Solutions to Knizhnik-Zamolodchikov equations with coefficients in non-bounded modules

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Abstract

We explicitly write down integral formulas for solutions to Knizhnik-Zamolodchikov equations with coefficients in non-bounded - neither highest nor lowest weight - \( sl_{n+1} \)-modules. The formulas are closely related to WZNW model at a rational level.

1 Introduction

Let \( g \) be a finite-dimensional simple Lie algebra, \( \hat{g} \) the corresponding non-twisted affine Lie algebra. Let \( \lambda \) be a weight of \( g \), \( M(\lambda, k) (M(\lambda, k)^\ast) \) be the Verma (contragredient Verma) module over \( \hat{g} \) with the central charge \( k \); for a \( g \)-module \( V \) be denote by \( V((z)) \) the module of formal Laurent series in \( z \) with coefficients in \( V \), regarded as a \( \hat{g} \)-module with the central charge equal to 0.

Vertex operator is a \( \hat{g} \)-linear map

\[
\Phi(z) : M(\lambda_1, k) \to M(\lambda_2, k)^\ast \otimes V((z)).
\] (1)

If highest weights \( (\lambda_1, k), \ldots, (\lambda_{N+1}, k) \) are generic then
\( M(\lambda_i, k) \approx M(\lambda_i, k)^\ast \), \( 1 \leq i \leq N+1 \) and one may consider a product of vertex operators \( \Phi_N(z_N) \cdots \Phi_1(z_1) \). Matrix element \( \langle u_{\lambda_{N+1}}^\ast, \Phi_N(z_N) \cdots \Phi_1(z_1) u_{\lambda_1} \rangle \)

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related to vacuum vectors is called a correlation function. One of the central results of conformal field theory (see [1]) is that a correlation function satisfies a remarkable system of Knizhnik-Zamolodchikov equations. We prepare notations in order to write down the trigonometric form of Knizhnik-Zamolodchikov equations.

Let

\[ g = h \oplus \bigoplus_{\alpha \in \Delta} g_{\alpha} \]

be a root space decomposition. Fix an invariant inner product on \( g \) and a basis \( \{ h_i \in h, \ g_{\alpha} \in g_{\alpha} : 1 \leq i \leq n, \ \alpha \in \Delta \} \) of \( g \) so that \((h_i, h_j) = \delta_{i,j}, \ (g_{\alpha}, g_{\beta}) = \delta_{\alpha,-\beta}\). For each \( \mu \in h^* \) denote by \( h_\mu \) an element of \( h \) satisfying (and uniquely determined) by the condition \((h_\mu, h) = \mu(h)\).

Set

\[ r = \frac{1}{2} \sum_{i=1}^{n} h_i \otimes h_i + \sum_{\alpha \in \Delta_+} g_{\alpha} \otimes g_{-\alpha}. \]

Being an element of \( U(g) \otimes U(g) \) \( r \) naturally acts on a tensor product of 2 \( g \)-modules. There are \( N^2 \) different ways to make it act on a tensor product of \( N \) \( g \)-modules via the following \( N^2 \) embeddings of \( U(g)^{\otimes 2} \) in \( U(g)^{\otimes N} \): each of them is associated to a pair of numbers \( 1 \leq i, j \leq N \) and sends

\[ U(g)^{\otimes 2} \ni \omega \mapsto \omega_{ij} \in U(g)^{\otimes N}, \]

so that

\[ \text{if } \omega = \sum_s a_s \otimes b_s \text{ then } \omega_{ij} = \sum_s 1 \otimes \cdots 1 \otimes a_s \otimes 1 \otimes \cdots 1 \otimes b_s \otimes 1 \otimes \cdots 1. \]

For a pair \( 1 \leq i, j \leq N \) introduce the following function in 2 complex variables with values in \( U(g)^{\otimes N} \):

\[ r(z_i, z_j) = \frac{r_{ij} z_i + r_{ji} z_j}{z_i - z_j}. \]

**Theorem 1.1 (Knizhnik, Zamolodchikov)** The correlation function

\[ \Psi(z) = (\nu_{N+1}^* \Phi_N(z_{N+1}) \Phi_1(z_1) \cdots \Phi_1(z_1)) \]

satisfies the following system of differential equations

\[ (k + h^\vee) \frac{\partial \Psi}{\partial z_i} = \left\{ \sum_{j \neq i} r_{ij}(z_i, z_j) - \frac{1}{2} \left( \lambda_1 + \lambda_{N+1} + 2l \right)(l) \right\} \Psi, \quad 1 \leq i \leq N, \]

where \( h^\vee \) is the dual Coxeter number of \( g \) and for each \( \mu \in h^* \) \( \mu^{(i)} \) stands for the operator acting on \( V_1 \otimes \cdots \otimes V_N \) as \( h_\mu \) applied to the \( i \)-th factor of \( V_1 \otimes \cdots \otimes V_N \).
Solutions to KZ equations

To keep track of the parameters we will be referring to (2) as $KZ(\lambda_{N+1}, \lambda_1)$.

The deep theory of KZ equations has been developed by several authors (see e.g. [2,3,4,5]) in the case when $V_i$ are highest weight modules. It has also been realized that this theory is relevant to physics applications in the case when $(\lambda_i, k)$ is either integral or generic. Indeed, if conflicting with the above assumptions, some of $M(\lambda_i, k)$ are reducible then the product $\Phi_N(z_N) \circ \cdots \circ \Phi_1(z_1)$ does not exist unless each of the operators $\Phi_i(z_i)$ can be pushed down to a map

$$\Phi_i(z_i) : L(\lambda_i, k) \to L(\lambda_{i+1}, k) \otimes V_i((z_i)),$$

where $L(\lambda, k)$ stands for an irreducible highest weight module with the highest weight $(\lambda, k)$. In the case when $(\lambda, k)$ is an admissible weight (for example, dominant integral weight) [6] the last condition reduces to the singular vector decoupling condition: matrix elements of $\Phi_i(z_i)$ related to singular vectors of $M(\lambda_i)$ vanish. It is known [7] that if each $(\lambda_i, k)$ is dominant integral then everything goes through nicely; in particular, the Schechtman-Varchenko integral solutions to (2) come from products of vertex operators (2). However if the central charge is not integral it has been realized ([10], see also [8,9]) that the singular vector decoupling condition implies that $V_i$ is neither highest nor lowest weight module. Though some results for such models were obtained in [8,9], where in particular the connection to quantum hamiltonian reduction was revealed, not much is known about KZ equations in this case.

In [11] a new method of constructing solutions to (2) was proposed which seems to be relevant to the problem. Let $G$ be a complex Lie group related to $g$, $F = G/B$ be a flag manifold and $F^0 \subset F$ be the big cell. There is a family of embeddings of $g$ into the algebra of order 1 differential operators on $F^0$

$$\pi_\mu : g \to Diff^1(F^0), \mu \in h^*.$$

This makes the space of analytic functions on $F^0$ into a huge $g$-module. Different $g$-closed subspaces give realization of different $g$-modules. For example, contragredient Verma modules are realized in the space of polynomials on $F^0$, $\mu$ being the highest weight and a constant function being a highest weight vector; this observation has been extensively used recently with regards to Wakimoto modules [12,13,14]. The spaces of multi-valued functions give modules with quite different properties, the simplest example being that of $\mathfrak{sl}_2$: in this case the big cell is $C$, contragredient Verma modules are realized in $C[x]$; the space $x^{\nu}C[x, x^{-1}], \nu \in C$ is also closed under the action of $\mathfrak{sl}_2$ and the embedding $\pi_\mu, \mu \in C$ makes it into generically irreducible $\mathfrak{sl}_2$-module. This module is transparently neither highest nor lowest weight one.

Regarding $V$ in (1) as a $g$-module realized in functions on the big cell one identifies elements of $V((z))$ with functions of 2 groups of variables: $x$ and $z$, where $x$ stands for a (vector) coordinate on the big cell and $z$ is a coordinate on $C$. Likewise, the correlation function

$$\Psi(z) = \langle u_{\lambda_{N+1}}, \Phi_N(z_N) \circ \cdots \circ \Phi_1(z_1) u_{\lambda_1} \rangle$$
is identified with a function of $x^{(1)}, \ldots, x^{(N)}; z_1, \ldots, z_N$ where $x^{(i)}$ is a coordinate on the $i$-th copy of the big cell, $z_i \in \mathbb{C}$, $1 \leq i \leq N$. One of the advantages of this functional realization is that the embedding $\pi_\lambda : \mathfrak{g} \rightarrow Diff^1(F^0)$ lifts to the mapping of the group $G$: for $g \in \mathfrak{g}$ the exponent $\exp(-t g)$ is a well-defined operator.

Let $W$ be the Weyl group of $\hat{\mathfrak{g}}$, $w = r_{m_1} r_{m_2} \cdots r_{m_l} \in W$ be a decomposition (not necessarily reduced), where $r_m$ denotes the reflection at the corresponding simple root. Set,

$$\beta_j = \frac{2(r_{m_1} \cdots r_{m_{l-j}} \cdot \lambda_1, \alpha_{m_{l-j+1}})}{(\alpha_{m_{l-j+1}}, \alpha_{m_{l-j+1}})} + 1, \ 1 \leq j \leq l.$$ 

Given

$$\Psi_{\text{old}}(z) = \Phi_N(z_N) \circ \cdots \circ \Phi_1(z_1) \circ \lambda_1,$$

set

$$\Psi_{\text{new}} = \prod_{j=1}^l \Gamma(-\beta_j)^{-1} \int \{ \exp(-t_1 F_{m_1}) \cdots \exp(-t_l F_{m_l}) \Psi_{\text{old}} \} \prod_{j=1}^l t_j^{\beta_j-1} \ dt_1 \cdots dt_l,$$

where the integration is carried out over an arbitrary cycle of the highest homology group related to the multi-valued integrand. In (3) it is set that $E_i, F_i, H_i, 0 \leq i \leq r_k \mathfrak{g}$ are canonical Cartan generators of $\mathfrak{g}$ and $E_i, F_i, H_i, 1 \leq i \leq r_k \mathfrak{g}$ are the ones coming from the inclusion $\mathfrak{g} \subset \hat{\mathfrak{g}}$.

**Theorem 1.2** [11] $\Psi_{\text{new}}$ is a solution to $KZ(\lambda_{N+1}, w \cdot \lambda_i)$. 

Theorem 1.2 works as follows: given a solution to $KZ$ it generates new ones labelled by elements of the affine Weyl group. In our notations the simplest solution to $KZ(\lambda_{N+1}, \lambda_1)$ is given by

$$\Psi_{\text{old}} = \prod_{i<j} (z_i - z_j)^2(\mu_i, \mu_j)/(k+\hbar') (z_i z_j)^{-2}(\mu_i, \mu_j)/(k+\hbar') \times$$

$$\prod_i z_i^{(\lambda_i + \lambda_{N+1} + 2\mu_i)/2(k+\hbar')},$$

where $\mu_i$ is a highest weight of $V_i, 1 \leq i \leq N$. In particular, $\Psi_{\text{old}}$ is independent of $x$'s. The purpose of this paper is to explicitly write down the integral

$$\Psi_{\text{new}} = \Psi_{\text{old}} \times$$

$$\prod_{j=1}^l \Gamma(-\beta_j)^{-1} \int \{ \exp(-t_1 F_{m_1}) \cdots \exp(-t_l F_{m_l}) \} \prod_{j=1}^l t_j^{\beta_j-1} \ dt_1 \cdots dt_l.$$

for $g = sl_{n+1}$, generalizing the calculation carried out in [11] for $sl_2$. 
**Remark.** The integral representation of $\Psi_{new}$ in Theorem 1.2 is nothing but the conventional definition of $F_{m_1}^{\beta_1} \cdots F_{m_t}^{\beta_t} \cdot \Psi_{old}$. The latter comes from looking at the "matrix element"

$$\langle \psi_{\lambda_{N+1}}^* \Phi_N(z_N) \cdots \Phi_1(z_1) F_{m_1}^{\beta_1} \cdots F_{m_t}^{\beta_t} \psi_{\lambda_1} \rangle.$$ 

Though the expression $F_{m_1}^{\beta_1} \cdots F_{m_t}^{\beta_t} \psi_{\lambda_1}$ is not understood as an element of $M(\lambda_1, k)$, the powers are chosen in such a way that it formally satisfies the singular vector conditions [11,15], which makes the statement of Theorem 1.2 almost obvious. One can similarly consider an expression

$$\langle \psi_{\lambda_{N+1}}^* E_{\mu_1} \cdots E_{\mu_t} \Phi_N(z_N) \cdots \Phi_1(z_1) \psi_{\lambda_1} \rangle,$$

for appropriate $\beta_1, \ldots, \beta_t$ and write down another solution in the form close to (3) but with $F$'s replaced with $E$'s or combine both methods or, finally, apply them to other solutions obtained in [4,8,9].

As to relation of our solution (3) to correlation functions, we have been able to verify in simplest cases that (3) indeed gives a matrix element of a product of vertex operators and hope that (3) will prove useful for investigation of other rational level models.

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## 2 Integral formulas for solutions of $KZ(\lambda_{N+1}, \lambda_1)$

### 2.1 Main result

Here we are going to write down the integral (4) in the case of $g = s_{n+1}$, $\hat{g} = s_{n+1}$. In this case there are $3(n+1)$ Cartan generators $E_i, F_i, H_i$, $0 \leq i \leq n$, where $E_i, F_i, H_i$, $1 \leq i \leq n$ are the ones coming from the inclusion $g \subset \hat{g}$. Explicitly the generators are described as follows. If $e_{ij} = (a_{ij})$ is an $(n+1) \times (n+1)$ matrix then $E_i = e_{i0+1}$, $F_i = e_{0i+1}$, $H_i = e_{ii} - e_{i+1i+1}$, $1 \leq i \leq n$ and $E_0 = e_{n+10} \otimes z$, $F_0 = e_{0n+1} \otimes z^{-1}$ (see [16] for details). The $g$-weight $\mu$ is considered as a vector $(\mu_1, \ldots, \mu_n)$, $\mu_i = \mu(H_i)$. The embedding

$$\pi : s_{n+1} \rightarrow Diff^1(F^0), \quad \mu = (\mu_1, \ldots, \mu_n)$$

is calculated in [12] (see also [13]). To recall this result we choose coordinates of the big cell $F^0$ to be $\{x_{ij} : 1 \leq i < j \leq n\}$ identifying as usual the big cell
with the subgroup of matrices

\[
\begin{pmatrix}
1 & x_{11} & \cdots & \cdots & x_{1n} \\
1 & \ddots & & \ddots & \vdots \\
\vdots & \ddots & \ddots & \ddots & \vdots \\
\vdots & & \ddots & \ddots & \vdots \\
0 & \cdots & \cdots & 1 & x_{nn}
\end{pmatrix}
\]

For \( 1 \leq i \leq n \) set

\[ \partial x_{ii} := \frac{\partial}{\partial x_{ii}}. \]

Then \( \pi_\mu \) acts on Cartan generators by

\[ E_i \mapsto -\partial x_{ii} - \sum_{j=i+1}^{n} x_{ij} \partial x_{ij}, \]

\[ F_i \mapsto x_{ii} \left( \sum_{j=1}^{i} x_{ji} \partial x_{ji} - \sum_{j=1}^{i-1} x_{ji-1} \partial x_{ji-1} \right) - \sum_{j=i+1}^{n} x_{ij} \partial x_{ij+1} + \sum_{j=1}^{i-1} x_{ii} \partial x_{ij-1} + \mu_i x_{ii}. \]

Here \( x_{ij} = 0 \) unless \( 1 \leq i \leq j \leq n \).

The matrix \( e_{1n+1} \) may be written as

\[ e_{1n+1} := [\cdots [E_1, E_2], \cdots], E_n]. \]

Using the above formulas one proves the following

**Lemma 2.1**

\[ \pi_\mu (e_{1n+1}) = -\partial x_{1n}. \]

From now on till the end of this section we omit writing \( \pi_\mu \) identifying Lie algebra elements with their images under \( \pi_\mu \).

The action of the Lie algebra \( \mathfrak{g} \) on a function on \( F^n \times \mathbb{C}^* \) is determined by the evaluation map \( g \otimes z^k \mapsto z^k g \). In particular

\[ F_0 = e_{1n+1} \otimes z^{-1} \mapsto -z^{-1} \partial x_{1n}. \]

The result of exponentiation of these formulas is given by

**Lemma 2.2**

(i) If \( \mu = 0 \) then

1) \( \exp(-tF_0) : z_{kl} \mapsto \begin{cases} 
    z^{-1} t + x_{ln} & (k,l) = (1,n) \\
    x_{kl} & \text{otherwise}
\end{cases} \)

2) \( \exp(-tF_i) : x_{kl} \mapsto \begin{cases} 
    \frac{x_{kl}}{1 + z_{ii} t} & \text{for } l = i \\
    -(x_{kl} - x_{k-1} z_{ii}) t + x_{kl-1} & \text{for } l = i - 1 \\
    x_{ii} t + x_{i+1} & \text{for } k = i + 1 \\
    x_{kl} & \text{otherwise}
\end{cases} \)
(ii) Generically

\[ \exp(-t F_i) \psi(x) = (1 + x_i t)^{\mu_i} \psi(x'), \quad 1 \leq i \leq n \]

where \( x' \) is given by the substitution of the item (i) while action of \( F_0 \) is independent of \( \mu \).

Proof

If \( \mu = 0 \) then all \( F_i \)'s are vector fields. The problem of evaluating an exponent of a vector field is, actually, a problem of the theory of ordinary differential equations: the exponent of a vector field is an element of a 1 parametric family of diffeomorphisms generated by the vector field and, therefore, is given by a general solution to the corresponding system of o.d.e.'s. In our case the system and the solution are (resp.)

1):

\[ \dot{x}_{kl} = \delta_{kl} \delta_{ln} z^{-1} \Rightarrow x_{1n} = z^{-1} t + x_{1n} \]

2):

\[ \begin{align*}
\dot{x}_{ii} &= x_{ii}^2 &\Rightarrow& & x_{ii} = \frac{x_{ii}(0)}{1-x_{ii}(0)t} \\
\dot{x}_{ji} &= x_{jj} x_{ji} &\Rightarrow& & x_{ji} = \frac{x_{ji}(0)}{1-x_{ji}(0)t} \\
\dot{x}_{ji-1} &= x_{ji} - x_{ji-1} x_{ij} &\Rightarrow& & x_{ji-1} = \frac{x_{ji}(0) - x_{ji-1}(0)x_{ij}(0)}{1-x_{ji}(0)t} t + x_{ji-1}(0) \\
\dot{x}_{i+1j} &= -x_{ij} &\Rightarrow& & x_{ij} = x_{ij}(0) \\
\Rightarrow& & x_{i+1j} &= -x_{ij}(0)t + x_{i+1j}(0)
\end{align*} \]

which completes proof of the item (i). As to the item (ii), one shows that any order 1 differential operator is conjugated to a vector field by the multiplication by a function it annihilates. This implies (ii) since \( F_i \cdot (x_i^{-\mu_i}) = 0 \). Q.E.D.

Now by using all this one can calculate the integrand of (4). But to formulate the result it is convenient to give some more notations.

Set

\[ T = (t_{ij}) := \begin{pmatrix}
  x_{11} & \cdots & \cdots & x_{1n} \\
  1 & \ddots & & 0 \\
  & \ddots & \ddots & \vdots \\
  0 & 1 & \cdots & x_{nm}
\end{pmatrix} \]

for \( 1 \leq i_k \leq i_{k-1} \leq \cdots \leq i_1 \leq j, \ j + k \leq n + 1 \).

\[ T_{i_1, i_2, \ldots, i_k}^j := \{ j + 1 - i_1, j + 2 - i_2, \ldots, j + k - i_k \} \]

\[ J_k^j := \{ j, j + 1, \ldots, j + k - 1 \} \]

\[ T_{i_1, i_2, \ldots, i_k}^{j_k} := (t_{ij})_{i \in T_{i_1, i_2, \ldots, i_k}^j, j \in J_k^j} \]

\[ Q_{i_1, i_2, \ldots, i_k}^{j_k} := \begin{cases} 
\det(T_{i_1, i_2, \ldots, i_k}^{j_k}) & \text{for } k > 0 \\
1 & \text{for } k = 0
\end{cases} \]
Introduce a collection of functions on the big cell along with an ordering on it.

**Definition** We write
\[
Q_{j_1,j_2,\ldots,j_k}^l \xrightarrow{\text{def}} \exp(-tF_l)Q_{j_1,j_2,\ldots,j_k}^l = \begin{cases} \frac{1}{(1+t)^{l+1}} \left\{ Q'^{l}_t + Q_{j_1,j_2,\ldots,j_k}^l \right\} & \text{for } l = j \\ Q'^{l}_t + Q_{j_1,j_2,\ldots,j_k}^l & \text{for } l \neq j \end{cases}
\]
In the definition it is assumed that \( \mu = 0 \).

**Lemma 2.3**

1) \( Q_{j_1,j_2,\ldots,j_k}^l \xrightarrow{F_{l+r-1}} Q_{j_1,j_2,\ldots,j_r+1,\ldots,j_k}^l \) for \( 1 \leq r \leq k \)

2) \( Q_{j_1,j_2,\ldots,j_k}^l \xrightarrow{F_k} Q^l_{j_1,j_2,\ldots,j_k,1} \) for \( k \geq 0 \)

3) \( Q_{j_1,j_2,\ldots,j_k+1,j_{k+1},\ldots}^l \xrightarrow{F_{k'}} (-1)^{n-j}Q^l_{j_1,j_2,\ldots,j_{k+1},1}z^{-1} \) where \( k' = \#\{ r : r > 2, i_r > 1 \} \)

Otherwise, \( Q \xrightarrow{F_1} 0 \)

The proof of this lemma is a standard calculation of linear algebra using Lemma 2.2; in particular we use Laplace expansion of a certain determinant to prove 2).

The above definition suggests to introduce the following \( n+1 \)-colored graph \( \Gamma \). The set of vertices of \( \Gamma \) is the set of all \( Q \neq 0 \) such that
\[
1 \xrightarrow{F_{j_1}} Q_1 \xrightarrow{F_{j_2}} Q_2 \longrightarrow \cdots \longrightarrow Q_{r-1} \xrightarrow{F_{j_k}} Q
\]
for some \( j_1,\ldots,j_k \). It follows from Lemma 2.3 that each vertex is of the form \( (-1)^{l}Q^l_{j_1,j_2,\ldots,j_k}z^{-r} \). Define a function on the set of vertices by
\[
l((-1)^{l}Q^l_{j_1,j_2,\ldots,j_k}z^{-r}) = (n+1)r + \sum_{p=1}^{k} j_p.
\]

Two vertices \( P,Q \) are connected by an edge of the color \( i \) if and only if
\[
P \xrightarrow{F_{i}} Q.
\]

With any vertex \( Q \in \Gamma \) associate a set \( \mathcal{P}(Q) \) of all oriented paths connecting 1 and \( Q \).

**Lemma 2.4** All \( \gamma \in \mathcal{P}((-1)^{l}Q^l_{j_1,j_2,\ldots,j_k}z^{-r}) \) are of the same length
\[
l((-1)^{l}Q^l_{j_1,j_2,\ldots,j_k}z^{-r}) \]
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Proof
Lemma 2.3 shows that if there is an edge going from $P$ to $Q$ then $l(Q) = l(P) + 1$. The lemma now follows from the obvious remark that $l(1) = 0$.

Q.E.D

We are in a position to write down the integral (4). Recall that $W$ is the Weyl group of $\mathbf{g}$ and $w = r_{m_1} r_{m_2} \cdots r_{m_l} \in W$ is a decomposition (not necessarily reduced), where $r_{m}$ denotes the reflection at the corresponding simple root $(\alpha_m)$. $m$ can be viewed as a map from $I_1(=\{1, 2, \cdots, l\})$ to $I_2(=\{0,1,\cdots,n\})$. Therefore, $m^{-1}(j)$, $j \in I_2$, is a subset of $I_1$. Set,

$$
\beta_j = \frac{2(r_{m_1} \cdots r_{m_l} \cdot \lambda_1, \alpha_{m_i-1-j})}{(\alpha_{m_i+1-j}, \alpha_{m_i+1-j})} + 1,
$$

$1 \leq j \leq l$

and

$$
K_w(t_1, t_2, \cdots, t_l) = \prod_{j=1}^{l} \Gamma(-\beta_j)^{-1} \times \{ \exp(-t_1 F_{m_1}) \cdots \exp(-t_l F_{m_l}) \prod_{j=1}^{l} t_j^{-\beta_j-1} \},
$$

where $l$ is viewed as an element of $V_1 \otimes \cdots \otimes V_N$ equal to the unit function on the product of $N$ copies of the flag manifold. With any path

$$
\gamma: 1 \xrightarrow{F_{t_1}} Q_1 \xrightarrow{F_{t_2}} Q_2 \cdots \xrightarrow{F_{t_{r-1}}} Q_{r-1} \xrightarrow{F_{t_r}} Q, \ r = l(Q)
$$

associate a polynomial in $t$'s:

$$
f_\gamma(t) = \sum_{p_1 < \cdots < p_r, p_i \in m^{-1}(j_i)} t_{p_1} t_{p_2} \cdots t_{p_r}.
$$

(This is the only point where the decomposition $w = r_{m_1} r_{m_2} \cdots r_{m_l}$ enters the calculation.) Denote by $I^\gamma$ the subgraph of $\Gamma$ consisting of all vertices connected with $Q_1$ by an oriented path. It is equivalently defined as a subgraph generated by all vertices $(-1)^{(n-2) \times I^\gamma} Q_{i_1, i_2, \cdots, i_k}$ with the fixed superscript $j$. Set

$$
P^\gamma_w(x, z; t_1, t_2, \cdots, t_l) = \sum_{P=0}^{l} \sum_{Q \in I^{\gamma}: l(Q) = l'} \sum_{\gamma \in P_{(q)}} f_\gamma(t).
$$

(5)

Theorem 2.5

$$
K_w(t_1, t_2, \cdots, t_l) = \prod_{j=1}^{l} \Gamma(-\beta_j)^{-1} \prod_{p=1}^{n} \prod_{j=1}^{l} (P^\gamma_w(x^{(p)}, z_{p}; t_1, t_2, \cdots, t_l))^{\mu(p)} \prod_{j=1}^{l} t_j^{-\beta_j-1},
$$

where $\mu(p) = (\mu_1^{(p)}, \cdots, \mu_n^{(p)})$, $1 \leq p \leq N$, is a highest weight of $V_p$ and $x^{(p)}$, $1 \leq p \leq N$, is a coordinate in the $p$-th copy of the flag manifold.
This theorem can be proved by induction on $l$ using Lemma 2.3 and Lemma 2.4.

Let $\mathcal{M}$ be the local system of continuous branches of $K_w(t_1, \cdots, t_l)$ over the domain of $K_w(t_1, \cdots, t_l)$ (say $\mathcal{D}$). Then finally we obtain

**Theorem 2.6** For any $\sigma \in H_1(\mathcal{M}, \mathcal{D})$, the integral

$$\int_{\sigma} K_w(t_1, t_2, \cdots, t_l) dt_1 dt_2 \cdots dt_l$$

satisfies the system $KZ(\lambda_{N+1}, w \cdot \lambda_1)$

**Remark.** Theorem 2.6 gives solutions as an integral over a certain cycle depending on parameters $(x, z)$. These cycles belong to a homology group of a complement to a collection of hypersurfaces $K_w(t_1, t_2, \cdots, t_l) = 0$ with coefficients in a local system defined over this complement. Note that generically ($l > 2$), and much unlike the case of Schechtman-Varchenko integral formulas, $K_w(t_1, t_2, \cdots, t_l) = 0$ is a union of hypersurfaces not isomorphic to hyperplanes and, therefore, investigation of the integral cannot be carried out by usual methods. We have already encountered with the same phenomenon in a different but related framework. As we argued in the Introduction, our integral formulas are intimately related to $\mathfrak{g}$ or $\mathfrak{g}$-modules extended by complex powers of a Lie algebra generators. Rigorous treatment of such modules requires consideration of a Lie algebra action on sections of a local system defined over a complement to a highly non-linear set of "shifted" Schubert cells on a flag manifold; for details see [11].

Note also that if $l = 1, 2$ then $K_w(t_1, t_2, \cdots, t_l) = 0$ is isomorphic to a union of affine hyperplanes and the number of cycles can be calculated using results of [5].

### 2.2 Some examples

Theorem 2.5 produces rather an algorithm to write down the kernel of the integral (4) than a completely explicit formula for it: (6) relies on (5), while the latter is a linear combination of explicitly given polynomials $Q_{l_1, \cdots, l_s} z^{-\tau}$ with coefficients in the form $\sum \lambda_{i_1} t_{i_1} \cdots t_{i_s}$, determined by the combinatorial data. We have been able to "resolve" the combinatorial part of the formula in the cases $\mathfrak{g} = \mathfrak{sl}_2, \mathfrak{sl}_3$. Although the $\mathfrak{sl}_2$-case was treated in [15], we discuss here both in a unified way for completeness.

**The $\mathfrak{sl}_2$-case.** In this case the flag manifold is $\mathbb{C}P^1$, the big cell is $C \subset \mathbb{C}P^1$. Fix a coordinate $x$ on $C$. Then the matrix $T$ (via which the polynomials $Q_1^{l_1, \cdots, l_s}$ are defined) is given by $T = (z)$. The set of all $Q_{l_1, \cdots, l_s}$ consists of 2 elements: $Q_1^1 = 1, Q_1^1 = x$. The vertices of the graph $\Gamma$ are all of the form: $A_i^\epsilon(x, z) = z^{-\epsilon} x^\epsilon, \epsilon = 0, 1, i = 0, 1, 2, 3, \cdots$. Further, $\Gamma$ coincides with $\Gamma^1$ and is given by

$$1 \xrightarrow{F_1} x \xrightarrow{F_0} z^{-1} x \xrightarrow{F_1} z^{-1} x \cdots$$
Solutions to KZ equations

Observe that the Weyl group of \( \widehat{sl}_2 \) is a free group generated by 2 reflections \( r_0, r_1 \) and, therefore, each element is uniquely expanded as either

\[
\cdots r_0 r_1 \cdots
\]

or

\[
\cdots r_1 r_0 \cdots
\]

the second one being relevant to our calculation. Setting

\[
w = r_1 \cdots r_0
\]

one obtains

\[
P_w(x, z; t_1, \ldots, t_l) = P_w^1(x, z; t_1, \ldots, t_l) = \sum_{c=0}^{l-1} \sum_{i=0}^{l-1} z^{-c} \sigma_i \epsilon (t_1, \ldots, t_l),
\]

\[
\sigma_i (t_1, \ldots, t_l) = \sum_{0 \leq i_1 < i_2 < \cdots < i_l < l/2} t_{2i_1+1} t_{2i_2+1} \cdots t_{2i_l+1},
\]

completing the \( sl_2 \)-case.

The \( sl_3 \)-case. The big cell is \( C^3 \) with coordinates \( x_{11}, x_{22}, x_{22} \). The matrix \( T \) is given by

\[
T = \begin{pmatrix} x_{11} & x_{12} \\ 1 & x_{22} \end{pmatrix}.
\]

The set of all \( Q_{i_1, \ldots, i_k} \) consists of 5 elements:

\[
Q^1 = 1, \ Q^1_{j} = x_{jj} (j = 1, 2) \ Q^1_{11} = x_{11} x_{22} - x_{12}, \ Q^2_{2} = x_{12}
\]

The graph \( \Gamma \) and its subgraphs \( \Gamma^1, \Gamma^2 \) are given by

\[
\Gamma_1 : \quad \begin{array}{cccccccc}
Q^1 & F_2 & Q^1_{11} & F_3 & -z^{-1} & F_1 & -z^{-1} Q^1_{11} & \cdots \\
1 & F_2
\end{array}
\]

\[
\Gamma_2 : \quad \begin{array}{cccccccc}
Q^2 & F_1 & Q^2 & F_3 & z^{-1} & F_2 & z^{-1} Q^2 & \cdots \\
1 & F_2
\end{array}
\]

The Weyl group \( W \) of \( \widehat{sl}_3 \) is realized as a group generated by reflections at a certain collection of affine lines on the plane (see [16]). These lines produce a covering of the plane by triangles, called alcoves, which \( W \) acts on effectively. Looking at this action one obtains a collection of elements of \( W \) so that a reduced decomposition of any element of \( W \) is contained in it.

Put \( c := r_0 r_1 r_2 \) (is called a Coxeter element), then any \( w \in W \) can be written as \( w = s c^k t c^{-1} u \) where \( s = c, r_2, r_1 r_2, u = c, r_2, r_2 r_1, t = r_0, r_0 r_1, r_0 r_1 r_0 \) if \( kl \neq 0 \) and if \( kl = 0 \), then \( t \) can also be equal to \( c \).
Further one obtains
\[ p^j_w(x, z; t_1, t_2, \ldots, t_l) = \sum_{l'=0}^{l} A^j_{l'} f^j_{l'}(t), \quad j = 1, 2, \]
where
\[ f^j_{l'}(t) = \sum_{p_1 < p_2 < \cdots < p_{l'}, p \equiv -1 (mod 3)} t_{p_1} t_{p_2} \cdots t_{p_{l'}}. \]

\( (k)_3 \in \{0, 1, 2\} \) signifies the residue of \( k \) modulo 3 and \( m : \{1, \ldots, l\} \rightarrow \{0, 1, 2\} \) is a function determining a reduced decomposition of \( w \).

\[ A^j_{l'} = \begin{cases} 
(\frac{1}{z})^{l'} & \text{if } l' = 3l \\
(\frac{1}{z})^{l'} z_{11} & \text{if } l' = 3l + 1 \\
(\frac{1}{z})^{l'} (z_{11} z_{22} - z_{12}) & \text{if } l' = 3l + 2.
\end{cases} \]

\[ A^3_{l'} = \begin{cases} 
(\frac{1}{z})^{l'} & \text{if } l' = 3q \\
(\frac{1}{z})^{l'} z_{22} & \text{if } l' = 3q + 1 \\
(\frac{1}{z})^{l'} z_{12} & \text{if } l' = 3q + 2.
\end{cases} \]

References


REFERENCES


