Eigenvalues of the Weyl Operator as Observables of General Relativity

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We consider the eigenvalues of the three-dimensional Weyl operator defined in terms of the (Euclidean) Ashtekar variables, and we study their dependence on the gravitational field. We notice that these eigenvalues can be used as gravitational variables, and derive explicit formulas for their Poisson brackets and their time evolution.

Abstract
A longstanding open problem in general relativity is the problem of finding a complete set of diffeomorphism invariant quantities; or quantities that have vanishing Poisson brackets with the canonical constraints (to our knowledge, the problem was first posed by Peter Bergman in reference [1]). Quantities of these kind could be related to physical measurements quite directly, and might be effective tools for studying Einstein’s equations and for the quantization of the theory [2]. Considerable effort has been expended in the past [3], and recently [4], on this problem. Alain Connes has recently observed that a natural set of diffeomorphism invariant objects is provided by the spectrum of the Dirac operator of the metric manifold, and has suggested that the operator’s eigenvalues might play a role as natural gravitational variables [5]. We begin here a simple preliminary exploration of this suggestion.

We consider the spectrum of the three-dimensional self-dual Weyl operator constructed in terms of the Ashtekar formalism. This operator, and thus its spectrum, depends on the geometry, namely on the (Ashtekar’s) gravitational fields $A_a(x)$ and $\tilde{\sigma}^a(x)$. We can think at the Weyl operator’s eigenvalues $\lambda_n$ as diffeomorphism invariant variables describing the geometry of the spacetime manifold; or, equivalently, as functions $\lambda_n[A_a, \tilde{\sigma}^a]$ on the phase space of general relativity, which commute with the diffeomorphism constraint and with the internal gauge constraint. In order to find the explicit form of these functions, we should solve the Weyl operator eigenvalue problem on a general geometry, a task clearly beyond our capacities. In spite of this, a surprising amount of progress can be made in analysing the properties of the $\lambda_n$ variables. In particular, we show in this paper that their evolution equation in coordinate time, as well as the Poisson brackets
between them can be obtained explicitly.

Ideally, one would like to re-express general relativity entirely in terms of the $\lambda_n$ variables. This would yield an interesting result: a fully 3-d diffeomorphism invariant formulation of the theory. In practice, we are far from such a result, because we have no control on the injectivity and surjectivity of the map that sends $[A_\alpha, \tilde{\sigma}^\alpha]$ in $\lambda_n$. More seriously, we are not able to express the formulas we obtain for the Poisson brackets and for the time evolution of the $\lambda_n$, solely in terms of the eigenvalues themselves. In spite of these clear limits, however, we view our preliminary results as somehow encouraging. The time evolution equation that we obtain is very simple, and the theory seems to be easily adapted to a formulation in terms of eigenvalues. There are several directions in which one may proceed further. For instance, we have not explored the possibility of studying a four-dimensional Dirac-like operator. More substantially, operator-algebraic techniques may exploit the idea of describing the geometry in terms of the spectrum of a Dirac-like operator [6] with much more powerful instruments, perhaps even sidestepping the surjectivity problem by exploring the possibility of non-commutative extensions of the theory [6]. From this last point of view, our work can be seen as a preliminary explorations of an important structure underlying general relativity, which may deserve substantial consideration.

We begin by considering the canonical formulations of Euclidean general relativity in the Astekar formalism [7]. This is given as follows. We fix a three dimensional compact manifold $M$ and two real fields $A^i_\alpha(x)$ and $\tilde{E}^i_\alpha(x)$
on M. We use \(a, b, \ldots = 1, 2, 3\) for (abstract) spatial indices and \(i, j, \ldots = 1, 2, 3\) for internal su(2) indices. We indicate coordinates on M with \(x\). As is well known, the relation between these fields and the conventional metric gravitational variables is as follows: the Ashtekar connection \(A^i_a(x)\) is the projection on a constant time surface of the self-dual part of the gravitational spin connection; its conjugate momentum \(\tilde{E}^i_a(x)\) is the (densitized) inverse triad, which is related to the three dimensional metric \(g_{ab}(x)\) of the constant time surface (which raises and lowers the spatial \(a, b\) indices) by

\[
g_{ab} \tilde{E}^a_i \tilde{E}^b_i = g_{ij} A^i_a(x) \tau_i,
\]

where \(g\) is the determinant of \(g_{ab}\). We recall that taking \(A^i_a(x)\) and \(\tilde{E}^i_a(x)\) as real fields yields the Euclidean theory. We have chosen a compact three-space in order to simplify the spectral properties of the Weyl operator.

We shall use a spinorial formalism, which is more appropriate for dealing with the Weyl operator. The spinorial version of the Ashtekar variables is given in terms of the Pauli matrices \(\tau_i, i = 1, 2, 3\), by

\[
\tilde{\sigma}^i(x) = -\frac{i}{\sqrt{2}} \tilde{E}^i_a(x) \tau_i,
\]

\[
A^i_a(x) = -\frac{i}{2} A^i_a(x) \tau_i.
\]

Thus, \(A^i_a(x)\) and \(\tilde{\sigma}^i(x)\) are 2x2 complex matrices. In some equations we will need to write the matrix indices explicitly: we use upper case indices \(A, B = 1, 2\) for the spinor space on which the Pauli matrices act. Thus, the components of the gravitational fields are \(A^A_B(x)\) and \(\tilde{\sigma}^A_B(x)\). We indicate the curvature of \(A^a\) as \(F_{ab}\). We shall need also the non-densitized version of
the soldering form

\[ \sigma_a = g^{-1/2} \bar{\sigma}_a. \]  

In terms of this notation the dynamics of Euclidean general relativity is given by the fundamental Poisson brackets

\[ \{ A^{AB}_a(x), \bar{\sigma}_{CD}^b(y) \} = \frac{1}{\sqrt{2}} \delta^b_a \delta_C^a \delta_D^B \delta^3(x,y); \]

and the constraints

\[ C_b = \sqrt{2} \text{Tr}[\bar{\sigma}^a F_{ab}] \approx 0, \]
\[ C = \sqrt{2} \mathcal{D}_a \bar{\sigma}^a \approx 0, \]
\[ S = \text{Tr}[\bar{\sigma}^a \bar{\sigma}^b F_{ab}] \approx 0. \]

Here \( C \) is a matrix in spinor space, and \( S \) is a scalar density. These constraints generate the infinitesimal gauge transformations

\[ \delta_{N^a} f = \int d^3y \left[ \{ f, C_a(y) \} - \text{Tr}[\{ f, C(y) \} A_a(y)] \right] N^a(y), \]
\[ \delta_{\rho} f = \int d^3y \text{Tr}[\{ f, C(y) \} \rho(y)], \]
\[ \delta_{\bar{\mathcal{N}}} f = \int d^3y \left\{ f, C(y) \right\} \bar{\mathcal{N}}(y); \]

where \( N^a \) is the Shift function, which generates spatial diffeomorphisms; \( \bar{\mathcal{N}} \) is the Lapse function, which generates coordinate time evolution; and \( \rho \) is a matrix field in spinor space, which generates the local internal rotations. The action of these generators on the fundamental variables is

\[ \delta_{\bar{\mathcal{N}}} \bar{\sigma}^a = \frac{1}{\sqrt{2}} [D_b \left( \bar{\mathcal{N}} \bar{\sigma}^b \bar{\sigma}^a - \bar{\mathcal{N}} \bar{\sigma}^a \bar{\sigma}^b \right)], \]
The problem that we consider in this work is the definition of gauge invariant quantities. More precisely, we are interested in three-dimensional and four-dimensional observables. We denote a functional of the elementary fields $F[A_a, \bar{\sigma}^a]$ as three-dimensional observable if

$$\delta_{\mathcal{N}} A_a = \frac{1}{\sqrt{2}} \left[ T \bar{\sigma}^b F_{a_b} - F_{a_b} \bar{\sigma}^b \right],$$

$$\delta_{\rho} \bar{\sigma}^a = -\rho \bar{\sigma}^a + \bar{\sigma}^a \rho,$$

$$\delta_{\rho} A_a = D_a \rho,$$

$$\delta_{N^a} \bar{\sigma}^a = -N^b \partial_b \bar{\sigma}^a + (\partial_a N^b) \bar{\sigma}^a,$$

$$\delta_{N^a} A_a = -N^b \partial_b A_a - (\partial_a N^b) \partial_b A_a.$$

and we denote it as four-dimensional observable if

$$\delta_{\mathcal{N}} F[A_a, \bar{\sigma}^a] \simeq \delta_{\rho} F[A_a, \bar{\sigma}^a] \simeq 0;$$

and we denote it as four-dimensional observable if it is three-dimensional observable and

$$\delta_{\mathcal{N}} F[A_a, \bar{\sigma}^a] \simeq 0.$$

Consider spinor fields on $\mathcal{M}$, that is, two-components complex fields $\lambda$, with components $\lambda^A(x)$. We follow the standard convention [8] of raising and lowering spinor indices as in $\lambda_A = \lambda^B \epsilon_{BA}$ and $\lambda^A = \epsilon^{AB} \lambda_B$, where $\epsilon^{AB}$ and $\epsilon_{AB}$ are antisymmetric and $\epsilon_{12} = \epsilon^{12} = 1$. The spinors in a point $x$ in $\mathcal{M}$ form a two-dimensional complex space. We consider the scalar product defined on this space by

$$(\lambda(x), \eta(x)) \equiv \bar{\lambda}^1(x) \delta_{AB} \eta^B(x) \equiv \bar{\lambda}^1(x) \eta^1(x) + \bar{\lambda}^2(x) \eta^2(x).$$
We use also the Ashtekar’s *dagger* notation in order to indicate this scalar product:

\[
\begin{align*}
(\eta^\dagger)^1(x) & \equiv (\bar{\eta})^2(x), \\
(\eta^\dagger)^2(x) & \equiv -(\bar{\eta})^1(x);
\end{align*}
\]

in terms of which

\[
(\lambda(x), \eta(x)) = (\lambda^\dagger)^A(x) \epsilon_{AB} \eta^B(x).
\]  

(21)

Given the triad field $\tilde{\sigma}^a$ on $M$, we can construct a volume form, and therefore a scalar product on the space of the spinor fields $\lambda$. This is given by

\[
\langle \lambda, \eta \rangle \equiv \int d^3x \sqrt{g(x)} (\lambda(x), \eta(x)).
\]  

(22)

We write this also as

\[
\langle \lambda, \eta \rangle \equiv \int_M \lambda \cdot \eta.
\]  

(23)

where the volume form defined by the triad and the scalar product are understood. Equipped with the product $\langle \ , \ \rangle$, the spinor fields form an Hilbert space $\mathcal{H}$. The spinorial Ashtekar connection naturally defines an $\text{SU}(2)$ covariant derivative on the spinor fields

\[
\mathcal{D}_a \eta = \partial_a \eta + A_a \eta,
\]  

(24)

where $A_a$ acts on $\eta$ by matrix multiplication. We also recall that the Ashtekar connection can be decomposed as

\[
A_a = \Gamma_a[\sigma] - \frac{1}{\sqrt{2}} \Pi_a,
\]  

(25)
where $\Gamma_a$ is the (unique) symmetric spinor connection compatible with the SU(2) soldering $\sigma_a$; and on the constraints surface we have $\Pi_a = K_{ab} \sigma^b$ where $K_{ab}$ is the extrinsic curvature of the three manifold.

Let us now introduce the main object we will deal with. We consider the operator $\hat{H}$ defined as

$$\hat{H} \eta = \sqrt{2} \sigma^a D_a \eta.$$  \hfill (27)

The operator $\hat{H}$ is the 3-dimensional self-dual Weyl operator; it has a precise physical interpretation, as the operator that generates the dynamics of a left-handed massless fermion, as a neutrino, on a gravitational background. Indeed, a massless spinor test-particle satisfies [9]

$$\frac{d}{dt} \eta(x, t) = \hat{H} \eta(x, t).$$  \hfill (28)

We will not make any explicit use here of this dynamical interpretation of the Weyl operator $\hat{H}$.

The Weyl operator is symmetric in $\mathcal{H}$, that is

$$\langle \hat{H} \lambda, \eta \rangle = \langle \lambda, \hat{H} \eta \rangle.$$  \hfill (29)

This can be proven by integrating by part, using the reality of the fields $A^i_a$ and $E^a_i$ and the hermiticity properties of the Pauli matrices, and making use of the constraint $C$ (equation 7). Neglecting domain's difficulties, we assume that $\hat{H}$ is self-adjoint. Then we can consider its eigenvalue problem

$$\hat{H} \eta_n = \lambda_n \eta_n.$$  \hfill (30)

The eigenspinors $\eta_n$ form an orthonormal basis of (possibly generalized) vectors in $\mathcal{H}$. We assume that they form a countable basis (this is not unreasonable, since $M$ is compact), and we take the indices $n$ as integers (see below).
The eigenvalues $\lambda_n$ are the objects on which we focus. Since $\hat{H}$ depends on the gravitational variables $A_a$ and $\bar{\sigma}^a$, so do its eigenvalues. Thus, the eigenvalue equation implicitly defines a countable family of functionals of the gravitational fields

$$\lambda_n = \lambda_n[\bar{\sigma}^a, A_a].$$

The eigenvalues $\lambda_n$ can thus be seen as a family of variables describing the gravitational field.

The functionals (31) are determined by the eigenvalue equation only implicitly. To find their explicit form we should solve the spectral problem for the operator $\hat{H}$ on arbitrary geometries. We can nevertheless obtain a large amount of important information about these functionals. In particular, we can compute their first variation, namely their derivative with respect to the gravitational variables $A_a$ and $\bar{\sigma}^a$. This allows us to check their gauge invariance explicitly and, more importantly, to write explicit formulas for their Poisson brackets and their time evolution. These are substantial steps towards the task of expressing general relativity entirely in terms of the $\lambda_n$ variables alone. The computation of the first variation of the $\lambda_n$'s is an application of the technology of quantum mechanics' time-independent perturbation theory. Consider an operator $\hat{O}$ with a complete set of eigenstates $v_n$ and eigenvalues $\lambda_n$

$$\hat{O} \; v_n = \lambda_n \; v_n.$$  

Let $\hat{O}$, and thus its eigenvectors and eigenvalues, depend on a parameter $\tau$. We want to compute $d\lambda_n/d\tau$. To this aim, consider a small variation
\( \tau \to \tau + \delta \tau \), which induces the variations \( \delta O, \delta v_n \) and \( \delta \lambda_n \). The variation of the eigenvalue equation gives

\[
\delta \dot{O} v_n + \dot{O} \delta v_n = \delta \lambda_n v_n + \lambda_n \delta v_n. \tag{33}
\]

By taking the scalar product of this equation with the eigenstate \( v_n \) (on the left), and using the orthogonality property and the self-adjointness of \( \dot{O} \), we obtain the formula, well known from quantum mechanics,

\[
\delta \lambda_n = (v_n, \delta \dot{O} v_n). \tag{34}
\]

Let us apply this to the Weyl operator. Consider a small variation \( A_a(x) \to A_a(x) + \delta A_a(x) \) and \( \tilde{\sigma}^a(x) \to \tilde{\sigma}^a(x) + \delta \tilde{\sigma}^a(x) \). From the definition of the Weyl operator we have immediately

\[
\frac{\delta \dot{H}}{\delta A_a(x)} = \sqrt{2} \tilde{\sigma}^a(x) \tag{35}
\]

and

\[
\frac{\delta \dot{H}}{\delta \tilde{\sigma}^a(x)} = \sqrt{2} \mathcal{D}_a. \tag{36}
\]

We need the derivative with respect to the density triad. This is obtained using

\[
\delta \sigma^a = \delta (g^{-1/2} \tilde{\sigma}^a) = g^{-1/2} \delta \tilde{\sigma}^a + \frac{1}{2} \tilde{\sigma}^a \text{Tr}[g^{-1} \tilde{\sigma}, \delta \tilde{\sigma}] \tag{37}
\]

By inserting these equations in the relation (34) for the Weyl operator, we obtain with some simple algebra the key technical result

\[
\delta \lambda_n = -\sqrt{2} \int_M \left[ \eta_n \cdot \delta \tilde{\sigma}^a \mathcal{D}_a \eta_n + \frac{1}{2} \eta_n \cdot \text{Tr}[\sigma_a \delta \tilde{\sigma}^a] \sigma^b \mathcal{D}_b \eta_n \right] - \sqrt{2} \int_M \eta_n \cdot \tilde{\sigma}^a \delta A_a \eta_n. \tag{38}
\]
This equation allows us to compute explicitly the variation of the eigenvalues under arbitrary variations of the Ashtekar variables. Equivalently, we can write the functional derivatives

\[
\frac{\delta \lambda_n}{\delta \sigma^a_{AB}} = -\sqrt{2} \left[ \eta_n^a (D_a \eta_n)_A + \frac{1}{2} \eta_n^a \sigma^a_{CD} (D_a \eta_n)_D \sigma^B_{BA} \right] \\
\frac{\delta \lambda_n}{\delta A_a^B} = -\sqrt{2} \eta_n^a \sigma^a_{CA} \eta_n^B. 
\]

(39)  
(40)

As a first use of equation (38), let us check the invariance of the eigenvalues under diffeomorphisms and SU(2) transformations explicitly. By replacing \( \delta \sigma^a \) and \( \delta A_a \) with the gauge variations given in equation (17), we have with simple algebra

\[
\delta_{\sigma} \lambda_n = 0, \\
\delta_{A_a} \lambda_n = 0; 
\]

(41)  
(42)

namely the eigenvalues are gauge invariant under three-dimensional diffeomorphisms and internal SU(2) gauge transformations.

By substituting the variations \( A_a \) and \( \sigma^a \) generated by the Hamiltonian constraint in equation (38), we obtain the formula for the evolution of the eigenvalues in coordinate time, with an arbitrary Lapse function \( N \)

\[
\delta_N \lambda_n = \sqrt{2} \int d^3 x \sqrt{g} \eta_n^A \epsilon_{AB} \left[ (\partial_a N) \epsilon^{ab} \sigma^b_{CA} + K^b_{a} \sigma^b_{CD} \right] D_b \eta_n^C. 
\]

(43)

If we chose a spatially constant Lapse function, the time evolution of the eigenvalues (which we can indicate now as \( \dot{\lambda} \equiv d/dt \lambda \)) is given by

\[
\dot{\lambda}_n = \sqrt{2} \int_M \eta_n \cdot K^a D_a \eta_n. 
\]

(44)
This is our main result.

Similarly, a straightforward calculation yields the Poisson brackets between two \( \lambda \)'s. From the definition of the bracket

\[
\{ \lambda_n, \lambda_m \} = \int d^3x \frac{\delta \lambda_n}{\delta A_a(x)} \frac{\delta \lambda_m}{\delta \tilde{\sigma}^a(x)} - \frac{\delta \lambda_n}{\delta \tilde{\sigma}^a(x)} \frac{\delta \lambda_m}{\delta A_a(x)}
\]

we obtain by inserting the derivatives (39) and (40), and with some simple algebra

\[
\{ \lambda_n, \lambda_m \} = \frac{1}{2} (\lambda_n - \lambda_m) \int d^3x \sqrt{q} \left[ \left( \eta_{mA} \eta_n^A \right) \left( \eta_{nB} \eta_m^B \right) - \frac{3}{2} \left( \eta_{nA} \eta_{mB} \right) \left( \eta_{mB} \eta_{mB} \right) \right]
\]

\[
+ \frac{1}{2} \int d^3x \sqrt{q} \left( \eta_{mA} \eta_m^A \eta_{nB} \right) \left[ \eta_{mB} D_a \eta_n^C - \eta_{nB} D_a \eta_m^C \right]
\]

A final property of the eigenvalues that we note is that if \( \lambda \) is an eigenvalue of the Weyl operator, so is \(-\lambda\). In fact, it is easy to verify that

\[
\dot{H} \eta = \lambda \eta
\]

implies

\[
\dot{H} \eta^\dagger = -\lambda \eta^\dagger
\]

We can therefore choose \( \lambda_{-n} = -\lambda_n \).

As indicated above, we would like to consider the map \((A_a, \tilde{\sigma}^a) \rightarrow \lambda_n [A_a, \tilde{\sigma}^a] \) as a change of variables on the phase space of general relativity. Given the dynamical interpretation of the Weyl operator (see equation 28), the variables \( \lambda_n \) are related to the energy spectrum of the massless left-handed fermions.
They capture geometrical features of the spacetime manifold via dynamical properties of these test fields. Many problems, however, remain open.

We have little control on the properties of this change of variables. First, we do not know the constraints satisfied by the $\lambda_n$ variables; namely the properties that a generic set of numbers $\lambda_n$ should have in order to be interpretable as the spectrum of a Weyl operator (surjectivity). Second, the other way around, it would be important to show that generically the map is non-degenerate; namely that the vanishing of the right hand side of equation (38) implies (generically) that the variations of $A_a$ and $\tilde{\sigma}^a$ are the gauge variations $\delta_\rho$ and $\delta_N^a$ given in (17) (injectivity). We suspect this is true, but we have not been able to prove it. If the injectivity of the map could be proven, we would be assured that the map can be inverted, and therefore that the right hand side of equations (44) and (47), which represent our main results, can be expressed solely in terms of the $\lambda_n$ variables. If this could be done explicitly we would have a three-dimensional diffeomorphism-invariant formulation of general relativity.

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