Algebraic structure of the Green’s ansatz and its $q$–deformed analogue

T. D. Palev*

Arnold Sommerfeld Institute for Mathematical Physics, Technical University of Clausthal, 38678 Clausthal-Zellerfeld, Germany

Abstract. The algebraic structure of the Green’s ansatz is analyzed in such a way that its generalization to the case of $q$–deformed para-Bose and para-Fermi operators is becoming evident. To this end the underlying Lie (super)algebraic properties of the parastatistics are essentially used.

Already in his first paper on parastatistics [1] Green developed a technique, the Green’s ansatz technique [2], appropriate for constructing new (reducible) representations for any set of para-Bose (pB) or para-Fermi (pF) operators. Although quite clear as a mathematical device, the inner structure of the Green’s ansatz remains somehow not completely understood up to now. Just to give an example, consider $m$ pairs of para-Bose creation and annihilation operators $b^\pm_r(p)$, $r = 1, \ldots, m$ of order $p$. Then each such operator is represented by a sum

$$b^\pm_r(p) = \sum_{k=1}^{p} b^\pm_{r,k},$$

where for any value of $k$ the $b^\pm_{r,k}$ operators obey the Bose commutation relations (here and throughout $[x, y] = xy - yx$, $\{x, y\} = xy + yx$)

$$[b^-_r, b^+_{s,k}] = \delta_{rs}, \quad [b^-_r, b^-_{s,k}] = [b^+_r, b^-_{s,k}] = 0, \quad \forall \ r, s,$$

and for $i \neq j$ all operators anticommute.

$$\{b^\pm_{r,i}, b^\pm_{r,j}\} = 0, \quad \forall \ \xi, \eta = \pm, \quad i \neq j, \quad r, s.$$  

One natural question that arises in relation to the above construction is why the Bose operators partially commute and partially anticommute. Is there any deeper reason behind? The purpose

* Permanent address: Institute for Nuclear Research and Nuclear Energy, Boul. Tsarigradsko Chausse 72, 1784 Sofia, Bulgaria; E-mail palev@bgearn.bitnet
of the present note is to answer questions like that and in fact to show that the Green’s ansatz construction (1)-(3) is a very natural one from a Lie superalgebra point of view.

Much of the motivation for the present work stems from the recent interest in deformed para-Bose and para-Fermi operators from various points of view: deformed paraoscillators [3-10] and, more generally, deformed oscillators (see [11, 12] also for collection of references in this respect), supersingleton Fock representations of $U_q[osp(1/4)]$ and its singleton structure [13], integrable systems [14-18] and $q-$parasuperalgebras [19].

In the applications we mentioned above one of the questions is how to construct representations of the deformed paraoperators. In the nondeformed case the Green’s ansatz gives in principle an answer to this question. Therefore it is natural to try to extent the same technique to the deformed case. In the present paper we will analyze the algebraic structure of the Green’s ansatz in such a way that its generalization to the quantum case will become evident. To this end we essentially use the circumstance (see also Corollary 1) that any $n$ pairs $F^\pm_1, \ldots, F^\pm_n$ of pF operators generate the simple Lie algebra $so(2n + 1)$ [20, 21], whereas $m$ pairs of pB operators $B^\pm_1, \ldots, B^\pm_m$ generate a Lie superalgebra [22], which is isomorphic to the basic Lie superalgebra $osp(1/2m)$ [23], denoted also as $B(0/m)$ [24].

In order to be slightly more general and to treat the pB and the pF operators simultaneously, denote by $G(n/m)$ a $2(m + n)$-dimensional $\mathbb{Z}_2$-graded linear space ($\mathbb{Z}_2 \equiv \{0, 1\}$) with a basis as follows:

\begin{align}
\text{even basis vectors} & \quad C^\pm_j(0) \equiv F^\pm_j, \quad j = 1, \ldots, n, \\
\text{odd basis vectors} & \quad C^\pm_i(1) \equiv B^\pm_i, \quad i = 1, \ldots, m.
\end{align}

Let $U(n/m)$ be the free associative unital (= with unity) superalgebra with generators (4) and (5), grading induced from the grading of the generators and relations

\begin{equation}
[[C^\xi_j(\alpha), C^\eta_j(\beta)], C^\zeta_k(\gamma)] = 2\varepsilon^\zeta \delta_{jk} \delta_{\alpha \beta} \delta_{\eta \xi} C^\xi_j(\alpha) - 2\varepsilon^\gamma (-1)^{\beta \eta} \delta_{\alpha \beta} \delta_{\eta \xi} C^\eta_j(\beta),
\end{equation}

where $\xi, \eta, \varepsilon = \pm$, $\alpha, \beta, \gamma \in \mathbb{Z}_2$ and $i, j, k$ take all possible values according to (4) and (5). In (6) and throughout $[,]$ is a supercommutator, defined on any two homogeneous elements $a, b$ from $U(n/m)$ as

\begin{equation}
[a, b] = ab - (-1)^{\delta_{\alpha \beta}(a)\delta_{\gamma}(b)} ba.
\end{equation}

In the case $\alpha = \beta = \gamma = 0$ (6) reduces to

\begin{equation}
[[F^\xi_i, F^\eta_j], F^\zeta_k] = 2\delta_{jk} \delta_{\xi \eta} F^\xi_i - 2\delta_{ik} \delta_{\xi \eta} F^\eta_j,
\end{equation}

\begin{equation}
[[F^\xi_i, F^\eta_j], F^\zeta_k] = 2\delta_{jk} \delta_{\xi \eta} F^\xi_i - 2\delta_{ik} \delta_{\xi \eta} F^\eta_j,
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[[F^\xi_i, F^\eta_j], F^\zeta_k] = 2\delta_{jk} \delta_{\xi \eta} F^\xi_i - 2\delta_{ik} \delta_{\xi \eta} F^\eta_j.
\end{equation}
whereas for $\alpha = \beta = \gamma = 1$ it gives

$$[[B_k^\xi, B_j^\eta]], B_k^\eta] = 2\varepsilon \delta_{ij} \delta_{\xi, -\eta} B_k^\xi + 2\varepsilon \delta_{ij} \delta_{\xi, -\eta} B_j^\eta, \quad (9)$$

Equations (8) and (9) are the defining relations for the para-Fermi and para-Boole operators, respectively [1].

The relations (6) define a structure of a Lie-super triple system [25] on $G(n/m) \subset U(n/m)$ with a triple product $G(n/m) \otimes G(n/m) \otimes G(n/m) \rightarrow G(n/m)$ defined as

$$[[x, y], z] = 2 < y|z > x - 2(-1)^{\deg(x) \deg(y)} < x|z > y \in G(n/m), \forall x, y, z \in G(n/m), \quad (10)$$

where the bilinear form $< x|y >$ is defined in agreement with (6) to be [25]

$$< C_k^\xi(\alpha)|C_j^\eta(\beta) > = \eta^\alpha \delta_{ij} \delta_{\xi, -\eta}, \quad \xi, \eta = \pm, \quad \alpha, \beta \in \mathbb{Z}_2. \quad (11)$$

Consider $U(n/m)$ as a Lie superalgebra (LS) with a supercommutator (7). Then it is straightforward to check that (lin.env. = linear envelope)

$$B(n/m) = \text{lin.env.}[[C_i^\xi(\alpha), C_j^\eta(\beta)], C_k^\gamma(\gamma)|\forall i, j, k, \xi, \eta, \gamma = \pm, \alpha, \beta, \gamma \in \mathbb{Z}_2] \quad (12)$$

is a subalgebra of the LS $U(n/m)$.

**Proposition 1** [26]. The LS $B(n/m)$ is isomorphic to the orthosymplectic LS $osp(2n + 1/2m)$. The associative superalgebra $U(n/m)$ is its universal enveloping algebra $U[osp(2n + 1/2m)]$.

The first part of the proposition was proved in [26]. The second part follows from the following two observations:

(i) The supercommutation relations between all generators of $osp(2n + 1/2m)$ (which constitute a basis in the underlying linear space) follow from the relations (6) between only the Lie-super triple generators (4) and (5).

(ii) The universal enveloping algebra of a given LS is the free associative unital algebra of its generators and the supercommutation relations they satisfy.

Observe that everywhere in the above considerations the para-Fermi operators appear as even (i.e. bosonic) variables, whereas the parabosons are odd (i.e. fermionic) operators. Moreover the parabosons do not commute with the parafermions. To the same conclusion arrived recently also Okubo [25] and Macfarlane [15].

As an immediate consequence of the above proposition we have

**Corollary 1.**
(a) [20, 21] The free associative unital algebra of the pF operators (4) is isomorphic to the universal enveloping algebra \( U[so(2n + 1)] \) of the orthogonal Lie algebra \( so(2n + 1) \):

\[
so(2n + 1) = \text{lin.env.}\{[F^\xi_i, F^\eta_j], F^\xi_i | i, j = 1, \ldots, n; \xi, \eta, \varepsilon = \pm\} \subset U[so(2n + 1)];
\]

(b) [23] The free associative unital algebra of the pB operators (5) is isomorphic to the universal enveloping algebra \( U[osp(1/2m)] \) of the orthosymplectic Lie superalgebra \( osp(1/2m) \):

\[
osp(1/2m) = \text{lin.env.}\{[B^\xi_i, B^\eta_j], B^\xi_i | i, j = 1, \ldots, m; \xi, \eta, \varepsilon = \pm\} \subset U[osp(1/2m)].
\]

Corollary 2. The representation theory of the Lie-superalgebra \( G(n/m) \) with generators (4), (5) and relations (6) is completely equivalent to the representation theory of the orthosymplectic Lie superalgebra \( osp(1/2m) \). In particular, the problem to construct the representations of \( n \) pairs of pF operators (4) is equivalent to the problem to construct the representations of the Lie algebra \( so(2n + 1) \); similarly, the representation theory of \( m \) pairs of pB operators is the same as the representation theory of the LS \( osp(1/2m) \).

The finite-dimensional representations of \( so(2n + 1) \) are known, they have been explicitly constructed [27]. All representations of the pF operators corresponding to a fixed order of the para-statistics are among the finite-dimensional representations. The pF operators have however several other representations [28], including representations with degenerate vacua. In a more practical aspect the results of [27] are unfortunately not so useful for the para-Fermi statistics. The point is that the transformation relations of the Gel’fand-Zetlin basis in [27] are given for a set of 2\( n \) operators (generating the rest of all 2\( n^2 \) + \( n \) generators), which are different from the pF operators and also different from the 3\( n \) Chevalley generators. The relations between the pF operators and the operators used in [27] are not linear.

The above corollaries are not of big practical use also for the representations of the para-Bose operators or, more generally, of the Lie-superalgebra \( G(n/m) \). The finite-dimensional representations of \( osp(2n + 1/2m) \) have been classified only so far [29]. Explicit expressions for the matrix elements are available only for low rank algebras (see [30] and the references therein). Moreover the interesting representations of the pB operators are infinite-dimensional. The Lie-superalgebra \( G(n/m) \) and hence \( osp(2n + 1/2m) \) have however one simple but important representation, the Fock representation, which is of particular interest for our considerations.

Proposition 2 [26]. Denote by \( W(n/m) \) the antisymmetric Clifford-Weyl superalgebra, namely the associative algebra generated by \( n \) pairs of Fermi creation and annihilation operators (CAOs) \( f^\pm_j \equiv c^\pm_j(0), i, j = 1, \ldots, n, \)
\[
\{f_i^\xi, f_j^\eta\} \equiv \{c_i^\xi(0), c_j^\eta(0)\} = \delta_{ij} \delta_{\xi,-\eta}, \quad \xi, \eta = \pm
\]

and \(m\) pairs of Bose CAOs \(b_i^\xi\) \(\equiv c_i^\xi(1), i, j = 1, \ldots, m,\)

\[
[b_i^\xi, b_j^\eta] \equiv [c_i^\xi(1), c_j^\eta(1)] = \eta \delta_{ij} \delta_{\xi,-\eta}, \quad \xi, \eta = \pm
\]

under the condition that the Bose operators anticommute with the Fermi operators,

\[
\{f_i^\xi, b_j^\eta\} \equiv \{c_i^\xi(0), c_j^\eta(1)\} = 0, \quad \xi, \eta = \pm, \quad i = 1, \ldots, m, \quad j = 1, \ldots, m.
\]

\(W(n/m)\) is an associative superalgebra with grading induced from the requirement that the Fermi operators are even elements and the Bose operators are odd. Consider \(W(n/m)\) as an algebra of (linear) operators in the corresponding Fock space \(H \equiv H(n/m), W(n/m) \subset \text{End}(H)\). Then the map

\[
\pi : \text{osp}(2n + 1/2m) \to W(n/m) \quad \text{defined as} \quad \pi(C_i^\xi(\alpha)) = c_i^\xi(\alpha), \quad \forall \xi = \pm \text{ and } i
\]

is a representation, a Fock representation, of \(\text{osp}(2n + 1/2m)\) or, which is the same, a representation of the Lie-superalgebra \(G(n/m)\).

In order to prove the proposition one has simply to check that the relations (6) remain valid after the replacement \(C_i^\xi(\alpha) \to c_i^\xi(\alpha)\).

In case of \(so(2n+1)\) or equivalently of \(n\) pairs of pF operators (resp. of \(osp(1/2m)\) or equivalently of \(m\) pairs of pB operators) the proposition 2 reduces to the usual representation of the pF operators with Fermi operators (resp. of the pB operators with Bose operators).

The relevant for us conclusion is that the operators (4) and (5) generate an associative superalgebra, namely \(U[\text{osp}(2n + 1/2m)]\) (Proposition 1) and that we know at least one representation of \(U[\text{osp}(2n + 1/2m)]\), namely its Fock representation (Proposition 2).

Set for simplicity \(L = \text{osp}(2n + 1/2m), U = U[\text{osp}(2n + 1/2m)]\) and let \(L^{\otimes p}\) and \(U^{\otimes p}\) be their \(p^{th}\) tensorial powers. Introduce the following notation (\(\epsilon\) is the unity of \(U\)):

\[
L^k = \{e_1 \otimes \ldots \otimes e_{i-1} \otimes a \otimes e_{i+1} \otimes \ldots \otimes e_p | a \in L, e_i = \epsilon \forall i \neq k\}. \quad (19)
\]

\[
U^k = \{e_1 \otimes \ldots \otimes e_{i-1} \otimes u \otimes e_{i+1} \otimes \ldots \otimes e_p | u \in U, e_i = \epsilon \forall i \neq k\}. \quad (20)
\]

Then the map \(\tau^k : L \to L^k \subset U^k\)
\[ \tau^k(a) = \epsilon_1 \otimes \ldots \otimes \epsilon_{i-1} \otimes a \otimes \epsilon_{i+1} \otimes \ldots \otimes \epsilon_p, \quad a \in L, \quad \epsilon_i = \epsilon \forall i \neq k. \]  

(21)

is a Lie superalgebra morphism of \( L \) onto \( L^k \); the same map (21) considered for all \( a \in U \) is an associative algebra morphism of \( U \) onto \( U^k \).

The set of the elements (21) generate \( U^k \) and, since

\[ U^{\otimes p} = U^1 U^2 \ldots U^p, \]

(22)

the elements (21) considered for all \( k = 1, \ldots, p \) generate \( U^{\otimes p} \).

The sum

\[ \Delta^{(p)} = \tau^1 + \tau^2 + \ldots + \tau^p : L \longrightarrow L^{\otimes p} \]

(23)

is a Lie superalgebra morphism, the "diagonal" LS morphism, of \( L \) into \( L^{\otimes p} \), which is extended to a morphism of the associative algebra \( U \) into the associative algebra \( U^{\otimes p} \) in a natural way:

\[ \Delta^{(p)}(a_1 a_2 \ldots a_m) = \Delta^{(p)}(a_1) \Delta^{(p)}(a_2) \ldots \Delta^{(p)}(a_m) \quad \forall a_1, a_2, \ldots, a_m \in L. \]

(24)

Let \( \tau^1, \tau^2, \ldots, \tau^p \) be (not necessarily different) representations of \( L = \text{osp}(2n + 1/2m) \) (and hence of \( U = U[\text{osp}(2n + 1/2m)] \)) in the \( \mathbb{Z}_2 \)-graded linear spaces \( H^1, H^2, \ldots, H^p \), respectively, i.e., the operators \( \tau^k[C^{\pm}_{\alpha} \otimes \cdots \otimes C^{\pm}_{\alpha}] \in \text{End}(H^k) \), \( k = 1, \ldots, p \), satisfy the Lie-super triple relations (6) and

\[ \text{deg} \{ \tau^k[C_{\alpha}^\pm] \} = \alpha. \]

(25)

Then

\[ \tau^1 \otimes \tau^2 \otimes \ldots \otimes \tau^p : U^{\otimes p} \longrightarrow \text{End}(H^1 \otimes H^2 \otimes \ldots \otimes H^p) \]

(26)

gives a representation of both the LS \( L^{\otimes p} \) and of the associative algebra \( U^{\otimes p} \).

The composition maps \( (k = 1, \ldots, p) \)

\[ (\tau^1 \otimes \tau^2 \otimes \ldots \otimes \tau^p) \circ \tau^k : U[\text{osp}(2n + 1/2m)] \longrightarrow \text{End}(H^1 \otimes H^2 \otimes \ldots \otimes H^p), \]

(27)

\[ (\tau^1 \otimes \tau^2 \otimes \ldots \otimes \tau^p) \circ \Delta^{(p)} : U[\text{osp}(2n + 1/2m)] \longrightarrow \text{End}(H^1 \otimes H^2 \otimes \ldots \otimes H^p) \]

(28)

give representations of both the LS \( \text{osp}(2n + 1/2m) \) and the associative algebra \( U[\text{osp}(2n + 1/2m)] \).

Therefore the operators

\[ \hat{\tau}^k_{\pm 1}(\alpha) = [(\tau^1 \otimes \tau^2 \otimes \ldots \otimes \tau^p) \circ \tau^k]C_{\alpha}^\pm = id^1 \otimes \ldots \otimes id^{k-1} \otimes \tau^k[C_{\alpha}^\pm] \otimes id^{k+1} \otimes \ldots \otimes id^p \]

\[ \in \text{End}(H^1 \otimes H^2 \otimes \ldots \otimes H^p), \quad k = 1, \ldots, p, \quad \alpha = \pm, \]

(29)
and
\[
\hat{c}_r^\pm(p, \alpha) = [(\pi^1 \circ \pi^2 \circ \ldots \circ \pi^n) \circ \Delta^{(p)}]C_r^\pm(\alpha) = \sum_{k=1}^p \hat{c}_r^{\pm k}(\alpha)
\]  
(30)
satisfy the Lie-supercg triple relations (6). From the very definition of a tensor product of associative algebras [31] we obtain (for all \(r, s\) according to (4) and (5) and \(\xi, \eta = \pm\):

\[
[[\hat{c}_r^{\xi i}(\alpha), \hat{c}_s^{\eta j}(\beta)] \equiv \hat{c}_r^{\xi i}(\alpha)\hat{c}_s^{\eta j}(\beta) - (-1)^{\alpha\beta}\hat{c}_s^{\eta j}(\beta)\hat{c}_r^{\xi i}(\alpha) = 0, \quad i \neq j = 1, \ldots, p. \tag{31}
\]

In particular for \(\alpha = 1\)

\[
\hat{b}_r^{\pm k} \equiv \hat{c}_r^{\pm k}(1) = id^1 \circ \ldots \circ id^{k-1} \circ \pi^k(B_r^{\pm}) \circ id^{k+1} \circ \ldots \circ id^p, \tag{32}
\]

\[
\hat{b}_r^{\pm}(p) \equiv \hat{c}_r^{\pm}(p, 1) = \sum_{k=1}^p \hat{b}_r^{\pm k}, \quad r = 1, \ldots, m, \tag{33}
\]

and

\[
\{\hat{b}_r^{\xi i}, \hat{b}_s^{\eta j}\} = 0 \quad i \neq j = 1, \ldots, p, \quad r, s = 1, \ldots, m, \quad \xi, \eta = \pm, \tag{34}
\]

whereas the operators \(\hat{b}_r^{\pm k}\) with the same upper case index \(k\) satisfy the pB relations (9) and may be also other, particular for the representation \(\pi^k\), relations.

Similarly for \(\alpha = 0\)

\[
\hat{f}_r^{\pm k} \equiv \hat{c}_r^{\pm k}(0) = id^1 \circ \ldots \circ id^{k-1} \circ \pi^k(F_r^{\pm}) \circ id^{k+1} \circ \ldots \circ id^p, \tag{35}
\]

\[
\hat{f}_r^{\pm}(p) \equiv \hat{c}_r^{\pm}(p, 0) = \sum_{k=1}^p \hat{f}_r^{\pm k}, \quad r = 1, \ldots, n, \tag{36}
\]

and

\[
[\hat{f}_r^{\xi i}, \hat{f}_s^{\eta j}] = 0 \quad i \neq j = 1, \ldots, p, \quad r, s = 1, \ldots, n, \quad \xi, \eta = \pm. \tag{37}
\]

Consider now the important case when all representations \(\pi^1, \pi^2, \ldots, \pi^n\) are the same and coincide with the Fock representation, namely \(\pi\) is a morphism of \(U[osp(2n + 1/2m)]\) onto the Clifford-Weyl algebra \(W(n/m)\) defined in (18),

\[
\pi^1 = \pi^2 = \ldots = \pi^n = \pi. \tag{38}
\]

In order to distinguish this particular case we do not write any more hats over the operators. Then from (30) we obtain
\[ c_r^\pm(p, \alpha) = [\pi^{(p)} \circ \Delta]c_r^\pm(\alpha) = \sum_{k=1}^p c_r^{\pm k}(\alpha) \in \text{End}(H^{(p)}), \]  

(39)

where according to (29) and (18)

\[ c_r^{\pm k}(\alpha) = \text{id}^1 \otimes \ldots \otimes \text{id}^{k-1} \otimes c_r^\pm(\alpha) \otimes \text{id}^{k+1} \otimes \ldots \otimes \text{id}^p, \quad k = 1, \ldots, p, \quad \alpha = \pm, \]  

(40)

and the operators \( c_r^{\pm k}(\alpha) \) satisfy according to (15) - (17) and (31) the relations (see (7)):

\[ [c_r^{\pm i}(\alpha), c_r^{\pm j}(\beta)] = \eta^\alpha \delta_{r,s} \delta_{i,j} \delta_{\alpha, \beta} \xi_{-\eta}, \quad \xi, \eta = \pm, \quad \alpha, \beta \in \mathbb{Z}_2. \]  

(41)

Setting \( b_r^\pm(p) = c_r^\pm(p, 1) \) we obtain from (39) and (41) the Green’s ansatz for the para-Bose operators of order \( p \),

\[ b_r^\pm(p) = [\pi^{(p)} \circ \Delta]B_r^\pm = \sum_{k=1}^p b_r^{\pm k}, \]  

(42)

where

\[ b_r^{\pm k} = \text{id}^1 \otimes \ldots \otimes \text{id}^{k-1} \otimes b_r^\pm \otimes \text{id}^{k+1} \otimes \ldots \otimes \text{id}^p, \quad k = 1, \ldots, p, \quad r = 1, \ldots, m. \]  

(43)

As it follows from (40) (or immediately from (43), taking into account that \( b_r^\pm \) are odd operators) the Bose operators \( b_r^{\pm k} \) partially commute and partially anticommute. More precisely,

\[ [b_r^{\pm k}, b_s^{\pm}] = \delta_{r,s}, \quad [b_r^{-1}, b_s^{-1}] = [b_r^{\pm}, b_s^{\pm}] = 0, \quad \forall k, r, s \]  

(44)

and for \( i \neq j \) all operators anticommute,

\[ \{b_r^{\pm i}, b_s^{\pm j}\} = 0, \quad \xi, \eta = \pm, \quad i \neq j. \]  

(45)

Similarly setting \( f_r^\pm(p) = c_r^\pm(p, 0) \) we obtain from (39) and (41) the Green’s ansatz for the para-Fermi operators of order \( p \),

\[ f_r^\pm(p) = [\pi^{(p)} \circ \Delta]F_r^\pm = \sum_{k=1}^p f_r^{\pm k}, \]  

(46)

where

\[ f_r^{\pm k} = \text{id}^1 \otimes \ldots \otimes \text{id}^{k-1} \otimes f_r^\pm \otimes \text{id}^{k+1} \otimes \ldots \otimes \text{id}^p, \quad k = 1, \ldots, p, \quad r = 1, \ldots, n. \]  

(47)
Setting in (40) $\alpha = 0$ (or directly from (47), taking into account that $f_r^k$ are even operators) one obtains:

$$\{f_r^{-k}, f_r^k\} = \delta_{rs}, \quad \{f_r^{-k}, f_s^{-k}\} = \{f_r^k, f_s^k\} = 0, \quad \forall \ k, r, s,$$

(48)

and for $i \neq j$ all operators commute,

$$[f_r^{\xi_i}, f_r^{\eta_j}] = 0, \quad \xi, \eta = \pm, \quad i \neq j.$$  

(49)

From the above considerations it is clear that the Green’s ansatz representation (46) of the pF operators of order $p$ is simply given as a representation of the pF operators $F_r^p (p)$, considered as generators of the universal enveloping algebra $U[so(2n + 1)]$, in the tensor product of $p$ copies of (irreducible, finite-dimensional) Fock representations of the Lie algebra $so(2n + 1)$.

Similarly, the Green’s ansatz representation (42) of the pB operators of order $p$ gives a representation of the pB operators in the tensor product of $p$ copies of (irreducible, infinite-dimensional) Fock representations of the Lie superalgebra $osp(1/2m)$.

The eqs (39), (41) generalize the concept of a Green’s ansatz to the case of Lie-super triple operators (4) and (5), which are free generators with relations (6) of the universal enveloping algebra $U[osp(2n + 1/2m)]$. The representation of the generators $C_r^\pm (\alpha)$ (= the representation of $osp(2n + 1/2m)$) is realized in the tensor product space $H^{\otimes p}$ of $p$ copies of Fock representations (18) of $osp(2n + 1/2m)$. Therefore the Green’s ansatz gives highly reducible representation of the Lie-super triple operators. In particular this is the case if only pB or pF operators are present. If $[0 \in H$ is the highest weight vector in $H$, then the irreducible subspace, containing $[0 \in H^{\otimes p}$ carries a representation, corresponding to an order of statistics $p$. The other irreducible components of $H^{\otimes p}$ contain also vacuum like states and among them are the highest weight vectors. The corresponding representations however do not correspond anymore to those with a fixed order of the statistics, namely to representations with a unique vacuum states (see for example [28]). The problem to decompose $H^{\otimes p}$ into a direct sum of irreducible subspaces with respect to the paraoperators (= with respect to $osp(2n + 1/2m)$ or even the simpler problem - to extract the irreducible submodule, carrying only the representation with an order of statistics $p$ - has not been solved so far. The problem was not solved also for the case of only pF operators $(m = 0)$ or only pB operators $(n = 0)$.

Passing to a short discussion of a possible generalization of the Green’s ansatz (39) to the case of deformed operators, we first observe that in all cases the Green’s ansatz is obtained (see (39), (42) and (46)) as a two step procedure, namely as a composition of two (associative algebra) morphisms:

$$\Delta^{(p)} : U[osp(2n + 1/2m)] \longrightarrow U[osp(2n + 1/2m)]^{\otimes p}$$

$$\pi^{\otimes p} : U[osp(2n + 1/2m)]^{\otimes p} \longrightarrow End(H^{\otimes p}).$$  

(50)
In the following we consider only such (one parameter) deformations of the Lie-super triple generators (4) and (5), which generate a Hopf deformation $U_q[osp(2n + 1/2m)]$ of $U[osp(2n + 1/2m)]$. By a Hopf deformation we mean a deformation of $U[osp(2n + 1/2m)]$, which preserves its Hopf algebra structure (as defined, for instance, in [32]).

Denote by

$$ C^\pm_j(0)_q \equiv F^\pm_{j_q}, \quad j = 1, \ldots, n, \quad \quad (51) $$

$$ C^\pm_i(1)_q \equiv B^\pm_{i_q}, \quad i = 1, \ldots, m. \quad \quad (52) $$

a set of $2(n+m)$ deformed Lie-super triple generators, which at $q \to 1$ reduce to (4) and (5). Suppose that they satisfy one or more defining relations

$$ \phi_q(C^\pm_j(0)_q, \ldots, C^\pm_n(0)_q, C^\pm_1(1)_q, \ldots, C^\pm_m(1)_q) = 0, \quad \quad (53) $$

which in the case $q \to 1$ reduce to the Lie-super triple relations (6). According to what we have said above, we require that the free unital associative algebra of the generators (51) and (52) with relations (53) is a Hopf deformation $U_q[osp(2n+1/2m)]$ of the universal enveloping algebra $U[osp(2n+1/2m)]$ of $osp(2n + 1/2m)$. Such operators do exist. The operators generating $U_q[osp(3/2)]$ (the case $n = m = 1$) were constructed in [33]. The Hopf deformation of one pair of pB operators was carried out in [5], of two pairs - in [34], of any number of pB operators - in [8-10]. The deformation of any number of pF operators was obtained in [35].

The deformed version of the Green’s ansatz of order $p$ we are going to present is be based on the relation (see (39))

$$ c^\pm(p, \alpha)_q = [\pi^{\otimes p} \circ \Delta^{(p)}] C^\pm_r(\alpha)_q, \quad \quad (54) $$

In order to define a deformed analogue of the operator $\Delta^{(p)}$ we use the circumstance that the superalgebra $U = U[osp(2n + 1/2m)]$ is a Hopf superalgebra with a comultiplication $\Delta$, defined as

$$ \Delta(a) = a \otimes \varepsilon + \varepsilon \otimes a, \quad \forall a \in L, \quad \Delta(\varepsilon) = \varepsilon \otimes \varepsilon. \quad \quad (55) $$

From (21) and (23) we deduce that

$$ \Delta^{(2)} = \Delta, \quad \Delta^{(3)} = (id \otimes \Delta) \circ \Delta^{(2)}, \quad \Delta^{(k)} = [(id^{\otimes (k-2)}) \otimes \Delta] \circ \Delta^{(k-1)}. \quad \quad (56) $$

The important point is that the operators $\Delta^{(k)}$ preserve the property to be morphisms also after the quantization (=Hopf deformation) of $U(L)$, i.e., the map

$$ \Delta^{(p)} : U_q[osp(2n + 1/2m)] \longrightarrow U_q[osp(2n + 1/2m)]^{\otimes p} \quad \quad (57) $$
is an associative algebra morphism. Certainly in the deformed case the eq. (55) has to be replaced with the corresponding expression for the comultiplication on \( U_q[osp(2n + 1/2m)] \).

In order to determine the deformed analogue of the operator \( \pi^{\odot p} \) we first observe that if \( \pi : U_q[osp(2n + 1/2m)] \rightarrow \text{End}(H) \) is a representation of \( U_q[osp(2n + 1/2m)] \) in the linear space \( H \), then

\[
\pi^{\odot p} : U_q[osp(2n + 1/2m)]^{\otimes p} \rightarrow \text{End}(H^{\otimes p})
\]

is a representation of \( U_q[osp(2n + 1/2m)]^{\otimes p} \). In the nondeformed case \( \pi \) is a morphism of \( U[osp(2n + 1/2m)] \) onto the Clifford-Weyl algebra \( W(n/m) \) (see (18)). Therefore it is natural to assume that in the deformed case \( \pi \) is a morphism of \( U_q[osp(2n + 1/2m)] \) onto a deformed algebra \( W_q(n/m) \). The deformed Clifford-Weyl algebra was defined in [33]. In case \( n = 0 \) \( W_q(0/m) \) is the associative superalgebra, generated by \( m \) triples \( b^\pm_r, k_r = q^{N_r}, \ r = 1, \ldots, m \) of commuting deformed Bose operators as defined in [36-38]. The morphism \( \pi \) of \( U_q[osp(1/2m)] \) onto \( W_q(0/m) \), namely the operators \( \pi(B^\pm_r) \in W_q(0/m) \) were constructed in [9]. In case \( m = 0 \) \( W_q(n/0) \) is the associative algebra of \( 3n \) triples deformed Fermi operators [39]. The morphism \( \pi \) of \( U_q[so(2n + 1)] \) onto \( W_q(n/0) \) is given in [35]. Thus, we are ready to state the following result.

**Proposition 3.** If \( \pi \) is the Fock representation of \( U_q[osp(1/2m)] \) (resp. of \( U_q[so(2n + 1)] \)) [39] and \( \Delta^{(p)} \) is the operator (57) then eq. (54) defines the deformed analogue of the Green ansatz of order \( p \) for \( m \) pairs of deformed para-Bose operators (resp. for \( n \) pairs of deformed para-Fermi operators).

The proof is evident since the composition of the morphisms \( \pi^{\odot p} \) and \( \Delta^{(p)} \) is also a morphism. Hence the operators \( c^\pm_a(p, \alpha)_q \), defined in (54), satisfy the same relations as \( C^\pm_a(\alpha)_q \); in the limit \( q \rightarrow 1 \) they reduce to the corresponding nondeformed para-Bose or para-Fermi operators of order \( p \).

The proposition 3 could be extended also to deformed Lie-super triple systems. So far however such a deformation was carried out only for the case \( n = m = 1 \) [33].

As an example we write down the Green’s ansatz of order \( p = 2 \) related to \( U_q[osp(1/4)] \), namely the ansatz corresponding to two pairs of deformed para-Bose operators \( b^\pm_1(2)_q \) and \( b^\pm_2(2)_q \) [34]:

\[
\begin{align*}
b^+_1(2)_q &= b^+_1 \odot q^{N_1-2N_2-1/2} + q^{-N_1-2N_2-3/2} \odot b^+_1 + (q^{1/2} - q^{-1/2})b^+_2 q^{N_1-N_2} \odot b^-_1 b^-_2 q^{-N_2}, \\
b^-_1(2)_q &= b^-_1 \odot q^{N_1+2N_2+3/2} + q^{-N_1+2N_2+1/2} \odot b^-_1 + (q^{-1/2} - q^{1/2})b^-_2 q^{N_1+N_2}, \\
b^+_2(2)_q &= b^+_2 \odot q^{N_2+1/2} + q^{-N_2-1/2} \odot b^+_2, \quad \xi = \pm.
\end{align*}
\]

This example indicates that the structure of the deformed Green’s ansatz is more involved. There is in particular a big asymmetry between the first pair of operators \( b^\pm_1(2)_q \) and the second pair \( b^\pm_2(2)_q \). As a result the relatively simple problem to decompose the tensor product of two Fock representation into a direct sum of irreducible representations of \( osp(1/4) \) [40] becomes very difficult in the deformed case and so far we were not able to solve it.
The asymmetry that appears in (59) and (60) is a consequence of the very different expressions for the comultiplication acting on different pairs of deformed para-Bose operators (see eq. (3.7) in [34]). The latter have been derived from the quite symmetrical expressions for the comultiplication defined on the Chevalley generators [32]. We believe it will be possible to write down new, symmetric expressions for the comultiplication and hence for the deformed Lie-super triple generators (54). To this end one has to use, may be, multiparametric deformations of \( U[osp(2n + 1/2m)] \) as this was done for \( gl(n) \) [41]. But this is certainly another open problem.

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