in /turns, it is sometimes more practical to speak in terms of rise time, related to the growth rate by the revolution period $T : \tau = T \times k[s]$

The eigenvalues and eigenmodes of the unperturbed synchro-betatron motion can be derived analytically. Assuming $N_s$ to be odd for simplicity, the unperturbed tune spectrum is given by:

$$Q_0 = \left\{ Q_{coh} = \pm (Q_x + n_a Q_s); n_a \in \mathbb{Z}, |n_a| < \frac{N_s - 1}{2} \right\}, (4.14)$$

where $Q_x$ and $Q_s$ are here the fractional parts of the betatron and synchrotron tunes. Eq.(4.14) indicates that the eigenvalues are centered around $Q_x$ spaced by a distance of $Q_s$. It is worth noting that each eigenvalue is degenerated $N_r$ times, those degenerated modes with identical azimuthal number are called the radial modes. The azimuthal order of an unperturbed mode is therefore given by

$$n_a = \frac{Q_{coh} - Q_x}{Q_s} = \Delta Q_x / Q_s$$

This degeneration is quickly lifted when considering the actual perturbed one turn matrix. The unperturbed modes of the CMM form a sort of angular "Fourier basis" on each ring of the decomposition.

Fig.5 illustrates the profile of the first mode of the unperturbed spectrum, pointing the fact that the azimuthal number defines the angular aspect, while the radial number indicates the ring on which the mode is located. A positive azimuthal number indicates a mode that will rotate in the trigonometric direction, while a negative one will rotate in the opposite direction.

Another important observation has to be stated about the eigenvalues of the unperturbed one-turn matrix: its spectrum is composed by two symmetric positive and negative parts, representing the fact that the particles rotating in the opposite direction are also valid physical solutions of the synchro-betatron motion.

When slightly perturbing the one turn matrix (for example when considering a very weak electron cloud), one can use degenerate perturbation theory to interpret what happens to the eigenmodes. Without diving into a precise analysis, one can see that a slightly perturbed eigenmodes of azimuthal number $n_a$ will therefore be composed of a linear combinations of mode belonging to the same degenerate azimuthal subspace, but with different radial numbers $n_r$ (at 0th order). This can be seen on the left figures of 16 and 17, where we see a slightly perturbed azimuthal mode 4 or 5 composed of a linear combination of different radial modes with identical azimuthal number.
4.3 Derivation of the Electron Cloud Matrix

Following the developments of the previous section, one as yet to derive an expression of the Electron Cloud Matrix, i.e. the matrix accounting for the effect of the electron cloud over a complete revolution in the accelerator on the cells of the CMM. More explicitly, we want to compute the following matrix $M_{EC}$:

$$
\begin{pmatrix}
  x_{1,k+1} \\
x'_{1,k+1} \\
x_{2,k+1} \\
x'_{2,k+1} \\
\vdots \\
x_{N_r N_s,k+1} \\
x'_{N_r N_s,k+1}
\end{pmatrix}
= M_{EC} \cdot
\begin{pmatrix}
  x_{1,k} \\
x'_{1,k} \\
x_{2,k} \\
x'_{2,k} \\
\vdots \\
x_{N_r N_s,k} \\
x'_{N_r N_s,k}
\end{pmatrix}.
$$

(4.15)

The first step is to compute the so-called coherent kick, i.e. the kick received by an entire cell of the CMM from the electron cloud. This requires to integrate the single particle kick obtained in equation (3.12) over the span of the cell $j$ we are considering, with coordinate $(x_j, x'_j)$. While we should formally integrate the kick over the 4 dimensional phase space, we make here the assumption that the slice will only sample the kick originating from the position $z_j$ of its center. Recalling the
assumption that every cell was transversely distributed as a Gaussian

\[ \Psi_j(x) = w_j \frac{1}{\sqrt{2\pi\sigma_x}} \exp\left(-\frac{(x - x_j)^2}{2\sigma_x^2}\right), \]

the integrated kick, or coherent kick, can be expressed as the following integral:

\[ \Delta x'_\text{coh}(x_j; z) = \int_{-\infty}^{\infty} dX \Delta x'(X, z) \Psi(X - x) \]

\[ = -w_j \frac{2\lambda_e L_e r_e}{\gamma^2} \int_{-\infty}^{\infty} dX \frac{1}{(X - \langle x \rangle_c(z))^2} \left[ 1 - \exp\left(-\frac{(X - \langle x \rangle_c(z))^2}{2D(z)}\right) \right] G(X; x_j, \sigma_x). \]

We can see that this integral is nothing but a slightly different form of the beam beam kick integral computed in [3], the differences being the fact that we consider here an electron cloud of r.m.s size \( \sqrt{D} \), centered around \( \langle x \rangle_c(z) \). Following the developments of [3], and using the result of the convolution of two Gaussian distributions:

We get for the coherent electron cloud kick over cell \( j \):

\[ \Delta x'_\text{coh}(x_j; z) = -w_j \frac{2\lambda_e L_e r_e}{\gamma^2} \frac{1}{(x_j - \langle x \rangle_c(z))^2} \left[ 1 - \exp\left(-\frac{(x_j - \langle x \rangle_c(z))^2}{2(D(z) + \sigma_x^2)}\right) \right] \]

(4.17)

The CMM being a linear model, it requires to make use of a linearised form of the coherent kick (valid for \( |x| \ll \sigma_x, \sqrt{D(z)} \)):

\[ \Delta x'_\text{coh}(x_j; z) = -w_j \frac{\lambda_e L_e r_e}{\gamma^2} \frac{1}{[D(z) + \sigma_x^2]^2} \]

\[ = -w_j K_{EC}(z) [x_j - \langle x \rangle_c(z)] \]

(4.18)

where we defined the kick coefficient \( K_{EC}(z) = \frac{\lambda_e L_e r_e}{\gamma^2} \frac{1}{[D(z) + \sigma_x^2]^2} / m \). The effect of the electron cloud on both coordinates can be summarised by the following equations:

\[
\begin{cases}
  x_{j,k+1}(z) = x_{j,k}(z) \\
  x'_{j,k+1}(z) = x'_{j,k}(z) - w_j K_{EC}(z) [x_j(z) - \langle x \rangle_c(z)]
\end{cases}
\]

(4.19)

Which shows that the linearised coherent electron cloud kick exactly reproduces the effect of a thin focusing lens of strength \( K_{EC} \) distributed along the entire accelerator. The focusing character of the electron was obviously expected from the attractive force they exert on the proton bunches.
Figure 6: Electron cloud linearised kick coefficient $K_{EC}(z)$: it is linearly influenced by $\rho_e$ while the influence of the intensity $N_b$ is to increase the number of pinches. (The cloud comes from positive values of $z$)

Fig.6 presents the influence of the intensity $N_b$ and the cloud initial density $\rho_e$ on the kick coefficient $K_{EC}$, it varies linearly with $\rho_e$ while the influence of $N_b$ is to increase the number of focalisation peaks and sharpen them.

Eq.(4.19) is obviously not yet implementable in the CMM, because the cloud and bunch centroids $\langle x \rangle_e(z), \langle x \rangle_b(z)$ needs to be expressed in terms of the cells coordinates. This will be the objective of next section.

**Bunch centroid CMM operator**

The calculation of the section centroid $\langle x \rangle_b(z)$ is not trivial, since we need to find a way of exploiting the polar decomposition of the CMM to express the value of the section centroid at the Cartesian coordinate $z$.

Recalling the definition of the bunch Gaussian distribution of Eq.(3.1), the section centroid is expressed in the continuous case as

$$\langle x \rangle_b(z) = \frac{\int_{-\infty}^{\infty} d\delta x(z,\delta)\Psi(z,\delta)}{\int_{-\infty}^{\infty} \Psi(z,\delta)} = \int_{-\infty}^{\infty} d\delta x(z,\delta) \frac{e^{-\delta^2/2\sigma_\delta^2}}{\sqrt{2\pi\sigma_\delta}}. \tag{4.20}$$

In order to apply this expression to our phase space decomposition, one needs to find an estimate expression of $\langle x \rangle_b(z)$.
The approach consists in considering \( x(z, \delta) \) to be constant over every cells and to be equal to the cell centroid \( x_i \). This allows us to write the integral as a sum that can be evaluated analytically, as long as we know every intersection of the line \( z = z_i \) with the cells boundaries, called \( \delta_j \), \((j = 1, \ldots, n)\).

The integral reads now as:

\[
\langle x \rangle_b(z) \approx x_0 \frac{\delta_1}{\int_{-\infty}^{\delta_1} e^{-\delta^2/2\sigma^2} \, d\delta} + \sum_{i=1}^{n-1} x_i \frac{\delta_{i+1}}{\int_{\delta_i}^{\delta_{i+1}} e^{-\delta^2/2\sigma^2} \, d\delta} + \frac{\delta_n}{\int_{\delta_n}^{\infty} e^{-\delta^2/2\sigma^2} \, d\delta}.
\] (4.21)

We see three types of integration domains appearing in the finite sum: from \([-\infty, \delta_1]\), \([\delta_i, \delta_{i+1}]\) and \([\delta_n, \infty]\), a computable solution can be stated for each domain as follows:

\[
\begin{align*}
\int_{-\infty}^{\delta_1} e^{-\delta^2/2\sigma^2} \, d\delta &= \frac{\text{erf} \left( \frac{\delta_1}{\sqrt{2}\sigma} \right) + 1}{2}, \\
\int_{\delta_i}^{\delta_{i+1}} e^{-\delta^2/2\sigma^2} \, d\delta &= \frac{\text{erf} \left( \frac{\delta_{i+1}}{\sqrt{2}\sigma} \right) - \text{erf} \left( \frac{\delta_i}{\sqrt{2}\sigma} \right)}{2}, \\
\int_{\delta_n}^{\infty} e^{-\delta^2/2\sigma^2} \, d\delta &= \frac{1 - \text{erf} \left( \frac{\delta_n}{\sqrt{2}\sigma} \right)}{2}.
\end{align*}
\] (4.22)

Where the error function is 
\( \text{erf}(u) = \frac{2}{\sqrt{\pi}} \int_0^u \exp(-t^2) \, dt \).

We can now explicitly rewrite Eq.(4.21) by defining the linear form \( \hat{X}_b \) acting on the space phase decomposition so that:

\[
\langle x \rangle_b(z) \approx \hat{X}_b(z) \cdot \vec{Y}.
\] (4.23)

**Electron Cloud Centroid CMM operator**

We know from Sec.3.2 that the electron cloud centroid oscillates with the varying frequency \( \omega_{e, coh}(z) \) around the bunch centroid as summarized in its coherent equation of motion:

\[
\langle \dot{x} \rangle_e(t) = -\omega^2_{e, coh}(t)(\langle x \rangle_e(t) - \langle x \rangle_b(t)),
\] (4.24)
As mentioned earlier, this equation does not generally possess an analytic solution. We choose to solve it numerically on a longitudinal grid composed by the $z$ coordinates of the cells composing the basis.

Figure 8: Example of a longitudinal grid of $z$ points obtained for a decomposition of 5 slices and 2 rings. The $z_i, i = 0, \ldots, 5$ position of the grid are marked with crosses.

Taking into account the fact that the integration steps are not constant along the grid, one can approximate the second order derivative of a function $f(x = x_n)$ on a non uniform grid as:

$$
\ddot{f}(x_n) \approx \frac{f(x_n + h_n/2) - f(x_n - h_{n-1}/2)}{\frac{1}{2}(h_n + h_{n-1})} - \frac{f(x_n) - f(x_{n-1})}{h_{n-1}} = \frac{2}{h_n + h_{n-1}} \left[ f(x_{n+1}) \frac{1}{h_n} + f(x_{n-1}) \frac{1}{h_{n-1}} - f(x_n) \left( \frac{1}{h_{n-1}} + \frac{1}{h_n} \right) \right],
$$

where $h_{n-1}$ is the grid spacing between $x_{n-1}$ and $x_n$, and $h_n$ is the spacing between $x_n$ and $x_{n+1}$. We can apply this expression to the equation of motion and obtain the following recurrent expression for the cloud centroid at position $z_i$:

$$
\langle x \rangle_{e,n} \approx c^2 \left[ \langle x \rangle_{c,n} h_n \left( -\frac{1}{2} \omega_{e,coh,n}^2 (h_n + h_{n-1}) + \frac{1}{h_n} + \frac{1}{h_{n-1}} \right) \right] - \langle x \rangle_{c,n-1} \frac{1}{h_{n-1}} \left[ h_n \frac{1}{2} \omega_{e,coh,n}^2 (h_n + h_{n-1}) h_n \right]
$$

where the $c^2$ prefactor comes from the change of variable applied on the second order derivative from time $t$ to longitudinal coordinate $z$ and $\langle x \rangle_{z,n} \equiv \langle x \rangle_{e}(z_n)$. We can now explicitly rewrite (4.26) as a linear form acting $\hat{X}_e$ on the basis vector $\tilde{Y}$:

$$
\langle x \rangle_{e}(z_{n+1}) \approx \hat{X}_e \left( \langle x \rangle_{c}(z_n), (x)(z_{n-1}), (x)b(z_n) \right) \tilde{Y}
$$

From this expression it is clear that the cloud centroid at position $z_{n+1}$ will be influenced by three terms: the bunch centroid at time $z_n$, obtained through the bunch operator derived in the previous section, and the two previous cloud centroids at position $z_n, z_{n-1}$. This behavior is to some extent similar to the effect of a wake field in the way that particles at the front of the bunch will influence the ones located further back.
A stability criterion on the integration steps size can be derived (chapter 9 of [7]) in the case of a constant oscillation frequency as:

$$ h_n \omega_{e,\text{coh}}(z_n) \leq 2. \tag{4.28} $$

As the coherent electron frequency $\omega_{e,\text{coh}}$ does not vary very rapidly over the bunch length, we will consider this stability criterion as a minimal requirement for the stability of our integrator.

Fortunately, the stability criterion is reached for computationally achievable discretisations.

One could also comment on the fact that the dynamics of the cloud centroid could be heavily influenced by the initial conditions, i.e. the conditions at the meeting point between the bunch and the cloud as well as the cloud centroid initial position and speed. Nevertheless, if one chose a basis that is sufficiently wide spread longitudinally (above $4\sigma_z$), the influence of the meeting point will be suppressed since the dynamics will start at points where the proton density is so low that it will not yet interact with the cloud, as shown on Fig.9.

![Figure 9: Electron cloud centroid motion (colored lines) for different longitudinal spans of the decompositions around an oscillating bunch centroid (dotted black line), bunch line density plotted in the lower plot: the influence of the longitudinal span of the discretisation vanishes as it reaches $4\sigma_z$, where the bunch line density is small. (electron cloud coming from negative values of $z$)](image)

From Eq.(4.27) we are now able to iteratively compute the operator approximating the cloud centroid at a given position $z_n$ given assumptions on the initial conditions of the cloud. In all of the subsequent computations, we consider the cloud centroid initial position to be given by the bunch centroid initial position, i.e. its position at $z = 4\sigma_z$. We also consider its initial velocity to be zero.

We have now derived all of the tools necessary to the systematic construction of the electron cloud matrix, which will be detailed in the following section.

## 5 BimBim

BimBim [6] is an implementation of the CMM in the Python programming language. It aims at systematically building the one turn matrix for both counter-circulating beams of a collider. The modelisation of several phenomenons is already implemented. Those phenomena include the interactions of both beams with each other (the beam-beam forces), the interaction of the beams with the pipe surrounding it (coupling impedance forces), transverse feedback and direct space charge. The code takes advantage of the object-oriented possibilities of Python and make use of sparse matrices for memory considerations.

As mentioned previously, the aim of this project is to extend BimBim by implementing a module accounting for the electron clouds effects. For simplicity of development, we only consider the horizontal transverse dimension of one beam composed of a single bunch. This should obviously
be extended to any beam configuration and to both dimensions of the transverse plane in future work.

The developments in the following sections will only be done accounting for this simplified beam configuration but can easily be extended to the general case.

5.1 Bunch centroid operator implementation

Recalling the derivation of the bunch centroid CMM operator done in Sec.4.3, the remaining objective is to implement an efficient way of systematically building the operator matrix (of dimension $N_z \times 2N$) for any given number of slices and rings.

The general procedure for generating the bunch centroid matrix can be presented as follows:

- **Step 1**: find the intersections between the cell boundaries and the vertical line $z = z_i$, remove the duplicates, and order them in a list of increasing order;
- **Step 2**: compute the integrals for the three types of integration domains accordingly to Eq.(4.22);
- **Step 3**: return the matrix line $\hat{X}_b(z_i)$;

As explained earlier, the bunch centroid operator is a matrix of dimension $N_z \times 2N$ that returns the vector representing the bunch centroid at every position $z_i$ existing in the basis when applied on a vector containing the configuration of the bunch.

**Step 1**: the implementation of this step is quite straightforward. The intersections can be of two nature: either with the radial boundaries or with the angular ones. Some computation can be avoided by noticing that the angular intersections only exist if $z_i$ has the same sign that $\cos(\theta_{b,j})$ and that the radial intersections only exist if $z_i^2 + \delta_j^2 \leq r_{b,j}^2$

**Step 3**: in order to return the vector line $\hat{X}_b(z_i)$, the integrals computed in step 2 must be filled in a vector line according to the cell they correspond to, i.e, we must retrieve what cell is contained between two intersections $\delta_j$ and $\delta_{j+1}$, with longitudinal coordinates $z_i$. This cell identification was implemented by checking for which cell $(m,n)$ the following inequalities are simultaneously respected:

$$
\begin{align*}
\begin{cases}
    r_{m,b} \leq r_j \leq r_{m+1,b} & \theta_{n,b} \leq \theta_j \leq \theta_{n+1,b} \\
    r_{m,b} \leq r_{j+1} \leq r_{m+1,b} & m = 0, \ldots, N_r - 1
\end{cases} & n = 0, \ldots, N_z - 1
\end{align*}
$$

where

$$
\begin{align*}
\begin{cases}
    r_j = \sqrt{z_i^2 + \delta_j^2} \\
    \theta_j = \arctan\left(\frac{\delta_j}{z_i}\right)
\end{cases}
\end{align*}
$$

The index of the retrieved cell is then given by $I = N_z(n - 1) + m$

This step is the most computationally costly of the algorithm since it needs to execute a number of checks equal to the number of cells in the basis, for every couple of intersections, and this again for every cell in the basis since $\hat{X}_b(z_i)$ needs to be computed for every cell.

One can notice that computation of $\hat{X}_b(z_i)$ for each cell depends only on the basis parameters $N_r$ and $N_z$ and does not depend in any way of the beam or machine parameters. This calculation can thus be executed only once for a certain basis and then kept in memory, saving a good amount of calculations. The part of the simulation requiring the most computations is thus still the diagonalization of the full one turn matrix.

5.2 Cloud centroid operator implementation

The construction of the cloud centroid operator is less tedious as we will see in this section. Similarly to the bunch centroid operator, the cloud centroid operator is a matrix of size $N_z \times 2N$. As indicated by the recurrent expression we obtained for the cloud centroid in Eq. (4.26), this matrix needs to be generated recurrently, starting from the head of the bunch. Therefore, one needs to generate the bunch centroid operator before the cloud one. Unlike the bunch operator, this operator cannot be stored once for all in memory as it depends also on other parameters than the dimensions of the basis. Fig.10 shows an example of the output of the bunch and cloud.
operators when applied on a mode-0 of the non perturbed spectrum. As expected, the cloud centroid oscillates around the bunch centroid with a frequency increasing with the proton line density and leaves the bunch following a straight trajectory. The bunch centroid looks consistent as well with the aspect of the mode on the longitudinal phase space.

![Figure 10](image)

Figure 10: Snapshots of the electron cloud and bunch centroids motion of an azimuthal \( \theta \) mode: the electron cloud meets the bunch at the right side of the plot. The behavior of the cloud centroid around the bunch centroid is consistent with its linearised equation of motion (3.9).

6 Benchmarking

6.1 Convergence analysis

![Figure 11](image)

Figure 11: Surface formed by the most unstable growth rates, in function of the number of slices and rings of the decomposition. Parameters taken from Tab. 1.

Before investigating the prediction of our model, one has to make sure that the decompositions we use is precise enough for ensuring reliable results. This criterion is reached when the predictions of the model do not vary wildly when modifying the number of slices or rings, i.e. when we reach convergence. The convergence of the results is somehow not trivial to estimate, as it can depend on the azimuthal order of the modes we consider (low order modes converge faster than high order ones). In order to quantify the required precision of the basis, we compute the growth rate of the most unstable mode in function of the number of slices and rings of the decomposition, in the case of the HL-LHC at injection (parameters listed in Tab. 1). We plot the surface composed of those maximal growth rates on Fig.11.
Before commenting the results, let’s inspect the different factors that could influence the convergence of the algorithm. First of all, there is an upper limit in the number of slices $N_s$ that is acceptable for a certain ratio of the betatron and synchrotron tunes $Q_x/Q_s$. Recalling the expression of the unperturbed spectrum (4.13), we see that the smallest tune of the spectrum will reach negative values when $N_s \geq 2 \frac{Q_x}{Q_s} + 1$. Due to other (still unexplained) effects, the actual limit in $N_s$ for the set of parameters listed in Tab. 1 is 71. The nature of those artifacts in the CMM is still relatively unclear and should be studied in future work.

Secondly, the longitudinal resolution of the basis must be sufficient for accurately resolving the pinching peaks of the cloud.

Lastly, the longitudinal resolution must be sufficient for the numerical integrator to be stable, this condition is satisfied in most cases and will not be a constraining factor.

![Figure 12: Gradient of the surface formed by most unstable modes growth rate obtained for different numbers of slices and rings. Convergence is not achieved even for low order modes (parameters listed in Tab.1).](image)

We represent on Fig.12 the norm of the gradient of the surface plotted on Fig.11 formed by those maximal growth rates, convergence is thus achieved when the norm of this gradient is considered to reach reasonably small values, meaning that an increase in the number of slices or rings will not cause a noticeable variation in the results. As the convergence is dependent on the azimuthal order of the modes, we decide to investigate until what order (i.e what value of $\Delta Q_x/Q_s$) we can reach convergence. For the consistency of the analysis, we chose to use a basis composed of equidistant (and not equipopulated) rings, so that the longitudinal span of the basis does not depend on the number of rings. Fig.12 clearly shows that convergence can not be reached with the considered set of parameters neither for low order modes ($|\Delta Q_x/Q_s| \leq 1$) nor high order ones ($|\Delta Q_x/Q_s| \leq 40$).

![Figure 13: Convergence analysis, with artificially fixed cloud r.m.s size. Results converge smoothly because the sharp pinching peaks are suppressed (parameters listed in Tab.1)](image)

Those irregularities are in fact caused by the very sharp peaks of the electron cloud kick caused by the pinching of the cloud (as mentioned in Sec.3.1 and shown on Fig.6), those peaks appeared to be so sharp that an excessively precise decomposition would be needed to achieve the necessary longitudinal resolution, hence convergence. Indeed, when plotting the same figure with an arbitrary fixed r.m.s. size cloud (no pinching), the results exhibit a logarithmic converging behavior, as shown of Fig.13.
Electron cloud effects in the Circulant Matrix Model

Figure 14: Gradient of the surface formed by most unstable modes growth rate obtained for different numbers of slices and rings, using parameters of Tab.1, with a cloud initial r.m.s. size equal to the beam r.m.s. size. Convergence is therefore achieved up to modes 37 for approximately 61 slices and 77 rings.

Those very sharp peaks are the consequence of the linearity of the dynamical model we used and are much milder when taking into account the non-linearities of the actual electron cloud dynamics. Unfortunately, this is the cost we have to pay for having considered a linear dynamical model for the electron motion. We choose now to use a reduced initial r.m.s. cloud size, equal to the initial r.m.s. size of the beam, with the effect of flattening and broadening the focalisation peaks. This situation is represented on the left plot of Fig.14, therefore achieved up to modes 37 for approximately 61 slices and 77 rings, while it is not achieved for higher order modes, as seen on the right plot of Fig.14. We can justify the choice of the cloud initial size by stating that our linear model is unlikely to be valid when considering clouds much larger than the beam size, since the electrons outside of the beam will not feel a linear force.

We present in the following section a parametric study of the simulated effect of electron clouds.

7 Numerical Results

We study the dependence of the instability on the electron density, in the case of the HL-LHC at injection. Fig.15 depict the evolution of the bunch spectrum in function of the electron density \( \rho_e \) (we only plotted the 100 most unstable modes per value of \( \rho_e \) for readability considerations). The growth rate of each mode is represented by the markers color. As expected, the bunch gets more unstable as the electron density increases. We can see from that the electron cloud seem to cause several mode couplings around densities of \( 7 \times 10^{12} \text{ m}^{-3} \) (namely modes 4 and 5, −1 and −2, −3 and −4).

From a model based on an equivalent impedance [2], we can obtain an estimation for critical electron density at which the mode coupling instability appears. This density is given by:

\[
\rho_{cr} = \frac{2\gamma Q_s}{\pi \tau_p C \beta_x}. 
\]

Our simulation seem to disagree with this estimate by 1 orders of magnitude, which might be due to the limited initial cloud size we had to use to ensure converged results, since this limited size reduces the number of electrons interacting with the bunch. This point should be investigated in future work in order to improve our model.
Figure 15: Evolution of the bunch spectrum with increasing electron density. Growth rates represented by the color scale. Modes 4 and 5, −1 and −2, −3 and −4 undergo coupling around densities of $7 \times 10^{12} \text{m}^{-3}$.

Fig. 16, 17 and 18 depict the aspect of modes 4 and 5 before and after the coupling. As one can see, the modes before coupling are a combination of unperturbed radial modes, as predicted by degenerate perturbation theory. Before reaching convergence, the modes exhibits a similar aspect (up to a phase).

Chromaticity and transverse feedback are commonly used as a mitigation tool of electron cloud instabilities [1]. We represent on Fig. 19 the evolution of the 100 most unstable modes for a high electron density of $1.10^{13} \text{m}^{-3}$ in function of the chromaticity $Q'$. One can see that chromaticity seems to stabilise certain mode that where unstable for a chromaticity of zero, while it induce instabilities in other modes. This confirms the important impact of chromaticity on the electron cloud instability. Nevertheless, since the model does not include the effect of external dipole and quadrupole fields on the electron cloud dynamics, these predictions are not directly comparable to the LHC data.

Figure 16: Mode 4, for low electron density (left) and just before coupling with mode 5 (right).
Electron cloud effects in the Circulant Matrix Model

Figure 17: Mode 5, for low electron density (left) and just before coupling with mode 4 (right).

Figure 18: Mode resulting from the coupling between modes 4 and 5.

Figure 19: Evolution of the bunch spectrum with chromaticity $Q'$ for a high density of electrons. Some mode that were initially unstable (red/yellow at the center of the plot) seem to be stabilized by increasing chromaticity (blue on the sides of the plot) while others seem to become unstable with increasing chromaticity (different parts of the spectrum represented on the left and right figures).
8 Conclusion and prospects

We derived an analytical expression for single bunch electron cloud effects in drift sections in the frame of the Circulant Matrix Model, considering one transverse dimension and using linearised equation of motions.

We implemented those electron clouds effects and their impact on the proton beam in the code BimBim by making use of a second order numerical integrator, which had never been done before in the CMM, and investigated the convergence of the results in the case of the high-Luminosity upgrade of the Large Hadron Collider at injection. Eventually we studied the impact of chromaticity and electron density on the bunch stability.

The possibility to study the electron cloud instability using the CMM was demonstrated for the first time, in particular recovering the strong variations of the instability growth rate with the chromaticity. While the results were encouraging, the full complexity of the instability mechanism is still out of reach.

The extension of the model to dipolar and quadrupolar sections of the accelerator could allow to benchmark the results with actual measurement datas. This extension would require to extend the code to a second transverse dimension in the first place, and then extend the electron cloud equation of motion in the case of dipolar and quadrupolar fields.

Another point to investigate is the impact of the initial cloud size $\sigma_0$ on the results of the simulation, as we show in section 6.1 that its influence can not be ignored as convergence was not achievable considering a large initial cloud size. This issue was caused by the very sharp focalisation peaks in the electron density arising from the linear nature of the model, this problem might be mitigated by adding a diffusion term in the equation of the electron cloud size in order to model the non-linearities of the dynamics.

The CMM exhibits significant artifacts when the frequency of the high order modes approach 0 or 1, leading to a limitation of its domain of validity. Investigation of this effect would be needed to study very large electron cloud densities, nevertheless we found that the most critical modes in the presence of realistic electron cloud densities for the LHC and HL-LHC are of sufficiently low order to be modeled within the CMM.

Acknowledgements

I would like to thank Dr. Xavier Buffat for the time and energy spent in helping me in this project, it was a real pleasure all year long!
A Parameters table

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Table 1: Set of parameters for the future High-Luminosity LHC upgrade at injection, beam and optics parameters taken from [8].
References


