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Off-Shell Bethe Ansatz for XXZ Vertex Model and Solution of Trigonometric Knizhnik-Zamolodchikov Equations

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Abstract

We prove that the wave vector of the off-shell Bethe Ansatz equation for the inhomogeneous XXZ vertex model in the quasiclassical limit gives the solution of the trigonometric Knizhnik-Zamolodchikov equation. We also observe that this solution is the quasiclassical limit of the solution of the quantum Knizhnik-Zamolodchikov equation corresponding to $U_q(\hat{sl}_2)$.

1 Introduction

Conformal field theory describes the critical behavior of the two dimensional physical systems many of which are exactly integrable. In this case we have found the structure connected to the Yang-Baxter equation, which guarantees exact integrability. Usually we considered the exact integrable homogeneous vertex model and their connection with the conformal field theory[4] or quantum field theory[5]. The next step in understanding this connection is the investigation of the inhomogeneous vertex models. Here we consider the inhomogeneous vertex model, where to each vertex we associate two parameters: the global spectral parameter λ and the disorder parameter z . The vertex weight matrix $\mathfrak{R}$ depends on $\lambda - z$. Hence the transfer matrix of the vertex model now depends on the disorder parameters $z_i, i = 1 \dots, N$. Due to the additivity of the spectral parameter transfer matrices with different values of spectral parameters commute with each other [1], which means that the model is integrable. If we have some rational solution of the Yang-Baxter equation $\mathfrak{R}(\lambda)$ and the transfer matrix $T(\lambda|\{z\})$, then by construction of the Algebraic Bethe Ansatz[6] we have an equation

$$T(\lambda|\{z\})\Phi(\lambda_1, \dots, \lambda_n|\{z\}) = \Lambda(\lambda, \lambda_1, \dots, \lambda_n|z)\Phi(\lambda_1, \dots, \lambda_n|\{z\}) - \sum_{\alpha=1}^n \frac{F_{\alpha}\Phi_{\alpha}}{\lambda - \lambda_{\alpha}}, \quad (1)$$

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where $\Phi(\lambda_1, \dots, \lambda_n | \{z\}) = \Phi(\lambda_1, \dots, \lambda_m | z_1, \dots, z_N)$ is a Bethe wave vector and $\Phi_\alpha = \Phi(\lambda_1, \dots, \lambda_{\alpha-1}, \lambda, \lambda_{\alpha+1}, \dots, \lambda_m | z_1, \dots, z_N)$. $F_\alpha(\lambda_1, \dots, \lambda_m | z_1, \dots, z_N)$ and $\Lambda(\lambda, \lambda_1, \dots, \lambda_m | z_1, \dots, z_N)$ are some functions. In the Bethe Ansatz we impose the condition $F_\alpha = 0$ and after this have the exact eigenvalue problem s. t. the vector Φ becomes an eigenvector and Λ an eigenvalue of $T(\lambda | z)$, respectively. In fact the condition $F_\alpha = 0$ is the mass-shell condition. If we keep all the “unwanted” terms in (1), then the vector Ψ in general satisfies the equation (1). We call equation (1) the Off-Shell Bethe Ansatz Equation (OSBAE). In [7, 8, 9] we have constructed the solution of the Knizhnik-Zamolodchikov equation, which is the differential equation for the N-point correlation function $\Psi(z_1, \dots, z_N)$ in the WZNW theory [10]

$$\mathfrak{a} \frac{\partial \Psi}{\partial z_k} = \sum_{i \neq k}^N \frac{t_k^a \otimes t_i^a}{z_k - z_i} \Psi \quad (2)$$

using the OSBAE (1) in the quasiclassical limit. In this limit we have the Off-Shell Bethe Ansatz problem for the Gaudin theory of nonlocal magnets [2, 3] $H_k, k = 1, \dots, N$

$$H_k = \sum_{i \neq k}^N \frac{t_k^a \otimes t_i^a}{z_k - z_i}. \quad (3)$$

In (2 and (3) the $t_i^a, a = 1, \dots, \dim(\mathfrak{g})$, represent the generators of the simple Lie algebra $\mathfrak{g}$ and act nontrivially in the representation spaces $V^i, i = 1, \dots, N$. The vector valued function $\Psi(z_1, \dots, z_N)$ is the correlation function of the primary fields in the WZNW theory and $\Psi(z_1, \dots, z_N) \in V^1 \otimes \dots \otimes V^N$. The solution of (2) as in [11] is

$$\Psi(z_1, \dots, z_N) = \oint \dots \oint \chi(\lambda | z) \varphi(\lambda | z) d\lambda_1, \dots, d\lambda_m. \quad (4)$$

Here $\varphi(\lambda | z) = \varphi(\lambda_1, \dots, \lambda_m | z_1, \dots, z_N)$ is the quasiclassical limit of the Bethe wave vector $\Phi(\lambda_1, \dots, \lambda_m | z_1, \dots, z_N)$ and in fact, it is the Bethe wave vector for Gaudin magnets (3) but off-mass shell. The scalar function $\chi(\lambda | z) = \chi(\lambda_1, \dots, \lambda_m | z_1, \dots, z_N)$ is constructed from the quasiclassical limit of the $\Lambda(\lambda = z_k, \lambda_1, \dots, \lambda_m | z_1, \dots, z_N), k = 1, \dots, N$ and $F_\alpha(\lambda_1, \dots, \lambda_m | z_1, \dots, z_N)$. This representation of the N-point correlation function in WZNW theory shows a deep connection between the inhomogeneous vertex models and the WZNW theory. The understanding of this fine structure seems to provide us with new knowledge of exact integrability and conformal field theory in two dimension. The application of this method in the case of trigonometric and elliptic solutions of the Yang-Baxter equation is also interesting in connection with the quantum Knizhnik-Zamolodchikov equation [12, 13, 14]. In this paper we consider the trigonometric solution of the Yang-Baxter equation and the corresponding trigonometric Knizhnik-Zamolodchikov equation [15]. Here we construct the Off-Shell Bethe Ansatz Equation for the XXZ inhomogeneous vertex model (sect. 2) and in the quasiclassical limit (sect. 3) we construct the solution of the trigonometric Knizhnik-Zamolodchikov equation (sect. 4). In sect. 5 we observe that the solution thus obtained is the quasiclassical limit of the solution of the quantum Knizhnik-Zamolodchikov equation corresponding to $U_q(\hat{\mathfrak{sl}}_2)$.

2 Inhomogeneous XXZ vertex model

The solution of the inhomogeneous XXZ vertex model is completely analogous to the homogeneous case. We briefly describe this use of the results in [16, 17]. We use three type

of $\mathfrak{R}$ matrices; $\mathfrak{R}_\sigma^{12}$, $\mathfrak{R}_{\sigma s}^{12}$ and $\mathfrak{R}_s^{12}$ acting in $V_\sigma \otimes V_\sigma$, $V_\sigma \otimes V_s$ and $V_s \otimes V_s$. These $\mathfrak{R}$ matrices satisfy the following three type of Yang-Baxter equations;

$$\mathfrak{R}_{\sigma s}^{12}(\lambda - \mu) \mathfrak{R}_\sigma^{13}(\lambda) \mathfrak{R}_{\sigma s}^{23}(\mu) = \mathfrak{R}_{\sigma s}^{23}(\mu) \mathfrak{R}_\sigma^{13}(\lambda) \mathfrak{R}_{\sigma s}^{12}, \quad (5)$$

$$\mathfrak{R}_{\sigma s}^{12}(\lambda - \mu) \mathfrak{R}_s^{13}(\lambda) \mathfrak{R}_s^{23}(\mu) = \mathfrak{R}_s^{23}(\mu) \mathfrak{R}_s^{13}(\lambda) \mathfrak{R}_{\sigma s}^{12}(\lambda - \mu), \quad (6)$$

$$\mathfrak{R}_s^{12}(\lambda) \mathfrak{R}_s^{13}(\lambda + \mu) \mathfrak{R}_s^{23}(\mu) = \mathfrak{R}_s^{23}(\mu) \mathfrak{R}_s^{13}(\lambda + \mu) \mathfrak{R}_s^{12}(\lambda). \quad (7)$$

In (5) $\mathfrak{R}_\sigma^{12}(\lambda)$ is a solution with $\dim V_\sigma = 2$, which is the well known XXZ spin half $\mathfrak{R}$ -matrix[3]

$$\mathfrak{R}_\sigma^{12}(\lambda) = \frac{1}{2} W_0(\lambda) I^1 \otimes I^2 + \frac{1}{2} \sum_{i=1}^3 W_i(\lambda) \sigma_i^1 \otimes \sigma_i^2, \quad (8)$$

$$\begin{aligned} W_0(\lambda) &= 2 \frac{\cos(\eta) \sin(\lambda + \eta)}{\sin(2\eta)}, \\ W_3 &= 2 \frac{\sin(\eta) \cos(\lambda + \eta)}{\sin(2\eta)}, \\ W_1 &= W_2 = 1. \end{aligned} \quad (9)$$

Here I^1, I^2 are unit operators and $\sigma_i = (\sigma_1, \sigma_2, \sigma_3)$ are the Pauli matrices acting on the first and second subspaces respectively. The matrix $\mathfrak{R}_{\sigma s}^{12}$ is given as

$$\mathfrak{R}_{\sigma s}^{12}(\lambda) = \sum_{i=0}^3 W_i(\lambda) \sigma_i^1 \otimes I_i^2. \quad (10)$$

In (10) $\sigma_0 = I$ is the unit operator and the operators I_i are given as follows

$$\begin{aligned} I_0 &= \frac{\cos(2\eta S^3)}{2 \cos(\eta)}, \\ I_3 &= \frac{\sin(2\eta S^3)}{2 \sin(\eta)}, \\ I_+ &= I_1 + i I_2 = S^+ f(S^3), \\ I_- &= I_1 - i I_2 = f(S^3) S^-, \\ f(S^3) &= \frac{1}{\sin(2\eta)} \sqrt{\frac{\sin(2\eta(S^3 + s + 1)) \sin(2\eta(s - S^3))}{(S^3 + s + 1)(s - S^3)}}. \end{aligned} \quad (11)$$

In fact I_3, I_+ and I_- are the generators of the $sl_q(2)$ algebra and together with I_0 they form the generators of the Sklyanin algebra[18]. In (10) $\mathfrak{R}_{\sigma s}^{12}(\lambda)$ acts on the space $V_\sigma \otimes V_s$ and $S_a = (S_1, S_2, S_3)$ are the matrices of the $SU(2)$ algebra in arbitrary dimension, which means that $\dim V_s = 2s + 1$ and $S_a S_a = s(2s + 1)$. It is possible to find $\mathfrak{R}_s^{12}(\lambda)$ from (6) by a recurrence relation. This $\mathfrak{R}$ matrix was found in explicit form in[19]. Now let us introduce the disorder parameters $z_1, \dots, z_N$ and as usual consider the monodromy operator[6]

$$J(\lambda|z) = \mathfrak{R}_{\sigma s_N}^{0N}(\lambda - z_N), \dots, \mathfrak{R}_{\sigma s_1}^{01}(\lambda - z_1), \quad (12)$$

$$T(\lambda|z) = \text{tr}_0 J(\lambda|z), \quad (13)$$

Here the trace is taken in the auxiliary space V_σ and is denoted as tr_0 . It is convenient to rewrite $\mathfrak{R}_{\sigma s}^{12}(\lambda)$ and $J(\lambda|z)$ as matrices in the auxiliary space V_σ

$$\mathfrak{R}_{\sigma s}(\lambda) = \begin{pmatrix} 2I_0 + 2\frac{W_3 I_3}{W_0} & \frac{2I_-}{W_0} \\ \frac{2I_+}{W_0} & 2I_0 - 2\frac{W_3 I_3}{W_0} \end{pmatrix} \quad (14)$$

$$J(\lambda|z) = \begin{pmatrix} A(\lambda|z) & B(\lambda|z) \\ C(\lambda|z) & D(\lambda|z) \end{pmatrix}. \quad (15)$$

In (14) we redefined the $\mathfrak{R}_{\sigma s}(\lambda)$ deviding by $\frac{W_0(\lambda)}{2}$. From equation (5) we then have

$$\mathfrak{R}^{12}(\lambda - \mu)(J_1(\lambda|z) \otimes J_2(\mu|z)) = (J_2(\mu|z) \otimes J_1(\lambda|z))\mathfrak{R}^{12}(\lambda - \mu) \quad (16)$$

and also

$$[T(\lambda|z), T(\mu|z)] = 0. \quad (17)$$

In (15) we introduced $\mathfrak{R}^{12}(\lambda) = P^{12}\mathfrak{R}_\sigma^{12}(\lambda)$, where P^{12} is the permutation operator. Rewriting (16) in components we get the commutation relations between the elements $J(\lambda|z)$. We need the following ones:

$$[A(\lambda|z), A(\mu|z)] = [D(\lambda|z), D(\mu|z)] = 0, [B(\lambda|z), B(\mu|z)] = [C(\lambda|z), C(\mu|z)] = 0, \quad (18)$$

$$A(\mu|z)B(\lambda|z) = \frac{1}{c(\lambda - \mu)}B(\lambda|z)A(\mu|z) - \frac{b(\lambda - \mu)}{c(\lambda - \mu)}B(\mu|z)A(\lambda|z), \quad (19)$$

$$D(\mu|z)B(\lambda|z) = \frac{1}{c(\mu - \lambda)}B(\lambda|z)D(\mu|z) - \frac{b(\mu - \lambda)}{c(\mu - \lambda)}B(\mu|z)D(\lambda|z), \quad (20)$$

where

$$b(\lambda) = \frac{\sin(2\eta)}{\sin(\lambda + 2\eta)}, c(\lambda) = \frac{\sin(\lambda)}{\sin(\lambda + 2\eta)}. \quad (21)$$

Using these commutation relations we find the Off-Shell Bethe Ansatz Equation(OSBAE)

$$T(\lambda|z)\Phi(\lambda_1, \dots, \lambda_m|z) = \Lambda(\lambda, \lambda_1, \dots, \lambda_m|z)\Phi(\lambda_1, \dots, \lambda_m|z) - \sum_{\alpha=1}^m \frac{F_\alpha \Phi_\alpha}{\sin(\lambda - \lambda_\alpha)}. \quad (22)$$

Here $\Phi(\lambda_1, \dots, \lambda_m|z)$ is the Bethe wave vector

$$\Phi(\lambda_1, \dots, \lambda_m|z) = \prod_{\alpha=1}^m B(\lambda_\alpha|z)|\Omega\rangle, \quad (23)$$

$$|\Omega\rangle = \prod_{i=1}^N \otimes |s_i, s_i\rangle, S_i^3 |s_i, s_i\rangle = s_i |s_i, s_i\rangle \quad (24)$$

and $\Phi_\alpha = \Phi(\lambda_1, \dots, \lambda_{\alpha-1}, \lambda, \lambda_{\alpha+1}, \dots, \lambda_m|z)$. The functions Λ and F_α are defined as follows:

$$\begin{aligned} \Lambda(\lambda, \lambda_1, \dots, \lambda_m|z) = & \prod_{i=1}^N \frac{\sin(\lambda - z_i + \eta(1 + 2s_i))}{\cos(\eta) \sin(\lambda - z_i + \eta)} \prod_{\alpha=1}^m \frac{\sin(\lambda_\alpha - \lambda + 2\eta)}{\sin(\lambda_\alpha - \lambda)} + \\ & + \prod_{i=1}^N \frac{\sin(\lambda - z_i + \eta(1 - 2s_i))}{\cos(\eta) \sin(\lambda - z_i + \eta)} \prod_{\alpha=1}^m \frac{\sin(\lambda - \lambda_\alpha + 2\eta)}{\sin(\lambda - \lambda_\alpha)}, \end{aligned} \quad (25)$$

$$\begin{aligned} F_\alpha(\lambda_1, \dots, \lambda_m|z) = & \sin(2\eta) \prod_{i=1}^N \frac{\sin(\lambda_\alpha - z_i + \eta(1 + 2s_i))}{\cos(\eta) \sin(\lambda_\alpha - z_i + \eta)} \prod_{\beta \neq \alpha}^m \frac{\sin(\lambda_\alpha - \lambda_\beta - 2\eta)}{\sin(\lambda_\alpha - \lambda_\beta)} - \\ & \sin(2\eta) \prod_{i=1}^N \frac{\sin(\lambda_\alpha - z_i + \eta(1 + 2s_i))}{\cos(\eta) \sin(\lambda_\alpha - z_i + \eta)} \prod_{\beta \neq \alpha}^m \frac{\sin(\lambda_\alpha - \lambda_\beta + 2\eta)}{\sin(\lambda_\alpha - \lambda_\beta)}. \end{aligned} \quad (26)$$

The next step in the Bethe Ansatz would consist in imposing the vanishing of the “unwanted” term in (22) s. t. $F_\alpha = 0$. The Bethe Ansatz equations $F_\alpha = 0$ classify the eigenvalues and eigenvectors of the $T(\lambda|z)$. One can say that when the Bethe Ansatz equations $F_\alpha = 0$ are satisfied, the wave function is on mass shell. If the conditions $F_\alpha = 0$ are not imposed, then, in general, we have equation (22), and the Bethe wave function is off mass shell. In this case, as in the rational situation[8, 9], we call equation (22) the Off-Shell Bethe Ansatz Equation(OSBAE).

3 OSBAE and Gaudin magnets

By quasiclassical expansion one commonly performs the expansion of the vertex weight $\mathfrak{R}(\lambda, \eta)$ around some point η_0 , such that $\mathfrak{R}^{12}(\lambda, \eta_0) = I^1 \otimes I^2$ [20]. One can parameterize η , such that $\eta_0 = 0$. It is easy to calculate the quasiclassical expansion of $\mathfrak{R}_{\sigma_s}(\lambda)$. From (9)-(11) we have

$$\mathfrak{R}_{\sigma_s}^{12}(\lambda) = I^1 \otimes I^2 + \eta r^{12}(\lambda) + O(\eta^2), \quad (27)$$

$$r^{12}(\lambda) = \frac{2}{\sin(\lambda)} [\sigma_1^1 \otimes S_1^2 + \sigma_1^2 \otimes S_2^2 + \cos(\lambda) \sigma_1^3 \otimes S_2^3] \quad (28)$$

The $r^{12}(\lambda)$ is the classical r-matrix and satisfies the classical Yang-Baxter equation

$$[r^{12}(\lambda - \mu), r^{13}(\lambda) + r^{23}(\mu)] + [r^{13}(\lambda), r^{23}(\mu)] = 0. \quad (29)$$

Let us introduce $J_k(\lambda) = J(\lambda = z_k|z)$, $T_k = T(\lambda = z_k|z)$ and $\Lambda_k = \Lambda(\lambda = z_k, \lambda_1, \dots, \lambda_m|z)$. From (12)-(15) and using (27)-(28) we find the quasiclassical expansion of all the objects in the OSBAE

$$T_k = 2 + 8\eta \sum_{j \neq k}^N \frac{1}{\sin(z_k - z_j)} [S_k^1 \otimes S_j^1 + S_k^2 \otimes S_j^2 + \cos(z_k - z_j) S_k^3 \otimes S_j^3] + O(\eta^2), \quad (30)$$

$$\lambda_k = 2 + 8\eta \sum_{i \neq k}^N s_k s_i \cot(z_k - z_i) + 8\eta \sum_{\alpha=1}^m s_k \cot(\lambda_\alpha - z_k) + O(\eta^2), \quad (31)$$

$$F_\alpha = 8\eta^2 \sum_{\beta \neq \alpha}^m \cot(\lambda_\alpha - \lambda_\beta) - 8\eta^2 \sum_{i=1}^N s_i \cot(\lambda_\alpha - z_i) + O(\eta^3), \quad (32)$$

$$B(\lambda|z) = 2\eta \sum_{i=1}^N \frac{S_i^-}{\sin(\lambda - z_i)} + O(\eta^2) = 2\eta S^-(\lambda) + O(\eta^2), \quad (33)$$

$$B(\lambda = z_k) = 2S_k^- + O(\eta), \quad (34)$$

$$\Phi(\lambda_1, \dots, \lambda_m|z) = (2\eta)^m \prod_{\alpha=1}^m S^-(\lambda_\alpha) |\Omega\rangle + O(\eta^{m+1}), \quad (35)$$

$$\Phi_\alpha = (2\eta)^{m-1} (2S_k^- \prod_{\beta \neq \alpha}^m S^-(\lambda_\beta) + O(\eta^m)). \quad (36)$$

After setting $\lambda = z_k$ in (22), then using (30)-(3) and combining the terms proportional to η^{m+1} we find the quasiclassical limit (the first nontrivial consequence) of the OSBAE (22)

$$H_k \varphi = h_k \varphi - \sum_{\alpha=1}^m \frac{f_\alpha S_k^- \varphi'_\alpha}{\sin(z_k - \lambda_\alpha)}. \quad (37)$$

Here the operators H_k are the Gaudin[2. 3]nonlocal magnets

$$H_k = \sum_{i \neq k}^N \frac{1}{\sin(z_k - z_i)} [S_k^1 \otimes S_i^1 + S_k^2 \otimes S_i^2 + \cos(z_k - z_i) S_k^3 \otimes S_i^3]. \quad (38)$$

In (37) we introduced the new notations

$$h_k = \sum_{i \neq k}^N s_k s_i \cot(z_k - z_i) + \sum_{\alpha=1}^m s_k \cot(\lambda_\alpha - z_k), \quad (39)$$

$$f_\alpha = \sum_{\beta \neq \alpha}^m \cot(\lambda_\alpha - \lambda_\beta) - \sum_{i=1}^N s_i \cot(\lambda_\alpha - z_i), \quad (40)$$

$$|\varphi\rangle = \prod_{\alpha=1}^m S^-(\lambda_\alpha) |\Omega\rangle. \quad (41)$$

In (37) we also introduced $\varphi = S^-(\lambda_\alpha) \varphi'_\alpha$. The equations (37)-(41) reproduce the Gaudin[2,3] results, which he found considering the spectral problem for the set of nonlocal commuting Hamiltonians H_k . If in (37) we impose the condition $f_\alpha = 0$, then we obtain φ as eigenvector of the operators H_k with eigenvalues h_k . The parameters $\lambda_1, \dots, \lambda_m$ have to be found from the quasiclassical Bethe Ansatz equations $f_\alpha = 0$.

4 Trigonometric Knizhnik Zamolodchikov Equation

Let us introduce the function $\chi(\lambda|z) = \chi(\lambda_1, \dots, \lambda_m | z_1, \dots, z_N)$ as in the rational case [7, 8, 9], obeying the following differential equations

$$\mathfrak{x} \frac{\partial \chi}{\partial z_j} = h_j \chi \quad (42)$$

$$\mathfrak{x} \frac{\partial \chi}{\partial \lambda_\alpha} = f_\alpha \chi. \quad (43)$$

Where $\mathfrak{x}$ is some constant. The zero curvature conditions are fulfilled

$$\frac{\partial h_j}{\partial \lambda_\alpha} = \frac{\partial f_\alpha}{\partial z_j}. \quad (44)$$

The solution of the equations (42)-(43) is given by

$$\chi(\lambda|z) = \prod_{i \neq j}^N (\sin(z_i - z_j))^{\frac{s_i s_j}{\mathfrak{x}}} \prod_{\alpha \neq \beta}^m (\sin(\lambda_\alpha - \lambda_\beta))^{\frac{1}{\mathfrak{x}}} \prod_{k \gamma} (\sin(z_k - \lambda_\gamma))^{-\frac{s_k}{\mathfrak{x}}}. \quad (45)$$

As in the rational case we define the vector-function $\Psi(z_1, \dots, z_N)$ through multiple contour integrals as follows

$$\Psi(z_1, \dots, z_N) = \oint \dots \oint \chi(\lambda|z) \varphi(\lambda|z) d\lambda_1 \dots d\lambda_m. \quad (46)$$

Here the integrations are to be taken over some cycles. Now it is easy to show that the vector-function (46) is a solution of the trigonometric Knizhnik-Zamolodchikov equation[15]

$$\mathfrak{x} \frac{\partial \Psi}{\partial z_k} = H_k \Psi. \quad (47)$$

Here H_k are the Gaudin trigonometric Hamiltonians (38). Substituting (46) into the Knizhnik-Zamolodchikov equation (47), using OSBAE (37) and (42), (43) and finally by taking into account the relation

$$\frac{\partial \varphi}{\partial z_k} = - \sum_{\alpha=1}^m \frac{\partial}{\partial \lambda_\alpha} \frac{S_k^- \varphi'_\alpha}{\sin(\lambda_\alpha - z_k)} \quad (48)$$

we find the equation

$$\sum_{\alpha=1}^m \oint \dots \oint \frac{\partial}{\partial \lambda_\alpha} \left[\frac{\chi S_k^- \varphi'_\alpha}{\sin(z_k - \lambda_\alpha)} \right] d\lambda_1 \dots d\lambda_m = 0 \quad (49)$$

It is evident that this equation is satisfied because the contours are closed.

5 Quantum Knizhnik-Zamolodchikov Equation

The quantum Knizhnik-Zamolodchikov equation was introduced in [13] as a difference equation for the correlation function of the intertwining operators and in [12] also as a difference equation for the form factor in integrable massive field theory. In formal language it is a difference equation for the vector-valued function $f(z_1, \dots, z_N) \in V^{l_1} \dots V^{l_N}$, where V^{l_i} is a representation space of the $U_q(\mathfrak{g})$. In this section we analyse the connection between the solution of the quantum Knizhnik-Zamolodchikov equation corresponding to $U_q(\hat{sl}(2))$ with our solution of the trigonometric Knizhnik-Zamolodchikov equation. The quantum Knizhnik-Zamolodchikov equation is the system of the following difference equations[13, 14]

$$f(z_1, \dots, z_j + \kappa, \dots, z_N) = A_j(z_1, \dots, z_N) f(z_1, \dots, z_N), \quad (50)$$

where the operators A_j act on $V^{l_1}, \dots, V^{l_N}$ are given by

$$A_j(z_1, \dots, z_N) = \mathfrak{R}^{l_{j-1}}(z_j - z_{j-1} + \kappa) \dots, \mathfrak{R}^{l_{j+1}}(z_j - z_{j+1} + \kappa), Z^{(l_j)}, \mathfrak{R}^{l_N}(z_j - z_N), \dots, \mathfrak{R}^{l_{j+1}}(z_j - z_{j+1}). \quad (51)$$

In (50), (51) κ is some constant, $\mathfrak{R}^{l_i}(\lambda)$ satisfies the Yang-Baxter equation (7), but the spaces 1, 2, 3 in general are different. The matrices $Z^{(l_j)}$ are defined as follows

$$[Z^l \otimes Z^m, \mathfrak{R}^{lm}(\lambda)] = 0. \quad (52)$$

In [14] the solution of the quantum Knizhnik-Zamolodchikov (50) was written as the following Jackson type integral

$$f(z_1, \dots, z_N) = \sum_{(\lambda_1, \dots, \lambda_m)} g(\lambda|z) \Phi(\lambda_1, \dots, \lambda_m|z), \quad (53)$$

where $\Phi(\lambda_1, \dots, \lambda_m|z)$ is the Bethe wave vector (23) and the function $g(\lambda|z)$ is given as

$$g(\lambda|z) = \prod_{i \neq j}^N P^{l_i, l_j}(z_i - z_j) \prod_{i=1}^N \prod_{\alpha=1}^m D^{l_i}(\lambda_\alpha - z_i) \prod_{\alpha \neq \beta}^m Q(\lambda_\alpha - \lambda_\beta) \exp(c \sum_{\alpha=1}^m \lambda_\alpha). \quad (54)$$

When all $l_i = 1, i = 1, \dots, N$ s. t. , the case that all the spins are half in our language, the functions $P^{l_i, l_j}(x)$, $D^{l_j}(x)$ and $Q(x)$ corresponding to $U_q(\hat{sl}(2))$ satisfy the following difference equations[14]

$$D(x + \kappa) = \frac{\exp(ix) - q^2}{\exp(ix) - 1} q^{-1} D(x), \quad (55)$$

$$Q(x + \kappa) = \frac{q \exp(ix) - q^{-1}}{q^{-1} \exp(ix + \kappa) - q} \frac{\exp(ix + \kappa) - 1}{\exp(ix) - 1} Q(x), \quad (56)$$

$$P(x + \kappa) = \frac{q \exp(ix) - q^{-1}}{\exp(ix) - 1} P(x). \quad (57)$$

If we put $\kappa = \eta \mathfrak{x}$ and $q = \exp(\frac{-i\eta}{2})$ in (55)-(57) and after we take the limit $\eta \rightarrow 0$ we obtain the differential equations

$$\mathfrak{x} \frac{dD}{dx} = \frac{1}{2} \frac{\cos(\frac{x}{2})}{\sin(\frac{x}{2})} D(x), \quad (58)$$

$$\mathfrak{x} \frac{dQ}{dx} = -\frac{\cos(\frac{x}{2})}{\sin(\frac{x}{2})} Q(x), \quad (59)$$

$$\varkappa \frac{dP}{dx} = -\frac{1}{2} \frac{\cos(\frac{x}{2})}{\sin(\frac{x}{2})} P(x). \quad (60)$$

The solutions are given as

$$\begin{aligned} D(x) &= \left(\sin\left(\frac{x}{2}\right)\right)^{-\frac{1}{\varkappa}}, \\ Q(x) &= \left(\sin\left(\frac{x}{2}\right)\right)^{\frac{2}{\varkappa}}, \\ P(x) &= \left(\sin\left(\frac{x}{2}\right)\right)^{\frac{1}{\varkappa}}. \end{aligned} \quad (61)$$

In equation (54) it is necessary to make the choice $c = c' \eta$ and then to take the limit $\eta \rightarrow 0$. So from (54), (61) we conclude that in the quasiclassical limit we have

$$g(\lambda|z) \rightarrow \chi(\lambda|z), \quad (62)$$

where $\chi(\lambda|z)$ is given by (45). Taking into account that the Bethe wave vector $\Phi(\lambda_1, \dots, \lambda_m|z)$ in the quasiclassical limit is of the form (35), we conclude that in the quasiclassical limit the solution of the quantum Knizhnik-Zamolodchikov equation is equal to the solution of the trigonometric Knizhnik-Zamolodchikov equation (46) in the sense that

$$f(z_1, \dots, z_N) \rightarrow (2\eta)^m \Psi(z_1, \dots, z_N). \quad (63)$$

In general, the quasiclassical limit of the solution (53) of the quantum Knizhnik-Zamolodchikov equation is connected with the solution of the trigonometric Knizhnik-Zamolodchikov equation as given by Cherednik[15]

$$\varkappa \frac{\partial \Psi}{\partial z_k} = \sum_{i \neq k}^N r_{ki}(z_k - z_i) \Psi, \quad (64)$$

where $r_{ki}(\lambda)$ is a classical r-matrix. In order to find a solution of equation (64) in our language it is necessary to use the OSBAE for the $T_s(\lambda)$ constructed from the solution $\mathfrak{R}_s(\lambda)$ (7). Note, that the $T_s(\lambda)$ obey the equation $[T(\lambda), T_s(\mu)] = 0$ and that they have the same Bethe wave vector $\Phi(\lambda_1, \dots, \lambda_m|z_1, \dots, z_N)$.

6 Conclusion

In this paper we have constructed a solution of the trigonometric Knizhnik-Zamolodchikov equation with use of the OSBAE for the inhomogeneous XXZ vertex model. The solution obtained here is the quasiclassical limit of the solution to the quantum Knizhnik-Zamolodchikov equation corresponding to $U_q(\hat{\mathfrak{sl}}_2)$. From this observation we conclude that all Bethe wave vectors, which satisfy to the OSBAE, generate the solutions of the quantum Knizhnik-Zamolodchikov equation. In the quasiclassical limit, when the Bethe wave vectors become equal to the Bethe wave vectors for the Gaudin magnets, they generate the solutions of the usual (undeformed) Knizhnik-Zamolodchikov equation (rational, trigonometric, and elliptic). It is intriguing to understand the physical meaning of this fine structure. Why do Bethe wave vectors for the inhomogeneous vertex model lead to the solution of the Knizhnik-Zamolodchikov equations (usual and deformed)? During the preparation of this paper we learned that a similar result (when all the spins $s_i = \frac{1}{2}, i = 1, \dots, N$) was obtained in [21].

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