CANONICAL COHERENT STATES FOR THE RELATIVISTIC HARMONIC OSCILLATOR\(^\dagger\)

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Abstract

A manifestly covariant, group-quantization of the 1+1-D relativistic harmonic oscillator has been given in both configuration and Bargmann-Fock space along with a generalized Bargmann transform relating Fock wave functions and a set of Relativistic Hermite polynomials. In this paper we construct manifestly covariant relativistic coherent states on the entire complex plane which reproduce others previously introduced on a given \(SL(2, \mathbb{R})\) representation, once a change of variables \(z \in C \rightarrow z_D \in \text{unit disk}\) is performed. We also introduce higher-order creation and annihilation operators, \(\hat{a}, \hat{a}^\dagger\), with canonical commutation relation \([\hat{a}, \hat{a}^\dagger] = 1\) rather than the covariant one \([\hat{z}, \hat{\xi}^\dagger] \approx \text{Energy and naturally associated with the } SL(2, \mathbb{R})\) group. The canonical coherent states are then defined as eigenstates of \(\hat{a}\). Finally, we construct a canonical, minimal representation in configuration space by mean of eigenstates of a canonical position operator.

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1 Introduction

Ordinary coherent states were introduced from the beginning of the developments of Quantum Mechanics and Radiation Theory in several different, yet equivalent, ways according to different interesting properties with direct applications to practical, mainly optical, systems (see the pioneer work by Glauber [1]). Essentially, these states can be characterized by a) giving minimal $q-p$ uncertainty relations, b) being eigenstates of the annihilation operator $\hat{a}$, and c) as the result of applying the displacement operator $e^{i(\alpha \hat{a} - \alpha^* \hat{a}^d)}$ on the vacuum $|0\rangle$, and among the practical properties, we mention the low noise in amplifying applications (as a consequence of a)) and optical coherence (as a consequence of b)) (see for instance [2, 3]).

Relativistic quantum mechanical systems in general are characterized by possessing manifestly covariant commutation relations of the form $[\hat{\imath}, \hat{\jmath}] \approx E$, so that the uncertainty relations are no longer $\Delta \hat{x} \Delta \hat{p} \geq \frac{\hbar}{2}$, but $\Delta \hat{\imath} \Delta \hat{\jmath} \geq \frac{1}{\hbar |E|}$, and, therefore, the absolute minimum can be reached only by the vacuum. In particular, the adopted commutation relations for the basic operators $\hat{\imath}, \hat{\jmath}$ and $\hat{E}$ corresponding to the quantum relativistic harmonic oscillator are:

$$[\hat{E}, \hat{\imath}] = -i\frac{\hbar}{m}\hat{\jmath}, \quad [\hat{E}, \hat{\jmath}] = i\hbar \omega^2 \hat{\imath}, \quad [\hat{\imath}, \hat{\jmath}] = i\hbar(1 + \frac{1}{mc^2} \hat{E}), \quad (1)$$

which implement a central (pseudo-)extension of the Lie algebra (see Ref. [4] for a study of the cohomology of Lie algebras) $SL(2, R) \approx$ Anti-deSitter group in 1+1 dimensions). This algebra, where $\hat{E}$ generates the time translations, $\hat{\jmath}$ the space translations and $\hat{\imath}$ the boosts, reproduces the pseudo-extended Poincaré algebra in the $\omega \to 0$ limit and the extended Newton (non-relativistic harmonic oscillator) algebra when $c \to \infty$. It should be recalled at this point that it is the pseudo-extended Poincaré group which regains the centrally extended Galilei group in the non-relativistic limit [5] (for a general study of central extensions of groups see Ref. [6]).

The deviation of the relativistic commutation relations between $\hat{\imath}$ and $\hat{\jmath}$ from the Galilean ones causes the different definitions of coherent states given above to be non-equivalent. The definition c) seems to be more widely adopted at least for those cases with an underlying group structure [7, 8].

In this paper we quantize the relativistic harmonic oscillator following a group approach to quantization [9, 10] which provides, in a natural way, a set of states -our relativistic coherent states- reproducing those previously introduced by Perelomov [7] after the change of variables

$$z_D = \frac{2}{\sqrt{N}} \frac{z}{\sqrt{1 + \frac{2z^2}{N^2}}}, \quad N \equiv \frac{mc^2}{\hbar \omega}, \quad (2)$$
from $C$ to the unit disk $D$, has been performed. Then, using a generalization of the concept of Polarization in Geometric Quantization [11, 12], we are able to find higher-order creation and annihilation operators $\hat{a}, \hat{a}^\dagger$ satisfying canonical (yet relativistic) commutation relations, and allowing for a conventional way of defining canonical, relativistic coherent states as eigenstates of the new higher-order annihilation operator $\hat{a}$. The new relativistic coherent states thus satisfy properties fully analogous to those of ordinary (non-relativistic) coherent states, although defined in terms of canonical or Darboux [13] co-ordinates.

The paper is organized as follows: Sec. 2 is an overview of the group quantization formalism applied to the relativistic harmonic oscillator in the Bargmann-Fock realization. In Sec. 3 we introduce a set of relativistic coherent states on the $SL(2, R)$ group in a natural way and compare them with others previously defined according to definition c) ([7, 8]). These states are related with the configuration space ones through a Relativistic Bargmann Transform. In Sec. 4 we introduce higher-order, canonical creation and annihilation operators mimicking the non-relativistic ones, and compute the eigenstates of the latter following definition b) which will accordingly be called canonical coherent states. Higher-order, canonical position and momentum operators are also defined acting on a canonical, minimal $q$-representation. The paper concludes with some remarks in Sec. 5.

2 Group Quantization and the Relativistic Harmonic Oscillator (RHO) in the Bargmann-Fock-like realization.

Our starting point will be a central pseudo-extension of the group $SL(2, R)$, denoted by $SL(2, R)\tilde{\otimes}U(1)$ [14], the coboundary of which is generated by a function which is an integer power of the parameter of the Cartan subgroup. The precise techniques of the group-quantization procedure [9, 10] will be explained on the way.

The $\tilde{G} \equiv SL(2, R)\tilde{\otimes}U(1)$ group law is:

\[
\begin{align*}
\zeta'' &= \zeta' \eta^{-2} + \zeta \kappa' + \frac{\zeta}{N(1 + \kappa)}(z'z'\eta^{-2} + z''z\eta^2) \\
z'' &= z'\eta^{-2} + z\kappa' + \frac{z'}{N(1 + \kappa)}(z'z'\eta^{-2} + z'z\eta^{-2}) \\
\eta'' &= \sqrt{\frac{2}{1 + \kappa}} \left[ \sqrt{\frac{1 + \kappa'}{2}} \sqrt{1 + \kappa} \eta' \eta + \sqrt{\frac{2}{1 + \kappa'}} \sqrt{1 + \kappa} \eta' \eta' \right] \\
\zeta'' &= \zeta' (\eta'' \eta'^{-1} \eta^{-1})^{-2N},
\end{align*}
\]
where
\[ \kappa \equiv \sqrt{1 + \frac{2zz^*}{N}} \]
\[ \kappa'' = \kappa' \kappa + \frac{1}{N} \left( z^* z \eta^{-2} + z^{**} z \eta^2 \right), \]
and \( z \in C, \eta \in U(1) \subset SL(2, R), \zeta \in U(1) \) and \( N \equiv \frac{m^2}{2\omega} \); the variables \( z, z^* \) parametrize the hyperboloid \( |z_1|^2 - |z_2|^2 = 1 \), \( (z_1 = \sqrt{1 + \frac{e}{2} \eta}, z_2 = \frac{1}{\sqrt{2N}} \sqrt{1 + \frac{e}{2} \eta} z) \). It must be noted that \( N \) is quantized (\( N = 1, 3/2, 2, 5/2, \ldots \)) on \( SL(2, R) \) but only restricted to be a positive number on the universal covering group.

The coboundary
\[ \Delta \equiv \left( \eta'' \eta'^{-1} \eta^{-1} \right)^{-2N} : SL(2, R) \times SL(2, R) \rightarrow U(1), \]
which is generated by
\[ \eta^{-2N} : SL(2, R) \rightarrow U(1), \]
realizes a pseudo-extension. We say that \( \Delta \) is a pseudo-cocycle and realizes a pseudo-extension rather than a trivial cocycle (coboundary) realizing a trivial extension, because in the \( e \to \infty \) limit, \( (\eta'' \eta'^{-1} \eta^{-1})^{-2N} \) goes to a true cocycle on the non-relativistic harmonic oscillator (Newton) group (see [5] for a general study of the contraction process under which a true cocycle is generated by a coboundary of the uncontracted group).

Group quantization uses the right-invariant vector fields (see, e.g. [13]) which act on \( U(1) \)-equivariant complex functions on \( \hat{G} \) as ordinary derivatives, to define a group representation (Bohr-Sommerfeld quantization). This representation is reducible, as can be stated from the existence of non-trivial operators (all the left-invariant vector fields) commuting with the representation.

The full quantization is achieved by reducing this representation in a way compatible with the action of right vector fields. The reduced Hilbert space is made of complex functions \( \Psi \) on \( \hat{G} \) such that
\[ \Psi(\zeta \ast g) = \zeta \cdot \Psi(g), \zeta \in U(1), g \in G \]
\[ \hat{X}^L \Psi = 0, \forall \hat{X}^L \in \mathcal{P}, \]
where a Polarization \( \mathcal{P} \) is a maximal left subalgebra containing the generators in the kernel of \( \Delta \) and excluding the central generator \( \Xi \equiv \hat{X}_\zeta^L \) of \( U(1) \).

The left- and right-invariant vector fields are:
\[ \hat{X}_z^L = \kappa \frac{\partial}{\partial z} + \frac{iz^*}{2N(1 + \kappa)} \left( i\eta \frac{\partial}{\partial \eta} \right) - \frac{iz^*}{1 + \kappa} \Xi \]
The operators are

\[
\begin{align*}
\hat{X}_z^L &= \kappa \frac{\partial}{\partial z^*} - \frac{iz}{2N(1 + \kappa)} \left( i\eta \frac{\partial}{\partial \eta} \right) + \frac{iz}{1 + \kappa} \Xi \\
\hat{X}_n^L &= i\eta \frac{\partial}{\partial \eta} - 2iz \frac{\partial}{\partial z} + 2iz^* \frac{\partial}{\partial z^*} \\
\hat{X}_\xi^L &= i\zeta \frac{\partial}{\partial \xi} \equiv \Xi,
\end{align*}
\]

(6)

\[
\begin{align*}
\hat{X}_z^R &= \frac{\eta^2}{(1 + \kappa)} \left[ \frac{(1 + \kappa)^2}{2} \frac{\partial}{\partial z} + \frac{z^2}{N} \frac{\partial}{\partial z^*} - \frac{iz^*}{2N} \left( i\eta \frac{\partial}{\partial \eta} \right) + iz^* \Xi \right] \\
\hat{X}_n^R &= \frac{\eta^2}{(1 + \kappa)} \left[ \frac{(1 + \kappa)^2}{2} \frac{\partial}{\partial z^*} + \frac{z^2}{N} \frac{\partial}{\partial z} + i \frac{z}{2N} \left( i\eta \frac{\partial}{\partial \eta} \right) - iz \Xi \right] \\
\hat{X}_\xi^R &= i\zeta \frac{\partial}{\partial \xi} \equiv \Xi.
\end{align*}
\]

(7)

The polarization is given by \( P = \langle \hat{X}_n^L, \hat{X}_z^L \rangle \), with solutions

\[
\hat{\Phi}(z, z^*, \theta) = \zeta \sum_n e^{-2i\pi \eta \frac{d}{dn}} \hat{\Phi}_n(z, z^*)
\]

(10)

\[
\hat{\Phi}_n(z, z^*) \equiv \frac{1}{\pi \sqrt{n!}} \sqrt{\frac{(2N)_n}{(2N)!}} \sqrt{\frac{2N - 1}{2N}} \left( 1 + \kappa \right)^{-N-n} z^n
\]

(11)

constituting the Fock-Bargmann-like space with the group invariant measure \( dz dz^*/\kappa^4 \). The expression \( (2N)_n \) in (11) stands for the Pochhammer symbol \( \Gamma((2N+n)/2) \).

\[\text{In reality the measure on the whole group is } \frac{dz dz^*}{\kappa^4} \text{ but the time variable (or } \theta \text{) can be factorized out.}\]
The relativistic Fock space is given by:

$$<0, N|N, 0> = 1, \quad |N, n> = \frac{(z^\dagger)^n|N, 0>}{\sqrt{n! (2N)^n}}$$

$$\hat{z}|N, n> = \sqrt{n(1 + \frac{n-1}{2N})}|N, n-1>$$

$$\hat{z}^\dagger|N, n> = \sqrt{(n+1)(1 + \frac{n}{2N})}|N, n+1>$$

$$\hat{H}|N, n> = n|N, n>$$

3 Relativistic coherent states (RCS).

In the group-quantization scheme, the coherent states (generalizing the standard non-relativistic coherent states [1]), as well as the wave functions given above, are defined by mean of infinitesimal relations (differential polarization equations), rather than a group action on the vacuum, associated with a previously given representation of the group [7, 8] (see [15, 16, 17] for a more general study of overcomplete families of states non-necessarily associated with groups). These are defined simply as:

$$|z> = \sum_{n=0}^{\infty} \Phi^N_n(z, z^*)|N, n> \iff \Phi^N_n(z, z^*) = <z|N, n>$$

The associated (time-independent) wave functions $$<z|z'> = \Phi_z(z')$$ correspond to the choice $$c_n \equiv c_n(z') = \Phi^N_n(z', z^*)$$ in (10). They are:

$$<z'|z> = \sum_{n=0}^{\infty} \Phi^N_n(z', z^*)\Phi^N_n(z, z^*)$$

$$= \frac{1}{\pi} \frac{2N-1}{2N} \left( \frac{1 + \kappa}{2} \right)^{-N} \left( \frac{1 + \kappa'}{2} \right)^{-N} \sum_{n=0}^{\infty} \frac{(2N)_n}{n! (2N)^n} \left( \frac{2z^* z}{1 + \kappa'} \right)^n$$

As in the non-relativistic case, the RCS constitute an overcomplete set and satisfy the reproducing kernel property with respect to the group measure:

$$I = \int \frac{d^2 z}{\kappa} |z><z|$$

$$|z'> = \int \frac{d^2 z}{\kappa} |z><z||z'>$$
The expectation values of $\hat{z}$ and $\hat{z}^\dagger$ in the coherent states are 

$\langle \hat{z} \rangle \equiv \frac{\langle \hat{z} \hat{z} \rangle}{\langle \hat{z}^\dagger \hat{z} \rangle} = z$

and $\langle \hat{z}^\dagger \rangle = z^*$, making the variables $z, z^* \in C$ especially suitable to describe the Bargmann-Fock-like representation. Defining the operators $\hat{x}$ and $\hat{p}$ in the usual way, i.e.

$$\hat{x} = \sqrt{\frac{\hbar}{2m\omega}}\left(\hat{z} + \hat{z}^\dagger\right), \quad \hat{p} = \sqrt{\frac{m\omega\hbar}{2}}\left(\hat{z}^\dagger - \hat{z}\right),$$

we get $\langle \hat{x} \rangle = x$, $\langle \hat{p} \rangle = p$, where $x$ and $p$ are defined in the same way, constituting the phase-space coordinates for Anti-deSitter space-time (see [18] where an adequate choice of time is discussed). We observe that these expectation values follow the classical trajectories (geodesics) of the motion.

Repeating the group quantization in the new variables we obtain the manifestly-covariant $\mathbf{x}$-representation. The states $|x, t\rangle$ are defined as:

$$|x, t\rangle \equiv \sum_{n=0}^{\infty} \Psi^N_n(x, t)^*|N, n\rangle,$$  

where

$$\Psi^N_n(x, t) \equiv e^{-i\omega t} \Phi^N_n(x)$$

and

$$\Phi^N_n(x) = \sqrt{\frac{\omega}{2\pi}} \left(\frac{m\omega}{\hbar}\right)^{1/4} \frac{1}{2\pi \sqrt{n!}} \frac{(2N)^n}{(2N)_n} \frac{\Gamma(n)}{\sqrt{\Gamma(N - 1/2)}} e^{-N\alpha} H^N_n(\eta),$$

$$\alpha \equiv \sqrt{1 + \frac{\omega^2}{\hbar^2} x^2}, \quad \eta \equiv \sqrt{\frac{m\omega}{\hbar} x} \text{ and } H^N_n(\eta) \text{ are the Relativistic Hermite polynomials} \ [19, 18]. \text{ These states are not eigenstates of the boost operator } \hat{x} \text{ in the same manner that the states } |z, \theta\rangle \equiv |e^{-i\theta} z\rangle \text{ are not eigenstates of the annihilation operator } \hat{z}. \text{ The integration measure is } dx \, dt, \text{ coming from the group measure once the } p\text{-integration has been regularized} \ [18].$$

Both representations are related through the Relativistic Bargmann transform [20], the kernel of which is nothing but the configuration-space wave function of the coherent states $|z, \theta\rangle$ defined above:

$$\langle x, t|z, \theta\rangle = \tilde{C}^N \left(\frac{1 + \kappa}{2}\right)^{-N} \alpha^N \left[1 + \frac{s_0^2}{N}\right]^{-N},$$

where

$$s_0 \equiv \sqrt{\frac{m\omega}{\hbar} x} - \frac{\sqrt{2} e^{-i(\omega t - \theta)} \alpha}{1 + \kappa},$$

and

$$\tilde{C}^N \equiv \frac{1}{\sqrt{\pi}} \left(\frac{m\omega}{\hbar}\right)^{1/4} \frac{1}{\sqrt{2\pi \sqrt{N\Gamma(N - 1/2)}}} \left(\frac{2N - 1}{2N}\right)^{1/2} \frac{\Gamma(N)}{\sqrt{\Gamma(N) - 1/2}}.$$


The time variable can be factorized out (non-trivially) from the manifestly-covariant $x$-representation, giving rise to a minimal $x$-representation $|x\rangle$. The new integration measure turns out to be $dx/a^2$ and it is this measure that makes the Relativistic Hermite polynomials (multiplied by the partial weights $a^{-N-n}$) a set of orthogonal functions [18, 21, 22].

In the non-relativistic limit we regain the usual coherent states in configuration space in both minimal and manifestly-covariant (except for a phase $e^{-i(\omega t)}$ factor in $z$) representations:

$$<x|z>^{NR} = \frac{1}{\sqrt{\pi}} \left( \frac{m\omega}{\hbar\pi} \right)^{\frac{x}{2}} \frac{1}{\sqrt{2m\omega/\hbar \omega}} e^{-\frac{1}{2}(\frac{m\omega}{\hbar x^2} + |z|^2)}$$

(24)

The uncertainty relations for the operators $\hat{x}$ and $\hat{p}$ are:

$$\Delta x \Delta \hat{p} = \frac{\hbar}{2} \sqrt{\kappa^2 + \frac{1}{4N^2}} \left[ |z|^4 - (z^2 + z^* z) \right] \geq \frac{1}{2} \hbar \kappa = \frac{1}{2} |<\hat{x}, \hat{p}>|$$

(25)

The equality holds for $z = |z| e^{i\pi \epsilon}$, i.e. $z \in R$, defining the so-called “intelligent states” [23], but only for $z = 0$ (the vacuum) we reach the absolute minimum.

Our RCS correspond to definition c) in Sec. 1, and can be identified with the generalized coherent states on the unit complex disk [7] once the change of variables $z_D = \sqrt{N} \frac{z}{1+|z|^2} \in D (z \in C)$, and the identification $k \equiv N$ have been made, where $D$ is the unit complex disk and $k$ is the Bargmann index characterizing the irreducible representations of $SL(2,R)$. For a calculation of the uncertainty relations in the unit Disk see [24].

It would be natural to ask whether a definition analogous to b) could also be given. In fact, there exists a solution to the relativistic eigenvalue problem:

$$\hat{z} |\rho> = |\rho>$$

(26)

Using the commutation relations (9) and the Fock-space representation (13), we obtain:

$$|\rho> = c_0 \sum_{n=0}^{\infty} \frac{(\sqrt{2N} \rho)^n}{\sqrt{n!(2N)_n}} |N, n>,$$

(27)

where $c_0$ is an arbitrary constant (a function of $\rho$ actually). In the $c \to \infty$ limit these states also reproduce the standard non-relativistic coherent states. They are related to others previously defined by Barut and Girardello [25] through the change $\rho \to \rho/\sqrt{N}$, and choosing $c_0 = 1$.

The scalar product of two $|\rho>$ states is:
where \( F_1 \) is a hypergeometric function and the norm is \( < \rho | \rho > = |c_0|^2 \sum_{n=0}^{\infty} \frac{(2N \rho^n \rho^*)^n}{n!(2N)_n} = |c_0|^2 F_1(2N; 2N |\rho|^2) \).

These constitute an overcomplete family but, unfortunately, the reproducing kernel property requires a measure other than the one provided by the \( SL(2, R) \) group (see [25]).

Very recently, [26], it has been shown that both sets of coherent states (Perelomov’s and Barut and Girardello’s) are particular cases of a more general definition of coherent states, the generalized intelligent states, which minimize the Robertson-Shrödinger uncertainty relation [27]. In the particular case of operators satisfying canonical commutation relations the states that minimize the Robertson-Shrödinger uncertainty relation have been called correlated states [28].

4 Canonical (higher-order) creation and annihilation operators: canonical, relativistic coherent states.

The definition of polarization in group quantization can be generalized so as to admit operators in the left enveloping algebra. This generalization has already been exploited in finding a position operator for the free relativistic particle [29] (as well as in solving anomalous problems [10]). In the present case it also makes sense to look for basic operators satisfying canonical (versus manifestly covariant) commutation relations. Let us then seek power series in \( \hat{X}_z^L \) and \( \hat{X}_z^R \),

\[
\hat{X}_z^{L,HO} = \hat{X}_z^L + \frac{\alpha}{N} \hat{X}_z^L \hat{X}_z^L \hat{X}_z^L + \ldots
\]

\[
\hat{X}_z^{R,HO} = \hat{X}_z^L - \mu \hat{X}_z^L \hat{X}_z^L - \frac{\nu}{N} \hat{X}_z^L \hat{X}_z^L \hat{X}_z^L \hat{X}_z^L + \ldots
\]

such that \( P^{HO} =< \hat{X}_z^{L,HO}, \hat{X}_z^{L,HO} > \) contains \( \hat{X}_z^L \) and excludes \( \hat{X}_z^L \). The coefficients of the power series are determined by the requirement that \( P^{HO} \) is a polarization and the corresponding right operators define a unitary action on the wave functions \( \Psi \) which fortunately are the same as before.

More specifically,

\[
\begin{align*}
[\hat{X}_z^{L,HO}, \hat{X}_z^{L,HO}] &= -2 \hat{X}_z^{L,HO} \\
[\hat{X}_z^{R,HO}, \hat{X}_z^{R,HO}] &= \hat{1}
\end{align*}
\]

The resulting higher-order (canonical) creation and annihilation operators are:

\[
\hat{a}^{HO} \equiv \hat{a} = \hat{\xi} - \left( \frac{1}{4N} - \frac{3}{32N^2} \right) \hat{\xi}^\dagger \hat{\xi} \hat{\xi} + \frac{7}{32N^2} \hat{\xi}^\dagger \hat{\xi}^\dagger \hat{\xi}^\dagger \hat{\xi}^\dagger + \ldots \equiv \sqrt{\frac{2}{1 + \hat{k}}} \hat{\xi}^\dagger
\]
\[
\hat{z}^\dagger_{\text{HO}} \equiv \hat{a}^\dagger = \hat{z}^\dagger \sqrt{\frac{2}{1 + \hat{\kappa}}}
\]

and the energy operator is:

\[
\hat{H}_{\text{HO}} = \hat{N} (\hat{\kappa} - 1) = \hat{a}^\dagger \hat{a}
\]

(32)

where \(\hat{\kappa} \equiv \sqrt{1 + \frac{2}{\hat{N}} (\hat{z}^\dagger \hat{z})}\) and the operator \(\sqrt{\frac{2}{1 + \hat{\kappa}}}\) must be considered to be functions of the single operator \((\hat{z}^\dagger \hat{z})\).

The commutation relations,

\[
\begin{align*}
[\hat{a}, \hat{a}^\dagger] &= \hat{1} \\
[\hat{H}_{\text{HO}}, \hat{a}] &= -\hat{a} \\
[\hat{H}_{\text{HO}}, \hat{a}^\dagger] &= \hat{a}^\dagger,
\end{align*}
\]

(33)

have the non-relativistic (canonical) form. The action of these new operators on the Fock space is:

\[
\begin{align*}
\hat{a}|N, n > &= \sqrt{n}|N, n - 1 > \\
\hat{a}^\dagger |N, n > &= \sqrt{(n + 1)}|N, n + 1 > \\
\hat{H}_{\text{HO}} |N, n > &= n|N, n >,
\end{align*}
\]

(34)

which reproduces the non-relativistic harmonic oscillator representation, even though the states \(|N, n >\) are the same relativistic energy eigenstates as before.

### 4.1 Canonical coherent states.

It seems quite natural to define canonical coherent states \(|a >\) as the eigenstates of the canonical annihilation operator, \(\hat{a}|a >= a|a >\), with solutions:

\[
|a > = e^{-|a|^2 / 2} \sum_n \frac{a^n}{\sqrt{n!}} |N, n >,
\]

(35)

and to introduce a “non-relativistic” Bargmann-Fock space in the usual way:

\[
< a|N, n > = < n, N|a >^* = e^{-|a|^2 / 2} \frac{a^n}{\sqrt{n!}} \equiv \tilde{\Phi}_n^{R,(a)},
\]

(36)

with measure just \(d\alpha d\bar{\alpha}\).
The connection to the relativistic Bargmann-Fock space is given by

\[ \hat{\Phi}_n(z) = \langle z | a \rangle = \sum_{n=0}^{\infty} \langle z | N, n > | n, N | a \rangle = \sum_{n=0}^{\infty} \hat{\Phi}_n(z) \hat{\Phi}_n^{N,R}(a)^* \]

\[ = \frac{1}{\pi} \sqrt{\frac{2N-1}{2N}} e^{-kP/2} \left( \frac{1 + \kappa}{2} \right)^{-N} \sum_{n=0}^{\infty} \frac{1}{n!} \sqrt{(2N)_n} \left( \frac{2az^*}{(2N)(1 + \kappa)} \right)^n \]

This series is upper bounded by the hypergeometric function \(_1F_0\) the general coefficient of which is \(\frac{1}{\pi}(2N)_n\). A power expansion in terms of \(\frac{1}{N}\) can be computed:

\[ < z | a > \approx \frac{1}{\pi} e^{-kP/2} e^{-x^2/2} \left\{ \sqrt{1 - \frac{1}{4N}} \left[ 1 - \frac{1}{2} \left( |x|^2 - az^* \right) \left( 3|z|^2 - az^* \right) \right] + \ldots \right\} \]

The expectation value \(< a | z | a >\) defines a classical function \(z = z(a)\) relating the variables \(a, a^*\) and \(z, z^*\) as follows:

\[ < a | z | a > = a \sum_{n=0}^{\infty} c_n < a | \left( \hat{a}^\dagger \hat{a} \right)^n | a > \],

where \(c_n\) are the coefficients of the power series of \(f(u) = \sqrt{1 + \frac{u}{2N}}\). Then we define:

\[ z(a) = \sqrt{1 + \frac{|a|^2}{2N}} a \]  

Note that although \(< a | \left( \hat{a}^\dagger \hat{a} \right)^n | a > \neq < a | \hat{a}^\dagger \hat{a}^n | a > = |a|^{2n}\), any operator of the form \(\hat{F} = \hat{G} \hat{a}^m\) (or \(\hat{G} = \hat{a}^p \hat{O}\)), where \([\hat{H}^{HO}, \hat{O}] = 0\), defines a classical function \(F(a)\) (or \(G(a)\)) by the formula:

\[ F(a) = a^m \sum_n o_n |a|^{2n}, \quad G(a) = a^p \sum_n o_n |a|^{2n}, \]

where \(< a | \hat{O} | a > = \sum_n o_n < a | \left( \hat{H}^{HO} \right)^n | a >\).

The functions

\[ a(z) = \sqrt{\frac{2}{1 + \kappa}} z, \quad a^*(z) = \sqrt{\frac{2}{1 + \kappa}} z^*, \]

the inverse relation of (40), turn out to be the Darboux coordinates taking the symplectic form \(\Omega \equiv \frac{1}{\kappa} da \wedge dz^*\) to the canonical form \(\Omega = da \wedge da^*\).

Finally, we define

\[ \hat{q} = \sqrt{\frac{\hbar}{2m\omega}} \left( \hat{a} + \hat{a}^\dagger \right) \]

\[ \hat{p} = \sqrt{\frac{m\omega\hbar}{2}} \left( \hat{a}^\dagger - \hat{a} \right), \]
satisfying
\[ [\hat{q}, \hat{\pi}] = i\hbar \hat{1}, \]
as well as their corresponding classical functions \( q \) and \( \pi \). For these operators we obviously obtain
\[ \Delta \hat{q} \Delta \hat{\pi} = \frac{\hbar}{2} \]
on the \(|a\rangle\) states.

A new minimal representation in configuration space can be introduced which will be called the canonical, minimal representation or the \( q \)-representation. The corresponding states, \(|q\rangle\), are the eigenstates of the position operator \( \hat{q} \). They prove to be
\[ |q\rangle = \left( \frac{m\omega}{\hbar\pi} \right)^{\frac{1}{4}} \sum_{n=0}^{\infty} \frac{1}{2^n n!} e^{-\xi^2/2} H_n(\xi) |N, n\rangle \]
where \( \xi \equiv \sqrt{\frac{m\omega}{\hbar}} q \) and \( H_n \) are the ordinary, non-relativistic Hermite polynomials. The integration measure is just \( dq \).

The analogue to the transformation kernel \( \langle z|a\rangle \), i.e. \( \langle x|q\rangle \), also makes sense and relates the Hermite polynomials and the Relativistic Hermite polynomials, and can be worked out in a similar way:
\[ \langle x|q\rangle = \sum_{n=0}^{\infty} \langle N, n| x, n\rangle |q\rangle = \sum_{n=0}^{\infty} \Phi_n^N(x) \Phi_n^{N,R}(q)^* \]
\[ = \sqrt{\frac{m\omega}{\hbar\pi}} e^{-\xi^2/2} \alpha^{-N} \sqrt{\frac{\Gamma(N)}{\sqrt{\Gamma(N-\frac{1}{2})}}} \sum_{n=0}^{\infty} \frac{\alpha^{-n}}{2^n n!} \left[ \frac{(2N)^n}{(2N)_n} H_n^N(\eta) H_n(\xi) \right] \]

Unlike in the case of \( \langle z|a\rangle \), the series above does not converge to an ordinary function, since in the \( \epsilon \to \infty \) limit \( \langle \xi|q\rangle = \delta(q - \xi) \). We can, however, compute the power series expansion in \( \frac{1}{N} \):
\[ \langle x|q\rangle \approx \delta(x - q) + \sqrt{\frac{m\omega}{\hbar}} \frac{1}{64N} \left\{ 12(1 + \eta^2) \delta(\xi - \eta) + 4 \left[ 9\xi + 3\eta + \xi^3 + \eta^3 \right] \times \delta'(\xi - \eta) + 6 \left[ (\xi^2 - \eta^2) + 2 \right] \delta''(\xi - \eta) + 4(\xi + \eta) \delta'''(\xi - \eta) \right\} + \ldots \]

The existence of Galilean-like creation and annihilation operators along with the \( SL(2, R) \) operators \( \hat{z}, \hat{z}^\dagger \), looks rather tricky at first sight and thus deserves some comment. First of all, the co-existence of both type of operators is possible only because the spectra of the non-relativistic and relativistic harmonic oscillator are the same and the Hamiltonian \( \hat{H} \) is shared by the two systems, although written in two different manners.
\[ \hat{H} = \hat{a}^\dagger \hat{a} = \sqrt{1 + \frac{2}{\hbar} \hat{z}^\dagger \hat{z}}. \]

The common (phase space) Poisson algebra contains two sub-algebras \((H, a^\dagger, a)\) and \((H, \hat{z}^\dagger, \hat{z})\) intersecting at \(H\) even though \((H, a^\dagger, a, \hat{z}^\dagger, \hat{z})\) does not close. The situation is in certain aspects similar to the case of the Schrödinger group \([30, 10]\) which (in 1+1 dimensions) is generated by an analogous set of operators with the only difference that the commutators between \(\hat{a}, \hat{a}^\dagger\) and \(\hat{z}, \hat{z}^\dagger\) close and, therefore, it is possible to find a quantum representation in which \(\hat{z}\) and \(\hat{z}^\dagger\) are written only as quadratic functions of \(\hat{a}\) and \(\hat{a}^\dagger\). This quantum representation is realized only for a special value of the \(SL(2, R)\) Bargmann index \(k = \frac{1}{4}, \frac{3}{4}\), with direct physical application in two-photon quantum optics \([2]\).

Furthermore, the fact that not only the \(SL(2, R)\) group but also the Newton group can be represented on the same Fock states simply demonstrates that it is the representation and not the states themselves which characterizes the physical system. Indeed, in the present case, the states \(|N, n\rangle\) support simultaneously the representation (13) and that corresponding to the non-relativistic harmonic oscillator (34), which could have been well denoted by just \(|n\rangle\). It is easily tested that

\[
\langle n|N, n\rangle = \int \frac{dzdz^*}{\kappa} dada^* \langle n|a^*<a|z><z|N, m\rangle = \delta_{n,m} \quad (49)
\]

Needless to say that the non-relativistic harmonic oscillator also support the construction of higher-order operators \(\hat{z}, \hat{z}^\dagger\) as functions of the operators \(\hat{a}, \hat{a}^\dagger\) (the inverse of (31)), thus realizing the \(SL(2, R)\) group on states \(|n\rangle\) and for any value of \(N\) (or Bargmann index \(k\)) and not just for \(N = \frac{1}{4}, \frac{3}{4}\) as in the case of the Schrödinger group.

5 Final Remarks

The construction of the canonical (higher-order) creation and annihilation operators \(\hat{a}^\dagger\) and \(\hat{a}\) in the 1+1-D relativistic harmonic oscillator is a matter of convenience rather than necessity, since a first-order polarization, the manifestly covariant one, \(\mathcal{P} = \langle X_L^n, X_L^z\rangle\) exists. However, the situation become quite different for the relativistic harmonic oscillator with spin, at least from a geometrical point of view. The reason is that the doubly pseudo-extended \(SO(3, 2)\) (anti-de Sitter) Lie algebra, containing the commutators

\[
[\hat{x}_i, \hat{p}_j] = i\hbar \delta_{ij}(\hat{1} + \frac{1}{mc^2} \hat{E}) \quad (50)
\]

accounting for the mass, and the commutator

\[
[\hat{J}_+, \hat{J}_-] = 2 \left( \hat{J}_3 + j\hat{1} \right) \quad (51)
\]
accounting for the spin, does not admit a consistent way (i.e. compatible with the rest of the symmetry) of defining two sets of first-order conjugated creation-annihilation (or co-ordinate-momentum) operators. In other words, the system does not admit a (first-order) polarization and, therefore, the Hilbert space of $U(1)$-equivariant complex functions on the group can be only partially reduced [18]. The full reduction then requires the introduction of higher-order operators in the polarization, generalizing those introduced here and accounting for proper intrinsic spin operators.

References