Non-Abelian Black Holes and Catastrophe Theory

I : Neutral Type

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Abstract

We re-analyze the globally neutral non-Abelian black holes and present a unified picture, classifying them into two types; Type I (black holes with massless non-Abelian field) and Type II (black holes with “massive” non-Abelian field). For the Type II, there are two branches: The black hole in the high-entropy branch is “stable” and almost neutral, while that in the low entropy branch, which is similar to the Type I, is unstable and locally charged. To analyze their stabilities, we adopt the catastrophe theoretic method, which reveals us a universal picture of stability of the black holes. It is shown that the isolated Type II black hole has a fold catastrophe structure. In a heat bath system, the Type I black hole shows a cusp catastrophe, while the Type II has both fold and cusp catastrophe.

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1 Introduction

The discovery by Bartnik and McKinnon of a non-trivial particle-like structure (BM particle) in the Einstein-Yang-Mills system is quite surprise[1], because it is known that there is no non-trivial solution in the Einstein theory and Yang-Mills(YM) theory respectively and because the work of Bekenstein[2], Hartle[3] and Teitelboim[4] shows that stationary black hole solutions are similarly hairless in a variety of theories coupling classical fields to Einstein gravity. After this solution, a variety of self-gravitating structure with non-Abelian fields have been found. Besides the BM particle, the colored black hole[5] in the same system, the Skyrmion[6, 7] or the Skyrme black hole[8, 7, 9], the dilatonic BM particle or dilatonic colored black hole[10, 9] the particle solution with massive Proca field (Procaon) or the Proca black hole[11], the monopole[12, 13, 14] or the black hole in monopole (monopole black hole)[13, 14, 15], and the sphaleron[16, 11] or sphaleron black hole[11] are discovered. Last two solutions (monopole and sphaleron) are obtained in Einstein-Yang-Mills-Higgs system.

However, the colored black hole is unstable against the radial perturbations[17]. While, the Skyrme black hole solution really stirred our interest for the non-Abelian black hole, since it turns out to be a stable solution[18]. Hence, it can be the likeliest candidate of a counterexample for the black hole no-hair conjecture[19]. There are two Skyrme black holes with the same horizon radius: one is stable and the other is unstable[9]. The Proca black hole and the sphaleron black hole have the similar properties, although the latter is always unstable. The monopole black hole is globally charged, and stable except for that with fine-tuned parameters. The dilatonic colored black hole is a natural extension of that in Einstein-Maxwell-dilaton system, which showed an interesting feature in thermodynamics[20]. We found that the dilatonic colored black hole has similar but more complicated properties[9].

Because of the non-Abelian gauge fields, each study revealed that in spite of a simple ansatz (spherical symmetry) those non-Abelian black holes have complicated structures and show a variety of properties, as shown in the above. Are there any common properties in those non-Abelian structures? Can we find any universal understanding for them? Answering these questions is the main purpose of the present paper. In order to reach our goal, we first classify non-Abelian black holes into two classes; one is globally neutral ones and the other is globally charged ones, because the charged monopole black hole is obviously different from others and should be discussed in a different context with the Reissner-Nordström black hole. We will discuss those two cases separately[21]. In this paper we just focus on neutral case. The neutral holes can be classified into two: one is a class of black holes with massless non-Abelian field (Type I), and the other is that with “massive” non-Abelian field (Type
II), i.e., the gauge or chiral field either has a mass in the original Lagrangian or get its mass through a symmetry breaking mechanism by Higgs fields. This classification provides us a unified picture for non-Abelian black holes and their particle solutions.

One of the most important questions about the above self- gravitating non-Abelian structure or black holes is, are they stable? We answer this question via a catastrophe-theoretic analysis, which shows its power over various phenomena. Catastrophe theory may provide us a clearer understanding than a linear perturbation method, because it visualizes the stability of non-Abelian black holes. Another merit in using this theory is that the way to investigate the stability becomes very simple. The elementary catastrophe of Type I black hole is found to be a fold type, which causes the existence of stable and unstable solutions. We also apply this to stability analysis of black holes in a heat bath and show that more complicated high order elementary catastrophe appears. The swallows tail will appear in the monopole black hole[7]. In §2, we shortly review non-Abelian black holes, and provide a unified picture. The stability analysis of non-Abelian black holes via a catastrophe theory are given in §3. Section 4 includes discussions and some remarks.

2 Summary of Non-Abelian Black Holes

We have re-analyzed five non-Abelian theories, which are listed in Table 1. Each theory has non-trivial self-gravitating structure (particle solution) and non-trivial black holes. We present a short review on those black hole solutions and summarize their properties in order to find out a universal picture, which will be obtained in the end of this section.

2.1 Colored Black Holes

We first consider the Einstein-Yang-Mills system, which action is described as[22];

\[ S = \int d^4x \sqrt{-g} \left[ \frac{1}{2\kappa^2} R(g) - \frac{1}{16\pi g_C^2} \text{Tr} F^2 \right], \tag{2.1} \]

where \( \kappa^2 = 8\pi G \), and \( F \) is the YM field expressed by its potential \( A \) as \( F = dA + A \wedge A \). \( g_C \) is a self-coupling constant of the YM field. In the spherically symmetric static spacetime, the metric is written as

\[ ds^2 = - \left( 1 - \frac{2Gm}{r} \right) e^{-2\xi} dt^2 + \left( 1 - \frac{2Gm}{r} \right)^{-1} dr^2 + r^2 \left( d\theta^2 + \sin^2 \theta d\phi^2 \right), \tag{2.2} \]
where a mass function \( m = m(r) \) and a lapse function \( \delta = \delta(r) \) depend only on the radial coordinate \( r \). In order to find an asymptotically flat black hole solution with a regular event horizon, we have to require the following boundary conditions,

(i) asymptotic flatness, i.e., as \( r \to \infty \),

\[
m(r) \to M = \text{finite} \tag{2.3}
\]
\[
\delta(r) \to 0. \tag{2.4}
\]

(ii) the existence of a regular horizon \( r_H \), i.e.,

\[
2GM_H = r_H, \tag{2.5}
\]
\[
\delta_H < \infty. \tag{2.6}
\]

(iii) nonexistence of singularity outside the horizon, i.e., for \( r > r_H \)

\[
2GM(r) < r, \tag{2.7}
\]

where the variables with a subscript \( H \) denote those values at the horizon.

The most generic form of a spherically symmetric SU(2) YM potential[23] is

\[
\mathbf{A} = a\mathbf{e}_r dt + b\mathbf{e}_r dr + [d\mathbf{e}_\theta - (1 + w)\mathbf{e}_\phi] d\theta + [(1 + w)\mathbf{e}_\theta + d\mathbf{e}_\phi] \sin \theta d\phi, \tag{2.8}
\]

where \( a, b, d \) and \( w \) are functions of time and radial coordinates, \( t \) and \( r \). We have adopted the polar coordinate description \((\mathbf{e}_r, \mathbf{e}_\theta, \mathbf{e}_\phi)\), i.e.

\[
\mathbf{e}_r = \frac{1}{2i}[\sigma_1 \sin \theta \cos \phi + \sigma_2 \sin \theta \sin \phi + \sigma_3 \cos \theta], \tag{2.9}
\]

\[
\mathbf{e}_\theta = \frac{1}{2i}[\sigma_1 \cos \theta \cos \phi + \sigma_2 \cos \theta \sin \phi - \sigma_3 \sin \theta], \tag{2.10}
\]

\[
\mathbf{e}_\phi = \frac{1}{2i}[-\sigma_1 \sin \phi + \sigma_2 \cos \phi], \tag{2.11}
\]

whose commutative relations are

\[
[\mathbf{e}_a, \mathbf{e}_b] = \mathbf{e}_c \quad a, b, c, = r, \theta, \text{ or } \phi. \tag{2.12}
\]

Here \( \sigma_i \) \((i = 1, 2, 3)\) denote the Pauli spin matrices. We can set the ansatz \( a \equiv 0 \) (’tHooft ansatz, i.e., purely magnetic YM field strength exists) and eliminate \( b \) using a residual gauge freedom. We can also set \( d \equiv 0 \) in this case[24]. In the static case, the remaining function \( w \) depend only on the radial coordinate \( r \). As a result, we obtain a simplified spherically symmetric YM potential as

\[
\mathbf{A} = (1 + w)[-\mathbf{e}_\phi d\theta + \mathbf{e}_\theta \sin \theta d\phi]. \tag{2.13}
\]
Substituting this into \( F = dA + A \wedge A \), we have an explicit form of the field strength:

\[
F = -w' \tau_\phi dr \wedge d\theta + w' \tau_{\theta} dr \wedge \sin \theta d\phi - (1 - w^2) \tau_\phi d\theta \wedge \sin \theta d\phi. \tag{2.14}
\]

As for the boundary conditions for YM field, we impose

\[
w \to \pm 1, \tag{2.15}
\]

which guarantees a finiteness of the energy of system. On account of these boundary conditions (2.3) \( \sim \) (2.7) and (2.15), the basic field equations have to be solved as an eigen value problem and the solutions are obtained only by means of numerical analysis except for trivial solutions such as the Schwarzschild black hole \( (w \equiv \pm 1) \) or the Reissner-Nordström black hole \( (w \equiv 0) \).

A colored black hole has the following properties: First, in the limit of \( r_H \to 0 \), we find a particle-like solution, which is the BM particle and cannot exist without gravity. For any values of mass, the colored black hole solutions exist, but when its mass increases, the structure gets similar to that of the Schwarzschild black hole. Because in the limit \( r_H \to \infty \), i.e., \( M \to \infty \), the energy contribution from the non-Abelian YM fields keeps still finite while the mass energy of the singularity at the center must get to infinity. The YM field makes no contribution to the black hole structure, although it can have nontrivial distribution on the Schwarzschild background. The colored black hole has discrete mass spectrum, which is characterized by the node number of YM potential. All solutions are unstable against the radial linear perturbations. The number of the unstable modes increases as the node number increases[26].

From the analysis of its temperature, we showed that the sign changes of the specific heat occur twice[9]. When the black hole is larger some critical mass scale \( (M_{1, cr} = 0.905mp/g_C) \) or smaller than the other critical mass scale \( (M_{2, cr} = 1.061mp/g_C) \), the specific heat is negative, but in between two critical mass scales, the effective charge, which is defined by a surface integration of the YM field at the horizon, becomes so large that the YM field seems to be dominated and consequently its specific heat becomes positive.

### 2.2 Dilatonic Colored Black Holes

Next, we consider models with the following action;

\[
S = \int d^4 x \sqrt{-g} \left[ \frac{1}{2\kappa^2} R(g) - \frac{1}{2\kappa^2} (\nabla \sigma)^2 - \frac{1}{16\pi g_C^2} e^{-\alpha \sigma} \text{Tr} F^2 \right], \tag{2.16}
\]

where \( \sigma \) is a dilaton field, and \( F \) and \( A \) are the YM field strength and its potential, respectively. \( \alpha(\geq 0) \) is a coupling constant of the dilaton field \( \sigma \) to the YM field \( F \). This
type of action arises from various unified theories including a superstring model[20]. For example, $\alpha = 1$ and $\alpha = \sqrt{3}$ are the cases arising from a superstring theory and from the 5-dimensional Kaluza-Klein theory, respectively. Setting $\alpha = 0$ with $\sigma \equiv 0$, the model (2.16) reduces to the Einstein-Yang-Mills system, which was discussed in §2.1.

The dilatonic colored black hole is an extension of the colored black hole, i.e., the colored black hole with a scale invariant massless scalar field $\sigma$. The spacetime and the YM potential ansatz are the same as those of the colored black hole. As for the asymptotic behavior of dilatonic field $\sigma$, we impose

$$\sigma \to 0, \quad \text{as} \quad r \to \infty,$$

(2.17)
in order to make the energy of system finite.

The properties of a dilatonic colored black hole are quite similar to the colored black hole besides its thermodynamical behavior. For a small coupling constant model ($\alpha < 0.5$) the sign change of specific heat occurs twice as the colored black hole does. However, for the larger the coupling constant $\alpha$, there is no sign change. The similar aspect was also seen for the Einstein-Maxwell-dilaton (EMD) system, although the criteria for the sign change of specific heat was somewhat different, i.e., $\alpha < 1[20]$. It is remarkable that the extension of a gauge field such as $U(1) \to SU(2)$ preserve the similar property. This may be caused by the fact that the (dilatonic) colored black hole has the similar spacetime structure to that of Reissner-Nordström black hole near the horizon and the strong coupling reduces its effective charge as in the EMD system.

### 2.3 Proca Black Holes

Next we consider a particle-like structure with massive “YM” field (Proca field) and its black hole solutions. Those are obtained from the action

$$S = \int d^4x \sqrt{-g} \left[ \frac{1}{2\kappa^2} R(g) - \frac{1}{4\pi g_C} \left( \frac{1}{4} \text{Tr} F^2 - \frac{\mu^2}{2} \text{Tr} A^2 \right) \right],$$

(2.18)

where $\mu$ is a mass of the vector field $F$. Although the mass term breaks a gauge invariance and such a theory is not renormalizable, the Proca model may be useful as an effective theory of massive spin 1 field. Furthermore, it may also be a simple and good model to understand common properties of black holes in Type II black hole which includes more realistic models discussed in the next two subsections, and to reveal its essence.

We assume a spherically symmetric static metric (2.2) and vector potential (2.13) and that their boundary conditions are the same as those of colored black hole.
Solving the Einstein-Proca equations, we show the $M - r_H$ relation in Fig. 1(a). The Proca black holes have the following properties: In the limit of $r_H \to 0$, we find two particle-like solutions. One corresponds to a self-gravitating particle solution (we shall call it Procaon), which can exist without gravity, i.e., in the Minkowski space, and the other has similar properties to those of the BM particle. Two branches of black hole solutions, which leave from those two particles, merged at some critical horizon radius. Beyond this critical point, where the black hole has a maximal mass $M_C$ and a maximal horizon radius $r_{HC}$, there exist no non-trivial structure. One interesting feature in Fig. 1(a), which turns out to be important later, is that there is a cusp structure at the critical point. Since the black hole entropy is given by

$$S = A/4 = \pi r_H^2,$$  

(2.19)

where $A$ is its area, the upper branch in Fig. 1(a) has larger entropy than that in the lower branch. Hence, we shall call each of them, the high- and low-entropy branches respectively. The low-entropy branch is similar to the colored black hole solution, and in fact it approaches to the colored black hole in the “low-energy” limit. The high-entropy branch approaches the Schwarzschild black hole in the “low-energy” limit. Here, the “low-energy” limit means that the mass of the non-Abelian field $\mu$ is much smaller than the Planck mass, $m_P = G^{-\frac{1}{2}}$. In the limit of “high-energy”, no solution exist. Both branches vanish around $\mu \sim m_P/g_C$.

From the stability analysis by a catastrophe theory given in the next section, we expect that the high-entropy branch is stable, while the low-entropy branch is unstable against radial perturbations. Since this was numerically proved for the Skyrme black holes (see §2.4), we believe that it is also true for the Proca black holes from our unified picture.

What makes the difference between the high-entropy and the low-entropy branches? To understand it from a stability point of view, we shall see the energy distribution of the YM field. The energy of YM fields consists of two terms, $\rho_{F^2}$ which is from the kinetic term and $\rho_{A^2}$ which is from the mass term. In Fig. 2 we show $\rho_{F^2}$ and $\rho_{A^2}$ of the Proca black hole separately. The total energy density is the sum of them, i.e. $\rho_{total} = \rho_{F^2} + \rho_{A^2}$. $\rho_{A^2}$ decays as $r^{-2}$ near the horizon because $w \sim 1$ and it drops very quickly at some radius around the Compton wave length of YM field ($\sim 1/\mu$) which is marked by an arrow in Fig. 2. As for $\rho_{F^2}$, it is complicated because various factors are tangles. But it also decay rapidly at the same radius as $\rho_{A^2}$ does so. The main difference between the high-entropy and low entropy branches is that $\rho_{A^2}$ is still dominated at the Compton wave length in the high-entropy branch while $\rho_{F^2}$ becomes dominant there in the low-entropy branch. We in general expect the mass term give a contribution to stabilize the structure. This is the case for the high-entropy branch because the typical size of the structure is about $1/\mu$, where $\rho_{A^2}$ is still
dominant. On the other hand, in the low-entropy branch $\rho_{F_2}$ becomes already dominant inside of the structure, then the mass term cannot stabilize the system. The extreme case is the colored black hole ($\rho_{M^t} \equiv 0$).

The specific heat in the high-entropy branch is always negative like the Schwarzschild black hole, while the specific heat in the low-entropy branch change its sign a few times. We show a numerical result in Fig. 3(a). When the mass of the non-Abelian field is small, i.e., in the “low-energy” case, we find the sign changes three times which occur at the critical mass scales of $M_{1,cr}$, $M_{2,cr}$, and $M_{3,cr}$ (In Fig. 3(a), $M_{1,cr} = 0.909 m_P / g_C$, $M_{2,cr} = 1.06 m_P / g_C$, and $M_{3,cr} = 1.62 m_P / g_C$, for $\mu = 0.05 m_P / g_C$). The critical point $M_{2,cr}$ is the same type as that of the Reissner-Nordström black hole and the smaller one ($M_{1,cr}$) corresponds to the lower critical point in the (dilatonic) colored black hole type. Between $M_{1,cr}$ and $M_{2,cr}$, the specific heat becomes positive, when the effective charge at horizon becomes large. The larger critical point ($M_{3,cr}$) appears only in Type II black holes. Beyond $M_{3,cr}$, the specific heat becomes positive again. As the mass of the non-Abelian field $\mu$ increases, however, $M_{1,cr}$ and $M_{2,cr}$ merge and the sign change of the specific heat changes occurs only once. This change may describe some important feature for the stability of black holes in a heat bath (see §3.2).

2.4 Skyrme Black Holes

In the following two subsections, we consider two realistic models. Here we deal with the Skyrme field as non-Abelian fields. The $\text{SU}(2) \times \text{SU}(2)$ invariant action coupled to gravity is given by

$$S = \int d^4x \sqrt{-g} \left( \frac{1}{2\kappa^2} R(g) + \frac{1}{4} f_S^2 \text{Tr} A^2 - \frac{1}{32g_S^2} \text{Tr} F^2 \right),$$

(2.20)

where $F$ and $A$ are the field strength and its potential, respectively, and $f_S$ and $g_S$ are coupling constants. We use the gauge coupling constant $g_S$ instead of $g_C$, both of which are the same up to a constant, i.e.,

$$g_S = \sqrt{4\pi} g_C.$$

(2.21)

For a spherically symmetric static solution, the spacetime metric is (2.2). As for the boundary conditions, we again require the asymptotic flatness (2.3), (2.4), the existence of a regular event horizon (2.5), (2.6), and the regularity of the spacetime (2.7).

$A$ and $F$ are described in terms of the $\text{SU}(2)$-valued function $U$ as

$$A = U^\dagger \nabla U, \quad F = A \wedge A.$$  

(2.22)
In a spherically symmetric static case, we make the hedgehog ansatz for $U$, i.e.,

$$U(x) = \cos \chi(r) + i \sin \chi(r) \sigma_i \hat{r}^i,$$

(2.23)

where the $\sigma_i$ denote the Pauli spin matrices and $\hat{r}^i$ is a radial normal. The finiteness of the energy of system yields the asymptotic boundary condition for $\chi$ as;

$$\chi \to 0, \quad \text{as} \quad r \to \infty. \quad (2.24)$$

For the particle-like solution (Skyrmion), the value of $\chi$ at the origin must be $2\pi n$, where $n$ is an integer and $|n|$ denotes the winding number of the Skyrmion. But in the case of the black hole solution, because it is topologically trivial, the winding number defined by

$$W_n \equiv \frac{1}{2\pi} |\chi_H - \sin(\chi_H)|,$$

(2.25)

is no longer an integer[8]. Nevertheless, since $W_n$ is close to $n$, we shall also call $n$ the “winding” number of the Skyrme black hole.

The properties of Skyrme black holes are quite similar to those of the Proca black holes. Replacing the mass parameter $\mu$ in the Proca model with that in the Skyrme model, i.e., $\mu = g_s f_s$, all discussions in §2.3 is applied to the Skyrme black hole except for a difference between symmetries of their vector fields. It is worth noting that a family of solutions provides a cusp structure in the $M - r_H$ plane as the Proca black holes (see Fig. 1(b)).

As we already mentioned in the Proca model, the high-entropy branch is stable, while the low-entropy branch is unstable[7, 8, 17, 18] against radial perturbations and this property is understood by a catastrophe theory. The specific heat is also quite similar to that of the Proca black holes (see Fig. 3(b)).

In addition, we should remark the following argument derived from its topological structure in the Skyrme model. For each exited state, i.e., each solution with larger winding number, we find the same cusp structure, which consists of the high- and low-entropy branch, in the $M - r_H$ plane[9]. Since the colored black hole has $n$ unstable modes for an $n$-node solution[17, 26], we expect that the low-entropy branch of the Skyrme black holes with the winding number $n$, has $n$ unstable modes. It is confirmed by the fact that the low-entropy branch approaches the colored black hole in the “low-energy” limit. While, the high-entropy branch may have no unstable mode, because the particle-like Skyrmion solution is topologically non-trivial, i.e., its homotopy group is not trivial. Although its black hole solution does not have such a topological invariance, they may be stable. There is no proof, but we believe that there is no stability change between a particle-like solution and the associated black hole solutions. In fact, we will show in the following section that a
catastrophe theory can be used to analyze the stability. Then we expect that there is no stability change between a particle-like solution and the associated black hole solutions and the Skyrme black hole solutions in high-entropy branch with any winding number must be stable.

2.5 Sphaleron Black Holes

As a final example, we start with the following bosonic part of the Lagrangian in the standard theory:

\[ S = \int d^4x \sqrt{-g} \left[ \frac{1}{2\kappa^2} R(g) - \frac{1}{16\pi g^2_C} \text{Tr} F^2 - \frac{1}{16\pi} (D_\mu \Phi) \right]^2 \Phi - \frac{1}{16\pi} V(\Phi) \right] \right] , \quad (2.26) \]

where

\[ D_\mu = \partial_\mu + \tau \cdot A_\mu, \quad (2.27) \]

\[ F = dA + A \wedge A, \quad (2.28) \]

\[ V(\Phi) = \lambda(\Phi \Phi - \Phi_0^2)^2. \quad (2.29) \]

\( F \) and \( A \) are the SU(2) YM field strength and its potential, respectively, and \( \Phi \) is the Higgs field. \( g_C \) is a coupling constant of the YM field, and \( \lambda \) and \( \Phi_0 \) are a self-coupling constant and a vacuum expectation value of the Higgs field, respectively.

The most general complex doublet of Higgs fields can be written as

\[ \Phi(x) = \frac{1}{\sqrt{2}} \exp \left[ -\tau \cdot \xi(x) \right] \left( \begin{array}{c} 0 \\ \Phi(x)/r \end{array} \right). \quad (2.30) \]

For spherically symmetric static solution we impose the usual ansatz[25]:

\[ \xi(x) = 2\pi \hat{r}; \quad \Phi(x) = \Phi(r), \quad (2.31) \]

The asymptotic behavior of the Higgs field is assumed to be

\[ \Phi(r) \to \Phi_0, \quad \text{as} \quad r \to \infty, \quad (2.32) \]

which means that the vacuum is in a symmetry broken state at infinity. As for the spacetime metric, the YM potential, and the boundary conditions we again set (2.2), (2.8), (2.3)~(2.7) and (2.15).

Using the above ansatz we solve the Einstein-Yang-Mills-Higgs equations numerically by means of shooting method. The shape of a family of solutions in the \( M - r_H \) plane (Fig.1(c)) and the behavior of the specific heat (Fig. 3(c)) may lead us to believe that the sphaleron
black holes have the same properties as the Skyrme black hole and the Proca black hole. The reason why we find the same properties is that the gauge field gets its mass, $\mu = g \Phi_0$, through a spontaneous symmetry breaking by the Higgs field. However, one important difference exists from the point of view of its stability. For the Skyrme black hole and the Proca black hole, high-entropy branch is stable against radial linear perturbations, but the sphaleron black hole is always unstable because of the topological reason. When we discuss stability in general, there are many modes to be investigated. A general argument about the instability of the sphaleron is based on a topological analysis\cite{[16]}, which does not specify any modes. For the sphaleron without gravity, the stability analysis with a spherically symmetric ansatz was done\cite{[27]}. It was explicitly shown that there is only one unstable mode. For the case of self-gravitation sphaleron or the sphaleron black hole, it is stable in the high-entropy branch against radial perturbations except for one unstable mode corresponding to the above. In the low-entropy branch, at least one more unstable mode appears\cite{[28]}. In this sense, the high entropy branch is more stable than the low-entropy branch. Hence, we argue that the high-entropy branch is “stable” while the low-entropy one is unstable. With this argument, we can make a unified picture for all neutral cases including the sphaleron black hole as well.

2.6 A Unified Picture

From the above summary, we can classify non-Abelian black holes by their properties into two types: Type I (black holes with massless non-Abelian field) and Type II (black holes with “massive” non-Abelian field). The colored black hole and the dilatonic colored black hole are classified as Type I. While the Proca black hole, the Skyrme black hole, and the sphaleron black hole are Type II. Those non-Abelian fields have a “mass” $\mu$. $\mu = g s \Phi_0$ for the Einstein-Skyrme system, and $\mu = g c \Phi_0$ for the Einstein-Yang-Mills-Higgs system. Each type of black hole has the following common properties.

1. Type I Black Hole

(1) This type of black hole is in an equilibrium state by balance between a repulsive force by the massless gauge field and the gravitational force. It has no upper bound of their mass. It may be essential for no upper bound of a mass that the non-Abelian field is massless. When the mass gets large, however, the spacetime approaches to the Schwarzschild black hole.

(2) The field strength at the horizon $B_H$, which is defined by

$$B_H \equiv (Tr F^2)^{\frac{1}{2}} \bigg|_{\text{horizon}},$$

may give us a naive idea about how much the black hole is locally charged. Here, the expres-
ion $B_H$ has been used because only the radial component of magnetic part of non-Abelian field is finite at the horizon. From Fig. 4, in which we show $B_H$ with respect to $M$, we find that the colored black hole is locally charged. Although they are globally neutral, their structure near the horizon is similar to that of the Reissner-Nordström black hole. When the black hole gets large, $B_H$ decreases, that is, it becomes effectively less charged because it approaches the Schwarzschild solution.

(3) These black holes are unstable against radial perturbations. The number of unstable modes increases as the node number gets large.

(4) The sign changes of the specific heat occur twice, but for the large the coupling constant ($\alpha \gtrsim 0.5$) in the dilatonic colored black holes, there is no sign change. This is because the strong coupling reduces its effective charge near the horizon. Those behaviors can be understood by a stability analysis of the black holes in a heat bath via a catastrophe theory (see §3.2).

2. Type II Black Hole

(1) This type of black hole has an upper bound of the mass scale, beyond which no non-trivial solution exist. There are two branches: one is “stable” and the other is unstable. In the “low energy” limit ($\mu \to 0$), the stable branch approaches the Schwarzschild black hole, while the unstable one converges to the colored black hole. The typical mass scale of non-trivial particle in the unstable branch is about $m_P/g_C$.

We suspect that the mass of the non-Abelian field causes the existence of the upper bound of the black hole mass, though it has not been proved. The non-trivial structure balanced between a repulsive force by the massive gauge field and the gravitational force may have a characteristic size $1/\mu$, which is the Compton wave length. The horizon of non-Abelian black hole must exist inside of the structure, otherwise the black hole swallows whole non-trivial structure, resulting in the trivial Schwarzschild black hole. The condition of $r_H \lesssim 1/\mu$ yields the upper bound of the black hole mass.

Furthermore, when a mass of the non-Abelian field $\mu$ is large enough, the horizon scale ($\lesssim m_P/g_C$) becomes larger than $1/\mu$, and then the non-Abelian field is again swallowed into the black hole beyond the event horizon. For this reason, there is no non-trivial solution in the “high-energy” limit ($\mu \sim m_P$).

(2) The high-entropy branch is “stable”, while the low-entropy branch is unstable. This behavior of stability is well understood by a catastrophe theory (see §3.1).

(3) As for the structure of Type II black hole, those in the low-entropy branch is quite
similar to the Type I. However, the structure of black holes in the high-entropy branch may be different from that of Type I. In the high-entropy branch, $\rho_{A^2}$ is always dominant in the whole structure, which exist effectively until the Compton wave length $\sim 1/\mu$. It stabilizes the black hole. On the other hand in the low-entropy branch $\rho_{F^2}$ becomes dominant inside of the structure. This is one of the reasons why the low-entropy branch is still unstable as the colored black hole.

(4) The specific heat in the high-entropy branch is always negative like the Schwarzschild black hole, while that in the low-entropy branch changes its sign a few times depending on the mass of the non-Abelian field $\mu$. When $\mu$ is small, i.e., in the “low-energy” case, we find the sign changes three times at $M_{1,cr}, M_{2,cr}, and M_{3,cr}$. The second critical point ($M_{2,cr}$) is the same type as that of the Reissner-Nordström black hole and the lowest one ($M_{1,cr}$) corresponds to the lower critical point in the Type I black hole. Between $M_{1,cr}$ and $M_{2,cr}$, the specific heat becomes positive. Beyond the largest critical point $M_{3,cr}$, which appears only in Type II black holes, the specific heat becomes positive again. But as $\mu$ increases, $M_{1,cr}$ and $M_{2,cr}$ merge and the sign change of the specific heat changes occurs only once. These changes may describe some important feature for the stability of black holes in a heat bath, which can be analyzed by a catastrophe theory (see §3.2).

(5) $B_H$ is still small for the high-entropy branch. This black hole is approximately neutral. On the other hand, for the low-entropy branch, $B_H$ is finite and rather large as seen from Fig. 4. This type of black hole is locally charged but globally neutral just as the colored black hole.

The negative specific heat in the high-entropy branch is also consistent with that of the Schwarzschild or Schwarzschild-de Sitter spacetime. On the other hand, for low-entropy branch, although the black hole is globally neutral, $B_H$ does not vanish at the horizon and the black hole is locally charged, which yields the sign change of the specific heat.

We summarize the above properties in Table 2.

3 Stability of Black Holes via Catastrophe Theory

As we mentioned, Type I black hole is unstable against radial perturbations for any coupling constant or in any mass scale, but in the case of Type II its aspect is rather complicated. They have two types of solutions; one of which is “stable” and the other is unstable, and they merge at a critical mass $M_C$ in the $M - r_H$ plane with a cusp. There are various
phenomena in nature whose stabilities change at this kind of cusp, and such a stability change is understood by means of a catastrophe theory.

The catastrophe theory is a new mathematical tool to explain a variety of change of states in nature, in particular a discontinuous change of states which occur eventually in spite of gradual changes of parameters of a system. It is widely applied today in various research fields, e.g., the structural stability, the crystal lattice, biology, embryology, linguistics etc., and as a matter of course, in astrophysics[29, 30]. Here we shall apply the catastrophe theory, which is different from a usual linear perturbation method, to analyze a stability of non-Abelian black holes.

In order to examine a stability with the catastrophe theory, we must first find control parameters, state variables and a potential function of the non-Abelian black hole system. They could be different depending on the type of the black hole (whether Type I or II) and on the environment around the black hole (whether the isolated system or that in a heat bath).

The next step to examine a stability of a certain system with the catastrophe theory is to draw the equilibrium space $M_V$, which consists of extrema of the potential function, i.e., of a family of solutions of the system, in the space of control parameters and state variables. Then we project the equilibrium space onto a control parameter space (which may also be called the control plane in the 2-dimensional case as the present models) by a catastrophe map $\chi_V : M_V \to R^N$, where $N$ is the number of the control parameters. There may exist singular points on the equilibrium space, where the Jacobian of the mapping $\chi_V$ vanishes. The image of the set of singular points is called the bifurcation set $B_V$. If such singular points exist, the mapping $\chi_V$ from $M_V$ to the control parameter space is singular and then the number of solutions with the same control parameters changes beyond the bifurcation set $B_V$ and then the stability also changes there. Hence, by looking at this bifurcation set, we can classify our models into several elementary catastrophe and show the properties about stability of the system as will be shown in below[31].

We have summarized our results in Table 3 together with the types of elementary catastrophe.

3.1 Isolated Black Holes

3.1.1 Type I non-Abelian black hole

Assuming that the state variable is the field strength at horizon $B_H$, the control parameters are the mass of black hole $M$ and the coupling constant $\alpha$, and the potential function
is the black hole entropy $S = \pi r_H^2$, we performed the above procedure for the dilatonic colored black hole. We find that its bifurcation set is an empty set, namely, all points in the equilibrium space are regular. Hence, the dilatonic colored black hole has no change of stability in any mass scale or for any coupling constant. This conclusion is consistent with the known fact that all dilatonic colored black holes are unstable, which is obtained from the linear perturbation method.

### 3.1.2 Type II non-Abelian black hole

Next we examine the Type II black hole, which is more interesting than Type I, because the cusp in $M - r_H$ plane provides us some catastrophe theoretic features. We show the equilibrium space and the bifurcation set of the Skyrme black holes in Fig. 5. We have here adopted $M$ and the mass of non-Abelian field $\mu (= g_S f_S)$ as control parameters, and $B_H$ as a state variable, and $S$ as a potential function, respectively. The equilibrium space looks like a piece of cloth folded and the catastrophe mapping from the equilibrium space to the control plane provides a line, which is the evidence of a fold catastrophe.

When a system has two control parameters like in the present case, two types of catastrophe are possible; one of which is a cusp catastrophe and the other is a fold catastrophe. To explain what will happen, however, the latter does not need two control parameters but only one parameter. Hence we have only to focus one parameter in the present model. We keep our eyes on the change of black hole mass $M$ leaving the mass of non-Abelian field $\mu$ fixed. There is two reasons why we choose $M$ as the essential control parameter instead of $\mu$.

One is that the fundamental parameters in the Lagrangian (the mass of the particle $\mu$, the coupling constants $g_C$, $g_S$, $f_S$, the vacuum expectation value $\Phi_0$ and so on) have certain fixed values for each theory and then it may be physically meaningless, though mathematically interesting, to consider a change of the mass of non-Abelian field $\mu$[32]. Another reason is that we are interested in what will happen on such a black hole, in particular, on how does its stability change when the mass of black hole changes through accretion of surrounded matter or by the Hawking radiation.

Fixing one control parameter $\mu$, we find a smooth solution curve on the equilibrium space (which is called the equilibrium curve). There exist two solutions on this curve for each mass. The solution with the smaller value of $B_H$ is stable against radial perturbations, i.e., in the high-entropy branch, while that with larger value of $B_H$ is unstable, i.e., in the low-entropy branch.

How does a catastrophe come in this system? Assume that there is a stable black hole
solution expressed by some point on the curve labeled $e$ in Fig.5. When its mass gets large, the point shifts to the right along the curve $e$, and reaches eventually to the end point (a singular point), beyond which there is no solution. Then the solution point is forced to jump to other stable solution discontinuously as a solid arrow in Fig.5. This is a catastrophe. This phenomenon is explained by watching a potential function. The schematic forms of the potential function (the black hole entropy $S$) depend on the control parameters as shown in Fig.6. In the region of small $\mu$ and small $M$ (the left hand side of $B_V$), the potential function has two extremal points, one of which is a maximal point corresponding to the stable high-entropy solution and the other is a minimal point corresponding to the unstable low-entropy type. Usually a minimal and a maximal points of a potential function denote a stable and an unstable solutions, respectively. However, since we use the entropy of the system as a potential function, the correspondence becomes reverse. When the mass of the black hole increases and the solution point reaches to the bifurcation set $B_V$, the maximal and the minimal points merge and turn to be a inflection point. Furthermore when the mass gets larger, i.e., the right hand side of $B_V$, there is no extremal point and no black hole solution. Catastrophe comes about with disappearance of a maximal and a minimal points.

We plot a family of the Skyrme black hole solutions in the 3- dimensional $M - B_H - S$ space and its projections onto each 2- dimensional plane. This 3-dimensional picture provides us clearer understanding and some new results (Fig. 7). In catastrophe theory, solutions are regarded as extremal points on the Whitney surface, $S = S(M, B_H)$, when the control parameter $M$ is fixed. At the maximal entropy in Fig. 7, the solution turns out to be an inflection point, beyond which there is no black hole solution. The projection curves onto the $M - B_H$ and $B_H - S$ plane and the original 3-dimensional curve are all smooth, while the projection onto the $M - S$ plane, and then that onto the $M - r_H$ plane shows a cusp structure.

A family of black hole solutions connects two particle-like solutions with a smooth curve like a “bridge”, though there is a “pit fall” of the catastrophe on it. One particle is “stable” and the other is unstable. This can be understood through the black hole solution with a catastrophe theory. Such a relationship between the particle-like solutions would not have been found out, if the associated black hole solutions were not investigated.

### 3.2 Black Holes in a Heat Bath

In thermodynamics, we often consider two complimentary situations: one is an isolated adiabatic system and the other is an isothermal state in a heat bath. For the former case, the entropy $S$ is the fundamental variable and the thermal equilibrium is realized at an
entropy maximum, while for the latter case the Helmholtz free energy $F$ is the fundamental variable and the equilibrium is obtained at a minimum point of the free energy.

In the black hole physics, we know that black holes have many analogous properties with the thermodynamical laws. After Hawking discovered a thermal radiation from a black hole, the thermodynamical interpretation of black hole physics by this analogy turns to be more realistic. The black hole may be regarded as the a thermodynamical object with the energy $M$, the entropy $S = A/4$ and the temperature $T = \kappa/4\pi$, where $A$ and $\kappa$ are the area and the surface gravity of the black hole, respectively. Although a black hole is not exactly a thermodynamical object, it may be very interesting to investigate what will happen when a black hole is put in a “heat reservoir”.

Here we shall consider an ideal situation in which a reservoir keeps the temperature of a black hole in a heat bath unchanged. The thermal radiation around a black hole plays a role of heat reservoir. The radiation goes into the black hole or the Hawking radiation comes out from the black hole to keep both temperatures same. For example, the isolated Schwarzschild black hole is stable, however the Schwarzschild black hole in a box, which is filled by radiation, is always unstable when the box is infinitely large[33]. The reason why it is so may be understood by the fact that the specific heat is negative. As for the Reissner-Nordström black hole, there is a sign change of the specific heat at $Q = \sqrt{3}M/2$. This sign change may correspond to a kind of phase transition[34], i.e., we expect that the black hole in a heat bath becomes stable when $Q > \sqrt{3}M/2$.

Although there is still a deep and unsolved problem about the interpretation of such a sign change of the specific heat and a stability in a “heat bath” [35], here we will deal with a black hole just as a conventional thermodynamical object. Then the Helmholtz free energy

$$F = M - TS$$

(3.1)

is a fundamental potential function. The extremal point describes an equilibrium state, i.e., the minimum and maximum correspond to a stable and an unstable equilibrium states, respectively. Using the Helmholtz free energy $F$, we shall consider what will happen when the non-Abelian black holes are put into a heat bath.

### 3.2.1 Type I non-Abelian black hole

In the catastrophe theory of the dilatonic colored black holes in a heat bath, we adopt the temperature $T$, rather than mass $M$, and the coupling constant $\alpha$ as control parameters and the Helmholtz free energy $F$ as a potential function instead of the entropy $S$ (see Table 3). We plotted the equilibrium space in Fig. 8. For the low coupling constant solutions, which
includes the colored black hole as the case of $\alpha = 0$, two folds appear, while no fold is seen in the case of high coupling constant. This configuration gives a cusp on the control plane through a catastrophe map. The bifurcation set is also shown in Fig. 8, which in fact has a cusp. Hence we can conclude that this type of stability belongs to a cusp catastrophe. As we can see from the behavior of the potential function shown in Fig. 9, a stable solution exists only in the region ABC lying between two curves AC and BC in Fig. 9, where two unstable solutions exist as well. The stable solution is characterized by the fact that the value of $\partial B_H/\partial T$ is negative in the equilibrium space, while in the the most part of the equilibrium space where its value is positive the solutions are unstable. This aspect is inverse of the normal cusp catastrophe, in which stable solutions exist for any value of control parameters and one unstable mode appears only in a folded region. Hence, strictly speaking, the system of Type I black hole in a heat bath may be classified into a dual cusp catastrophe.

This type of catastrophe reveals some interesting properties. Suppose that there is a stable Type I black hole which is expressed by a point on the curve $e$ in Fig. 8. When the temperature of the black hole either increases or decreases, the solution point reaches to either $C_1$ or $C_2$, beyond which there is no stable solution. Then the solution point may jump to the Schwarzschild black hole along one of the solid arrows. Though the final state are almost the same whichever the temperature increases or decreases, behaviors of the potential function are a little bit different as will be shown in below.

When the temperature of the black hole increases and the point reaches at $C_1$, the maximal point $m_1$ and the minimal point $m_2$ on the bifurcation set (the curve BC) merge, leaving one maximal point $m_3$. However, the black hole does not go to $m_3$ because it is unstable. Instead the solution will jump to more simple Schwarzschild black hole. When the temperature decreases, and the solution point crosses the curve AC, the minimal point $m_2$ and the maximal point $m_3$ merge, leaving the maximal point $m_1$. The solution again jumps to the Schwarzschild black hole.

As we mentioned, a catastrophe occurs on the bifurcation set. Is there any physical change which characterizes the bifurcation set in the present model? In the case of Reissner-Nordström black hole in a heat bath, the bifurcation set satisfies $M = \sqrt{Q}$, which is the point where the specific heat changes its sign and a kind of second order phase transition may take place. For the present non-Abelian black holes, from our numerical analysis, we also expect that the bifurcation set consists of the points where the specific heat changes its sign. This result is consistent with the fact that the sign change occurs twice for the small coupling constant $\alpha$ while no sign change occurs for the large values.
3.2.2 Type II non-Abelian black hole

The stability analysis of Type II black hole in a heat bath is more complicated. We plotted the equilibrium space of the Skyrme black hole and the bifurcation set on the control plane (Fig. 10) as an example. The bifurcation set consists of two components, one of which is the curve corresponding to a fold catastrophe and the other is the cusp shape corresponding to a cusp catastrophe.

We showed the behaviors of the potential function in Fig. 11. Fixing $\mu$, it is interesting to examine how the potential function looks like as the temperature of the black hole changes. There are qualitatively four cases (a)~(d) as shown by dotted lines in Fig. 11. The shape change of the potential function is shown in Fig. 12. For the case (a), first one minimum $m_1$ and one maximum $m_2$ exist for the low temperature. When the temperature increases (the solution point moves to right), the point eventually reaches to the curve AB, where an inflection point appears. Beyond this point, since there are two minima and two maxima in the region ABCD, another maximum $m_3$ and minimum $m_4$ appears. Increasing the temperature further, the minimum $m_2$ and the maximum $m_3$ merge and turn to be a inflection point on the curve CD. The maximum $m_1$ and the minimum $m_4$ are left. Suppose that an initially stable solution is at the minimum $m_2$. As the temperature increases, it will disappear and jumps to the minimum $m_4$ beyond the curve CD. A catastrophe occurs. Although the free energy at minimal point $m_4$ gets smaller than that of minimal point $m_2$ before the curve CD, the solution does not go to the point $m_4$, because the point $m_2$ is locally stable. Note that quantum effects break this rule because of a quantum tunneling.

The case (b) after the curve EC is qualitatively not different from the case (a), except that new phase appears before the curve EC. For the case (c), the appearance of the maximum $m_3$ and the minimum $m_4$ is accomplished at BF earlier than that of $m_1$ and $m_2$, which occurs on the line BC. The stable solution $m_4$, which exists initially at the low temperature, continues to exist and there happens no catastrophe even if the temperature increases. Hence, a stable black hole at the low temperature can be different depending on the coupling constant $\alpha$. When the temperature increases, some solutions continue to exist while a catastrophe occurs for other solutions and the spacetime structure changes discontinuously. This is interesting, because we do not know whether our solution is in the case (a) (or the case (b)) or in the case (c) only from the black hole structure at the low temperature. The last case (d) is easy to understand because the shape of a potential function changes only on the BF.

We found two types of elementary catastrophe (a fold and a cusp types) from the bifurcation set. Unless the number of control parameters is more than two, however, such a catastrophe can never happen at the same time according to Thom’s theorem[36]. Hence
we expect that this system may have higher dimensional elementary catastrophe and the bifurcation set in Fig. 10 is just a cross section of a higher bifurcation set. It may be a swallow tail catastrophe, a butterfly catastrophe or else.

4 Concluding Remarks

We have re-analyzed five known globally neutral non-Abelian black holes and presented a unified picture. Those are classified into two types depending on whether the non-Abelian field is massless (Type I) or massive (Type II). The Type I non-Abelian black holes (the colored black hole and the dilatonic colored black hole) have one particle-like solution (the BM particle) in zero entropy limit. Although those black holes can have any large mass, the spacetime approaches to the Schwarzschild black hole when its mass gets large. The specific heat changes its sign twice for the low coupling constant or no changes for the high coupling constant. The Type II non-Abelian black holes have two types of solutions: one belongs to the “stable” high-entropy branch and the other to the unstable low-entropy branch. These two branches merge at a critical mass $M_C$ and provide a cusp there in the $M - S$ plane. There is no solution beyond the cusp on which the solution has maximal mass and maximal entropy. The high-entropy branch has no change of the sign of specific heat and resembles to the Schwarzschild de-Sitter black hole, while the low-entropy branch changes the sign of specific heat a few times and is similar to the Type I black hole. This is consistent with the fact that the high-entropy branch tends to a family of the Schwarzschild solutions and the low-entropy branch converges to a family of the colored black hole solutions as the mass of the non-Abelian field decreases.

In order to analyze the stability of non-Abelian black hole, we adopt a catastrophe theoretic method. The Type II non-Abelian black hole shows a fold type of elementary catastrophe and the Type I black hole in a heat bath has a cusp catastrophe. In case of the Type II black hole in a heat bath, we find a fold type and a cusp type structures in the bifurcation set. Since we may not have both in one elementary catastrophe, this structures may be unified into a higher dimensional elementary catastrophe.

Although a catastrophe theory may be a good tool to examine a stability in nature, we should give some critical comments as well. First, when we discuss a stability change using a catastrophe theory, we usually focus on some specific modes though many modes have to be investigated. Hence even if we find a stable branch from the catastrophe theoretic
analysis, it does not always mean that such a system is completely stable. For example, the equilibrium space of the sphaleron black hole is similar to that of the Skyrme black hole in Fig. 5. From its structure we might conclude that the lower (small $B_H$) solutions in the equilibrium space are stable while the upper (large $B_H$) solutions are unstable. But this is wrong for the sphaleron black holes because both of solutions are unstable for the topological reason. What is true is that the lower solutions are stable for a certain mode while the upper solutions become unstable for the same mode. Hence although the analysis is very easy, we must be very careful to conclude about a stability of a system by using a catastrophe theory.

Secondly, we have not succeeded to include the most stable Schwarzschild solution in our catastrophe theoretic analysis. We can expect that after a catastrophe occurs, the solution will jump to the Schwarzschild black hole branch. But since the Schwarzschild solution does not appear in the equilibrium space, we cannot expect which mass of the black hole will be found after the catastrophe. We have to solve the dynamical equations. However, since we have still the area theorem in the present models[26], the area must increase. When the non-Abelian black hole collapses into the Schwarzschild solution, a part of the energy of the YM or other fields may be emitted into infinity. In that case, the mass energy is not conserved and the final Schwarzschild black hole becomes less massive. As a result, we can conclude that the non-Abelian black hole will jump into the range between the same mass state, which corresponds to no emission of the non-Abelian fields, and the same entropy state, which corresponds to a maximal emission of the fields. If the entropy of black hole is not conserved, this catastrophic jump is a kind of first order transition.

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(see Bizon in [5]). We can set $C = 0$ without loss of generality for the colored black hole. However, in the case of the Einstein-Yang-Mills-Higgs system, we have found 
new sphaleron solution keeping this freedom of $d$ (see [25]).

[25] Under more general YM field configuration (2.8) and Higgs field ansatz (2.30), we find 


[32] However, if we consider the case that the coupling constants change with the temperature or the energy scale as running coupling constants, we may find something different in the history of the universe from the discussion here.


Figure Captions

FIG 1: The mass-horizon radius diagrams for (a) the Proca black hole with $\mu/g_{CM} = (i)$ 0.05, (ii) 0.10, and (iii) 0.15, (b) the Skyrme black hole with $\mu(= g_S f_S)/g_{SM} = (i)$ 0.01, (ii) 0.02, and (iii) 0.03, (c) the sphaleron black hole with $\mu(= g_{C} \Phi_0)/g_{CM} = (i)$ 0.1, (ii) 0.2, and (iii) 0.3. $C$ is a cusp, where the black hole has a maximal entropy. $C$ is a cusp, where the black hole has a maximal entropy. Beyond its entropy there is no non-Abelian black hole. The Schwarzschild black hole (the dot-dashed line) and the colored black hole (the dotted line) are also shown as references. The high-entropy (large horizon radius) branch is “stable”, while the low-entropy (small horizon radius) branch is unstable.

FIG 2: The distributions of energy density for (a) the colored black hole (Type I), (b) low-entropy branch, and (c) the high-entropy branch of the Proca black hole (Type II) with $\mu/g_{CM} = 0.05$. The horizon radius of each black hole is $0.01 g_{C}/l_{P}$. We plot the enetry densities $\rho_{F^2}$ from the kinetic term and $\rho_{A^2}$ from the mass term separately. At the Compton Wave length of the massive YM field ($\sim 1/\mu$), which marked by an arrow, $\rho_{A^2}$ is still dominant in the high-entropy branch while $\rho_{F^2}$ becomes dominant in the low-entropy branch just the same as the colored black hole.

FIG 3: The mass-temperature relations for (a) the Proca black hole with $\mu/g_{CM} = (i)$ 0.05, (ii) 0.10, and (iii) 0.15, (b) the Skyrme black hole with $\mu/g_{SM} = (i)$ 0.01, (ii) 0.02, and (iii) 0.03, (c) the sphaleron black hole with $\mu/g_{CM} = (i)$ 0.1, (ii) 0.2, and (iii) 0.3. The specific heat in the high-entropy branch is always negative. While, that in the low-entropy branch changes its sign three times for small values of $\mu$ and once for large values of $\mu$. 
FIG 4: The field strength at the horizon $B_H$ with respect to $M$ for the colored black hole (Type I) and the Proca black hole with $\mu = 0.05g_cm_p$ (Type II). For the Type I and the low-entropy branch of the Type II, we find rather large value of $B_H$, which means that the black hole is locally charged near the horizon. When the black hole gets large its value decreases because the spacetime approaches to the Schwarzschild solution. For the high-entropy branch of the Type II, however, the value of $B_H$ is very small. Such a black hole is almost neutral everywhere.

FIG 5: The equilibrium space of the Skyrme black hole. Upper side and lower side of the equilibrium space correspond to the unstable and stable solutions respectively. When the black hole mass gets large (along the curve e), the stable and the unstable solutions merge at a critical mass scale. A catastrophe occurs at this critical point and the Skyrme black hole will jump to the Schwarzschild black hole (along the solid arrow). On the control plane, we draw the bifurcation set $B_V$, which shows a fold catastrophe. To make it easy to see, we have moved down the level of the control plane lower than the original position.

FIG 6: The potential function $S$ (entropy) of the Skyrme black hole. A maximal point and a minimal point exist in the left hand side of the bifurcation set $B_V$, while in the right hand side there is no extremal point.

FIG 7: The solution curve in the three dimensional space of $(M, B_H, S)$ and its projections onto each two dimensional plane for the Skyrme black hole with $\mu = 0.02g_sm_p$. The cusp $C$ in $M - S$ plane is a critical point for stability. For the fixed control parameter $M$, there are two solutions at extremal points on the Whitney surface: the maximal one is stable, but the minimal one is unstable. Beyond the critical point $C$, there is no extremal point, i.e., no non-Abelian black hole.
FIG 8: The equilibrium space and the bifurcation set of the dilatonic colored black hole in a heat bath. From the shape of the bifurcation set, we classify this system into a cusp catastrophe.

FIG 9: The potential function $F$ (Helmholtz free energy) of the dilatonic colored black hole in a heat bath. In the interior of the region ABC, there are three extremal points $m_1, m_2,$ and $m_3,$ one of which ($m_2$) is stable and others ($m_1, m_3$) are unstable. In the other region, however, only one unstable solution. Such a configuration is not seen in a usual cusp catastrophe but may be regarded as a dual cusp catastrophe.

FIG 10: The equilibrium space and the bifurcation set of the Skyrme black hole in a heat bath. The bifurcation set consists of a cusp structure ECD and a smooth curve ABF. The appearance of both a smooth curve and a cusp at same time means that this system is be classified into one elementary catastrophe with two control parameters but may into higher type (a swallow’s tail or a butterfly whose number of the control parameters is higher than three).

FIG 11: The potential function $F$ (Helmholtz free energy) of the Skyrme black hole in a heat bath. In the region ABCD, there are two maxima and two minima points ($m_1 \sim m_4$). The four lines labeled (a)$\sim$(d) are used in the discussion about the evolution of the black holes. Each line correspond to different values of the mass $\mu$.

FIG 12: The behaviors of the potential function of the Skyrme black hole in a heat bath. There are four cases labeled (a)$\sim$(d) which correspond to the line labeled (a)$\sim$(d) in Fig.11. The characters at the bottom of several figures, e.g. AB, CD, mean that the figures are those on the curves EBC, CD, or ABF of the bifurcation set in Fig. 11.
Table Captions

**TABLE 1:** The models we re-analyzed and the names of the non-Abelian black holes and non-trivial particles. We classify them into two types; Type I (models with massless non-Abelian field) and Type II (models with massive non-Abelian field).

**TABLE 2:** The properties of Type I and II black holes. “small” or “low” in Type I means that its value is similar to small or low in Type II. See text about the meaning of “stable” for the sphaleron black hole. $C_\#$ denotes how many times the sign of the specific heat changes in the branch.

**TABLE 3:** The control parameters, the state variable and the potential function of Type I and II black holes. The type of elementary catastrophe for each system is also given.