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Functional Integral of QED in Terms of
Gauge Invariant Quantities

by

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Abstract

We reduce the functional integral of quantum electrodynamics to an integral containing only local gauge invariant quantities. The set of (bosonic) invariants contains bilinear combinations of the spinor field and a real-valued covector field.

1 Introduction

This Letter is a continuation of [1], where we have shown that the “classical” Dirac-Maxwell system can be reformulated in a spin-rotation covariant way in terms of gauge invariant quantities. For earlier attempts to solve this problem we refer to [2]-[5] and further references therein. Here we show that similar constructions can be performed on the quantum level, where fermion fields are treated as anticommuting quantities. The present work is part of a programme of formulating the quantum theory of gauge fields in terms of gauge invariant quantities, see [6]-[9].

The Letter is organized as follows: In section 2 we define gauge invariant quantities, built from the gauge potential and the (anticommuting) spinor field components. Next we prove certain identities, which enable us to formulate the physical action (under the functional integral) in terms of these invariants. In section 3 we reduce the functional integral of QED to an integral in terms of these quantities. The extremely difficult problem of making the functional integral rigorous is not touched.

2 Gauge Invariants

A field configuration of this model consists of a U(1)-gauge potential (A_μ) and a four-component spinor field (ψ^a), where $a, b, \dots = 1, 2, \dot{1}, \dot{2}$ denote bispinor indices and $\mu, \nu, \dots = 0, 1, 2, 3$ spacetime indices. The components of (ψ^a) anticommute, the (real) dimension of the Grassmann-algebra built by these quantities is (pointwise) 8. A bispinor is represented by a pair of Weyl spinors:

$$\psi^a = \begin{pmatrix} \phi^K \\ \varphi_{\dot{K}} \end{pmatrix}. \quad (2.1)$$

The Lagrangian is given by

$$\begin{aligned} \mathcal{L} &= \mathcal{L}_{gauge} + \mathcal{L}_{mat} \\ &= -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} - m\bar{\psi}^a\beta_{ab}\psi^b - Im\left\{\bar{\psi}^a\beta_{ab}(\gamma^\mu)^b{}_c D_\mu\psi^c\right\}, \end{aligned} \quad (2.2)$$

where

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu, \quad (2.3)$$

$$D_\mu\psi^a = \partial_\mu\psi^a + ieA_\mu\psi^a, \quad (2.4)$$

are the electromagnetic field strength and the covariant derivative. The bar denotes complex conjugation, β_{ab} denotes the Hermitean metric in bispinor space and $(\gamma^\mu)^b{}_c$ are the Dirac matrices. For the representation used for these quantities see [1].

Let us define the following gauge invariant Grassmann-algebra-valued quantities of second rank:

$$L^\mu = \frac{1}{2}\bar{\phi}^{\dot{K}}\phi^L\sigma^\mu_{\dot{K}L}, \quad (2.5)$$

$$R^\mu = \frac{1}{2} \bar{\varphi}_L \varphi_{\dot{K}} \sigma^{\mu \dot{K} L}, \quad (2.6)$$

$$H = \bar{\varphi}_K \phi^K, \quad (2.7)$$

$$B_\mu = \text{Im} \{ \bar{H} (\bar{\varphi}_K D_\mu \phi^K + \phi^K \overline{D_\mu \varphi_K}) \}. \quad (2.8)$$

Moreover, we define the following auxiliary invariants, which later on will be eliminated from the functional integral:

$$P = \frac{1}{2} \bar{\phi}^{\dot{1}} \phi^1 \bar{\phi}^{\dot{2}} \phi^2, \quad (2.9)$$

$$Q = \frac{1}{2} \varphi_{\dot{1}} \bar{\varphi}_1 \varphi_{\dot{2}} \bar{\varphi}_2. \quad (2.10)$$

We stress that P and Q are space time scalars. This follows from the antisymmetry of products of two spinor components of the same type, e.g. $\phi^1 \phi^2 = \frac{1}{2} \epsilon_{KL} \phi^K \phi^L$. But ϵ_{KL} is an $SL(2, C)$ -invariant scalar product on spinor space.

Lemma 1 *The following algebraic identities hold:*

$$2L^\mu R_\mu = -|H|^2 = -H\bar{H}, \quad (2.11)$$

$$L^\mu L^\nu = \eta^{\mu\nu} P, \quad (2.12)$$

$$R^\mu R^\nu = \eta^{\mu\nu} Q, \quad (2.13)$$

$$|H|^2 L^\mu R^\nu = -2\eta^{\mu\nu} PQ, \quad (2.14)$$

where $\eta^{\mu\nu}$ is the Minkowski metric. Moreover, the following differential identities hold:

$$\begin{aligned} (|H|^2)^2 F_{\mu\nu} &= \frac{1}{2e} (|H|^2 (\partial_{[\mu} B_{\nu]}) - (\partial_{[\mu} |H|^2) B_{\nu]}) \\ &\quad + \frac{2}{e} \epsilon^{\alpha\beta\gamma\delta} R_\alpha L_\beta (\partial_{[\mu} R_{\gamma]})(\partial_{\nu]} L_\delta). \end{aligned} \quad (2.15)$$

$$\begin{aligned} |H|^2 \mathcal{L}_{mat} &= -2m |H|^2 \text{Re}(H) - (L^\mu + R^\mu) B_\mu \\ &\quad + (L^\mu - R^\mu) \text{Im}(\bar{H} \partial_\mu H) - 2\epsilon^{\alpha\beta\mu\gamma} L_\alpha R_\beta \partial_\mu (R_\gamma + L_\gamma), \end{aligned} \quad (2.16)$$

$$|H|^2 R_\alpha (\partial_\mu L_\beta) = 2Q L_\beta (\partial_\mu L_\alpha) - 2\eta_{\alpha\beta} (\partial_\mu P) Q + 2\eta_{\alpha\beta} P (\partial_\mu Q), \quad (2.17)$$

$$|H|^2 L_\alpha (\partial_\mu R_\beta) = 2P R_\beta (\partial_\mu R_\alpha) - 2\eta_{\alpha\beta} (\partial_\mu P) Q + 2\eta_{\alpha\beta} Q (\partial_\mu P). \quad (2.18)$$

Proof. The proof of the algebraic identities is by simple inspection, using the anti-commuting character of the fermion fields. To prove (2.15), we use

$$F_{\mu\nu} \phi^K = (D_\mu D_\nu - D_\nu D_\mu) \phi^K,$$

which implies

$$(|H|^2)^2 F_{\mu\nu} = |H|^2 \bar{H} \bar{\varphi}_K (D_\mu D_\nu - D_\nu D_\mu) \phi^K.$$

Now a lengthy, but straightforward calculation gives

$$\begin{aligned} (|H|^2)^2 F_{\mu\nu} &= \frac{1}{2e} (|H|^2 (D_{[\mu} B_{\nu]}) - (D_{[\mu} |H|^2) B_{\nu]}) \\ &\quad + \frac{2}{e} \epsilon^{\alpha\beta\gamma\delta} R_{\alpha} L_{\beta} (D_{[\mu} R_{\gamma]})(D_{\nu]} L_{\delta}). \end{aligned} \quad (2.19)$$

Using the fact that the covariant derivative acting on invariants reduces to the partial derivative we get (2.15). The proof of (2.16) is straightforward. To prove (2.17) one has to differentiate (2.13) and use identities (2.11) and (2.14). A similar calculation leads to (2.18). $\square$

Concerning formulae (2.12), (2.13) and (2.14), we stress that identities of this type do not exist in the case of commuting spinor fields. Here they occur due to the fact that P and Q are elements of maximal rank in the Grassmann subalgebras spanned by ϕ^K and $\varphi_{\bar{K}}$ respectively. Moreover, observe that formula (2.15) is an identity on the level of elements of maximal rank of the full Grassmann algebra.

Finally, we note that

$$B_\mu = 2e |H|^2 A_\mu + \text{Im} \{ \bar{H} (\bar{\varphi}_K \partial_\mu \phi^K + \phi^K \partial_\mu \bar{\varphi}_K) \}. \quad (2.20)$$

3 Functional Integral

We start with the functional integral given in terms of the gauge dependent set (A_μ, ψ^a) by

$$\mathcal{F} = \int \prod dA d\psi d\bar{\psi} e^{iS[A, \psi, \bar{\psi}]}, \quad (3.1)$$

where

$$S[A, \psi, \bar{\psi}] = \int d^4x \mathcal{L}[A, \psi, \bar{\psi}]$$

denotes the physical action. The integral over ψ and $\bar{\psi}$ is understood in the sense of Berezin [10], see also [11] and [12]. Inserting a physical observable, i.e. a gauge invariant function, $\mathcal{O}(A, \psi, \bar{\psi})$ under the above integral (and normalizing it appropriately) one obtains the vacuum expectation value of this observable. For this purpose, in the conventional approach one has to invent a gauge fixing procedure. Here we propose a different approach based upon a reduction of (3.1) to an integral in terms of local invariants. This leads to the possibility of calculating vacuum expectation values for a certain class of observables, which will be characterized at the end of the Letter. For the sake of simplicity of notation, however, we will consider the “bare” integral (3.1). Finally, we

will see that our method works also for the calculation of integrals containing observables from the above mentioned class.

In what follows we will use the following notion of the δ -distribution on superspace

$$\delta(u - U) = \int e^{2\pi i \xi(u-U)} d\xi = \sum_{n=0}^{\infty} \frac{(-1)^n}{n!} \delta^{(n)}(u) U^n, \quad (3.2)$$

where u is a c-number variable and U an even-rank combination of Grassmann variables ψ and $\bar{\psi}$. Due to the nilpotent character of U the above sum is finite. This δ -distribution is a special example of a vector-space-valued distribution in the sense of [13]. From the above definition we have immediately

$$1 = \int \delta(u - U) du. \quad (3.3)$$

One easily shows the following

Lemma 2 *For an arbitrary smooth function f we have*

$$f(u)\delta(u - U) = f(U)\delta(u - U). \quad (3.4)$$

From this we get a technique, which frequently will be used in this chapter:

$$\begin{aligned} \int f(\alpha)\delta(u - U) du &= \int f\left(\alpha \frac{u^n}{\tilde{u}^n}\right) \delta(u - U) \delta(u - \tilde{u}) du d\tilde{u} \\ &= \int f\left(\frac{\alpha}{\tilde{u}^n} U^n\right) \delta(u - U) \delta(U - \tilde{u}) du d\tilde{u}. \end{aligned} \quad (3.5)$$

Here, α denotes any c-number or Grassmann-algebra-valued quantity and n is a positive integer, such that the rank of αU^n is smaller or equal to the maximal rank.

Now, for the invariants $L^\mu, R^\mu, B_\mu, H, \bar{H}$ defined in the previous chapter we introduce corresponding c-number mates $l^\mu, r^\mu, b_\mu, h, \bar{h}$, which by definition are gauge invariant.

Proposition 1 *The functional integral $\mathcal{F}$ in terms of invariants is given by*

$$\mathcal{F} = \int \prod dv dl dr dh d\bar{h} K(l, r, h) e^{iS[v, l, r, h]}, \quad (3.6)$$

where the integral kernel $K[l, r, h]$ is given by

$$K = \frac{1}{2i} \left\{ \frac{1}{16} \frac{\partial^2}{\partial r^\nu \partial r_\nu} \frac{\partial^2}{\partial l^\mu \partial l_\mu} + \frac{\partial^4}{\partial h^2 \partial \bar{h}^2} - \frac{1}{2} \frac{\partial^2}{\partial r^\mu \partial l_\mu} \frac{\partial^2}{\partial h \partial \bar{h}} \right\} \delta(h) \delta(l) \delta(r), \quad (3.7)$$

and

$$\mathcal{L}[v, l, r, h] = -\frac{1}{4} (\partial_{[\mu} v_{\nu]})^2 - 2m \operatorname{Re}\{h\} - 2e(l^\mu + r^\mu)v_\mu - 2(l^\mu - r^\mu) \frac{\operatorname{Im}(\bar{h} \partial_\mu h)}{2|h|^2}. \quad (3.8)$$

Proof. Using (3.3) we obtain:

$$\mathcal{F} = \int \prod dA d\psi d\bar{\psi} \int \frac{1}{2i} \prod dh d\bar{h} dl dr db e^{iS[A_\mu, \psi, \bar{\psi}]} \delta(h-H) \delta(l-L) \delta(r-R) \delta(b-B), \quad (3.9)$$

where $\delta(h-H) = \delta(\text{Re}(h)-\text{Re}(H)) \delta(\text{Im}(h)-\text{Im}(H))$. Due to (3.5), under the functional integral we rewrite

$$\mathcal{L} = \frac{|h|^4}{|\tilde{h}|^4} \mathcal{L}_{\text{gauge}} + \frac{|h|^2}{|\tilde{h}|^2} \mathcal{L}_{\text{mat}}, \quad (3.10)$$

insert $\delta(h-\tilde{h}) d\tilde{h}$ and replace h by H everywhere. Now we apply identities (2.15) and (2.16) to $|H|^4 \mathcal{L}_{\text{gauge}}$ and $|H|^2 \mathcal{L}_{\text{mat}}$ respectively and, subsequently, replace all Grassmann-algebra-valued quantities in $\mathcal{L}$ by corresponding c-number quantities and integrate over $\tilde{h}$. Applying the same procedure to the second $F_{\mu\nu}$ -factor in $\mathcal{L}_{\text{gauge}}$ leads to $\mathcal{L} = \mathcal{L}[b, r, l, h]$. Next, using again (3.5), we observe that because of (2.17) and (2.18) all terms containing the totally antisymmetric symbol $\epsilon^{\alpha\beta\gamma\delta}$ vanish under the functional integral. The Lagrangian under the functional integral takes the form

$$\begin{aligned} \mathcal{L} = & -\frac{1}{4} \left(\partial_\mu \frac{b_\nu}{2e|h|^2} \right)^2 - 2m \text{Re}\{h\} \\ & - 2e(l^\mu + r^\mu) \frac{b_\mu}{2e|h|^2} - 2(l^\mu - r^\mu) \frac{\text{Im}(\bar{h}\partial_\mu h)}{2|h|^2}. \end{aligned} \quad (3.11)$$

Now we observe that the gauge potential A_μ enters $\mathcal{F}$ only under the δ -distribution $\delta(b-B)$, see formula (2.20), where - due to the above procedure - H has been already replaced by h . Hence, we can trivially integrate out A_μ . Due to (2.20), this integration gives the factor $2e|h|^2$ under the integral. Thus, after performing a change of variables from b_μ to $v_\mu := \frac{b_\mu}{2e|h|^2}$ we obtain

$$\mathcal{F} = \int \prod d\psi d\bar{\psi} \frac{1}{2i} dh d\bar{h} dl dr dv e^{iS[v, l, r, h, \bar{h}]} \delta(h-H) \delta(l-L) \delta(r-R), \quad (3.12)$$

with the Lagrangian given by (3.8).

We stress that the above procedure of integrating out A_μ is equivalent to performing a change of variables in superspace, given by (2.20), using theorem 2.1 of [11].

Now, the fields ψ and $\bar{\psi}$ occur only in the δ -distributions. To integrate them out we use the integral representation (3.2). This way we obtain:

$$\begin{aligned} \mathcal{F} = & \int \prod dv \frac{1}{2i} dh d\bar{h} dl dr e^{iS[v, l, r, h, \bar{h}]} \int \prod d\rho d\lambda d\chi d\bar{\chi} \\ & e^{2\pi i(\chi h + \bar{\chi} \bar{h} + \rho r + \lambda l)} \int \prod d\psi d\bar{\psi} e^{-2\pi i \bar{\psi}^a C_{ab} \psi^b}, \end{aligned} \quad (3.13)$$

where C_{ab} is given by

$$C_{ab} = \begin{pmatrix} \lambda_{LK} & \delta_{\bar{L}}^{\bar{K}} \bar{\chi} \\ \delta_K^L \chi & \rho^{KL} \end{pmatrix}, \quad (3.14)$$

with $\lambda_{LK} = \frac{1}{2} \lambda_\mu \sigma_{LK}^\mu$ and $\rho^{KL} = \frac{1}{2} \rho_\mu \sigma^{\mu KL}$. The last (Gaussian) integral [10] in (3.13) is equal to the determinant of C_{ab} :

$$\begin{aligned} \det C_{ab} &= \det(\lambda_{LK}) \det(\rho^{MN}) + |\chi|^2 (|\chi|^2 - \lambda_{LK} \rho^{LK}) \\ &= \frac{1}{16} \lambda_\mu \lambda^\mu \rho_\nu \rho^\nu + |\chi|^2 (|\chi|^2 - \frac{1}{2} \lambda_\nu \rho^\nu). \end{aligned} \quad (3.15)$$

Thus, integration over χ , $\bar{\chi}$, ρ and λ yields the final result. $\square$

We see that our methods of changing variables used in the proof work in the same way for vacuum expectation values of observables of the type $\mathcal{O} = \mathcal{O}(f, B, L, R, H, \bar{H})$, yielding an additional factor $\mathcal{O}(\text{curl } v, |h|^2 v, l, r, h, \bar{h})$ in (3.6). For instance, the current-current propagator may be calculated this way.

We stress that in this approach the photon and the electron are no longer fundamental fields. Instead, we obtained a description in terms of the electric current $2e(l + r)$, the chiral current $2(l - r)$, the covector field v , which is a certain combination of the photon field A and the spinor field ψ , and a complex scalar field h . The quantity v is a sort of “hydrodynamical velocity”, see [2].

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