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ENTROPY LAW FROM INHOMOGENEOUS THERMO FIELD DYNAMICS

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Abstract

In the framework of thermo field dynamics (TFD) for spatially inhomogeneous time-dependent nonequilibrium situations, we derive the kinetic equation, which is different from the Boltzmann equation due to quantum effects. It is shown that this kinetic equation leads to the entropy law. An expression for the entropy flow is found.

The development of thermo field dynamics (TFD) within a recent few years [1, 2, 3, 4] made it a powerful and systematic tool with a solid foundation to tackle time-dependent nonequilibrium systems of quantum fields. We see the reasons for this from the following aspects of TFD: (i) In TFD time-dependent thermal situation is described by time-dependent unperturbed (quasi particle) number density, whereas thermal vacuum is kept *time-independent* [3, 4]. Thus the whole time-dependent process can be represented in a single Hilbert space using a single thermal vacuum, because of which TFD calculation of Green's functions is a mathematically well-defined procedure. (ii) There are redundant parameters in thermal Bogoliubov transformation. It was shown that under a particular choice of the redundant parameters the Feynman diagram method is available in calculating Green's functions of Heisenberg operators without relying on the Gell-Mann-Low relation [3, 4, 5]. (iii) Therefore, it is possible to study general structures of Green's functions in time representation, as was done in [4]. Moreover, a simple method of perturbative calculation of diagrams, called TFD recipe, was invented [6, 7] and is very helpful for studying Green's functions. (iv) As TFD is formulated properly as an operator formalism of quantum fields because of (i) and as the Green's functions in time representation are put into a compact form (allowing us a physical interpretation of each part) owing to (ii) and (iii), the renormalization becomes a transparent procedure in time-dependent TFD. As witnessed in [3, 4], the renormalization condition in the spatially homogeneous time-dependent thermal situation derived an equation for the unperturbed number density, which coincides with the kinetic equation of Boltzmann type and consequently leads to the entropy increase law, i.e., the second law of thermodynamics.

Then we ask whether the spatially homogeneous time-dependent TFD can be extended to the inhomogeneous one including the entropy law. Such an extension is vital for comparison with experiments, since most nonequilibrium phenomena are associated with inhomogeneity. The formulation of inhomogeneous TFD for the unperturbed system is already given in [8, 9]. An attempt to derive a kinetic equation from the renormalization was made in [10]. In the present paper we derive a kinetic equation using a renormalization condition

different from the previous one [10] and show the entropy law from it.

The spatial inhomogeneity in TFD is introduced through the momentum-mixing thermal Bogoliubov transformation [8, 9]: It reads for bosonic oscillator variables a_k (the discussions in this paper is restricted bosonic operators for simplicity, as extension to fermionic ones is straightforward)

$$\begin{aligned} a_k(t)^\mu &= \mathbf{B}^{-1}(t)_{kq}^{\mu\nu} \xi_q^\nu e^{-i \int^t ds \omega_q(s)} \\ \bar{a}_k(t)^\mu &= e^{i \int^t ds \omega_q(s)} \bar{\xi}_q^\nu \mathbf{B}(t)_{qk}^{\nu\mu} \end{aligned} \quad (1)$$

where the suffices k, q stand for momentum variables, μ, ν are thermal indices, i.e., $a^1 = a, a^2 = \tilde{a}^\dagger, \bar{a}^1 = a^\dagger, \bar{a}^2 = -\tilde{a}$. The ξ_k -operators define the time-space-independent thermal vacua $|0\rangle$ and $\langle 0|$ as $\xi_k|0\rangle = \tilde{\xi}_k|0\rangle = 0$ and $\langle 0|\xi_k^\dagger = \langle 0|\tilde{\xi}_k^\dagger = 0$. We take the following form for the thermal Bogoliubov matrix $\mathbf{B}(t)$,

$$\mathbf{B}(t)_{kq}^{\mu\nu} = \begin{bmatrix} \delta(\vec{k} - \vec{q}) + \mathbf{N}(t)_{kq} & -\mathbf{N}(t)_{kq} \\ -\delta(\vec{k} - \vec{q}) & \delta(\vec{k} - \vec{q}) \end{bmatrix}^{\mu\nu}, \quad (2)$$

with

$$\mathbf{N}(t)_{kq} = \langle 0|\bar{a}(t)_q^1 a(t)_k^1|0\rangle. \quad (3)$$

This parameterization of the thermal matrix is the same as that chosen in homogeneous case [4],

$$B[n_k(t)]^{\mu\nu} = \begin{bmatrix} 1 + n_k(t) & -n_k(t) \\ -1 & 1 \end{bmatrix}^{\mu\nu}, \quad (4)$$

except for momentum mixing, and is crucial for the availability of Feynman method later.

We recall from [8] the expression for unperturbed bosonic field of Schrödinger type using $a_k(t)^\mu$ in (1),

$$\varphi(t, \vec{x})^\mu = \frac{1}{(2\pi)^{3/2}} \int d^3k a_k(t)^\mu e^{i\vec{k}\cdot\vec{x}}, \quad (5)$$

which yields the unperturbed 2×2 matrix propagator,

$$\begin{aligned} \Delta(x, x')^{\mu\nu} &\equiv -i\langle 0|T[\varphi(x)^\mu \bar{\varphi}(x')^\nu|0\rangle \\ &= \int \frac{d^3k d^3k' d^3q}{(2\pi)^3} e^{i\vec{k}\cdot\vec{x}} \mathbf{B}^{-1}(t)_{kq}^{\mu\mu'} \mathcal{G}(t, t' : \vec{q})^{\mu'\nu'} \mathbf{B}(t')_{qk'}^{\nu'\nu} e^{-i\vec{k}'\cdot\vec{x}'} \end{aligned} \quad (6)$$

where

$$\mathcal{G}(t, t' : \vec{q})^{\mu\nu} = e^{-i \int_{t'}^t ds \omega_q(s)} \begin{bmatrix} -i\theta(t-t') & 0 \\ 0 & i\theta(t'-t) \end{bmatrix}^{\mu\nu}. \quad (7)$$

The unperturbed Hamiltonian governing the time evolution of the above φ is given by

$$\begin{aligned} \hat{H}_Q(t) &= \int d^3x \bar{\varphi}(t, \vec{x})^\mu \omega(-i\vec{\nabla}) \varphi(t, \vec{x})^\mu - \hat{Q}_R(t) \\ \hat{Q}_R(t) &\equiv \int d^3x d^3y \bar{\varphi}(t, \vec{x})^\mu R(t, \vec{x}, \vec{y}) T_0^{\mu\nu} \varphi(t, \vec{y})^\nu, \end{aligned} \quad (8)$$

where

$$T_0 = \begin{bmatrix} 1 & -1 \\ 1 & -1 \end{bmatrix} \quad (9)$$

and

$$R(t, \vec{x}, \vec{y}) = \int \frac{d^3q}{(2\pi)^3} \left[i \frac{\partial}{\partial t} + \{ \omega(i\vec{\nabla}_y) - \omega(-i\vec{\nabla}_x) \} \right] n(t, \frac{\vec{x} + \vec{y}}{2} : \vec{q}) e^{i\vec{q} \cdot (\vec{x} - \vec{y})}. \quad (10)$$

Here the number density parameter $n(x : \vec{k})$ is related to $\mathbf{N}(t)_{kq}$ through the Fourier transformation

$$n(t, \vec{x} : \vec{k}) \equiv \int d^3q e^{i\vec{q} \cdot \vec{x}} \mathbf{N}(t)_{k+\frac{q}{2}, k-\frac{q}{2}} \quad (11)$$

or

$$\mathbf{N}(t)_{kk'} \equiv \int \frac{d^3x}{(2\pi)^3} e^{-i(\vec{k}-\vec{k}') \cdot \vec{x}} n(t, \vec{x} : \frac{\vec{k} + \vec{k}'}{2}). \quad (12)$$

The total Hamiltonian $\hat{H}$ in TFD, given by

$$\hat{H}(t) = H(t) - \tilde{H}(t), \quad (13)$$

is divided into the unperturbed part $\hat{H}_Q(t)$ in (8) and interaction one $\hat{H}_I(t)$,

$$\hat{H}(t) = \hat{H}_Q(t) + \hat{H}_I(t) \quad (14)$$

$$\hat{H}_I(t) = \hat{H}_{\text{int}}(t) + \hat{Q}_R(t). \quad (15)$$

The second term in (15) is the counter term, while $\hat{H}_{\text{int}}(t)$ consists of non-linear terms and the usual counter terms such as the energy counter term. For definiteness, the following model of the interaction density is taken in this letter,

$$\begin{aligned} \hat{H}_{\text{int}}(t) &= H_{\text{int}}(t) - \tilde{H}_{\text{int}}(t) \\ H_{\text{int}}(t) &= g \int d^3x \left(\varphi^\dagger(x) \varphi(x) \right)^2. \end{aligned} \quad (16)$$

Here and below the usual counter terms in $\hat{H}_{\text{int}}(t)$, which are not essential in our following discussions, are suppressed.

On basis of the division (14) of $\hat{H}(t)$, one can develop the interaction representation. As was shown in [4], the Feynman diagram method is available in the calculation of the full propagator,

$$G(x, x')^{\mu\nu} \equiv -i\langle 0|T[\varphi_H(x)^\mu \bar{\varphi}_H(x')^\nu]|0\rangle, \quad (17)$$

$\varphi_H(x)^\mu$ being the Heisenberg operator. For this the choice of the form for $\mathbf{B}(t)_{kq}$ in (2) is crucial [4].

The 2×2 -matrix self-energy $\Sigma(x, x')^{\mu\nu}$ is defined through the Dyson-Schwinger equation,

$$G(x, x')^{\mu\nu} = \Delta(x, x')^{\mu\nu} + \int d^4y d^4y' \Delta(x, y)^{\mu\mu'} \Sigma(y, y')^{\mu'\nu'} G(y', x')^{\nu'\nu}. \quad (18)$$

The loop self-energy without vertex corrections in our model (16), corresponding to the Feynman diagram in Fig. 1, is obtained as

$$\Sigma_{\text{loop}}(x, x')^{\mu\nu} = -8g^2 \begin{bmatrix} (\Delta(x, x')^{11})^2 \Delta(x', x)^{11} & -(\Delta(x, x')^{12})^2 \Delta(x', x)^{21} \\ -(\Delta(x, x')^{21})^2 \Delta(x', x)^{12} & (\Delta(x, x')^{22})^2 \Delta(x', x)^{22} \end{bmatrix}. \quad (19)$$

Substitute (6) into this, then we find in the Fourier space with respect to $\vec{x}$ and $\vec{x}'$,

$$\Sigma_{\text{loop}}(t, t')_{kk'}^{\mu\nu} \equiv \int \frac{d^3x d^3x'}{(2\pi)^3} e^{-i\vec{k}\cdot\vec{x}} \Sigma_{\text{loop}}(x, x')^{\mu\nu} e^{i\vec{k}'\cdot\vec{x}'}, \quad (20)$$

$$\begin{aligned} &= C_g \int \prod_{i=1}^3 (d^3k_i d^3k'_i) \delta(\vec{k} - \vec{k}_1 - \vec{k}_2 + \vec{k}_3) \delta(\vec{k}' - \vec{k}'_1 - \vec{k}'_2 + \vec{k}'_3) \\ &\times \left[-i\theta(t - t') e^{-i \int_{t'}^t ds [\omega_{k_1}(s) + \omega_{k_2}(s) - \omega_{k_3}(s)]} \right. \\ &\quad \times (\mathbf{M}_1(t')_{k_1 k'_1 k_2 k'_2 k'_3 k_3} T_+ + \mathbf{M}_0(t')_{k_1 k'_1 k_2 k'_2 k'_3 k_3} T_0) \\ &\quad + i\theta(t' - t) e^{-i \int_{t'}^t ds [\omega_{k'_1}(s) + \omega_{k'_2}(s) - \omega_{k'_3}(s)]} \\ &\quad \left. \times (\mathbf{M}_1(t)_{k_1 k'_1 k_2 k'_2 k'_3 k_3} T_- - \mathbf{M}_0(t)_{k_1 k'_1 k_2 k'_2 k'_3 k_3} T_0) \right], \end{aligned} \quad (21)$$

where

$$C_g \equiv \frac{8g^2}{(2\pi)^6}, \quad (22)$$

$$\begin{aligned}
M_o(t)_{k_1 k'_1 k_2 k'_2 k'_3 k_3} &= N(t)_{k_1 k'_1} N(t)_{k_2 k'_2} (I_{k'_3 k_3} + N(t)_{k'_3 k_3}), \\
M_1(t)_{k_1 k'_1 k_2 k'_2 k'_3 k_3} &= (I_{k_1 k'_1} I_{k_2 k'_2} + N(t)_{k_1 k'_1} I_{k_2 k'_2} + I_{k_1 k'_1} N(t)_{k_2 k'_2}) N(t)_{k'_3 k_3} \\
&\quad - N(t)_{k_1 k'_1} N(t)_{k_2 k'_2} I_{k'_3 k_3},
\end{aligned} \tag{23}$$

with the notation of $I_{kk'} = \delta(\vec{k} - \vec{k}')$. The expression in (21) can be rewritten in the form of $B^{-1} \times (\text{a diagonal matrix}) \times B$, using the B -matrix in (4), when the formulae,

$$\begin{aligned}
B^{-1}[n(t)]^{\mu\mu'} \left[\frac{1+\tau_3}{2} \right]^{\mu'\nu'} B[n(t')]^{\nu'\nu} &= T_+^{\mu\nu} + n(t') T_0^{\mu\nu} \\
B^{-1}[n(t)]^{\mu\mu'} \left[\frac{1-\tau_3}{2} \right]^{\mu'\nu'} B[n(t')]^{\nu'\nu} &= T_-^{\mu\nu} - n(t) T_0^{\mu\nu}
\end{aligned} \tag{24}$$

with (9) and

$$T_+ = \begin{bmatrix} 1 & 0 \\ 1 & 0 \end{bmatrix}, \quad T_- = \begin{bmatrix} 0 & 0 \\ -1 & 1 \end{bmatrix}, \tag{25}$$

are used, namely,

$$\begin{aligned}
\Sigma_{loop}(t, t')^{\mu\nu}_{kk'} &= C_g \int \prod_{i=1}^3 (d^3 k_i d^3 k'_i) \delta(\vec{k} - \vec{k}_1 - \vec{k}_2 + \vec{k}_3) \delta(\vec{k}' - \vec{k}'_1 - \vec{k}'_2 + \vec{k}'_3) \\
&\quad \times B^{-1}[\mathcal{N}(t)]^{\mu\mu'} \begin{bmatrix} 1 & 0 \\ 0 & \mathbf{M}_1(t) \end{bmatrix}^{\mu'\lambda} V(t, t')^{\lambda\lambda'} \begin{bmatrix} \mathbf{M}_1(t') & 0 \\ 0 & 1 \end{bmatrix}^{\lambda'\nu'} B[\mathcal{N}(t')]^{\nu'\nu}
\end{aligned} \tag{26}$$

where

$$V(t, t')^{\mu\nu} = \begin{bmatrix} -i\theta(t-t') e^{-i \int_{t'}^t ds [\omega_{k_1}(s) + \omega_{k_2}(s) - \omega_{k_3}(s)]} & 0 \\ 0 & i\theta(t'-t) e^{-i \int_{t'}^t ds [\omega_{k'_1}(s) + \omega_{k'_2}(s) - \omega_{k'_3}(s)]} \end{bmatrix}^{\mu\nu} \tag{27}$$

$$\mathcal{N}(t) = \frac{M_o(t)_{k_1 k'_1 k_2 k'_2 k'_3 k_3}}{M_1(t)_{k_1 k'_1 k_2 k'_2 k'_3 k_3}}. \tag{28}$$

Note that for simplicity we drop the dependencies of $\mathbf{M}_1$, $\mathcal{N}$ and V on k_i 's and k'_i 's.

To the loop self-energy is added the counter-term self-energy, denoted by $\Sigma_Q(x, x')^{\mu\nu}$, to make the total self-energy,

$$\Sigma(t, t')^{\mu\nu}_{kk'} = \Sigma_{loop}(t, t')^{\mu\nu}_{kk'} + \Sigma_Q(t, t')^{\mu\nu}_{kk'}. \tag{29}$$

The term $\hat{Q}_R(t)$ in (15) contributes to $\Sigma_Q(x, x')^{\mu\nu}$ as

$$\Sigma_Q(t, t')^{\mu\nu}_{kk'} = \mathbf{R}(t)_{kk'} \delta(t-t') T_0^{\mu\nu} \tag{30}$$

where

$$\mathbf{R}(t)_{kk'} = i\dot{\mathbf{N}}(t)_{kk'} + (\omega_{k'}(t) - \omega_k(t))\mathbf{N}(t)_{kk'}. \quad (31)$$

The next step is to extract the *on-shell* part from the total self-energy obtained above, on which a renormalization condition will be imposed. For this, consider the diagonal matrix function $V(t, t')^{\mu\nu}$ included in $\Sigma_{loop}(t, t')^{\mu\nu}_{kk'}$, representing the propagation of the quasi particles ξ . We assume that ω_k is independent of time, then V depends only on $t - t'$. Then one may define its Fourier transformation by

$$V(t)^{\mu\nu} = \int \frac{dk_0}{2\pi} e^{-ik_0 t} V(k_0)^{\mu\nu} \quad (32)$$

to find that

$$V(k_0)^{\mu\nu} = \begin{bmatrix} \frac{1}{k_0 - \omega_{k_1} - \omega_{k_2} + \omega_{k_3} + i\epsilon} & 0 \\ 0 & \frac{1}{k_0 - \omega_{k'_1} - \omega_{k'_2} + \omega_{k'_3} - i\epsilon} \end{bmatrix}^{\mu\nu}. \quad (33)$$

Let us define its on-shell by putting

$$\begin{aligned} k_0 \delta^{\mu\nu} &= \left(\omega_{(k+k')/2} - \omega_{(k_1+k'_1)/2} - \omega_{(k_2+k'_2)/2} + \omega_{(k_3+k'_3)/2} \right) \delta^{\mu\nu} \\ &+ \begin{bmatrix} \omega_{k_1} + \omega_{k_2} - \omega_{k_3} & 0 \\ 0 & \omega_{k'_1} + \omega_{k'_2} - \omega_{k'_3} \end{bmatrix}^{\mu\nu} \end{aligned} \quad (34)$$

there, meaning that the on-shell part of $V(t - t')^{\mu\nu}$ is given by

$$V^{(on)}(t - t')^{\mu\nu} = \delta(t - t') \left[\frac{1}{\omega_{(k+k')/2} - \omega_{(k_1+k'_1)/2} - \omega_{(k_2+k'_2)/2} + \omega_{(k_3+k'_3)/2} + i\epsilon\tau_3} \right]^{\mu\nu} \quad (35)$$

where τ_3 is the Pauli matrix. Thus the on-shell part of the loop self-energy, denoted by $\Sigma_{loop}^{(on)}(t, t')^{\mu\nu}_{kk'}$, follows from substituting this $V^{(on)}(t - t')^{\mu\nu}$ into $V(t, t')^{\mu\nu}$ inside $\Sigma_{loop}(t, t')^{\mu\nu}_{kk'}$,

$$\begin{aligned} \Sigma_{loop}^{(on)}(t, t')^{\mu\nu}_{kk'} &= C_g \int \prod_{i=1}^3 (d^3 k_i d^3 k'_i) \delta(\vec{k} - \vec{k}_1 - \vec{k}_2 + \vec{k}_3) \delta(\vec{k}' - \vec{k}'_1 - \vec{k}'_2 + \vec{k}'_3) \delta(t - t') \mathbf{M}_1(t) \\ &\times B^{-1}[\mathcal{N}(t)]^{\mu\mu'} \left[\frac{1}{\omega_{(k+k')/2} - \omega_{(k_1+k'_1)/2} - \omega_{(k_2+k'_2)/2} + \omega_{(k_3+k'_3)/2} + i\epsilon\tau_3} \right]^{\mu'\nu'} B[\mathcal{N}(t)]^{\nu'\nu} \end{aligned} \quad (36)$$

with (23) and (28). The on-shell counter term is $\Sigma_Q(t, t')^{\mu\nu}_{kk'}$ in (30) itself, therefore the on-shell part of the total self-energy is given by

$$\Sigma^{(on)}(t, t')^{\mu\nu}_{kk'} = \Sigma_{loop}^{(on)}(t, t')^{\mu\nu}_{kk'} + \Sigma_Q(t, t')^{\mu\nu}_{kk'}. \quad (37)$$

Extending the arguments in our previous papers of the spatially homogeneous case [3, 4], we require the renormalization condition that the total on-shell self-energy in (37) should be diagonal in terms of the quasi particle operators ξ_p^μ and $\bar{\xi}_p^\mu$, i.e.,

$$\mathbf{B}(t)_{pk}^{\mu\mu'} \boldsymbol{\Sigma}^{(on)}(t, t')_{kk'}^{\mu'\nu'} \mathbf{B}^{-1}(t)_{k'p'}^{\nu'\nu} = (\text{diagonal with respect to the thermal indices}). \quad (38)$$

Although this is a 2×2 -matrix relation, it turns out that the following single equation suffices to satisfy the condition (38):

$$\begin{aligned} \dot{\mathbf{N}}(t)_{kk'} - i(\omega_{k'} - \omega_k) \mathbf{N}(t)_{kk'} &= -i \int d^3q \{ \delta \mathbf{W}(t)_{kq} \mathbf{N}(t)_{qk'} - \mathbf{N}(t)_{kq} \delta \mathbf{W}^*(t)_{qk'} \} \\ &+ 2\pi C_g \int \prod_{i=1}^3 (d^3k_i d^3k'_i) \delta(\vec{k} - \vec{k}_1 - \vec{k}_2 + \vec{k}_3) \delta(\vec{k}' - \vec{k}'_1 - \vec{k}'_2 + \vec{k}'_3) \\ &\times \delta(\omega_{(k+k')/2} - \omega_{(k_1+k'_1)/2} - \omega_{(k_2+k'_2)/2} + \omega_{(k_3+k'_3)/2}) \mathbf{M}_o(t) \end{aligned} \quad (39)$$

where

$$\begin{aligned} \delta \mathbf{W}(t)_{kk'} &\equiv C_g \int \prod_{i=1}^3 (d^3k_i d^3k'_i) \delta(\vec{k} - \vec{k}_1 - \vec{k}_2 + \vec{k}_3) \delta(\vec{k}' - \vec{k}'_1 - \vec{k}'_2 + \vec{k}'_3) \\ &\times \frac{\mathbf{M}_1(t)}{\omega_{(k+k')/2} - \omega_{(k_1+k'_1)/2} - \omega_{(k_2+k'_2)/2} + \omega_{(k_3+k'_3)/2} + i\epsilon}. \end{aligned} \quad (40)$$

Equation (39) is the kinetic equation.

Let us translate (39) into that in $(\vec{x}, \vec{k})$ -representation. To do this, we prepare the following formula: When the $*$ -product in double $\vec{k}$ -representation is defined by

$$(\mathbf{F}_1 * \mathbf{F}_2)_{k_1 k_2} = \int d^3q \mathbf{F}_{k_1 q} \mathbf{F}_{2 q k_2}, \quad (41)$$

one has

$$\begin{aligned} F_1 * F_2(\vec{x} : \vec{k}) &= \int \frac{d^3q_1 d^3q_2 d^3y_1 d^3y_2}{(2\pi)^6} e^{i(\vec{q}_1 \cdot \vec{y}_1 + \vec{q}_2 \cdot \vec{y}_2)} \\ &\times F_1(\vec{x} + \frac{\vec{y}_2}{2} : \vec{k} + \vec{q}_1) F_2(\vec{x} - \frac{\vec{y}_1}{2} : \vec{k} + \vec{q}_2) \\ &= \exp \left[-\frac{i}{2} \left(\frac{\partial}{\partial \vec{x}_1} \cdot \frac{\partial}{\partial \vec{k}_2} - \frac{\partial}{\partial \vec{x}_2} \cdot \frac{\partial}{\partial \vec{k}_1} \right) \right] F_1(\vec{x} : \vec{k}) F_2(\vec{x} : \vec{k}) \end{aligned} \quad (42)$$

in terms of

$$F(\vec{x} : \vec{k}) \equiv \int d^3q e^{i\vec{q} \cdot \vec{x}} \mathbf{F}_{k+\frac{q}{2}, k-\frac{q}{2}} \quad (43)$$

In the last expression of (42), $\partial/\partial\vec{x}_i$ and $\partial/\partial\vec{k}_i$ ($i = 1, 2$) operate on arguments in $F_i(\vec{x} : \vec{k})$.

With help of (42), (39) is rewritten in $(\vec{x}, \vec{k})$ -representation as

$$\begin{aligned}
& \dot{n}(t, \vec{x} : \vec{k}) - i\{\omega(\vec{k} - \frac{1}{2i}\vec{\nabla}_x) - \omega(\vec{k} + \frac{1}{2i}\vec{\nabla}_x)\}n(t, \vec{x} : \vec{k}) \\
&= -i \exp\left[-\frac{i}{2}\left(\frac{\partial}{\partial\vec{x}_W} \cdot \frac{\partial}{\partial\vec{k}_n} - \frac{\partial}{\partial\vec{x}_n} \cdot \frac{\partial}{\partial\vec{k}_W}\right)\right] \delta W(t, \vec{x} : \vec{k}) n(t, \vec{x} : \vec{k}) \\
&+ i \exp\left[\frac{i}{2}\left(\frac{\partial}{\partial\vec{x}_W} \cdot \frac{\partial}{\partial\vec{k}_n} - \frac{\partial}{\partial\vec{x}_n} \cdot \frac{\partial}{\partial\vec{k}_W}\right)\right] n(t, \vec{x} : \vec{k}) \delta W^*(t, \vec{x} : \vec{k}) \\
&+ i \delta W_0(t, \vec{x} : \vec{k}),
\end{aligned} \tag{44}$$

where

$$\begin{aligned}
\delta W(t, \vec{x} : \vec{k}) &= C_g \int d^3q \prod_{i=1}^3 (d^3k_i d^3k'_i) e^{i\vec{q}\cdot\vec{x}} \\
&\times \delta(\vec{k} + \frac{\vec{q}}{2} - \vec{k}_1 - \vec{k}_2 + \vec{k}_3) \delta(\vec{k} - \frac{\vec{q}}{2} - \vec{k}'_1 - \vec{k}'_2 + \vec{k}'_3) \\
&\times \frac{\mathbf{M}_1(t)}{\omega_{(k+k')/2} - \omega_{(k_1+k'_1)/2} - \omega_{(k_2+k'_2)/2} + \omega_{(k_3+k'_3)/2} + i\epsilon}
\end{aligned} \tag{45}$$

$$\begin{aligned}
&= C_g \int \prod_{i=1}^3 d^3L_i \delta(\vec{k} - \vec{L}_1 - \vec{L}_2 + \vec{L}_3) M_1(t, \vec{x} : \vec{L}_i) \\
&\times \frac{1}{\omega_{L_1+L_2-L_3} - \omega_{L_1} - \omega_{L_2} + \omega_{L_3} + i\epsilon}
\end{aligned} \tag{46}$$

$$\begin{aligned}
\delta W_0(t, \vec{x} : \vec{k}) &= -2\pi i C_g \int d^3q \prod_{i=1}^3 (d^3k_i d^3k'_i) e^{i\vec{q}\cdot\vec{x}} \\
&\times \delta(\vec{k} + \frac{\vec{q}}{2} - \vec{k}_1 - \vec{k}_2 + \vec{k}_3) \delta(\vec{k} - \frac{\vec{q}}{2} - \vec{k}'_1 - \vec{k}'_2 + \vec{k}'_3) \\
&\times \delta(\omega_{(k+k')/2} - \omega_{(k_1+k'_1)/2} - \omega_{(k_2+k'_2)/2} + \omega_{(k_3+k'_3)/2}) \mathbf{M}_0(t)
\end{aligned} \tag{47}$$

$$\begin{aligned}
&= -2\pi i C_g \int \prod_{i=1}^3 d^3L_i \delta(\vec{k} - \vec{L}_1 - \vec{L}_2 + \vec{L}_3) M_0(t, \vec{x} : \vec{L}_i) \\
&\times \delta(\omega_{L_1+L_2-L_3} - \omega_{L_1} - \omega_{L_2} + \omega_{L_3})
\end{aligned} \tag{48}$$

with

$$\begin{aligned}
M_0(t, \vec{x} : \vec{L}_i) &= n(t, \vec{x} : \vec{L}_1) n(t, \vec{x} : \vec{L}_2) (1 + n(t, \vec{x} : \vec{L}_3)) \\
M_1(t, \vec{x} : \vec{L}_i) &= (1 + n(t, \vec{x} : \vec{L}_1) + n(t, \vec{x} : \vec{L}_2)) n(t, \vec{x} : \vec{L}_3) \\
&\quad - n(t, \vec{x} : \vec{L}_1) n(t, \vec{x} : \vec{L}_2).
\end{aligned} \tag{49}$$

For comparison, we write down the famous Boltzmann equation,

$$\dot{n}(t, \vec{x} : \vec{k}) + \vec{v} \cdot \vec{\nabla}_x n(t, \vec{x} : \vec{k}) = \text{St}[n](t, \vec{x} : \vec{k}) \quad (50)$$

where the collision integral is given by

$$\begin{aligned} \text{St}[n](t, \vec{x} : \vec{k}) \equiv & \int \prod_{i=1}^3 d^3 L_i w(\vec{k}, \vec{L}_3 : \vec{L}_1, \vec{L}_2) \\ & \left\{ n(t, \vec{x} : \vec{L}_1) n(t, \vec{x} : \vec{L}_2) (1 + n(t, \vec{x} : \vec{k})) (1 + n(t, \vec{x} : \vec{L}_3)) \right. \\ & \left. - n(t, \vec{x} : \vec{k}) n(t, \vec{x} : \vec{L}_3) (1 + n(t, \vec{x} : \vec{L}_1)) (1 + n(t, \vec{x} : \vec{L}_2)) \right\}. \end{aligned} \quad (51)$$

For our present model, the transition rate w in the Born approximation takes the form of

$$w(\vec{k}, \vec{L}_3 : \vec{L}_1, \vec{L}_2) = 2\pi C_g \delta(\vec{k} - \vec{L}_1 - \vec{L}_2 + \vec{L}_3) \delta(\omega_k - \omega_{L_1} - \omega_{L_2} + \omega_{L_3}). \quad (52)$$

As is well-known, the entropy law follows from the Boltzmann equation (50).

In spatially homogeneous case, the kinetic equation derived from time-dependent TFD completely coincides with the above Boltzmann equation without $\vec{x}$ -dependence [3, 4]. Therefore, it was easy to show that the entropy law holds for spatially homogeneous time-dependent TFD.

Our kinetic equation in inhomogeneous case (44) differs from the Boltzmann equation. We see the following three sources of such differences.

(i) The term $-i\{\omega(\vec{k} - \frac{1}{2i}\vec{\nabla}_x) - \omega(\vec{k} + \frac{1}{2i}\vec{\nabla}_x)\}n$ replaces $\vec{v} \cdot \vec{\nabla}_x n$ in the Boltzmann equation which represents the classical kinematical effect of particle flow. We expand it around $\vec{k}$ as

$$\begin{aligned} -i\{\omega(\vec{k} - \frac{1}{2i}\vec{\nabla}_x) - \omega(\vec{k} + \frac{1}{2i}\vec{\nabla}_x)\} &= \frac{\partial \omega}{\partial \vec{k}} \cdot \vec{\nabla}_x + \dots \\ &= \left[\sum_{j=0}^{\infty} \frac{(-1)^j}{(2j+1)!} \left(\frac{1}{2} \frac{\partial}{\partial \vec{k}_\omega} \cdot \vec{\nabla}_x \right)^{2j} \right] \frac{\partial \omega}{\partial \vec{k}} \cdot \vec{\nabla}_x, \end{aligned} \quad (53)$$

where the suffix ω in $\frac{\partial}{\partial \vec{k}_\omega}$ is put to remind that this differentiation operates on ω but not on n . If we identify the group velocity of the wave,

$$\vec{v}_g = \frac{\partial \omega}{\partial \vec{k}}, \quad (54)$$

with the classical velocity $\vec{v}$, neglecting the higher derivative terms, we see a correspondence between our kinetic equation and the Boltzmann equation regarding the kinematical effect of particle flow. Our kinetic equation includes the nature of quantum wave.

(ii) The presence of the factor $\exp \left[-\frac{i}{2} \left(\frac{\partial}{\partial \vec{x}_W} \cdot \frac{\partial}{\partial \vec{k}_n} - \frac{\partial}{\partial \vec{x}_n} \cdot \frac{\partial}{\partial \vec{k}_W} \right) \right]$ on r. h. s. of (44) is due to the effect of spontaneous change in momentum of propagating particle. As a matter of fact, in (39) we get the terms like $\int d^3q \delta \mathbf{W}(t)_{pq} \mathbf{N}(t)_{qp'}$, from which the factor on the top of this paragraph originates.

(iii) The presence of $1/(\omega_{L_1+L_2-L_3} - \omega_{L_1} - \omega_{L_2} + \omega_{L_3} + i\epsilon)$ in (46) reflects the effect the energy uncertainty.

First consider a very crude approximation of our kinetic equation (44), namely, neglect (ii) (then automatically no (iii)), and keep only the $\vec{v}_g$ in (i). Then (44) simply reduces to the Boltzmann equation (50) built on a classical picture. It is natural since all the effects neglected above are of quantum origin.

Let us treat (44). We just rewrite it as

$$\dot{n}(t, \vec{x} : \vec{k}) - i \left\{ \omega(\vec{k} - \frac{1}{2i} \vec{\nabla}_x) - \omega(\vec{k} + \frac{1}{2i} \vec{\nabla}_x) \right\} n(t, \vec{x} : \vec{k}) = \text{St}[n](t, \vec{x} : \vec{k}) + \Delta \text{St}[n](t, \vec{x} : \vec{k}) \quad (55)$$

where $\text{St}[n](t, \vec{x} : \vec{k})$ is found in (51) and

$$\begin{aligned} \Delta \text{St}[n](t, \vec{x} : \vec{k}) \equiv & -i \left(\exp \left[-\frac{i}{2} \left(\frac{\partial}{\partial \vec{x}_W} \cdot \frac{\partial}{\partial \vec{k}_n} - \frac{\partial}{\partial \vec{x}_n} \cdot \frac{\partial}{\partial \vec{k}_W} \right) \right] - 1 \right) \delta W(t, \vec{x} : \vec{k}) n(t, \vec{x} : \vec{k}) \\ & + i \left(\exp \left[\frac{i}{2} \left(\frac{\partial}{\partial \vec{x}_W} \cdot \frac{\partial}{\partial \vec{k}_n} - \frac{\partial}{\partial \vec{x}_n} \cdot \frac{\partial}{\partial \vec{k}_W} \right) \right] - 1 \right) n(t, \vec{x} : \vec{k}) \delta W^*(t, \vec{x} : \vec{k}). \end{aligned} \quad (56)$$

Expanding the exponential factors and using (46), we have

$$\begin{aligned} \Delta \text{St}[n](t, \vec{x} : \vec{k}) = & C_g \int \prod_{i=1}^3 d^3 L_i \\ & \left[\sum_{j=0}^{\infty} \frac{(-1)^{j+1}}{(2j+1)! 2^{2j}} \left(\frac{\partial}{\partial \vec{x}_W} \cdot \frac{\partial}{\partial \vec{k}_n} - \frac{\partial}{\partial \vec{x}_n} \cdot \frac{\partial}{\partial \vec{k}_W} \right)^{2j+1} \delta(\vec{k} - \vec{L}_1 - \vec{L}_2 + \vec{L}_3) \right. \\ & \left. \times \mathcal{P} \frac{1}{\omega_{L_1+L_2-L_3} - \omega_{L_1} - \omega_{L_2} + \omega_{L_3}} \right] \end{aligned}$$

$$\begin{aligned}
& + \Sigma_{j=1}^{\infty} \frac{(-1)^{j+1}}{(2j)! 2^{2j}} \left(\frac{\partial}{\partial \vec{x}_W} \cdot \frac{\partial}{\partial \vec{k}_n} - \frac{\partial}{\partial \vec{x}_n} \cdot \frac{\partial}{\partial \vec{k}_W} \right)^{2j} \delta(\vec{k} - \vec{L}_1 - \vec{L}_2 + \vec{L}_3) \\
& \quad \times 2\pi \delta(\omega_{L_1+L_2-L_3} - \omega_{L_1} - \omega_{L_2} + \omega_{L_3}) M_1(t, \vec{x} : \vec{L}_i) n(t, \vec{x} : \vec{k})
\end{aligned} \tag{57}$$

$$\begin{aligned}
& \simeq -C_g \int \prod_{i=1}^3 d^3 L_i \left(\frac{\partial}{\partial \vec{x}_W} \cdot \frac{\partial}{\partial \vec{k}_n} - \frac{\partial}{\partial \vec{x}_n} \cdot \frac{\partial}{\partial \vec{k}_W} \right) \delta(\vec{k} - \vec{L}_1 - \vec{L}_2 + \vec{L}_3) \\
& \quad \times \mathcal{P} \frac{M_1(t, \vec{x} : \vec{L}_i)}{\omega_{L_1+L_2-L_3} - \omega_{L_1} - \omega_{L_2} + \omega_{L_3}} n(t, \vec{x} : \vec{k}),
\end{aligned} \tag{58}$$

where only the lowest gradient term is picked up in the last expression.

Now introduce the entropy density $S(t, \vec{x})$ by

$$S(t, \vec{x}) \equiv \int d^3 k s(t, \vec{x} : \vec{k}) \tag{59}$$

$$s(t, \vec{x} : \vec{k}) \equiv (1 + n(t, \vec{x} : \vec{k})) \ln(1 + n(t, \vec{x} : \vec{k})) - n(t, \vec{x} : \vec{k}) \ln n(t, \vec{x} : \vec{k}). \tag{60}$$

Its time derivative becomes

$$\dot{S}(t, \vec{x}) = \int d^3 k \dot{n}(t, \vec{x} : \vec{k}) \ln \frac{1 + n(t, \vec{x} : \vec{k})}{n(t, \vec{x} : \vec{k})} \tag{61}$$

We substitute (55) with (53) and (57) or (58). Here we explicitly show the result up to the first non-trivial gradient terms, although *inclusion of higher spatial derivative terms does not change our final conclusion of the entropy law below*. Noting that $\partial n / \partial \vec{k} \ln \frac{1+n}{n} = \partial s / \partial \vec{k}$ and $\partial n / \partial \vec{x} \ln \frac{1+n}{n} = \partial s / \partial \vec{x}$ and performing integral in part with respect to $\vec{k}$, we manipulate

$$\begin{aligned}
& \int d^3 k \Delta \text{St}[n](t, \vec{x} : \vec{k}) \ln \frac{1 + n(t, \vec{x} : \vec{k})}{n(t, \vec{x} : \vec{k})} \\
& = \vec{\nabla}_x \cdot \left[-C_g \int d^3 k \prod_{i=1}^3 d^3 L_i \left\{ \delta(\vec{k} - \vec{L}_1 - \vec{L}_2 + \vec{L}_3) \right. \right. \\
& \quad \left. \left. \times \mathcal{P} \frac{M_1(t, \vec{x} : \vec{L}_i)}{\omega_{L_1+L_2-L_3} - \omega_{L_1} - \omega_{L_2} + \omega_{L_3}} \frac{\partial s(t, \vec{x} : \vec{k})}{\partial \vec{k}} \right\} + O(\vec{\nabla}_x) \right].
\end{aligned} \tag{62}$$

Thus we have the following equation for the entropy density,

$$\dot{S}(t, \vec{x}) + \vec{\nabla}_x \cdot \vec{J}_s(t, \vec{x}) = \text{Pr}[n](t, \vec{x}) \tag{63}$$

where

$$\vec{J}_s(t, \vec{x}) \equiv \int d^3 k \left[\frac{\partial s(t, \vec{x} : \vec{k})}{\partial \vec{k}} \left\{ \vec{v}_g + C_g \int \prod_{i=1}^3 d^3 L_i \delta(\vec{k} - \vec{L}_1 - \vec{L}_2 + \vec{L}_3) \right. \right.$$

$$\times \mathcal{P} \frac{M_1(t, \vec{x} : \vec{L}_i)}{\omega_{L_1+L_2-L_3} - \omega_{L_1} - \omega_{L_2} + \omega_{L_3}} \Big\} + O(\vec{\nabla}_x) \Big] \quad (64)$$

$$\text{Pr}[n](t, \vec{x}) \equiv \int d^3k \text{St}[n](t, \vec{x} : \vec{k}) \ln \frac{1 + n(t, \vec{x} : \vec{k})}{n(t, \vec{x} : \vec{k})}. \quad (65)$$

As $\text{St}[n](t, \vec{x} : \vec{k})$ inside $\text{Pr}[n](t, \vec{x})$, given by (51), is the collision integral of the classical Boltzmann equation, it can be proved that $\text{Pr}[n](t, \vec{x})$ is the production of the entropy per unit time and unit volume:

$$\text{Pr}[n](t, \vec{x}) \geq 0. \quad (66)$$

In summary, we have the equation of continuity (63) with the production source for entropy $\text{Pr}[n](t, \vec{x})$ in (65), which leads to the conclusion that the total entropy,

$$\mathcal{S}(t) \equiv \int d^3x S(t, \vec{x}), \quad (67)$$

increases in time,

$$\frac{d}{dt} \mathcal{S}(t) \geq 0. \quad (68)$$

The source term $\text{Pr}[n](t, \vec{x})$ remains the same as that in the classical Boltzmann equation. During this derivation of the entropy law we found a natural definition of the entropy flow $\vec{J}_s(t, \vec{x})$ given by (64).

The definition of the on-shell part of self-energy in (36) and (37) and the renormalization condition on it in (38) were crucial. The basis of an argument at more profound level to justify the choice of the on-shell part is open.

In spatially inhomogeneous case the momentum is not a good quantum number, which is the very reason for momentum-mixing in the present formalism. It is then a natural attempt to look for a good quantum number in inhomogeneous situation, by which quasi-particle states are labelled. This possibility should be pursued in future study.

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Figure Captions

Fig. 1 The loop self-energy diagram without vertex correction.

Fig. 1