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On the classical origin of the fermionic Schwinger term

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ABSTRACT

The fermionic Schwinger term which arises in the context of quasifree second quantization by normal ordering the fermionic charges is shown to have its origin in the “exotic” symplectic left action of the connected component of the restricted unitary group on its homogenous space: the Grassmannian.

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1. Introduction.

The aim of this letter is to shed light on the classical origin of the fermionic Schwinger term which appears in the commutation relations of the algebra of charges (Current algebra)[1],[2]. We give an explicit expression of the generating functions of the symplectic left action of the restricted unitary group on the fermionic state space – the infinite dimensional Grassmannian – which extends the one presented in [3]. In addition we compute the co-adjoint cocycle of the momentum mapping. This cocycle turns out to be non trivial, which implies that the symplectic action is “exotic” in the sense of [4]. Moreover the differential of this co-adjoint cocycle at the identity yields the fermionic Schwinger term, which therefore appears in the Poisson brackets of the generating functions. We thus have obtained a representation of the fermionic Current algebra in terms of smooth functions on the Grassmannian.

2. The infinite dimensional Grassmannian as a fermionic state space.

In this section we establish the notation by briefly reviewing the geometry of the infinite dimensional Grassmannian and by relating it to the algebraic framework of second quantization. Let $(\mathcal{H}, \langle \cdot, \cdot \rangle)$ be the separable Hilbert space of initial conditions (= fixed time field configurations) for the free Dirac equation in $d + 1$ dimensions. Then the phase $F_0 := s\text{-}\lim_{\epsilon \searrow 0} (H_0 + \epsilon)/|H_0 + \epsilon|^{-1}$ of the free Dirac Hamiltonian H_0 determines a polarization of $\mathcal{H}$. That is: $\mathcal{H}$ decomposes into a direct sum of two mutually orthogonal closed infinite dimensional subspaces W_0 and $W_0^\perp$ such that $F_0 = P_{W_0} - P_{W_0^\perp}$. Here P_{W_0} and $P_{W_0^\perp}$ denote the corresponding orthogonal projections.

According to [3] the infinite dimensional Grassmannian $\text{Gr}(\mathcal{H})$ which is associated to the polarization $\mathcal{H} = W_0 \oplus W_0^\perp$ is defined as the manifold of all complex closed subspaces $W \subset \mathcal{H}$ such that the orthogonal projection $P_{W_0} P_W : W \rightarrow W_0$ is Fredholm and the orthogonal projection $P_{W_0^\perp} P_W : W \rightarrow W_0^\perp$ is Hilbert Schmidt. With $F \equiv F_W = P_W - P_{W^\perp}$ the above characterization of subspaces is equivalent to the requirement $\|F - F_0\|_{\text{HS}} < \infty$, where $\|\cdot\|_{\text{HS}}$ denotes the Hilbert Schmidt norm. Since locally around W any other subspace can be written as a graph of a Hilbert Schmidt operator $W \rightarrow W^\perp$, $\text{Gr}(\mathcal{H})$ is an infinite dimensional complex Hilbert manifold modelled on the complex Hilbert space $\text{HS}(W_0, W_0^\perp)$ of Hilbert Schmidt operators $W_0 \rightarrow W_0^\perp$. We shall think of a generic $F \in \text{Gr}(\mathcal{H})$ as the phase of some Dirac Hamiltonian, but we emphasize that in what follows only its phase enters the discussion. This merely reflects the fact that there is no unique correspondence between the Fock vacuum and the dynamics in the single particle picture. We remark in addition that although our discussion is a priori independent of dimension, it is well known that for $d > 1$ Dirac Hamiltonians which depend on external gauge fields have $F - F_0$ in a higher Schatten Ideal [5].

Let $U_{\text{res}}(\mathcal{H})$ be the real Banach Lie group of all $U \in \mathfrak{B}(\mathcal{H})$ such that $[F_0, U]$ is Hilbert Schmidt¹ where $U_{\text{res}}(\mathcal{H})$ carries the metric topology induced by $\|U_1 - U_2\| +$

¹This is equivalent to the requirement that the odd part $1/2(U - F_0 U F_0)$ of U with respect to the $\mathbb{Z}_2$ grading on $\mathfrak{B}(\mathcal{H})$ induced by F_0 should be in the Hilbert Schmidt class.

$\| [F_0, U_1 - U_2] \|_{\text{HS}}$. Here $\mathfrak{B}(\mathcal{H})$ denotes the algebra of bounded linear operators on $\mathcal{H}$ and $U_{\text{res}}(\mathcal{H})$ is referred to as the restricted unitary group. Then $U_{\text{res}}(\mathcal{H})$ acts transitively on $\text{Gr}(\mathcal{H})$ by

$$F \mapsto l_U(F) := U F U^{-1} \quad (1)$$

and the stabilizer of F_0 is the contractible subgroup $U(W_0) \times U(W_0^\perp)$, such that $\text{Gr}(\mathcal{H})$ can be identified with the homogenous space $U_{\text{res}}(\mathcal{H}) / U(W_0) \times U(W_0^\perp)$. Since $U(W_0) \times U(W_0^\perp)$ is contractible there is a homotopy equivalence $\text{Gr}(\mathcal{H}) \simeq U_{\text{res}}(\mathcal{H})$ and as a consequence $\pi_0(\text{Gr}(\mathcal{H})) = \mathbb{Z}$ and $\pi_1(\text{Gr}_0(\mathcal{H}), F_0) = 0$, since the homotopy groups of $U_{\text{res}}(\mathcal{H})$ are well known [3],[4],[6]. Here $\text{Gr}_0(\mathcal{H})$ denotes the connected component of $\text{Gr}(\mathcal{H})$ which contains the polarization F_0 . In addition we remark that the connected component of the identity $U_{\text{res},0}(\mathcal{H})$ leaves each connected component of $\text{Gr}(\mathcal{H})$ invariant.

The Lie algebra $\mathfrak{u}_{\text{res}}(\mathcal{H})$ of $U_{\text{res}}(\mathcal{H})$ consists of all $X^* = -X$ such that $[F_0, X]$ is Hilbert Schmidt and as a Banach Lie algebra $\mathfrak{u}_{\text{res}}(\mathcal{H})$ is equipped with the norm $\|X\| + \|[F_0, X]\|_{\text{HS}}$. By identifying tangent vectors at F with selfadjoint Hilbert Schmidt operators $\dot{F}^* = \dot{F}$ which in addition satisfy $\dot{F}F + F\dot{F} = 0^2$, we can write down the explicit expression for the fundamental fields ζ_X , $X \in \mathfrak{u}_{\text{res}}(\mathcal{H})$ of the $U_{\text{res}}(\mathcal{H})$ left action on $\text{Gr}(\mathcal{H})$ as

$$\zeta_X(F) = [X, F]. \quad (2)$$

As a complex manifold there exists a complex structure $\mathcal{J}$ on $\text{Gr}(\mathcal{H})$ which restricts on each tangent space to the map $\dot{F} \mapsto -iF\dot{F}$ and together with the hermitian metric g which is given by $g(\dot{F}_1, \dot{F}_2)_F := 1/4 \text{Tr } \dot{F}_1 \dot{F}_2$ we can define the Kaehler form Ω according to [7] by $\Omega(\dot{F}_1, \dot{F}_2)_F := g(\dot{F}_1, \mathcal{J}\dot{F}_2)_F = -i/4 \text{Tr } \dot{F}_1 F \dot{F}_2$. Evaluation of Ω on the fundamental fields (2) leads to

$$\Omega(\zeta_X, \zeta_Y)_F = \frac{i}{4} \text{Tr } F[F, X][F, Y]. \quad (3)$$

Since $l_{U*}\zeta_X = \zeta_{UXU^{-1}}$ we deduce from (3) that $l_U^*\Omega = \Omega$, and therefore Ω is left invariant. Moreover it is straightforward to check $d\Omega = 0$ on fundamental fields by using $[\zeta_X, \zeta_Y] = \zeta_{[X, Y]}$ together with (3), which implies that $\text{Gr}(\mathcal{H})$ is a Kaehler manifold. In what follows we restrict our attention to $\text{Gr}_0(\mathcal{H})$ and consider $(\text{Gr}_0(\mathcal{H}), \Omega)$ as a symplectic manifold with a symplectic left action action $l : U_{\text{res},0}(\mathcal{H}) \times \text{Gr}_0(\mathcal{H}) \rightarrow \text{Gr}_0(\mathcal{H})$.

In order to relate the infinite dimensional geometry of $\text{Gr}(\mathcal{H})$ to the algebraic procedure of quasifree fermionic second quantization we recall [2] that to the classical relativistic phase space $\mathcal{H}$ there is associated a unique unital C^* algebra $\text{CAR}(\mathcal{H})$ which is generated as a C^* algebra by the range of an antilinear map $f \mapsto a(f)$ such that $a(f)^*a(g) + a(g)a(f)^* = \langle g, f \rangle \mathbf{1}$ and $a(f)a(g) + a(g)a(f) = 0$ for all $f, g \in \mathcal{H}$. The unitary group and in particular the restricted unitary group act on $\text{CAR}(\mathcal{H})$ by Bogoliubov automorphisms $a(f) \mapsto a(Uf)$. Since every $F \in \text{Gr}(\mathcal{H})$ determines via

$$\omega_F(a(f_n)^* \dots a(f_1)^* a(g_1) \dots a(g_m)) := \delta_{mn} \det(\langle g_i, \frac{1}{2}(\mathbf{1} - F)f_j \rangle) \quad (4)$$

a unique quasifree gauge invariant Fock state ω_F over $\text{CAR}(\mathcal{H})$, $\text{Gr}(\mathcal{H})$ may be identified with a specific part of the state space of charged relativistic fermions. By construction

²We adapt here the kinematical definition of tangent space.

$\|F_1 - F_2\|_{\text{HS}} < \infty$ for any two points F_1 and F_2 in $\text{Gr}(\mathcal{H})$ and due to [8] The associated GNS representations are mutually unitarily equivalent.

Let us denote by $\mathcal{F}_0, \pi_{F_0}$ and $|F_0\rangle$ the Hilbert space, the representation space and the cyclic unit vector associated with ω_{F_0} via the GNS construction. Then according to [1] there is a linear map $u_{\text{res}}(\mathcal{H}) \ni X \mapsto Q_0(X)$ where $Q_0(X)$ is an essentially selfadjoint operator on a common invariant dense domain $\mathcal{D} \subset \mathcal{F}_0$ which contains $|F_0\rangle$ such that $[Q_0(X), \pi_{F_0}(a(f))^*] = \pi_{F_0}(a(Xf))^*$ and in addition

$$i[Q_0(X), Q_0(Y)] = Q_0([X, Y]) + \left(\frac{i}{4} \text{Tr } F_0[F_0, X][F_0, Y] \right) \mathbf{1} \quad (5)$$

are valid on $\mathcal{D}$. In the algebraic treatment the Schwinger term on the right hand side of (5) has its origin in the normal ordering prescription $\langle F_0 | Q_0(X) | F_0 \rangle = 0$ of the fermionic charges $Q_0(X)$. It is well known that this term represents a non trivial Lie algebra two cocycle on $u_{\text{res}}(\mathcal{H})$ which gives rise to the non split central extension

$$0 \rightarrow \mathbb{R} \rightarrow \tilde{u}_{\text{res}}(\mathcal{H}) \rightarrow u_{\text{res}}(\mathcal{H}) \rightarrow 0 \quad (6)$$

of $u_{\text{res}}(\mathcal{H})$ by $\mathbb{R}$ [3],[5]. Here $\tilde{u}_{\text{res}}(\mathcal{H}) \simeq u_{\text{res}}(\mathcal{H}) \oplus \mathbb{R}$ as a Banach Lie algebra with bracket $[(X, \lambda), (Y, \mu)] = (XY - YX, i/4 \text{Tr } F_0[F_0, X][F_0, Y])$. In the physical language $\tilde{u}_{\text{res}}(\mathcal{H})$ is the algebra of fermionic charges (Current algebra). Comparing (5) with (3) shows that the above Schwinger term can equally well be obtained by evaluating the symplectic form over F_0 on the fundamental fields of the symplectic $U_{\text{res}}(\mathcal{H})$ action on $\text{Gr}(\mathcal{H})$. This point of view will be exploited systematically in the following paragraph. We shall show that this term not only arises by a simple normal ordering prescription, but has its origin in the exotic symplectic left action l .

3. The momentum map and the co-adjoint one cocycle.

In this chapter we refer to the notation in [4]. Since $\pi_1(\text{Gr}_0(\mathcal{H}), F_0) = 0$ we have also in infinite dimensions $H^1(\text{Gr}_0(\mathcal{H}), \mathbb{R}) = 0$ and therefore each locally hamiltonian vector field is hamiltonian. So we might expect to find the momentum mapping J of the symplectic left action (1). Let $\text{HS}(\mathcal{H})$ denote the Hilbert space of Hilbert Schmidt operators in $\mathfrak{B}(\mathcal{H})$. Then for $X \in \text{HS}(\mathcal{H}) \cap u_{\text{res}}(\mathcal{H})$ the explicit expression for the generating function $\hat{J}(X)$ is given by

$$\hat{J}(X)(F) := -\frac{i}{2} \text{Tr}(F - F_0)X, \quad (7)$$

[3] and it is fairly easy to check that in this case $d\hat{J}(X)(\zeta_Y)_F = -\frac{i}{2} \text{Tr}[Y, F]X = \frac{i}{4} \text{Tr } F[F, X] \times [F, Y] = \iota_{\zeta_X} \Omega(\zeta_Y)_F$. However (7) cannot be true for all $X \in u_{\text{res}}(\mathcal{H})$. Take for instance $X = i\mathbf{1}$ which generates the global $U(1)$ gauge transformations $\mathcal{H} \ni f \mapsto e^{i\theta} f$, then $-i/2 \text{Tr}(F - F_0)(i\mathbf{1}) = 1/2 \text{Tr}(F - F_0)$ which well defined and finite only for $\text{Tr } |F - F_0| < \infty$. But in general $F - F_0$ is Hilbert Schmidt, and so (7) has to be modified. This can be done in the following manner

Proposition 1. Let $X \in \mathfrak{u}_{\text{res}}(\mathcal{H})$. Then the smooth function $\hat{J}(X) : \text{Gr}_0(\mathcal{H}) \rightarrow \mathbb{R}$ defined by

$$\hat{J}(X)(F) := -i \text{Tr } P_W P_{W_0}^\perp X P_W + i \text{Tr } P_W^\perp P_{W_0} X P_W^\perp$$

exists for all $X \in \mathfrak{u}_{\text{res}}(\mathcal{H})$ and satisfies $d\hat{J}(X) = \iota_{\zeta_X} \Omega$, where $F = P_W - P_{W_0}^\perp$. Moreover since the linear map $X \rightarrow \hat{J}(X)(F)$ is bounded in X for fixed F , the assignment $X \mapsto \hat{J}(X)$ determines the momentum mapping

$$J : \text{Gr}_0(\mathcal{H}) \rightarrow \mathfrak{u}_{\text{res}}(\mathcal{H})^*$$

of the symplectic $U_{\text{res},0}(\mathcal{H})$ action via $J(F) \cdot X := \hat{J}(X)(F)$.

Proof. First we note that $P_W P_{W_0}^\perp$ and $P_W^\perp P_{W_0}$ are Hilbert Schmidt by construction of the Grassmannian and since $[X, F_0]$ and $[X, F] = [X, F_0] + [X, F - F_0]$ are Hilbert Schmidt, $P_{W_0}^\perp X P_{W_0}$ and $P_W X P_W^\perp$ must be Hilbert Schmidt. Therefore the operators $P_W P_{W_0}^\perp X P_W = (P_W P_{W_0}^\perp)(P_{W_0}^\perp X P_{W_0})P_W + (P_W P_{W_0}^\perp)X(P_W P_{W_0}^\perp)^*$ and $P_W^\perp P_{W_0} X P_W^\perp = (P_W^\perp P_{W_0}) \times (P_{W_0} X P_{W_0}^\perp)P_W^\perp + (P_W^\perp P_{W_0})X(P_W^\perp P_{W_0})^*$ are traceclass. So $\hat{J}(X)$ exists for all $X \in \mathfrak{u}_{\text{res}}(\mathcal{H})$. Moreover

$$\begin{aligned} d\hat{J}(X)(\zeta_Y)_F &= i \text{Tr } P_{W_0}^\perp X P_W [Y, P_W^\perp] + i \text{Tr } [Y, P_W^\perp] P_W P_{W_0}^\perp X \\ &\quad - i \text{Tr } P_{W_0} X P_W^\perp [Y, P_W] - i \text{Tr } [Y, P_W] P_W^\perp P_{W_0} X \\ &= i \text{Tr } P_{W_0}^\perp X P_W Y P_W^\perp - i \text{Tr } P_W^\perp Y P_W P_{W_0}^\perp X \\ &\quad - i \text{Tr } P_{W_0} X P_W^\perp Y P_W + i \text{Tr } P_W Y P_W^\perp P_{W_0} X \\ &= i \text{Tr } P_W Y P_W^\perp P_{W_0}^\perp X P_W - i \text{Tr } P_W^\perp Y P_W P_W P_{W_0}^\perp X P_W^\perp \\ &\quad - i \text{Tr } P_W^\perp Y P_W P_{W_0} X P_W^\perp + i \text{Tr } P_W Y P_W^\perp P_{W_0} X P_W \\ &= i \text{Tr } P_W Y P_W^\perp (P_{W_0}^\perp X P_W + P_{W_0} X P_W) \\ &\quad - i \text{Tr } P_W^\perp Y P_W (P_{W_0}^\perp X P_W^\perp + P_{W_0} X P_W^\perp) \\ &= i/4 \text{Tr } F[F, X][F, Y] \\ &= \iota_{\zeta_X} \Omega(\zeta_Y)_F, \end{aligned}$$

where we have used $[Y, P_W] = -[Y, P_W^\perp]$ together with $\text{Tr } AB = \text{Tr } BA$ for A, B Hilbert Schmidt. Finally

$$\begin{aligned} |\text{Tr } P_W P_{W_0}^\perp X P_W| &\leq |\text{Tr}(P_W P_{W_0}^\perp)(P_{W_0}^\perp X P_{W_0})P_W| + |\text{Tr}(P_W P_{W_0}^\perp)X(P_W P_{W_0}^\perp)^*| \\ &\leq \max\left\{\frac{1}{2}\|P_W P_{W_0}^\perp\|_{\text{HS}}, \|P_W P_{W_0}^\perp\|_{\text{HS}}^2\right\}(\|X\| + \|[F_0, X]\|_{\text{HS}}) \end{aligned}$$

which implies that $X \mapsto -i \text{Tr } P_W P_{W_0}^\perp X P_W$ is a bounded linear functional on $\mathfrak{u}_{\text{res}}(\mathcal{H})$. Together with a similar estimate for $i \text{Tr } P_W^\perp P_{W_0} X P_W^\perp$ we have $\hat{J}(\cdot)(F) \in \mathfrak{u}_{\text{res}}(\mathcal{H})^*$ for each fixed F . $\square$

Remark. For $X \in \mathfrak{u}_{\text{res}}(\mathcal{H}) \cap \text{HS}(\mathcal{H})$ we have $\text{Tr } P_W P_{W_0}^\perp X P_W = \text{Tr}(X P_W)(P_W P_{W_0}^\perp) = \text{Tr } X P_W P_{W_0}^\perp = \text{Tr } P_W P_{W_0}^\perp X$ and therefore we find $\hat{J}(X)(F) = -i \text{Tr } P_W P_{W_0}^\perp X P_W$

$+ i \operatorname{Tr} P_W^\perp P_{W_0} X P_W^\perp = -i \operatorname{Tr} P_W P_{W_0}^\perp X + i \operatorname{Tr} P_W^\perp P_{W_0} X = -i/2 \operatorname{Tr}(F - F_0)X$, which coincides with 7. In general neither X nor $X P_W$ are Hilbert Schmidt and the above manipulations under the trace are no more valid. So the expression for $\hat{J}(X)$ in proposition 1 extends (7) to all of $\mathfrak{u}_{\text{res}}(\mathcal{H})$.

Given the momentum mapping J we may compute the associated co-adjoint cocycle $\sigma : \mathfrak{U}_{\text{res},0}(\mathcal{H}) \rightarrow \mathfrak{u}_{\text{res}}(\mathcal{H})^*$ which is defined by $\sigma(U) \cdot X := \text{constant value of } ((J \circ l_U) \cdot X - \operatorname{Ad}_{U^{-1}}^* J \cdot X) \text{ on } \operatorname{Gr}_0(\mathcal{H})$. By definition σ measures the non Ad^* equivariance of the momentum mapping and furthermore determines a class $[\sigma]$ in $H^2(\mathfrak{U}_{\text{res},0}(\mathcal{H}), \mathfrak{u}_{\text{res}}(\mathcal{H})^*)$ which only depends on the symplectic action and not on the specific momentum mapping. We find

Proposition 2. *The coadjoint cocycle $\sigma : \mathfrak{U}_{\text{res},0}(\mathcal{H}) \rightarrow \mathfrak{u}_{\text{res}}(\mathcal{H})^*$ of the momentum map J is given by*

$$\sigma(U) \cdot X = \frac{i}{4} \operatorname{Tr} F_0 [F_0, U^{-1}] [F_0, XU].$$

It determines a non trivial class in $H^1(\mathfrak{U}_{\text{res},0}(\mathcal{H}), \mathfrak{u}_{\text{res}}(\mathcal{H})^*)$. Therefore J cannot be redefined to be Ad^* equivariant.

Proof. Since $J \circ l_U - \operatorname{Ad}_{U^{-1}}^* J$ is constant on $\operatorname{Gr}_0(\mathcal{H})$ we may take its evaluation at F_0 to obtain $\sigma(U) \cdot X$. Hence

$$\begin{aligned} \sigma(U) \cdot X &= J(U F_0 U^{-1}) \cdot X - J(F_0) \cdot U^{-1} X U \\ &= -i \operatorname{Tr} P_{W_0} [U^{-1}, P_{W_0}^\perp] X U P_{W_0} + i \operatorname{Tr} P_{W_0}^\perp [U^{-1}, P_{W_0}] X U P_{W_0}^\perp \\ &= -i \operatorname{Tr} P_{W_0} U^{-1} P_{W_0}^\perp X U P_{W_0} + i \operatorname{Tr} P_{W_0}^\perp U^{-1} P_{W_0} X U P_{W_0}^\perp \\ &= \frac{i}{4} \operatorname{Tr} F_0 [F_0, U^{-1}] [F_0, XU]. \end{aligned}$$

We defer the proof of non triviality for the moment. $\square$

Remark. Note that both $[F_0, U^{-1}]$ and $[F_0, XU] = X[F_0, U] + [F_0, X]U$ are in the Hilbert Schmidt class. Using commutativity of Hilbert Schmidt operators under the trace σ can be rewritten as $\sigma(U) \cdot X = i/4 \operatorname{Tr}(F_0 X l_U(F_0) - X)(l_U(F_0) - F_0)$. For $X \in \operatorname{HS}(\mathcal{H}) \cap \mathfrak{u}_{\text{res}}(\mathcal{H})$ this simplifies to $-i/2 \operatorname{Tr}(l_U(F_0) - F_0)X$ and the latter expression is what we would have obtained for σ by using (7). It is obvious that the latter expression cannot hold on all of $\mathfrak{u}_{\text{res}}(\mathcal{H})$.

4. Comomentum map and Schwinger term.

The differential of the co-adjoint cocycle σ at the identity yields a two cocycle on $\mathfrak{u}_{\text{res}}(\mathcal{H})$. Denoting the latter by Σ_σ we have $\Sigma_\sigma(X, Y) := \left. \frac{d}{dt} \right|_{t=0} \sigma(e^{tX}) \cdot Y$, and this cocycle appears as a Schwinger term in the Poisson brackets of the generating functions [4]:

$$\{\hat{J}(X), \hat{J}(Y)\} - \hat{J}([X, Y]) = -\Sigma_\sigma(X, Y). \quad (8)$$

Moreover the linear map $\sigma \mapsto \Sigma_\sigma$ induces a homomorphism $H^1(U_{\text{res},0}(\mathcal{H}), u_{\text{res}}(\mathcal{H})^*) \rightarrow H^2(u_{\text{res}}(\mathcal{H}), \mathbb{R})$ of abelian groups, since coboundaries are mapped into coboundaries. Hence $[\Sigma_\sigma] \neq 0 \Rightarrow [\sigma] \neq 0$ and we have non triviality of $[\sigma]$ provided Σ_σ is not a coboundary. But $\{\hat{J}(X), \hat{J}(Y)\} - \hat{J}([X, Y]) = \text{constant}$ on $\text{Gr}_0(\mathcal{H})$ together with $\hat{J}(Z)(F_0) = 0$ for all $Z \in u_{\text{res}}(\mathcal{H})$ imply

$$\begin{aligned} -\Sigma_\sigma(X, Y) &= \{\hat{J}(X), \hat{J}(Y)\}(F_0) - \hat{J}([X, Y])(F_0) \\ &= \{\hat{J}(X), \hat{J}(Y)\}(F_0) \\ &= \Omega(\zeta_X, \zeta_Y)_{F_0} \\ &= \frac{i}{4} \text{Tr } F_0[F_0, X][F_0, Y]. \end{aligned} \tag{9}$$

Since the Schwinger term $i/4 \text{Tr } F_0[F_0, X][F_0, Y]$ is known to represent a non trivial two cocycle on $u_{\text{res}}(\mathcal{H})$ we have $[\sigma] \neq 0$ and thus the proof of proposition 1 is complete.

A consequence of (9) is that the generating functions $\hat{J}(X)$ do not close under Poisson bracket. That is: The comomentum map is not a homomorphism of Lie algebras and due to $[\Sigma_\sigma] \neq 0$ cannot even be redefined to be one. However by adding the constant functions the smallest subalgebra in $(C^\infty(\text{Gr}_0(\mathcal{H}), \mathbb{R}), \{\cdot, \cdot\})$ which contains all $\hat{J}(X)$ is isomorphic to $\tilde{u}_{\text{res}}(\mathcal{H})$, where the isomorphism is given by $(X, \lambda) \mapsto \hat{J}(X) + \lambda$. We thus have recovered the algebra of fermionic charges in an entirely classical context as a subalgebra of smooth functions on the Grassmannian.

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