Gauge-Symmetry Breakdown at the Horizon of Extreme Black Holes

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November 1997

Abstract

Static solutions of the Einstein-Yang-Mills-Higgs system contain-
ing extreme black holes are studied. The field equations imply strong restrictions on boundary values of all fields at the horizon. If the Yang-Mills radial electric field $E$ is non-zero there, then all fields at the horizon take values in the centralizer of $E$. For the particular case of SU(3), there are two different kinds of centralizers: two-dimensional abelian (Cartan subalgebra) and four-dimensional ($\text{su}(2) \times \text{u}(1)$) ones. The two-dimensional centralizer admits only constant fields: even the geometry of the horizon is that of constant curvature. If the cosmological constant $\Lambda$ is negative, a two-surface of any genus is possible; for positive curvature, only spherically symmetrical horizons are allowed. For the four-dimensional centralizer, all spherically symmetrical horizons are explicitly given.

Finally, some complete spacetime solutions are constructed whose horizons have the structure found by our method. There are also abelian solutions of a new type. In some cases there are different spacetimes having the same type of horizon.
1 Introduction and Summary

Classical solutions to Einstein-Yang-Mills equations have attracted much attention recently. Spherically symmetrical nonabelian solutions (so-called “solitons”) have been found ([1]) and studied ([2, 3]). It is remarkable that gauge fields and gravitational singularities can cancel and finite-energy, regular solutions exist. It is hoped that such non-perturbative effects may have some consequences on quantum level as well. Similar (but singular) solutions that contain event or Killing horizons (“black holes”) also exist ([4, 5, 6]). For reviews, see [7] and most recently [8].

The existence of such solutions reveals an unexpected richness of the Einstein-Yang-Mills system. In particular, the non-abelian black holes represent counterexamples to the “no hair conjecture”. In addition, there are speculations about the role that these solutions could play in microphysics ([9, 10]).

There has been much interest in the extreme Reissner-Nordström black holes within the standard Einstein-Maxwell theory. They admit surprisingly simple solutions of the perturbation equations [11]; some of them seem to be stable with respect to both classical and quantum processes and there are attempts to interpret them as solitons ([7, 12]); also, they admit supersymmetry ([13]). Very recently extreme Reissner-Nordström black holes were discussed in the context of the Einstein-Maxwell theory with a cosmological constant $\Lambda$ [14]. In particular, it is possible to analyze collisions of black holes analytically and study the cosmic censorship hypothesis by considering charged black holes with $\Lambda \neq 0$ ([15, 16]).

The geometry of extreme black hole spacetimes exhibits some character-
istic features. There is no Einstein-Rosen bridge (or “wormhole”) joining two asymptotically flat regions and containing a minimal 2-sphere. Instead, there is one asymptotically flat and one asymptotically cylindrical region on each $t = \text{const.}$ hypersurface. We call the boundary of the cylindrical region “internal infinity”. Such an internal infinity is a compact two-dimensional spacelike surface. The hypersurfaces $t = \text{const.}$ do not intersect the horizon, but only approach such an intersection at the internal infinity.

These properties yield a tool for a systematic study of extreme black holes ([17, 18]). As the space approaches a cylinder at an internal infinity, all radial derivatives of smooth fields must vanish there, and the field equations degenerate in the limit to a system of some algebraical, some first-order and some second-order differential equations on a compact two-dimensional surface; the second order equations are elliptic. Such a system has few solutions. For example, there are only spherically symmetrical solutions in most cases [19].

In the present paper, we start an investigation of the structure of internal infinity for the Einstein-Yang-Mills-Higgs system with the gauge group SU(3). We choose the Higgs field with values in the adjoint representation of the group and with the usual simple biquadratic potential. Other potentials (see, e.g. [20]) could be analyzed similarly and will lead to analogous results.

The plan of the paper is as follows.

In Section 2, we write down the Lagrangian, explain the character of the limit to the internal infinity and write down the limiting system of equations. In Section 3, we observe that a non-zero “electric” charge $q$ of the black hole leads to a kind of “symmetry breaking”: the values of all fields must lie in
the centralizer of $q$. All gauge non-equivalent charges lie in a sector of the two-dimensional Cartan subalgebra. The direction at the boundary of this sector has a four-dimensional non-abelian centralizer corresponding to the subgroup $U(1) \times SU(2)$ of $SU(3)$. For all other directions in the sector, the centralizers coincide with the two-dimensional Cartan subalgebra itself, so the solution is necessarily abelian.

In Section 4, we find all solutions for the two-dimensional centralizers. There are only abelian Higgs vacuum (true or false) fields with maximal symmetry. For negative values of the cosmological constant there are internal infinities of all orientable topologies (arbitrary genus) as the curvature scalar can be negative. If the curvature is positive, a spherically symmetrical solution results.

We also find some complete spacetime solutions with these boundaries. The corresponding geometry is the Reissner-Nordström spacetime. The gauge field is characterized by four gauge invariant charges (two electric and two magnetic). We observe that the known Reissner-Nordström “embeddings” ([21, 22]) have at most two charges (electric and magnetic).

In Section 5, we study the spherically symmetrical solutions on the four-dimensional centralizer. As $SU(3)$ is broken to $U(1) \times SU(2)$, we can use the well-known $SU(2)$ ansatz. We find two classes of solutions. 1) The Higgs field has its true vacuum value and its component into the “hypercharge” ($T_8$) direction does not vanish. This leads to the Reissner-Nordström spacetime with a magnetic $SU(2)$-charge as well as an electric and magnetic “hypercharge”. 2) Only the $SU(2)$-part of the Higgs field is non-zero, but smaller than the true vacuum value. We show that these are the internal infinities
either to the countable family of extreme black holes described in [23] or to those found in [24].

In the Appendix it is indicated how the existence of extreme black hole solutions leads to restrictions of the ranges of the free parameters in the Lagrangian of the model.

In this way all static “electric” solutions (non-zero electric charge) are likely to be found: it is very plausible that static solutions have maximal symmetry and higher genus may lead to only asymptotically locally anti-de Sitter spacetimes. Moreover, Reissner-Nordström-type solutions are expected to be stable, in contrast to more complicated solutions. Further study is necessary especially for “magnetic” solutions (zero electric charge).

Our conventions: the metric signature is +2, the Riemann tensor $R^\mu_{\nu\rho\sigma}$ is defined by

$$R^\mu_{\nu\rho\sigma} = \Gamma^\mu_{\nu\sigma,\rho} - \Gamma^\mu_{\nu\rho,\sigma} + \ldots,$$

and the Ricci tensor $R_{\mu\nu}$ is

$$R_{\mu\nu} = R^\rho_{\mu\rho\nu}.$$

The units are chosen such that $\hbar = c = 1$.

## 2 The Model

### 2.1 The Lagrangian

We consider a four-dimensional manifold $\mathcal{M}$ and a system of fields consisting of a spacetime metric $g_{\mu\nu}$, a Yang-Mills field $W_\mu$ and a Higgs scalar field $Q$;
\( W_\mu \) and \( Q \) have their values in the Lie algebra \( l\mathcal{G} \) of a simple, matrix, compact Lie group \( \mathcal{G} \).

The Lagrangian is given by

\[
L = L_G + L_M, \quad (1)
\]

where

\[
L_G = -\frac{1}{16\pi\gamma^2} \int d^3 x \sqrt{-4g} (4R - 2\Lambda),
\]

\[
L_M = \frac{1}{4\pi} \int d^3 x \sqrt{-4g} \left[ \frac{1}{4e^2} (G_{\mu\nu}, G^{\mu\nu}) + \frac{1}{2} (D_\mu Q, D^\mu Q) + V(Q) \right],
\]

\( \gamma^2 \) is the Newton constant (\( \gamma \) is the Planck length),

\[
4g = \det(g_{\mu\nu}),
\]

\( 4R \) is the curvature scalar of \( g_{\mu\nu} \), \( \Lambda \) is the cosmological constant, \( e \) is the coupling constant of the Yang-Mills field,

\[
G_{\mu\nu} = \partial_\mu W_\nu - \partial_\nu W_\mu - [W_\mu, W_\nu]
\]

is the gauge field strength, \( (\cdot, \cdot) \) is an invariant positive definite quadratic form on \( l\mathcal{G} \)

\[
(\cdot, \cdot) = -\frac{1}{\nu} (\cdot, \cdot)_K,
\]

proportional to the Killing form \( (\cdot, \cdot)_K \) in such a way that most of the structure constants of an orthonormal basis are small integers,

\[
D_\mu Q = \partial_\mu Q - [W_\mu, Q]
\]

is the covariant derivative defined by the gauge field,

\[
V(Q) = \frac{1}{8} k [(Q, Q) - F^2]^2
\]
is the Higgs potential and \( k \) and \( F \) are positive constants. Under a gauge transformation \( U(x), x \in \mathcal{M}, U \in \mathcal{G} \), the fields transform as follows:

\[
W'_\mu = U^{-1}W_\mu U - U^{-1}\partial_\mu U,
\]

\[
Q' = U^{-1}QU.
\]

The field equations resulting from the Lagrangian (1) read

\[
R_{\mu\nu} - \frac{1}{4}Rg_{\mu\nu} = 8\pi\gamma^2\tilde{T}_{\mu\nu},
\]

(2)

\[
(-4g)^{-1/2}D_\nu \left[ (-4g)^{1/2}G^\mu\nu \right] + e^2[D^\mu Q, Q] = 0,
\]

(3)

\[
(-4g)^{-1/2}D_\nu \left[ (-4g)^{1/2}D^\nu Q \right] - \frac{k}{2} \left[ (Q, Q) - F^2 \right] Q = 0,
\]

(4)

where

\[
\tilde{T}_{\mu\nu} = T_{\mu\nu} - \frac{\Lambda}{8\pi\gamma^2}g_{\mu\nu}
\]

(5)

and

\[
T^{\mu\nu} = \frac{1}{4\pi\epsilon^2} \left[ (G^{\mu\rho}, G^\rho_\nu) - \frac{1}{4}g^{\mu\nu} (G^{\rho\sigma}, G_{\rho\sigma}) \right] + \frac{1}{4\pi} \left[ (D^\mu Q, D^\nu Q) - \frac{1}{2}g^{\mu\nu} (D_\rho Q, D^\rho Q) - g^{\mu\nu}V(Q) \right].
\]

(6)

2.2 The Ansatz for the Solution

We will look for solutions \((\mathcal{M}, g, W, Q)\) to eqs. (2)–(6) describing a static extremal black hole. Thus, we will assume the following properties.

1. \((\mathcal{M}, g, W, Q)\) is static: there is a timelike vector field \( \xi^\mu(x) \) with the norm \( N(x) \),

\[
N^2 = -g_{\mu\nu}\xi^\mu\xi^\nu,
\]
and a map $X : \mathcal{M} \to l\mathcal{G}$ such that

$$
\mathcal{L}_\xi g_{\mu\nu} = 0
$$

$$
\mathcal{L}_\xi W_\mu = [W_\mu, X] - \partial_\mu X,
$$

$$
\mathcal{L}_\xi Q = [Q, X],
$$

where $\mathcal{L}_\xi$ is the Lie derivative with respect to $\xi^\mu(x)$.

2. Let $\Sigma$ be an inextendable $\xi$-orthogonal hypersurface in $(\mathcal{M}, g)$. Then $\Sigma$ is a Cauchy hypersurface for $(\mathcal{M}, g)$. Further, $\Sigma$ is complete with respect to the distance function associated with the positive definite metric induced on $\Sigma$ by $g_{\mu\nu}$.

3. The function $N(x)$ on $\Sigma$ is positive,

$$
N(x) > 0 \quad \forall x \in \Sigma
$$

not bounded below away from zero. There is $\epsilon > 0$ such that each component $\mathcal{K}$ of the set

$$
N(x) < \epsilon
$$

in $\Sigma$ satisfies the conditions

(i) $N(x)$ has no critical points in $\mathcal{K}$,

(ii) the surfaces $N(x) = \text{const.}$ are compact.

Thus $\mathcal{K}$ is diffeomorphic to $\mathbb{R} \times \mathcal{H}$, where $\mathcal{H}$ is a compact two-dimensional manifold and there are coordinates $N, x^2, x^3$ in $\mathcal{K}$ such that the metric has the form

$$
ds^2 = -N^2 dt^2 + \rho^2 dN^2 + g_{AB} dx^A dx^B,
$$

(7)

$A, B = 2, 3,$ and $\rho$ and $g_{AB}$ depend only on $N$ and $x^A$. 

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Each component \( K \) contains an internal infinity as \( N(x) \to 0 \). As \( \Sigma \) is complete, the distance of any point of \( \Sigma \) to the infinity must diverge. Thus

\[
\int_0^a \rho dN = \infty,
\]

if \( a < \epsilon \), and we must have

\[
\lim_{N \to 0} \rho^{-1} = 0.
\]

4. Let us choose a particular gauge in each \( K \):

(i) The gauge frame is parallel along the \( N \)-curves in \( \Sigma \):

\[ W_1 = 0. \]

(ii) The gauge frame is propagated by the static group along the orbits of \( \xi^\mu \):

\[ \partial_0 W_\mu = \partial_0 Q = 0. \]

Then, the functions \( \rho, g_{AB}, W_0, W_A \) and \( Q \) of the variables \( N, x^2, x^3 \) are continuous in \( K \) together with all their derivatives up to order 2. The function

\[
(N\rho)^{-1} \partial_1 W_0
\]

is continuous together with its all first order derivatives. The limits \( N \to 0 \) of all these functions and derivatives exist and are smooth functions of \( x^2 \) and \( x^3 \).

This condition of regularity of the fields at the internal infinity is an essential part of the ansatz. It is a technical point: physically, one can only justify the regularity at all points of the spacetime, in particular of the
horizon. Some examples show that the regularity at the horizon does not imply the regularity at the internal infinity ([13]).

The limits $N \rightarrow 0$ of the fields $g_{AB}, W_0, W_A$ and $Q$ form fields on the manifold $\mathcal{H}$. Eqs. (2)-(6) imply a system of equations for these fields on $\mathcal{H}$ that has been derived [17]. In the present special case these equations take the following form (we omit the limit sign):

$$\frac{1}{(N\rho)^2} = \gamma^2 \left[ \frac{(E,E)}{e^2} + \frac{(B,B)}{e^2} - 2V - \frac{\Lambda}{\gamma^2} \right],$$

$$R = 2\gamma^2 \left[ \frac{(E,E)}{e^2} + \frac{(B,B)}{e^2} + g^{AB}(D_AQ,D_BQ) + 2V + \frac{\Lambda}{\gamma^2} \right],$$

$$\frac{(E,E)}{e^2} + \frac{(B,B)}{e^2} - 2V = \text{const.},$$

$$(D_AQ,D_BQ) = \frac{1}{2}g_{AB}g^{KL}(D_KQ,D_LQ),$$

$$D_AE = 0,$$

$$[E,Q] = 0,$$

$$\frac{\epsilon^{AB}}{\sqrt{g}}D_BB + g^{AB}e^2[Q,D_BQ] = 0,$$

$$\frac{1}{\sqrt{g}}D_A(\sqrt{g}g^{AB}D_BQ) - \frac{\partial V}{\partial Q} = 0$$

where $E$ and $B$, given by

$$G_{10} = N\rho E, \ G_{AB} = \epsilon_{AB}\sqrt{g}B$$

are radial electric and magnetic fields on $\mathcal{H}$, $R$ is the curvature scalar of $g_{AB}$,

$$g = \det(g_{AB}),$$

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\[ \epsilon_{AB} \text{ and } \epsilon^{AB} \text{ are the antisymmetric densities, and the internal index at } \partial V/\partial Q \text{ is raised by the metric } (\cdot, \cdot). \]

The structure of eqs. (8)–(15) leads naturally to two different sorts of solutions: those with \( E \neq 0 \) at least at one point of \( \mathcal{H} \), and those with \( E = 0 \) everywhere on \( \mathcal{H} \). The former solutions are called “electric”, the latter “magnetic”. The “symmetry breaking” is a property of the electric solutions.

3 Electric Solutions

3.1 The Role of Centralizers

Suppose that there is a point \( p \in \mathcal{H} \) such that \( E(p) \neq 0 \). Then eq. (12) implies that there is a gauge along \( \mathcal{H} \) (consisting possibly of more patches) such that \( E \) is a fixed constant element of \( l\mathcal{G} \) everywhere along \( \mathcal{H} \) (equal to \( E(p) \)). Let us choose this gauge. Then,

\[ \partial_A E = 0, \]

and eq. (12) implies

\[ [W_A, E] = 0. \tag{16} \]

It follows, in particular, that

\[ [B, E] = 0. \tag{17} \]

Let us denote by \( Z_X \) the centralizer of the element \( X \) in the Lie algebra \( l\mathcal{G} \):

\[ Z_X = \{ Y \in l\mathcal{G} \mid [Y, X] = 0 \}. \]
Then, eqs. (16), (17) and (13) imply the following proposition.

**Proposition:** For the above choice of gauge along $\mathcal{H}$, the values of all fields $W_A, E, B$ and $Q$ lie in $Z_E$.

Thus, centralizers will play a crucial role for the electric solutions: the gauge group will be “broken” to a subgroup which is generated by a centralizer.

### 3.2 General Properties of Centralizers

Let us collect those properties of centralizers that hold generally for simple groups $G$ and that will be relevant to our problem. (They might be already given in the literature; we were not able to find a suitable reference. However, the proofs are short and a mere list of the properties, which would be necessary in any case, is not much shorter than our Sections 3.2 and 3.3.)

Directly from the definition, we have

$$aX \in Z_X \quad \forall a \in \mathbb{R},$$

(so $Z_X$ is always at least one-dimensional), and

$$Z_{aX} = Z_X \quad \forall a \in \mathbb{R}.$$

Thus, it is sufficient to look for normalized elements of $kG$:

$$(X, X) = 1.$$

Then the elements $X$ and $-X$ of this Killing sphere $S$ have the same centralizer.
Suppose that

\[ Y = ad(g)X, \quad g \in \mathcal{G}. \]

Then,

\[ Z_Y = ad(g)Z_X. \]

This follows immediately from the relation:

\[ [ad(g)U, ad(g)V] = ad(g)[U, V]. \]

The adjoint representation of \( \mathcal{G} \) on \( \mathfrak{g} \) leaves the Killing sphere \( S \) invariant. Centralizers of two elements of the same orbit of \( \mathcal{G} \) in \( S \) can be mapped on each other by \( ad(g) \) for some \( g \in \mathcal{G} \); the corresponding solutions of eqs. (8)-(15) will, therefore, be gauge related. However, elements from different orbits will give gauge non-equivalent solutions. Thus, it is sufficient for our purposes to study only some typical representatives of the orbits. Let us show how this can be done for \( SU(3) \).

### 3.3 The Case of SU(3)

Let us choose the following basis for the Lie algebra \( su(3) \) (Gell-Mann matrices times “i”):

\[
T_1 = \begin{pmatrix}
0 & i & 0 \\
 i & 0 & 0 \\
0 & 0 & 0
\end{pmatrix}, 
T_2 = \begin{pmatrix}
0 & 1 & 0 \\
-1 & 0 & 0 \\
0 & 0 & 0
\end{pmatrix},
\]

\[
T_3 = \begin{pmatrix}
i & 0 & 0 \\
0 & -i & 0 \\
0 & 0 & 0
\end{pmatrix}, 
T_4 = \begin{pmatrix}
0 & 0 & i \\
0 & 0 & 0 \\
i & 0 & 0
\end{pmatrix},
\]

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\[
T_5 = \begin{pmatrix}
0 & 0 & 1 \\
0 & 0 & 0 \\
-1 & 0 & 0
\end{pmatrix}, \quad T_6 = \begin{pmatrix}
0 & 0 & 0 \\
0 & 0 & i \\
0 & i & 0
\end{pmatrix}, \\
T_7 = \begin{pmatrix}
0 & 0 & 0 \\
0 & 0 & 1 \\
0 & -1 & 0
\end{pmatrix}, \quad T_8 = \frac{1}{\sqrt{3}} \begin{pmatrix}
i & 0 & 0 \\
0 & i & 0 \\
0 & 0 & -2i
\end{pmatrix},
\]

In this basis, the form \((\cdot, \cdot)\) is given by

\[(X, Y) = X_a Y_a.\]

Observe that the diagonal basis elements \(T_3\) and \(T_8\) span the Cartan subalgebra \(C\) of \(\text{su}(3)\).

Let \(X \in lG\). As \(X\) is skew-symmetric, there is an element \(U \in \text{SU}(3)\) such that

\[U^{-1}XU = \text{ad}(U)X\]

is a diagonal matrix, or

\[\text{ad}(U)X \in C.\]

Hence: each orbit of \(G\) in \(lG\) has a representative in \(C\), and we can restrict ourselves to centralizers of the elements of \(C \cap S\).

Suppose that \(X, Y \in C\), and that there is \(U \in \text{SU}(3)\) such that

\[Y = U^{-1}XU.\]

Then \(Y\) must have the same eigenvalues as \(X\). As both matrices are diagonal, they can differ only in the order of their diagonal elements. Moreover, we can achieve any permutation of any diagonal elements of any \(X \in C\) by the
corresponding permutation matrix $U$ (an orthogonal matrix with 0 and 1 as elements). For even permutations, $U \in SU(3)$; for odd permutations, $-U \in SU(3)$. Hence exactly those elements of $C$ that differ by a permutation of their diagonal elements lie in the same orbit of $SU(3)$ in $su(3)$.

Let us introduce the coordinates $\rho$ and $\alpha$ on $C$ by

$$X = T_3 \rho \sin \alpha + T_8 \rho \cos \alpha.$$

For $X \in C \cap S$, $\rho = 1$. The six permutations of the diagonal elements of $X$ leave $\rho$ invariant and send $\alpha$ to

$$\alpha, \quad \alpha + \frac{2\pi}{3}, \quad \alpha + \frac{4\pi}{3}, \quad -\alpha, \quad -\alpha + \frac{2\pi}{3}, \quad -\alpha + \frac{4\pi}{3}.$$

Hence for each orbit of $SU(3)$ in $su(3)$, there is exactly one representative in the sector

$$0 \leq \alpha < \frac{\pi}{3}. \quad (18)$$

Finally, we must find the centralizers to all elements in the sector (18). This is easy: all matrices $Y$ which commute with a given diagonal matrix $X$ have a block-diagonal form: the non-diagonal elements $Y_{ij}$ must vanish, if

$$X_{ii} \neq X_{jj}.$$ 

Thus, all $X$’s with three different diagonal elements commute only with diagonal matrices. This is the case for

$$0 < \alpha < \frac{\pi}{3},$$

and $Z_X = C$ in this case. Only for $\alpha = 0 - T_8$ direction, $X_{11} = X_{22}$, so $Z_X = su(2) \times u(1)$. Hence, we have a two-dimensional abelian centralizer $C$ if $0 < \alpha < \frac{\pi}{3}$, generating the subgroup $U(1) \times U(1)$, and a four-dimensional non-abelian centralizer if $\alpha = 0$, generating the subgroup $SU(2) \times U(1)$.
4 Two-Dimensional Centralizers

4.1 Solutions at the Internal Infinity

Let us assume that

\[ E = E_\alpha T_\alpha \neq 0, \]

where greek indices run through two values: 3 and 8. All different solutions correspond to the range

\[ 0 < \frac{E_3}{E_8} < \sqrt{3}. \]

Then, we must have

\[ W_A = W^\alpha_A T_\alpha, \]
\[ Q = Q_\alpha T_\alpha, \]
\[ B = B_\alpha T_\alpha, \]

and eqs. (8)-(15) become

\[ \frac{1}{(N\rho)^2} = \gamma^2 \left[ \frac{E_\alpha E_\alpha}{e^2} + \frac{B_\alpha B_\alpha}{e^2} - \frac{k}{4} (Q_\alpha Q_\alpha - F^2)^2 - \frac{\Lambda}{\gamma^2} \right], \] (19)

\[ R = 2\gamma^2 \left[ \frac{E_\alpha E_\alpha}{e^2} + \frac{B_\alpha B_\alpha}{e^2} + \frac{k}{4} (Q_\alpha Q_\alpha - F^2)^2 + g^{AB} \partial_A Q_\alpha \partial_B Q_\alpha + \frac{\Lambda}{\gamma^2} \right], \] (20)

\[ \frac{E_\alpha E_\alpha}{e^2} + \frac{B_\alpha B_\alpha}{e^2} - \frac{k}{4} (Q_\alpha Q_\alpha - F^2)^2 = \text{const.}, \] (21)

\[ \partial_A Q_\alpha \partial_B Q_\alpha = \frac{1}{2} g_{AB} g^{KL} \partial_K Q_\alpha \partial_L Q_\alpha \] (22)
\[ \partial_A E_\alpha = 0, \] (23)
\[ \partial_A B_\alpha = 0, \] (24)
\[ \frac{1}{\sqrt{g}} \partial_A (\sqrt{g} g^{AB} \partial_B Q_\alpha) - \frac{k}{2} (Q_\alpha Q_\alpha - F^2) Q_\alpha = 0. \] (25)
We obtain immediately

\[ E_\alpha = \text{const.}, \quad B_\alpha = \text{const.} \]  \hspace{1cm} (26)

Then eq. (21) yields

\[ Q_3^2 + Q_8^2 = \text{const.} \]

This means that the function \( Q_3(x^2, x^3) \) is not independent of \( Q_8(x^2, x^3) \), and so the matrix on the l.h.s. side of eq. (22) is degenerate. However, it must be proportional to \( g_{AB} \) which is non-degenerate. Hence the proportionality factor must be zero, and we obtain

\[ Q_\alpha = \text{const.} \]  \hspace{1cm} (27)

Substituting eq. (27) in eq. (25), we find that

\[ (Q_\beta Q_\beta - F^2)Q_\alpha = 0. \]

Hence there are only two cases:

A) \( Q_\alpha = 0 \),

B) \( Q_3^2 + Q_8^2 = F^2 \),

A) being the “false” and B) the “true” Higgs vacuum. The remaining eqs. (19) and (20) become

\[ \frac{1}{(N\rho)^2} = \gamma^2 \left( \frac{E_\alpha E_\alpha}{e^2} + \frac{B_\alpha B_\alpha}{e^2} - \epsilon \frac{kF^4}{4} - \frac{\Lambda}{\gamma^2} \right), \]  \hspace{1cm} (28)

\[ R = 2\gamma^2 \left( \frac{E_\alpha E_\alpha}{e^2} + \frac{B_\alpha B_\alpha}{e^2} + \epsilon \frac{kF^4}{4} + \frac{\Lambda}{\gamma^2} \right), \]  \hspace{1cm} (29)

where \( \epsilon = 1 \) for the case A and \( \epsilon = 0 \) for the case B. Eq. (29) implies that \( (\mathcal{H}, g_{AB}) \) is a compact two-dimensional space of constant curvature. If \( R > 0 \),
$(\mathcal{H}, g_{AB})$ is a sphere of radius $r_H$,

$$R = \frac{2}{r_H^2}.$$  

If $R = 0$, we will have toroidal topology, and if $R < 0$, the genus can have any value larger than 1. However, $R \leq 0$ only if $\Lambda < 0$. The corresponding black hole spaces are then likely to be only asymptotically locally anti-de Sitter (with some identifications made at anti-de Sitter infinity). This question must be further studied.

Let us limit ourselves to spherical holes. If we denote by $q_\alpha$ the electric and by $p_\alpha$ the magnetic charges of the hole, then

$$E_\alpha = \frac{q_\alpha}{r_H^2}, \quad B_\alpha = \frac{p_\alpha}{r_H^2}.$$  

Let us choose $x^2 = \theta$ and $x^3 = \phi$, the ordinary spherical coordinates on $\mathcal{H}$. Then, we can set

$$W_\theta^\alpha = 0, \quad W_\phi^\alpha = -p_\alpha \cos \theta.$$  

Observe that eq. (29) implies then an analogy to the “extremality” condition

$$m^2 = q^2 + p^2,$$

namely

$$r_H^2 = \gamma^2 \left( \frac{q_\alpha q_\alpha}{e^2} + \frac{p_\alpha p_\alpha}{e^2} \right) + \left( \Lambda + \frac{e^2 \gamma^2 k F^4}{4} \right) r_H^4.$$  

In the case A, different solutions can be parametrized by the angle $\alpha$ between the vector $q_\alpha$ and hypercharge axis, and by three further parameters: lengths of the vectors $q_\alpha$ and $p_\alpha$, and the angle $\beta$ between $q_\alpha$ and $p_\alpha$. In the case B, the solutions are classified by the same angle $\alpha$, and by four further parameters: three as the case A, and the angle between $q_\alpha$ and $Q_\alpha$.

This type of solution has been described for any gauge group in [17].
4.2 Spacetime Solutions

Let us look for a whole asymptotically flat spacetime with the fields $W_\mu$ and $Q$, whose internal infinity has the structure found in the previous section. To this end, we choose the following ansatz:

1) spherical symmetry everywhere,

2) the fields $W_\mu$ and $Q$ take values from $C$ everywhere,

3) the Higgs field $Q$ is in its false (case A) or true (case B) vacuum state everywhere.

The assumptions 2 and 3 for $Q$ lead to a) vanishing of the $Q$-part in $T^{\mu\nu}$ with the exception of the term

$$\frac{1}{4\pi}V(Q)g^{\mu\nu} = \frac{k\epsilon}{32\pi}F^4 g^{\mu\nu}$$

(cf. eq. (6)), b) eq. (4) being identically satisfied, and c) vanishing of the source term in eq. (3).

The Yang-Mills field splits into two “Maxwell fields” $W_3^{\mu}$ and $W_8^{\mu}$, and eq. (3) becomes

$$\partial_\nu \left[ (-4g)^{1/2}G_3^{\mu\nu} \right] = 0,$$

$$\partial_\nu \left[ (-4g)^{1/2}G_8^{\mu\nu} \right] = 0,$$

(30) (31)

where

$$G_\mu^{\alpha} = \partial_\mu W_\nu^{\alpha} - \partial_\nu W_\mu^{\alpha}, \quad \alpha = 3, 8.$$

Finally, Eq. (2) becomes

$$R_{\mu\nu} - \frac{1}{2} 4 R g_{\mu\nu} + \bar{\Lambda} g_{\mu\nu} = 8\pi\gamma^2(T_3^{\mu\nu} + T_8^{\mu\nu}),$$

(32)
where the “effective” cosmological constant $\bar{\Lambda}$ is given by

$$\bar{\Lambda} = \Lambda + \frac{1}{4} \epsilon k \gamma^2 F^4.$$ 

The solution to the system (30)-(32) is just the Reissner-Nordström spacetime with the metric

$$ds^2 = -\phi dt^2 + \phi^{-1} dr^2 + r^2 d\Omega^2,$$  

(33)

where

$$\phi(r) = 1 - \frac{2\xi}{r} + \frac{\eta^2}{r^2} - \frac{1}{3} \zeta r^2;$$  

(34)

$\xi, \eta$ and $\zeta$ being some constants. An extreme horizon will exist only if the function $\phi r^2$ has a double-root $r_H$. This happens if and only if the following two relations are satisfied

$$\zeta r_H^4 - r_H^2 + \eta^2 = 0,$$  

(35)

$$\xi = r_H (1 - \frac{2}{3} \zeta r_H^2).$$  

(36)

Then

$$r^2 \phi(r) = \left( -\frac{1}{3} \zeta r^2 - \frac{2}{3} \zeta r_H r - \zeta r_H^2 + 1 \right) (r - r_H)^2,$$  

(37)

and we have two cases

(i) Cosmological double horizon:

$$\zeta \in \left( 0, \frac{1}{4\eta^2} \right),$$

$$r_H = \frac{1}{\sqrt{\zeta}} \sqrt{\frac{1}{2} + \frac{1}{2} \sqrt{1 - 4\zeta \eta^2}},$$  

(38)

$$\xi = r_H \left( \frac{2}{3} - \frac{1}{3} \sqrt{1 - 4\zeta \eta^2} \right).$$  

(39)
The spacetime is not static near this double horizon so that the method of this paper does not seem applicable.

(ii) Black-hole double horizon:

\[ \zeta \in \left( -\infty, \frac{1}{4\eta^2} \right), \]

\[ r_H = \frac{|\eta|}{\sqrt{\frac{1}{2} + \frac{1}{2}\sqrt{1 - 4\zeta\eta^2}}}, \quad (40) \]

\[ \xi = r_H \left( \frac{2}{3} + \frac{1}{3}\sqrt{1 - 4\zeta\eta^2} \right). \quad (41) \]

For \( \zeta \leq 0 \), only black-hole double horizon can arise. Consider \( \zeta > 0 \). Then it is easy to see that the function \( r^2\phi(r) \), if \( \eta \neq 0 \), must always have one negative root and one largest positive root \( r_c \). In general, it may also have two different real roots, \( r_1, r_2 \) such that \( 0 < r_1 < r_2 < r_c \); \( r_1, r_2 \) and \( r_c \) correspond then to a single inner, outer and cosmological horizon respectively. If

\[ 0 < r_1 = r_2 = r_H < r_c, \]

we have the black-hole double horizon. If \( 0 < r_1 < r_2 = r_c = r_H \), we use the term “cosmological” double horizon. Clearly, both black-hole double horizon and cosmological double horizon can never occur simultaneously. For the black-hole double horizon

\[ (r^2\phi)'' |_{r=r_H} \geq 0 \]

so that \( r_H < \frac{1}{\sqrt{2\zeta}} \). For the cosmological double horizon,

\[ (r^2\phi)'' |_{r_H} < 0 \]

and

\[ r_H > \frac{1}{\sqrt{2\zeta}}. \]
A special case arises when

\[ r_1 = r_2 = r_c = \frac{1}{\sqrt{2}\zeta}. \]

Then all three horizons coincide,

\[ (r^2 \phi)'' |_{r_H} = 0, \]

\[ \eta^2 = \frac{9}{8} \xi^2, \zeta = \frac{1}{4\eta^2}. \]

In [25], some conformal diagrams corresponding to the double horizons are drawn. Very recently, special Reissner-Nordström spacetimes with the cosmological double horizons were analyzed in connection with a possible violation of the cosmic censorship in [16]. In our case, the non-vanishing cosmological constant – which plays a crucial role in [16] – can entirely arise from the Higgs field.

Eq. (29) will be satisfied, if

\[ \zeta = \Lambda + \frac{1}{4} \epsilon \gamma^2 kF^4 = \bar{\Lambda} \]

and

\[ \eta^2 = \frac{\gamma^2}{\epsilon^2} (q_\alpha q_\alpha + p_\alpha p_\alpha). \]

Comparing eqs. (7) and (33), we obtain

\[ N = a \sqrt{\phi}, \]

\[ \rho^2 = \frac{1}{\phi N^2} = \frac{4}{a^2 \phi'/2}, \]

and

\[ \lim_{N \to 0} (N\rho)^2 = \lim_{r \to r_H} \frac{4\phi}{\phi'^2}, \]

22
where $a$ is an arbitrary positive constant. Setting

$$\phi = H(r)(r - r_H)^2,$$

we have

$$\lim_{N \to 0} (N\rho)^2 = \frac{1}{H(r_H)},$$

and eq. (39) yields

$$H(r_H) = \frac{1 - 2\zeta r_H^2}{r_H^2}.$$  

With this value of $(N\rho)^{-2}$, eq. (28) is identically satisfied. Hence our solution at the internal infinity corresponds to the given spacetime solution, as claimed.

## 5 Four-Dimensional Centralizer

### 5.1 Solutions at the Internal Infinity

The four-dimensional centralizer is spanned by the basis elements $T_1, T_2, T_3$ and $T_8$. We assume, therefore, the fields of the form

\begin{align*}
W^a_0 &= (0, \ldots, 0, W_0^8), \\
W^a_1 &= 0, \\
W^a_A &= (W^1_A, W^2_A, W^3_A, 0, \ldots, 0, W^8_A), \\
E_a &= (0, \ldots, 0, E_8), \\
B_a &= (B_1, B_2, B_3, 0, \ldots, 0, B_8), \\
Q_a &= (Q_1, Q_2, Q_3, 0, \ldots, 0, Q_8).
\end{align*}  

(42) 

(43)
Thus the fields split in a U(1) field $W^8$, $E_8$, $B_8$, and $Q_8$, and an SU(2) field $W^k$, $E_k = 0$, $B_k$, and $Q_k$. We will use the latin indices $a, b, c$ for the whole field, i.e., they assume the values 1, \ldots, 8; the indices $i, j, k$ will take only the values 1, 2, 3 and will distinguish the components of the SU(2) field. We will also use the vector notation $\vec{W}_\mu, \vec{E}, \vec{B}$ and $\vec{Q}$ for the SU(2) field.

Eqs. (12) to (15) split also into U(1) equations – for the 8th components, and SU(2) equations – for the first 3 components.

Non-trivial U(1) equations:
\begin{align}
\partial_A E_8 &= 0, \\
\partial_A B_8 &= 0, \\
\frac{1}{\sqrt{g}}\partial_A (\sqrt{g}g^{AB}\partial_B Q_8) &= \frac{1}{2} k (\vec{Q}^2 + Q_8^2 - F^2) Q_8.
\end{align}

Nontrivial SU(2) equations:
\begin{align}
\epsilon^{AB} \sqrt{g} \nabla_B \vec{B} + 2e^2 g^{AB} \vec{Q} \times \nabla_B \vec{Q} &= 0, \\
\frac{1}{\sqrt{g}} \nabla_A (\sqrt{g}g^{AB} \nabla_B \vec{Q}) - \frac{1}{2} k (\vec{Q}^2 + Q_8^2 - F^2) \vec{Q} &= 0, \\
\vec{B} = \frac{1}{\sqrt{g}} \epsilon^{AB} (\partial_A \vec{W}_B + \vec{W}_A \times \vec{W}_B),
\end{align}

where we have introduced the SU(2) covariant derivative,
\[\nabla_A \vec{X} = \partial_A \vec{X} + 2\vec{W}_A \times \vec{X}.
\]
The components $a = 4, 5, 6, 7$ of all eqs. (12)-(15) are satisfied identically.

The remaining eqs. (8)-(11) take the form
\[\frac{1}{(N\rho)^2} = \gamma^2 \left[ \frac{1}{e^2}(E_8^2 + B_8^2) + \frac{1}{e^2}B^2 - \frac{1}{4} k(\vec{Q}^2 + Q_8^2 - F^2)^2 - \frac{\Lambda}{\gamma^2} \right],\]
\[ R = 2\gamma^2 \left[ \frac{1}{\epsilon^2} (E_s^2 + B_s^2) + \frac{1}{\epsilon^2} \vec{B}^2 + \frac{\Lambda}{\gamma^2} + g^{AB} (\nabla_A \vec{Q} \cdot \nabla_B \vec{Q} + \partial_A Q_s \partial_B Q_s) + \frac{1}{4} k (\vec{Q}^2 + Q_s^2 - F^2)^2 \right], \quad (51) \]

\[ \frac{1}{\epsilon^2} (E_s^2 + B_s^2) + \frac{1}{\epsilon^2} \vec{B}^2 - \frac{1}{2} k (\vec{Q}^2 + Q_s^2 - F^2)^2 = \text{const.}, \quad (52) \]

and

\[ \nabla_A \vec{Q} \cdot \nabla_B \vec{Q} + \partial_A Q_s \partial_B Q_s = \frac{1}{2} g_{AB} g^{KL} (\nabla_K \vec{Q} \cdot \nabla_L \vec{Q} + \partial_K Q_s \partial_L Q_s). \quad (53) \]

In contrast to the case of the two-dimensional (abelian) centralizers treated in the previous section, it does not now appear easy to deduce from the system of eqs. (44)–(53) that the internal infinity must necessarily be a compact space of constant curvature. Therefore, we make a spherically symmetric ansatz for both the metric and the fields at \( \mathcal{H} \). We thus assume

\[ g_{AB} dx^A dx^B = r_H^2 (d\theta^2 + \sin^2 \theta d\varphi^2), \quad (54) \]

so that

\[ R = \frac{2}{r_H^2}. \]

The spherically symmetric ansatz for the SU(2) fields (see, e.g. [20, 26]) is given by

\[ \vec{W}_r = 0, \]
\[ \vec{W}_\theta = w(\sin \varphi, -\cos \varphi, 0), \]
\[ \vec{W}_\varphi = w \sin \theta (\cos \theta \cos \varphi, \cos \theta \sin \varphi, -\sin \theta), \quad (55) \]
\[ \vec{Q} = Q(\sin \theta \cos \varphi, \sin \theta \sin \varphi, \cos \theta), \quad (56) \]
where \( w \) and \( Q \) are constants. Substituting for \( \vec{W}_A \) into eq. (49), we obtain

\[
\vec{B} = b(\sin \theta \cos \varphi, \sin \theta \sin \varphi, \cos \theta),
\]

where

\[
b = 2w(w - 1)r_H^{-2}.
\]

The spherically symmetric ansatz for the U(1)-fields is simply

\[
E_8 = \text{const.}, B_8 = \text{const.}, \quad Q_8 = \text{const}.
\]

Let us turn to eqs. (44)–(53) and substitute our ansatz into them. Eqs. (44) and (45) are satisfied identically. Eq. (46) admits only the following two types of solutions:

\[
Q_8 \neq 0, \quad \vec{Q}^2 + Q_8^2 = F^2,
\]

and

\[
Q_8 = 0.
\]

We shall discuss the two solutions separately.

A) \( Q_8 \neq 0, \quad \vec{Q}^2 + Q_8^2 = F^2 \)

This is the “true Higgs vacuum”, \( V(Q) = 0 \). Using eqs. (55) and (56), we find eq. (48) to be satisfied only if

\[
w = \frac{1}{2}.
\]

This, however, implies that

\[
\nabla_A \vec{Q} = \nabla_A \vec{B} = 0,
\]
so that eqs. (47) and (53) are also satisfied. Clearly the same is true with eq. (52) since \( \vec{B}^2 = b^2 = \text{const.} \) and \( \vec{Q}^2 = Q^2 = \text{const.} \). Denoting the electric and magnetic hypercharges by \( q \) and \( p \), we have

\[
E_8 = \frac{q}{r_H}, \quad B_8 = \frac{p}{r_H},
\]

and in view of eqs. (58) and (61), we can rewrite eqs. (50) and (51) in the form

\[
\left( \frac{\gamma N \rho}{r_H} \right)^2 = \gamma^4 \frac{e^2 r_H^4}{r_H^4} (q^2 + p^2 + \frac{1}{4}) - \gamma^2 \Lambda, \tag{64}
\]

\[
\left( \frac{\gamma}{r_H} \right)^2 = \gamma^4 \frac{e^2 r_H^4}{r_H^4} (q^2 + p^2 + \frac{1}{4}) + \gamma^2 \Lambda. \tag{65}
\]

The magnitudes of the Higgs fields \( \vec{Q}^2 = Q^2 \) and \( Q_8 \) are only constrained by

\[
Q^2 + Q_8^2 = F^2.
\]

Hence, these solutions can be described by three parameters: \( q, p \) and \( Q \).

**B) \( Q_8 = 0 \)**

Starting again from the ansatz (55) and (56) (and assuming in general non-vanishing \( E_8 \) and \( B_8 \)), we find in this case that eq. (47) yields the relation

\[
b = -2e^2 Q^2, \tag{66}
\]

if \( w \neq \frac{1}{2} \). As a consequence of eq. (48) we obtain

\[
\frac{1}{4} \frac{k r_H^2}{(1 - 2w)^2} (F^2 - Q^2) = 1. \tag{67}
\]

Therefore, solutions exist only if

\[
F^2 \geq Q^2. \tag{68}
\]
Expressing \( w \) from eq. (67), one finds

\[
w = \frac{1}{2} \pm \frac{1}{4} \sqrt{kr_H} \sqrt{F^2 - Q^2}.
\]  

(69)

Clearly, the equality in (68) can arise only for \( w = \frac{1}{2} \) (cf. (67)), which is the case of the true Higgs vacuum discussed above. Hereafter we assume that \( F^2 > Q^2 \).

It can be easily checked that eq. (53) is satisfied as well as the condition (52). The remaining eqs. (50) and (51) can be rewritten into the form

\[
\left( \gamma \frac{N_p}{r_H} \right)^2 = \frac{\gamma^4}{e^2} (E_8^2 + B_8^2) + 4e^2 (\gamma Q)^4 - \frac{1}{4} k \left[ (\gamma F)^2 - (\gamma Q)^2 \right]^2 - \gamma^2 \Lambda (70)
\]

\[
\left( \gamma \frac{r_H}{N_p} \right)^2 = \frac{\gamma^4}{e^2} (E_8^2 + B_8^2) + \frac{1}{4} (16e^2 - k)(\gamma Q)^4 + \frac{1}{4} k(\gamma F)^4 + \gamma^2 \Lambda.
\]  

(71)

These represent restrictions on the parameters. All possible classes of the parameters are briefly mentioned in the Appendix.

Summarizing, the solutions (55), (56), (57) with \( b \) and \( w \) given by eqs. (66) and (69), with in general non-vanishing \( E_8 \) and \( B_8 \), represent a gravitating 't Hooft-Polyakov-type monopole in the SU(2) fields which has additional electric and magnetic U(1) fields in the 8th direction and an extreme black hole “in the middle”. However, these fields are defined at the internal infinity only, and it remains to be seen whether they can be extended to global spacetime solutions.

### 5.2 Spacetime Solutions

As in the case of the two-dimensional centralizers, we are going to construct some solutions in the whole spacetime that, at the internal infinity, go over to the fields found in Section 5.1.
I. Solutions of the Reissner-Nordström Type

Ia. The True-Higgs-Vacuum Solutions

Assume everywhere in a static spherically symmetric spacetime that

(i) the Yang-Mills field $\vec{W}_k$ has the form (55) with $w = \frac{1}{2}$, and $\vec{W}_0 = 0$;
(ii) the Higgs field $\vec{Q}$ is given by eq. (56) with $Q = \text{const.}$;
(iii) the field $Q_8 = \text{const.}$ satisfying together with $\vec{Q}$ the true vacuum condition:

$$Q^2 + Q_8^2 = F^2,$$

(iv) the U(1) field $W_0^8$ has the form

$$W_0^8 = -\frac{c}{r},$$

where $c = \text{const.}$ is to be determined by the condition

$$E_8 = \frac{1}{N}\partial_r W_0^8$$

and

$$W_2^8 = 0, \ W_3^8 = -p \cos \theta.$$

Under these conditions it is easy to see that $D_\mu Q = 0$ everywhere. Since, in addition, the Higgs field is in the true vacuum state, it neither contributes to $T^{\mu\nu}$ (eq. (6)), nor to the source term in the Yang-Mills eq. (3), and eq. (4) for the Higgs field is identically satisfied. The gauge fields split everywhere into SU(2) and U(1) fields, and eq. (3) becomes

$$\frac{1}{\sqrt{-g}}\partial_{\nu}\left(\sqrt{-g}G^{\mu\nu}_{i}\right) - [W_{\nu}, G_{i}^{\mu\nu}] = 0 \quad (74)$$

$$\partial_{\nu}\left(\sqrt{-g}G^{\mu\nu}_{8}\right) = 0. \quad (75)$$
Under our assumptions for $W^a_{\mu}$, one easily finds the components of $G^a_{\mu\nu}$, the only non-vanishing ones being

$$\vec{G}_{\theta\varphi} = -\frac{1}{2} \sin \theta (\sin \theta \cos \varphi, \sin \theta \sin \varphi, \cos \theta), \quad (76)$$

$$G^8_{tr} = \frac{c}{r^2}, \quad G^8_{\theta\varphi} = p \sin \theta. \quad (77)$$

The form of the sources indicates that the metric will again be of the Reissner-Nordström type. Indeed, assuming the metric (33), we have $\sqrt{-g} = r^2 \sin \theta$, and one can immediately check that the Yang-Mills eqs. (74), (75) are satisfied by the fields (76) and (77) everywhere.

Hence, we can proceed as in Section 4.2 and write the function $\phi(r)$ in the form (34) and find again the expressions for the location $r_H$ of the black hole or cosmological double horizons and the mass parameters $\xi$. Putting then

$$\eta^2 = \left(\frac{\gamma}{e}\right)^2 (q^2 + p^2 + \frac{1}{4}), \quad (78)$$

$$\zeta = \Lambda, \quad (79)$$

we find eqs. (65) and (35) to be satisfied. Finally, we can check that eq. (64) is also satisfied in an analogous way as in Section 4.2. By using eqs. (72), (73) and the expression for $N\rho$ found at the end of Section 4.2, we obtain

$$E_8 = \frac{\sqrt{1 - 2\Lambda r_H^2}}{r_H^2} c$$

so that the constant $c$ appearing in eqs. (72) and (77) is given by

$$c = \frac{r_H}{\sqrt{1 - 2\Lambda r_H^2}} q. \quad (80)$$
To summarize, the global solution is of the Reissner-Nordström-de Sitter form (33) and (34) with $\eta$ and $\zeta$ given by eqs. (78) and (79), $\xi$ and $r_H$ by eqs. (38), (39), (40) and (41) in Section 4.2. The Yang-Mills fields are determined by eqs. (55), (61), (72), (73), (76), (77) and (80), the Higgs field by eqs. (56) with $Q = \text{const.}$ and by $Q_8 = \text{const.}$ such that $Q^2 + Q_8^2 = F^2$.

Setting $q = p = Q_8 = 0$, we obtain the field of a pure magnetic monopole with the magnetic charge $\eta^2 = \frac{\gamma^2}{4e^2}$ (cf. eq. (78)). This corresponds to the solutions found in [27] and [28] within the SU(2) theory (due to our different choice of the su(2)-basis one has to replace their $e^2$ by our $4e^2$).

Ib. The False-Higgs-Vacuum Solutions

Putting $Q^a = 0$ (but keeping $F \neq 0$), and choosing $w = 0$ or $w = 1$ in eq. (55) so that $\vec{B} = 0$ (cf. eqs. (57), (58)), we easily find the Reissner-Nordström solution with a non-vanishing cosmological constant. An example of such a solution is in fact found in Section 4.2 ($q_3 = 0, \epsilon = 1$).

II Recent Numerical Solutions

Let us set everywhere in a static spherically symmetric spacetime

(i) $Q_a = 0$, but in general

$$V(Q) = \frac{1}{8} k F^4 \neq 0;$$

(ii) the fields $W^8_0, W^8_A$ and $E_8$ as in eq. (72), (73) and at the internal infinity, $E_8$ and $B_8$ as in eq. (63);

(iii) $W^a_1 = 0, \vec{W}_A$ given by eq. (55) with $w = w(r)$;
(iv) the metric in the following form (often used for spherically symmetric spacetimes):

\[
ds^2 = - \left(1 - \frac{2\gamma^2 m(r)}{r}\right) \sigma^2(r) dt^2 + \left(1 - \frac{2\gamma^2 m(r)}{r}\right)^{-1} dr^2 + r^2 (d\theta^2 + \sin^2 \theta d\varphi^2).
\]

(81)

The Higgs field equation (4) is identically satisfied by this ansatz, the source term in eq. (3) vanishes and the Higgs field contribution to the stress tensor (6) is just the term \(-\frac{1}{4\pi} g^{\mu\nu} V\), which can be interpreted as a cosmological constant. The fields now correspond precisely to the well-known ansatz (see, e.g. [20]) that was recently used also in [23] (with vanishing \(B_8\)), and we can use the spacetime solutions found in [23] by numerical methods. They are labelled by \(r_H\), by a parameter \(\alpha\), and by a positive integer \(n\). The condition of extremality determines \(\alpha\), and it is easy to see that it is equivalent to our eq. (71) in which \(Q = 0\) and

\[
\Lambda = -\frac{1}{4} k \gamma^2 F^4.
\]

(82)

For a more detailed discussion of the properties of these solutions we refer to [23].

Let us just note that one solution at the internal infinity corresponds here to a number of spacetime solutions (with different \(n\)).

Acknowledgements. The authors appreciate the discussions with D. Maison, N. Straumann and M. Volkov. J.B. acknowledges the support of the Tomalla Foundation and of the grants GACR-0503 and GAUK-318 of the Czech Republic and Charles University. A.H. acknowledges the support of the Swiss National Science Foundation.
Appendix

Here, we analyze the range of the parameters describing the internal-infinity solutions on the four-dimensional centralizer given in Section 5.1B. That is $Q_8 = 0$, but in general all other fields $\vec{Q}, \vec{B}, E_8, B_8$ are non-vanishing. $\vec{Q}$ and $\vec{B}$ are given by eqs. (56), (57) and (58).

First, as a consequence of eqs. (58), (66) and (69) we find the relation

$$\left( \frac{\gamma}{r_H} \right)^2 = \frac{1}{4} \left[ k(\gamma F)^2 + (16e^2 - k)(\gamma Q)^2 \right]. \quad (A1)$$

Recall (cf. eq. (68) and the discussion following eq. (69)) that we assume

$$Q^2 < F^2. \quad (A2)$$

Further two relations are given by eqs. (70) and (71).

Let us discuss some consequences of the relations (A1), (A2), (70) and (71). It is easy to see that the compatibility of eqs. (71) and (A1) requires that

$$(16e^2 - k)(\gamma Q)^4 - (16e^2 - k)(\gamma Q)^2 + 4\gamma^2 \Lambda + k \left[ (\gamma F)^4 - (\gamma Q)^2 \right] + \frac{4\gamma^4}{e^2} (E_8^2 + B_8^2) = 0. \quad (A3)$$

Combining eq. (A3) with the condition that the r.h.s. of eq. (70) has to be positive, we obtain the inequality

$$\left[ 2(\gamma Q)^2 - 1 \right] \left\{ 16e^2(\gamma Q)^2 + k \left[ (\gamma F)^2 - (\gamma Q)^2 \right] \right\} + \frac{8\gamma^4}{e^2} (E_8^2 + B_8^2) \geq 0 \quad (A4)$$

which does not involve the cosmological constant. Conversely, if (A3), (A4) and (A2) are satisfied, then (71), (A1) and the positivity of the r.h.s. of (70)
are guaranteed. Hence, the possible values of the parameters $Q, E_8$ and $B_8$ are determined by relations (A2), (A3) and (A4); the horizon radius is given by eq. (A1).

As in [19], it is useful to discuss separately three cases: (i) $16e^2 - k > 0$, (ii) $16e^2 - k = 0$, and (iii) $16e^2 - k < 0$. Since for $E_8 = B_8 = 0$ our equations go over to those of [19] – except that one has to replace $e^2$ in [19] by $4e^2$ here – we discover admissible values of $Q$ in all three cases. However, more possibilities arise now as in general $E_8 \neq 0$ and $B_8 \neq 0$. A detailed discussion will be given elsewhere.

**References**


