Quantization of Field Theories
Generalizing Gravity-Yang-Mills Systems on the Cylinder

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Abstract

Pure gravity and gauge theories in two dimensions are shown to be special cases of a much more general class of field theories each of which is characterized by a Poisson structure on a finite dimensional target space. A general scheme for the quantization of these theories is formulated. Explicit examples are studied in some detail. In particular gravity and gauge theories with equivalent actions are compared. Big gauge transformations as well as the condition of metric nondegeneracy in gravity turn out to cause significant differences in the structure of the corresponding reduced phase spaces and the quantum spectra of Dirac observables. For $R^2$ gravity coupled to $SU(2)$ Yang Mills the question of quantum dynamics (‘problem of time’) is addressed.

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1 Introduction

Non-abelian gauge theories as well as several models of gravity on two-dimensional space-time manifolds have become an active field of research in recent years [1],[2],[3],[4],[5],[6]. Both types of theories are closely related to each other. This is best illustrated by the fact that the Lagrangian of a gravity theory with vanishing torsion and constant curvature (Jackiw Teitelboim model [7]) can be rewritten as the one of a nonabelian gauge theory with vanishing field strength ($BF$-theory) [8].

Indeed, as shown in section 2 of the present article pure gravity and gauge theories in two dimensions may be seen as special cases of a more general class of models each of which is characterized by an antisymmetric tensorfield on a finite dimensional target space $L$ (more precisely by a Poisson structure on $L$). A general scheme for the quantization of these models in a Hamiltonian formulation (restricting the topology of the space time manifold to the one of a cylinder) is presented. The heart of this scheme is the reinterpretation of the constraints as horizontality conditions on $U(1)$-bundles over loop spaces. In the special case of non abelian gauge theories the manifolds underlying the loop spaces are the coadjoint orbits (the orbits generated by the gauge group on the dual of its Lie algebra) equipped with the standard symplectic structure [9],[10]. In any case the result is a finite dimensional quantum mechanical system generically including discrete degrees of freedom of topological origin.

The considerations in section 2 are rather formal and abstract. Explicit examples are given in section 3, including several theories of quantum gravity. The reader may find it helpful to have a look at these examples while reading section 2.

The constraints generate those symmetry transformations only which are connected to the unity. The effect of large symmetry transformations is studied in section 4 at the example of gauge theories based on the $so(2,1)$ Lie algebra. We argue that the quantum theory obtained by the implementation of the constraints corresponds to an $\widetilde{SL}(2, R)$ gauge theory ($\widetilde{ }$ denoting the universal covering). For an $SO(2, 1)$ gauge theory the implementation of big gauge transformations yields a one parameter family of unitarily inequivalent quantum theories. The results obtained are checked for both theories by investigating the topological structure of the reduced phase spaces. The latter are compared to the reduced phase space (RPS) of the Jackiw Teitelboim model on a cylinder characterized by the same action. Inequivalences found are traced to the fact that the action of the constraints generates diffeomorphisms only for space time manifolds with nondegenerate metric and thus connect gravitationally inequivalent solutions of the equations of motion.

In section 5 we investigate $R^2$-gravity coupled to an $SU(2)$-Yang Mills theory. The Hilbert space and the operators corresponding to a set of independent Dirac observables are constructed explicitly. As in any quantum theory of gravity the Dirac observables are space-time independent and the Hamiltonian vanishes on
physical quantum states. Strategies to resolve this apparent 'problem of (space-


time' [11] are developed at the example of the reparametrization invariant nonrelativistic free particle. Realizing these strategies in the gravity-Yang-Mills

system, one finds some partial confirmation of them through the fact that a gravity flat limit reproduces the usual SU(2) quantum dynamics.

The material covered in this work is based on two talks delivered in Helsinki


and St Petersberg (cf. [12],[13],[14]). To allow for a comprehensive treatment, more recent developments have been included as well (cf. also [15]).

2 The General Formalism

The action $S$ of a non abelian gauge theory with a finite dimensional semisimple
gauge group is given by

$$S = \int \langle F, \ast F \rangle, \quad F = dA + A \wedge A.$$  \hfill (1)

Here $\langle \ldots \rangle$ denotes the Killing metric on the Lie algebra of the gauge group, $A = A_\mu dx^\mu$ a Lie algebra valued one form (gauge connection), and the Hodge dual $\ast$ is to be taken with respect to a fixed metric on the space time manifold. If the latter is a cylinder $S^1 \times R$, one may parametrize it by a coordinate $x^0 \in R$ and a $2\pi$ periodic coordinate $x^1$. For the Hamiltonian formulation one may choose $x^0$ as the evolution parameter of the Hamiltonian system. One


then finds the zero component $A_0$ of the gauge connection to play the role of a Lagrange multiplier giving rise to the system of first class constraints (Gauss law constraints)

$$\partial B(x^1) + ad^*_{A_1(x^1)} B(x^1) \approx 0,$$ \hfill (2)

where $B$ is the momentum conjugate to the one component $A_1$ of the gauge


connection and takes its values in the dual space of the Lie algebra. The symbol $ad^*$ denotes the coadjoint action of the Lie algebra on its dual space and $\partial$ the derivative with respect to $x^1$.

All the physical systems considered in this paper have a Hamiltonian structure generalizing the one of the non abelian gauge theories. They are obtained by modifying the constraints (2) due to

$$G_i(x) = \partial B_i(x) + \nu^j_i(B) A_j^1(x) \approx 0 \quad x \in S^1$$ \hfill (3)

where $B$ takes values in a linear space $L$, $A_1$ takes values in the dual space $L^*$,


and $\nu : L \rightarrow L \wedge L$ is a map from $L$ into the space of antisymmetric tensors over $L$. [Indices refer to an arbitrary basis in $L$ and the dual basis in $L^*$. Summation over pairs of upper and lower indices is assumed. Throughout this section we will abbreviate $x^1$ by $x$, as done already in (3). The fundamental Poisson brackets are given by $\{ A_i^1(x), B_j(y) \} = \delta_i^j \delta(x-y)$. The Gauss law of the nonabelian gauge theory is recovered from (3) if $L^*$ is identified with the
Lie algebra of the gauge group and $v$ is chosen due to $v_{ij}(B) = f_{ij}^k B_k$, where $f_{ij}^k$ denote the structure constants of the Lie algebra.

For general $v$, (3) will not define a system of first class constraints. Calculating the commutator of two constraints, we find (3) to be first class, iff

$$\frac{\partial v_{ij}}{\partial B_k} v_{kl} + v_{ij} \epsilon_{kl} = 0.$$  \hspace{1cm} (4)

This is precisely the condition for $v$ to generate a Poisson structure on $L$.\footnote{Cf., e.g., \cite{16}; cf. also the article of A. Alekseev and A. Malkin in the present Lecture Notes as well as \cite{18}}. The constraint algebra reads

$$\{G_i(x), G_j(y)\} = \delta(x - y) \frac{\partial v_{ij}}{\partial B_k} G_k(x). \hspace{1cm} (5)$$

The aim of the rest of this section is to investigate the quantization of the system under the restriction (4).

To quantize the system in a momentum representation we consider quantum wave functions as complex valued functionals on the space $\Gamma_L$ of smooth parametrized loops in $L$:

$$\Gamma_L = \{ B: S^1 \to L, x \to B(x) \}. \hspace{1cm} (6)$$

Following the Dirac procedure \cite{17}, we consider the kernel of the quantum constraints

$$\tilde{G}_i(x) \Psi[B] = \left( \partial B_i(x) + i\hbar v_{ij}(B) \frac{\delta}{\delta B_j(x)} \right) \Psi[B] = 0 \hspace{1cm} (7)$$

as the space $\mathcal{H}$ of physical states.

Let us consider two simple examples: For $v \equiv 0$ the support of wave functions in $\mathcal{H}$ is restricted to constant loops. Thus there is a natural identification of $\mathcal{H}$ with the space of constant valued functions on $L$. If $v_{ij}(B)$ is a constant invertible matrix, we may rewrite (7) according to

$$\left( v^{-1} i \tilde{v}_j \partial B_j(x) - \frac{\hbar i}{1} \frac{\delta}{\delta B_i(x)} \right) \Psi[B] = 0$$

and the physical wave functions have the form

$$\Psi[B] = c \exp \left( \frac{i}{2\hbar} \oint B_k(x)(v^{-1})^{ij} \partial B_j(x) dx \right), \quad c \in C. \hspace{1cm} (9)$$

So in this case $\mathcal{H}$ can be identified with the complex plane.

In general, $v$ is neither trivial nor nondegenerate. For (4) the vector fields

$$V_i = v_{ij}(B) \frac{\partial}{\partial B_j} \hspace{1cm} (10)$$
are in involution. Thus they generate an integral surface\(^2\) (symplectic leaf) \(I_B^n\) through any point \(B_0 \in L\). Denote by \(J = \{I_B^n, B_0 \in L\}\) the space of these integral surfaces. In \(B_0 \in L\) the tangent vectors \(V_i\) span a subspace \(S_B^n\) of the tangent space \(T_B^n(L)\). Given a cotangent vector \(w = w^i dB_i \in T^*_B(L)\) in the kernel of \(S_B^n\) (i.e. \(v_{ij}(B_0)w^j = 0\)), we may use the antisymmetry of \(v\) to find \((B(x_0) := B_0)\)
\[
w^i \dot{G}_i(x_0) \Psi = w^i \partial B_i(x_0) \Psi (A) = 0. \tag{11}
\]
Thus in any point \(B_0\) the tangent vector \(\partial B\) along a loop \(B \in \Gamma_L\) is tangential to \(S_B^n\), if \(\Psi (A) \neq 0\). In other words: The support of \(\Psi\) is restricted to loops, which are entirely contained in some integral surface \(I \in J\).

Given a fixed element \(I_0 \in J\), let us denote by \(\Gamma_{I_0}\) the space of loops on \(I_0\). The vector fields
\[
V_i(x) = v_{ij}(B(x)) \frac{\delta}{\delta B_j(x)} \tag{12}
\]
form an overcomplete basis in the tangent space over \(\Gamma_{I_0}\). The ansatz \(\Psi |_{\Gamma_{I_0}} = \exp \Phi\) for the restriction of \(\Psi\) to \(\Gamma_{I_0}\) allows to rewrite the constraint equation according to
\[
\mathcal{A} = \frac{\eta}{i} d \Phi \tag{13}
\]
where \(\mathcal{A}\) denotes the one form on \(\Gamma_{I_0}\) given implicitly by
\[
\mathcal{A}(V_i(x)) = \partial B_i(x), \tag{14}
\]
and \(d\) denotes the exterior derivative on \(\Gamma_{I_0}\). The above ansatz is general, if we exclude the trivial solution \(\Psi |_{\Gamma_{I_0}} \equiv 0\). Locally eq. (13) is integrable, iff \(\mathcal{A}\) is closed. With the general identity
\[
d \mathcal{A}(V_i, V_j) = \mathcal{A}(V_j) \mathcal{A}(V_i) - \mathcal{A}(V_i) \mathcal{A}(V_j) + \mathcal{A}([V_i, V_j]) \tag{15}
\]
and (4) we can indeed verify \(d \mathcal{A} = 0\), if the constraints are first class. Still, there could be local obstructions to the integrability of (13), if the first homotopy group of the underlying space, \(\Pi_1(\Gamma_{I_0})\), is nontrivial. At this point one should note, however, that \(\mathcal{A}\) need not be exact, as \(\Phi\) is determined by \(\Psi\) up to transitions \(\Phi \rightarrow \Phi + i2\pi n, n \in \mathbb{Z}\), only. Therefore \(\Psi\) is well defined, iff \(\mathcal{A}\) is integral, i.e. iff (h \(\equiv 2\pi \hbar\))
\[
\int_\gamma \mathcal{A} = n \hbar, \quad n \in \mathbb{Z} \tag{16}
\]
for any (noncontractible) closed loop \(\gamma\) representing an element of \(\Pi_1(\Gamma_{I_0})\). This condition yields a restriction on the support of \(\Psi\) to a (possibly discrete) subset \(J_\gamma\) of \(J\). For \(I_0\) in this subset \(J_\gamma\) \(\Psi\) is determined up to a multiplicative integration constant on any connected component of \(\Gamma_{I_0}\). (The space \(\Pi_0(\Gamma_{I_0})\) of connected

\footnote{Possibly with singularities}
components of $\Gamma_I$ is in one to one correspondence with the first homotopy group $\pi_1(I_0)$, [18]). Let
\[ I = \cup_{I \in \mathcal{J}} \pi_0(\Gamma_I). \]
Then $\mathcal{H}$ is identified naturally with space of complex valued functions on $I$.

There is also a less abstract description of $\mathcal{H}$: Denote by $\{Q_{(\alpha)}\}, \alpha = 1, \ldots, r$ a maximal set of independent functions on $L$ invariant under the action of the vector fields $V_i$. We will denote those subspaces of $L$, where the $Q_{(\alpha)}$ are constant, as their level surfaces $M_{Q_{(\alpha)}}$. If the connected components of $M_{Q_{(\alpha)}}$ are elements of $\mathcal{J}$ (this is the generic situation in many examples, c.f. next section), the wave functions can be written as
\[ \Psi[I] = \Psi(Q_{(\alpha)}, mQ, nQ) \exp \left( i \frac{\hbar}{\pi} \int A \right), \]
where the discrete parameters $mQ, nQ$ characterize the zeroth and first homotopy group of the level surfaces described above. (18) yields the physical wavefunctions in terms of the variables $B(x)$ and thus allows to describe the action of quantum operators in $\mathcal{H}$.

An alternative formulation of the integrality condition (16) is provided by the relation between one forms on a loop space $\Gamma_M$ and two forms on the underlying space $M$: Any path $\gamma$ in $\Gamma_M$ corresponds to a one parameter family of loops in $M$ spanning a two dimensional surface $\sigma(\gamma)$. To any closed loop in $\Gamma_M$ the corresponding surface in $M$ is closed. Thus any two form $\omega$ on $M$ generates a one form $\alpha$ on $\Gamma_M$ via
\[ \int_{\gamma} \alpha = \int_{\sigma(\gamma)} \omega \]
and $\alpha$ is closed and integral, iff $\omega$ is closed and integral. (The latter means that the integral of $\omega$ over any closed surface is an integer multiple of $2\pi \hbar$).

Of course, not every one form on $\Gamma_M$ can be described in this way. In our case, however, the one form $A$ on $\Gamma_I$, $I \in \mathcal{J}$ is generated by a two form $\Omega$ on $I$ characterized by its contraction with the vector fields (10):
\[ \Omega(V_i, V_j) = \epsilon_{ij}. \]
To prove this let us choose a path $\gamma \in \Gamma_I$ parametrized by a parameter $\tau \in [0, 1]$. Any point in $\gamma$ corresponds to a loop $B(x)$. Thus $\gamma$ induces a map $S^1 \times [0, 1] \to L : (x, \tau) \mapsto B(x, \tau)$. Denote by $\hat{B}$ the tangent vector in the tangent space of $I$ corresponding to the derivative of this map with respect to $\tau$. $\hat{B}$ as well as $\partial B$ can be written as linear combination of the $V_i$:
\[ \hat{B}(x, \tau) = \epsilon^i(x, \tau) V_i, \quad \partial B(x, \tau) = \mu^i(x, \tau) V_i \Rightarrow \partial B = \mu^i \epsilon_{ij}. \]
The corresponding vectors in the tangent space of $\Gamma_I$ are now given by
\[ \hat{B} = \int_x \epsilon^i(x, \tau) V_i(x), \quad \partial B = \int_x \mu^i(x, \tau) V_i(x). \]
With (14) we have
\[ \int_A A = \int A(B) d\tau = \int \epsilon^i \partial_i B d\tau = \int \epsilon^i v^{ij} d\tau = \int \Omega(\partial_i B) d\tau = \int_{\gamma_0} \Omega. \] (23)

Thus our assertion is proven. So (16) is equivalent to the condition
\[ I \in \mathcal{J} \iff \Omega \text{ is integral on } I. \] (24)

\[ \Omega \] is an integral symplectic form and thus gives \( I \) the structure of an integrable space. Furthermore, \( \Omega \) is invariant under the flow of the vector fields \( V_i \) (i.e. \( \mathcal{L}_i \Omega = 0 \), where \( \mathcal{L} \) denotes the Lie derivative). So the vector fields \( V_i \) are locally Hamiltonian with respect to the symplectic form \( \Omega \) on \( I \).

Let us illustrate the formalism by the example of non abelian gauge theories (cf. also [6]). There \( L \) is the dual space \( g^* \) of the Lie algebra \( g \) of the gauge group. Condition (4) becomes the Jacobi identity. \( \mathcal{J} \) is the space of orbits generated by the action of the Lie algebra and \( \{ Q_{(a)} \} \) is the set of Casimirs on \( g \). \( \Omega \) is the standard symplectic form on the coadjoint orbits associated with the coadjoint action of the generators of the Lie algebra and \( \{ Q_{(a)} \} \) is the set of Casimirs on \( g \). \( \Omega \) is the standard symplectic form on the coadjoint orbits of a Lie group as introduced in [9]. The coadjoint orbits are quantizable spaces, if this symplectic form is integral and their quantization yields the unitary irreducible representations of \( g \). This observation establishes a connection of the momentum representation to the configuration space representation of quantum mechanics for non abelian gauge theories on a cylinder: In the configuration space representation, where wave functions are functionals on the space of gauge connections, the physical wave functions (i.e. the kernel of the constraints) can be identified with the functions on the space of unitary irreducible representations of \( g \) [1].

The generalization of our considerations to the case, where the \( B_i \) are local coordinates on a nonlinear space, is straightforward. In this case \( v \in T(L) \wedge T(L) \) is an skew symmetric two tensor over the tangent bundle and the \( v_{ij} \) are the components of \( v \) with respect to the coordinate basis in \( T(L) \). The formulation (7) of the constraints can be made coordinate independent:
\[ i_g [\partial \Psi(B) + \partial \Psi(B)] | \Psi[B] = 0 \forall f : L \rightarrow R. \] (25)

For \( I \in \mathcal{J} \) and \( B_0 \in I \) denote by \( v | _s(B_0) \) the restriction of \( v \) to \( S(B) = TB(I) \). We then have \( \Omega = (v | s)^{-1} \). To prove this let us choose coordinates \( \{ c_1, \ldots, c_r, Q_{(1)}, \ldots, Q_{(r)} \} \) on \( L \). In these coordinates we have
\[ v = \sum_{a=1}^s v_{a\bar{\beta}} \delta_{ca} \delta_{c\bar{\beta}} \] (26)
as the \( Q_{(a)} \) are constant on \( I \). Now (20) immediately implies
\[ \Omega = (v^{-1})^{a\bar{\beta}} dc_a d\bar{c}_\beta. \] (27)
3 Explicit Examples

The constraints (3) are induced by the action

\[ S = \int [B_d A^d + \frac{1}{2} v_{ij} (B) A^i \wedge A^j] \]  

(28)

where \( A \) is an \( L^* \) valued one-form and the zero-form \( B \) is a map into the corresponding dual space \( L \). The action (28) is already of first order, i.e., Hamiltonian: \( A_1 \) and \( B \) are seen to be canonical conjugates and \( A_0 \) enforces the constraints (3).

The simplest nontrivial examples for the considerations of the previous section can be formulated in a three-dimensional target space \( L \). For the rest of this section we will thus stick to such a space. In three dimensions any antisymmetric two-tensor \( v \) can be rewritten according to

\[ v_{ij}(B) = \varepsilon_{ijk} u^k(B) \]  

(29)

for some \( u^i \), \( \varepsilon_{ijk} \) being the standard antisymmetric \( \varepsilon \)-tensor with \( \varepsilon_{123} = 1 \). Indices may be raised and lowered by means of the metric \( \varepsilon_{ij} = \text{diag}(\pm 1, 1, 1) \). The \( \varepsilon_{ij} \) can be thought of as being the structure constants of \( L^* := \text{so}(3) \) or \( L^* := \text{so}(2, 1) \) in an appropriate basis \( \{ T_i \} \). If \( u \) is the identity map, (29) takes the form \( \int B_i F^i \), which is the weak coupling limit of (1).

There are several further possibilities to satisfy the generalized Jacobi identity (4). One is provided by \( u^i = u^i(B_i) \). Another choice, of more interest for the following, is given by

\[ u_a = B_a, \quad v_3 = v_3(B^2, B_3), \quad (B)^2 \equiv B_a B^a, \quad a \in \{ 1, 2 \}. \]  

(30)

Rewriting \( A \) in the latter case as

\[ A = \varepsilon^a T_a + \omega T_3, \]  

(31)

the action (28) takes the form

\[ S_G = \int [B_0 (d\varepsilon^a - \varepsilon^a_b \omega \wedge \varepsilon^b) + B_3 d\omega + v_3(\varepsilon^1 \wedge \varepsilon^2)], \]  

(32)

with \( \varepsilon_{ab} = \varepsilon_{a\beta} \delta^\beta_b \). Much of our interest in (32) stems from the fact that this action can be reinterpreted as the action of a gravitational theory: Viewing \( \varepsilon^a \) as a zweibein and \( \omega \) as a spin connection, the term following \( B_0 \) is identified with the torsion two-form \( D\varepsilon^a \), whereas \( d\omega \) becomes the curvature two-form. \( B_3 \) and \( B_3 \) are vector and scalar valued functions, respectively, living on the two-dimensional manifold characterized by the metric \( g = \varepsilon^a \varepsilon^b \kappa_{ab} \). The latter is of Euclidean or Minkowski type, corresponding to the respective signature of the frame metric \( \kappa_{ab} = \text{diag}(\pm 1, 1) \). Eliminating the \( B \) fields by means of
their equations of motion for some special choices of $u_3$, we can establish the equivalence of $S_G$ with purely geometrical actions of two dimensional gravity. E.g., $u_3 = (1/4\gamma)(B_3)^2 - \lambda + \frac{\gamma}{2}(B)^2$ with $\alpha \neq 0$ is easily seen to lead to 2D gravity with torsion [5]

$$S_G^{KV} = -\int [\pm \gamma d\omega \wedge *d\omega \pm \frac{1}{2\alpha} De^\theta \wedge *De_\theta + \lambda \varepsilon],$$

where $\varepsilon \equiv e^1 \wedge e^2$ is the metric induced volume form or $\varepsilon$-tensor and `$*$` denotes the Hodge dual operation ($*\varepsilon = \pm 1$). (33) is the most general Lagrangian yielding second order differential equations for $e^\theta$ and $\omega$ in two dimensions. With $\alpha = 0$ the same choice for $u_3$ leads to the similar action of torsionless $R^2$ gravity.

To find the set $J$ of integral surfaces $I$, on which physical wave functionals $\Psi$ [8] have their support, let us for a moment return to an arbitrary dimensional target space $L$ and a Poisson structure $v$ of the form $v = f_{ijk} u_k(B)(\partial/\partial B_j) \wedge (\partial/\partial B_i)$ where the $f$'s are the structure constants of some Lie algebra of rank $r$; (29) may be regarded as a special case of this. If $C(u) = C^{ijk...}u_iu_ju_k...$ denotes one of the $r$ independent Casimirs,

$$w = \frac{\partial C(u)}{\partial u_i} dB_i$$

will be annihilated by $v$. In the case of a non abelian gauge theory $u_i = B_i$ and $w = dC(B)$ so that according to (11) the (physical) wave functions $\Psi$ have support only on loops $B$ with constant values of the Casimirs, as has been noted already at the end of the previous section. In the present case of (29) the only independent Casimir is the Killing metric $\kappa_{ij}$, so that (11) with (34, 30) becomes

$$(B^a \partial B_a + u_3 \partial B_3) \Psi = 0.$$  

(35)

For reasons of calculational simplicity, let us specify $u_3$ to

$$u_3 = U(B_3) + \frac{\alpha}{2}(B)^2.$$  

(36)

Multiplying now (35) by the integrating factor $2 \exp(\alpha B_3)$, we obtain

$$\partial Q \Psi = 0, \quad Q = (B)^2 \exp(\alpha B_3) + 2 \int^{B_3} U(y) \exp(\alpha y) dy.$$  

(37)

Thus generically the level surfaces $M_Q$ generated by $Q = const$, and thus (generically) also the integral surfaces $I \in J$, will be two-dimensional for the considered class of examples (32). In the following we will specify the potential $u_3$ to study these surfaces in more detail.

Prototypes are provided by the $SO(3)$ and $SO(2,1)$ BF-theories based on the $so(3)$ and $so(2,1)$ algebra, respectively, resulting from $u_3 = B_3$.\footnote{Actually it is rather the BF-theories of the corresponding universal covering groups which have been quantized so far; the further steps necessary to quantize an $SO(2,1)$-BF-theory will be discussed in the following section.}

Fixed
values of the Casimir $Q \equiv B^i B_i$ yield the coadjoint orbits of the groups, i.e. in the former case spheres for $Q > 0$ and the origin for $Q = 0$ and in the latter case one sheet hyperboloids for $Q > 0$, two sheet hyperboloids for $Q < 0$, and the light cone in the target space for $Q = 0$. In the compact case, we see that all the level surfaces, which coincide with the integral surfaces, are two dimensional except for the zero dimensional origin. The spheres are connected and simply connected, but have $\Pi_2 = Z$. According to our general considerations of section 2 we therefore know that the spectrum of $Q$ becomes discrete. This (i.e. eq. (16) or (24)) was a necessary and sufficient condition for the integrability of the horizontality condition (7). (The determination of this spectrum shall be taken up at the end of this section, after having further analyzed the topological structure of the integral surfaces).

In the noncompact $sl(2, R)$ case $\Pi_2$ is always trivial and the spectrum of $Q$ remains continuous. For $Q < 0$, however, the level surfaces $M_Q$ consist of two (simply connected) parts, thus corresponding to two different integral surfaces $I$ of the vector fields $V_i$ defined in (10). For $Q > 0$ $\Pi_2(M_Q)$ is trivial, but $\Pi_1(M_Q) = Z$; thus the integral surfaces $I$ of $V_i$ coincide with the level surfaces $M_Q$ in this case, but loops $B$ with different winding number around the target space hyperboloid are not smoothly connected to each other in the space of loops on $I$ ($\Pi_2(I) = \Pi_1(I) = Z$); thus for any winding number of $B \sim B(x^1)$ we can prescribe an independent initial value for the solution of the first order differential equation (7). This illustrates the necessity for the two quantum numbers $m_Q$ and $n_Q$ within (18). Actually, they correspond also to invariant Dirac observables, if we allow the latter to become discontinuous: Clearly

$$m_Q := \Theta(-Q)\Theta(B_1), \quad n_Q := \Theta(Q) \int \delta(\phi)dx^1$$

where $\Theta$ is the Heaviside step function and $\phi$ is the angle variable of polar coordinates in the $(B_2, B_3)$-plane, are also Dirac observables in this extended sense, independent from the continuous invariant $Q$.

The $Q = 0$ level surface plays a special role: Since $V$ vanishes at the origin, the latter is an integral surface by itself and splits $Q = 0$ into three parts. (Note that (7) constrains the wave functionals to have support only on loops not passing through a target space point where $V_i$ vanishes; thus this splitting transfers consistently to the spaces of loops on the integral surfaces). This implies also that $\Pi_1$ becomes nontrivial for the future and the past target space light cones. Allowing also for invariant distributions, we can uniquely describe the integral surfaces of $V_i \equiv V_i[B(x^1)]$, i.e. the space $I$ of eq. (17) (with $\bar{J} = J$ here), by means of Dirac observables: \footnote{The integral surfaces of $V_i$ are characterized by the same quantities except for $n_Q$.} Defining $\Theta(0) := 1$ we have to add merely $\delta[B_1]$ to $Q$, $m_Q$, and $n_Q$ so as to get a complete set of independent commuting Dirac observables for the $sl(2, R)$-$BF$-theory. The space $J$ of integral surfaces $I$ as well as the space of loops on it, $\bigcup_{I \in J} \Gamma_I$, are, however, not Hausdorff at
$Q = 0$. As a consequence there might arise some ambiguity in gluing together the orbit spaces with $Q \neq 0$, an issue which is certainly closely related to the determination of an inner product.

Summing up the $sl_2$ case, we find the physical wave functionals to effectively become functions (possibly also generalized ones) on the space of the above Dirac observables, the corresponding spectra remain classical, and the phase factor becomes essentially superfluous, one can get rid of it by changing the basis in the U(1) quantum bundle.

Concerning the question of the inner product, let us remark only that on large parts of the phase spaces of any of the models (32) with (36) and Minkowski signature, the variable conjugate to $Q$ can be written as $(e^\pm := (e^2 \pm e^1)/\sqrt{2})$

$$P = -\frac{1}{2} \int \exp(-\alpha B_3) \frac{e_{1-}}{B_+} dx^1 \approx -\frac{1}{2} \int \exp(-\alpha B_3) \frac{e_{1+}}{B_-} dx^1. \quad (39)$$

Pulling through the phase factor of (18), which in local target space coordinates takes the form

$$\exp \left( -\frac{i}{\hbar} \oint \ln |B_+|^3 B_3 dx^1 \right) \sim \exp \left( \frac{i}{\hbar} \oint \ln |B_-|^3 B_3 dx^1 \right) \quad (40)$$

the Dirac observable $P$ acts via $(\hbar/i)(d/dQ)$ on $\Psi$. Requiring that it will become a Hermitean operator severely restricts the measure of the inner product, but, in the case that $\Psi$ depends also nontrivially on quantum numbers $m$ or $n$, this does not determine the inner product entirely. It is not quite clear, if one should require the ‘Dirac observables’ $mQ$ and $nQ$, introduced above for the $sl_2$-theory, to become hermitean as well. In this case the corresponding eigenspaces would be orthogonal.

Next let us find the space of integral surfaces for $R^2$-gravity, i.e. for (32) with potential $u_3 = -(B_3)^2 - \lambda$, yielding $Q((B)^2, B_3) = 3(B_3)^2 - 2(B_3)^2/3 - 2\lambda B_3$. For $\lambda > 0$, $Q = const$ allows to determine $B_3$ uniquely as a function of $(B)^2$.

Thus the resulting surfaces in the target space are diffeomorphic to a plane so that there is no quantization of the classical spectrum of $Q$ and there are also no additional quantum numbers within the wave functions (18). So for $\lambda > 0$ the resulting Hilbert space is the one of an ordinary particle on a line.

For $\lambda = 0$ the situation is similar, only that the value $Q = 0$ (critical value) plays a similarly exceptional role as in the $BF$ case: one gets a conic singularity of the plane at $(B)^2 = B_3 = 0$ for Euclidean signature ($k_{ab} = \delta_{ab}$), and for Minkowski signature ($k_{ab} = diag(-1, 1)$) additionally a non Hausdorff structure (of $J$) at this point.

For $\lambda < 0$ there are two critical values of $Q$: $Q_{(>)} = 0, Q_{(<)} = 0, \pm \sqrt{-\lambda}$ of $B_3$, corresponding to the zeros of $u_3$ resp. $U$. For $Q \in (-\infty, Q_{(<)} \cup (Q_{(>)} \infty)$ the resulting surfaces are again manifolds with trivial topology. For $Q \in (Q_{(<)}, Q_{(>)}$ and Euclidean signature we get two disconnected surfaces of the topology of a plane and a sphere, respectively. Thus
the continuous spectrum $Q \in R$ has a twofold degeneracy for some specific values of $Q \in (Q_-, Q_+)$. For $Q \in (Q_-, Q_+)$ and Minkowskian signature the level surfaces $M_Q$ are connected and of trivial second homotopy; however, there are two fundamental noncontractible loops, the winding numbers of which give rise to a quantum number $\eta Q \in \mathbb{Z}$.

To analyse the situation for general potential $U$, it is helpful to use a $(B)^2$ over $B_3$ diagram. Any fixed value of $Q$ induces a curve $C_Q$ in this diagram. The intersections of $C_Q$ with the $B_3$ axis are most crucial for the topology of $M_Q$. Let us first consider the Euclidean case, where only non-negative values of $(B)^2$ are admissible: Any part of $C_Q$ (in the positive of $(B)^2$) between two successive intersections with the $B_3$ axis leads to a spherical $M_Q$, any part of $C_Q$ with $(B)^2 \geq 0$ and exactly one point of vanishing $(B)^2$ on it yields a 'plane', and a $C_Q$ with no such points or intersections results in a cylindrical $M_Q$ (or an empty $M_Q$ for strictly negative $(B)^2$, as, e.g., in the $so(3)$-example for $Q < 0$). Changes of the topology of $M_Q$ (along the choice of $Q$) can happen only at sliding intersections of $C_Q$ with the $B_3$ axis; the latter are possible only at $B_3 = \beta_\ast, U(\beta_\ast) = 0$, and thus only for the 'critical values' $Q_\ast = Q(0, \beta_\ast)$ of $Q$. The critical points $(B_1, B_2, B_3) = (0, 0, \beta_\ast)$ (and only these) are then fixed points of the vector fields $V_i$ and constitute an (zero dimensional) integral surface by itself. For Minkowski signature the transition from $C_Q$ to $M_Q$ is a bit more cumbersome. The result is, however, quite simple: If $C_Q$ contains no points $(B)^2 = 0$, $M_Q$ consists of two disconnected 'planes'; if $C_Q$ contains $l$ points of (nonsliding) intersections with the $B_3$ axis, it has $l + 1$ fundamental non-contractible loops. At the critical values $Q = Q_\ast$ (sliding intersections) we again have fixed points $(0, 0, \beta_\ast)$, and the set $J$ of integral surfaces becomes non Hausdorff there.

For both signatures the fixed points correspond also to the distributional solutions $\delta[\beta_3] \delta[\beta_3 - \beta_\ast]$ of the quantum constraints and might be implemented via a point measure in the inner product. (A somewhat special case arises when choosing $u_3 \equiv 0$, describing ‘flat gravity’ on the cylinder, where the set of $\beta_\ast$ becomes uncountable and needs a separate treatment). Aside from these fixed point solutions the wave functions have the form (18). We further observe that in our class of examples (32) the integral surfaces have a non trivial second homotopy only for Euclidean signature and that non trivial $\Pi_1$ implies trivial $\Pi_2$ and vice versa.

The discrete part of the spectrum of the Dirac observable $Q$ is obtained most easily via the two-form $\Omega$ of section 2. According to (27) it is the inverse of $v$ restricted to the integral surfaces, which are (deformed) spheres in the case under study. By construction $v(dQ, \cdot) = 0$. Furthermore, due to (29) and (30) $v(dB_3, \cdot)$ is independent of the potential $u_3$. Thus it will be convenient to calculate the inverse of $v$ $\Omega_{\alpha}$ in coordinates $Q, B_3$ and e.g. $\varphi = \arctan(B_2/B_1)$; these cover the spheres up to the poles at $B_3 = 0$. Since $v(dB_3, d\varphi) = 1$ we obtain

$$\Omega = dB_3 \wedge d\varphi.$$ (41)
Integrating this two-form over the considered 'sphere', the integrability condition (16) becomes (cf. eqs. (24, (19))

\[ B_{3,\text{max}}(Q) - B_{3,\text{min}}(Q) = n\hbar, \quad n \in \mathbb{N} \]  

(42)

where \( B_{3,\text{max}}, B_{3,\text{min}} \) denote the values of \( B_3 \) at the poles. Given a curve \( CQ \) introduced above, it is then easy to decide if this value of \( Q \) allows for a spherical integral surface or not. For the case of \( u_3 = B_3 \), (41) becomes the rotation invariant Kostant-Souriau form \( \Omega = r \sin \theta d\theta d\varphi = (\varepsilon^{ijk} B_i dB_j dB_k / r^2) \bigg|_{r=\sqrt{Q}} \)

where \((r, \theta, \varphi)\) denote spherical target space coordinates, and the quantization condition (42) can be expressed also explicitly in terms of \( Q = r^2 \), namely as \( Q = n^2 \pi^2 / 4, n \in \mathbb{N} \). If we add to this \( Q = 0 \), corresponding to the distributional solution located at \( B = 0 \), this spectrum coincides precisely with the one obtained in the connection representation \([2]\).

4 Large Gauge Transformations and Metric Non-Degeneracy

The previous two sections have been devoted to the analysis of the models under consideration in a Hamiltonian formulation, where the symmetries of the system are expressed in terms of first class constraints. There are, however, some subtle points connected with this approach:

- The constraints are the generators of infinitesimal symmetry transformations. Large gauge transformations (i.e. symmetry transformations not connected to the unity) cannot be generated by infinitesimal transformations and thus they are not determined by the constraints.

- In the gravity theories presented in the previous sections the zero components of the zweibein and the spin connection played the role of Lagrange multiplier fields. We eliminated them from the phase space as unphysical degrees of freedom. But in a theory of gravity one usually requires the metric of the space time manifold to be non-degenerate (i.e. \( \det g \neq 0 \) everywhere). Obviously it is difficult to realize this condition after eliminating the zero components of the zweibein from the phase space. Even if we allow for a degenerate metric, the problem is not solved: The constraints (3) with (30) generate the symmetries of the gravity theory only for \( \det g \neq 0 \) and turn out to connect gravitationally inequivalent solutions separated in the phase space by regions with a degenerate metric.

In the present section we will illustrate the importance of these points by considering concrete examples. Our analysis will include the explicit calculation of the reduced phase space (i.e. the space of solutions of the equations of motion modulo the symmetries of the model) for gauge and gravity theories based on the \( \mathfrak{sl}(2, R) \) Lie algebra. All of the theories considered are characterized by the
same Lagrangian
\[ \int \langle R, F \rangle \]
and thus a naive calculation of the constraints yields equivalent Hamiltonian systems. Nevertheless we will find that the reduced phase spaces differ as the symmetry contents of the models differ.\(^5\)

In the first example let us regard the action \( \int \langle R, F \rangle \) as the one of a \( PSL(2,R) \) gauge theory. \( PSL(2,R) \) is the group obtained from \( SL(2,R) \) by the identification \( 1 \sim -1 \) and is isomorphic to \( SO_+(2,1) \), the component connected with the unity of \( SO(2,1) \). Thus its Lie algebra is given by
\[ [T_i, T_j] = \varepsilon_{ij}^k T_k, \]
where the last index in the \( \varepsilon \)-tensor has been raised by means of the Killing metric \( \varepsilon_{ij} = \text{diag}(-1, 1, 1) \). A possible matrix representation of (44) is provided by the real matrices \( T_1 = i\sigma_3/2, \ T_2 = -\sigma_1/2, \ T_3 = -\sigma_3/2 \), where the \( \sigma_i \) are the Pauli matrices. From this one finds \( \varepsilon_{ij} = 2tr(T_i T_j) \) so that, e.g., the Dirac observable \( Q = B_i B^i \) introduced in eq. (37) can be expressed alternatively as \( Q = 2tr B^2 = -4det B (B \equiv B_i T^i) \).

The group \( G \) of symmetry transformations is the group of smooth mappings from the cylinder into \( PSL(2,R) \):\(^6\)
\[ G_{PSL(2,R)} = \{ g : S^1 \times R \rightarrow PSL(2,R) \} \]
The equations of motion,
\[ F = 0, \quad dB + [A, B] = 0, \]
yield the connection to be flat and the Lagrange multiplier field \( B \) to be covariantly constant. Up to gauge transformations a flat connection \( A \) on a cylinder is determined by its monodromy \( M_A = \mathcal{P} \exp \oint A \in PSL(2,R) \) generating parallel transport around the cylinder (\( \mathcal{P} \) denotes path ordering and the integration runs over a closed curve \( C \) winding around the cylinder once). As the exponential map is surjective on \( PSL(2,R) \), any monodromy matrix can be generated by a connection of the form \( A = A_1 dx^1 \) where \( A_1 \) is constant:
\[ A = \begin{pmatrix} z & y + t \\ y - t & -z \end{pmatrix} dx^1, \quad t, y, z \in R \]
Constant gauge transformations act on \( A \) via the adjoint action leaving the determinant \( t^2 - y^2 - z^2 \) invariant and may be interpreted as Lorentz transformations in the three dimensional Minkowski space \( (t, y, z) \). Hyperbolic, elliptic
\[ ^5 \text{This is similar to the inequivalence of the symmetry generators (}d/dy\text{) and } \eta(d/dy) \text{ on a line even when disregarding } \eta = 0 \text{ [12].} \]
\[ ^6 \text{There are no nontrivial principal } G \text{-bundles on a cylindrical base manifold, if the chosen structure (gauge) group } G \text{ is connected.} \]
and parabolic elements, respectively, in the Lie algebra correspond to spacelike, timelike, and lightlike vectors, respectively, in this Minkowski space. By Lorentz transformations in the \((t, y, z)\) plane they can be brought into the form:

\[
A^{hyp} = \begin{pmatrix} 0 & \alpha \\ \alpha & 0 \end{pmatrix} \, dx^1, \quad A^{ell} = \begin{pmatrix} 0 & \vartheta \\ -\vartheta & 0 \end{pmatrix} \, dx^1, \quad A^{par} = \begin{pmatrix} 0 & 0 \\ \pm 1 & 0 \end{pmatrix} \, dx^1
\]

with \(\alpha, \vartheta \in \mathbb{R}\) and the identification \(\alpha \sim -\alpha\). Exponentiation yields the monodromy matrices

\[
M_{A^{hyp}} = \begin{pmatrix} \cosh 2\pi \alpha & \sinh 2\pi \alpha \\ \sinh 2\pi \alpha & \cosh 2\pi \alpha \end{pmatrix}, \quad M_{A^{ell}} = \begin{pmatrix} \cos 2\pi \vartheta & \sin 2\pi \vartheta \\ -\sin 2\pi \vartheta & \cos 2\pi \vartheta \end{pmatrix},
\]

\[
M_{A^{par}} = \begin{pmatrix} 1 & 0 \\ \pm 2\pi & 1 \end{pmatrix}
\]

inducing the further identification \(\vartheta \sim \vartheta + 1/2\) in the elliptic sector (remember \(\int dx^1 = 2\pi\) and \(1 \sim -1\)). The integration of the second eq. (46) gives \(B(x^1, x^1) = B(x^1, x^1 + 2\pi) = MA B(x^1, x^1)M_A^{-1}\) and thus choosing a connection from (48) \(B(x)\) has to commute with the corresponding monodromy matrix and consequently with the connection itself. Using (46) again one finds \(B(x)\) to be constant. We obtain:

\[
B^{hyp} = \begin{pmatrix} 0 & c_1 \\ c_1 & 0 \end{pmatrix}, \quad B^{ell} = \begin{pmatrix} 0 & c_2 \\ -c_2 & 0 \end{pmatrix}, \quad B^{par} = \begin{pmatrix} 0 & 0 \\ 0 & c_3 \end{pmatrix}, \quad c_i \in \mathbb{R}
\]

In the case \(A = 0\) (corresponding to \(\alpha = 0\) or \(\vartheta = 0\), respectively, in (48)) \(B(x)\) is constant, too, but it is not restricted by its commutator with the monodromy matrix. It is, however, subject to constant gauge transformations, as they leave \(A = 0\) invariant. Considerations similar to those above show that also in this case gauge representatives of the solutions are given by (50) with \(c_3 = \pm 1\) and the identification \(c_1 \sim -c_1\). (48) and (50) give a complete parametrization of the reduced phase space of the \(PSL(2, \mathbb{R})\)-gauge theory.

The group of gauge transformations \(G_{PSL(2, \mathbb{R})}\) as defined above is not connected; rather it consists of an infinite number of components not smoothly connected to each other: \(\Pi_0(G) = \Pi_1(PSL(2, \mathbb{R})) = \mathbb{Z}\). A complete set of representatives for the components of \(G_{PSL(2, \mathbb{R})}\) is given by

\[
g_{(n)} = \begin{pmatrix} \cos(nx^1/2) & \sin(nx^1/2) \\ -\sin(nx^1/2) & \cos(nx^1/2) \end{pmatrix}, \quad n \in \mathbb{Z}
\]

Parametrizing the phase space as in (48) - (50) we also implemented these gauge transformations. The action of the group elements \(g_{(n)}\) on the connections (48)
gives in the hyperbolic sector

\[
A^{hynn}_{(\alpha)} = \begin{pmatrix}
\alpha \sin(n x^1) & \alpha \cos(n x^1) + n/2 & -\alpha \sin(n x^1) \\
\alpha \cos(n x^1) - n/2 & \cos(n x^1) & \sin(n x^1)
\end{pmatrix}
\, dz^1
\]

\[
B^{hynn}_{(\alpha)} = c_1 \begin{pmatrix}
\sin(n x^1) & \cos(n x^1) \\
\cos(n x^1) & -\sin(n x^1)
\end{pmatrix}
\]  \quad (52)

An analogous result is obtained in the parabolic sector. In the elliptic sector
the \(g_{(\alpha)}\) generate a transformation \(\vartheta \rightarrow \vartheta + n/2\). They are responsible for the
previous identification \(\vartheta \sim \vartheta + 1/2\), which is removed now.

With this knowledge it is straightforward to find the RPS for an \(SL(2, R)\)
gauge theory: The action is the same as the one of the \(PSL(2, R)\) theory. Gauge
transformations of the type \(g_{(2i+1)}, i \in Z\) are not allowed, as we do not have
the identification \(1 \sim -1\). Consequently the hyperbolic sector of the RPS is
parametrized by \((A^{hyp}_{(\alpha)}, B^{hyp}_{(\alpha)})\) and \((A^{hyp}_{(\beta)}, B^{hyp}_{(\beta)})\). An analogous result holds for
the parabolic sector. In the elliptic sector we have \((A^{ell}, B^{ell})\) but with the
identification \(\vartheta \sim \vartheta + 1/2\) rather than \(\vartheta \sim \vartheta + 1/2\). In contrast to the \(PSL(2, R)\)-
case there are elements of the RPS which cannot be represented by a constant
connection. This is a consequence of the non surjectivity of the exponential map
between Lie algebra and group in the case of \(SL(2, R)\).

From the homotopical point of view \(SL(2, R)\) is the double covering of
\(PSL(2, R)\). Analogously, excluding all gauge transformations not connected
to the unity from \(\tilde{G}\) is equivalent to choosing the universal covering \(\tilde{S}L(2, R)\)
as the gauge group of the theory. (Note that \(\Pi_1(\tilde{G}_{S\ell} (2, R)) = \Pi_1(\tilde{S}L(2, R)) = \{1\}\)).

Thus the RPS of \(\tilde{S}L(2, R)\) is parametrized by \((A^{hyp}_{(\alpha)}, B^{hyp}_{(\alpha)}), n \in Z, \alpha \in R, \)the
analogous solutions in the parabolic sector, and \((A^{ell}, B^{ell}), \vartheta \in R\) (without
any identification).

To see the significance of the difference between the \(PSL(2, R)\) and the
\(\tilde{S}L(2, R)\) gauge theory, let us have a look on the quantization of the models:
In the first case the elliptic sector of the configuration space (i.e. the space of
gauge inequivalent connections) is compact and thus we expect the possibility
of unitarily inequivalent quantum theories with a discrete spectrum for the
momentum operator (i.e. the Dirac observable \(Q = -4 \det B\)). We may
compare this with the result obtained in the previous section: There we used the
Gauss law constraints to realize gauge transformations in the quantum theory.
As outlined above the constraints generate those gauge transformations only
which are connected to the unity. So the quantum theory we obtained
 corresponds to the \(\tilde{S}L(2, R)\) gauge theory. Indeed a continuous spectrum for \(Q\)
was found. Furthermore, the discrete parameter \(n_Q\) within the wave functions
is also readily identified with the parameter \(n\) of the hyperbolic sector in the
above parametrization of the RPS of the \(\tilde{S}L(2, R)\) theory.

To find the correct quantization of the \(PSL(2, R)\) theory we have to implement
large gauge transformations. To this end let us employ the exponen-
tial map in order to rewrite the Gauss law. Starting from an initial loop $B_0$ in some coadjoint orbit any loop $B$ may be written as $B = gB_0g^{-1}$ for some $g = \exp X \in \mathcal{G}$, $X : S^1 \to \mathfrak{sl}(2, \mathbb{R})$. (As we mentioned above the exponential map is surjective on $PSL(2, \mathbb{R})$). If $g$ is connected to the unity, we also have $g(t) = \exp tX \in \mathcal{G}$ for $t \in [0, 1]$. The Gauss law can then be rewritten as

$$
\int \langle X, \partial(e^{iX}B_0e^{-iX}) \rangle dx^1 \Psi[B] = \frac{i}{\hbar} \frac{\partial}{\partial t} \Psi[e^{iX}B_0e^{-iX}].
$$

With the identity

$$
\int dx^1 \langle X, \partial(e^{iX}B_0e^{-iX}) \rangle = -\int dx^1 \langle e^{-iX} \partial X e^{iX}, B_0 \rangle = -\int dx^1 \frac{\partial}{\partial t} \langle e^{-iX} \partial X e^{iX}, B_0 \rangle,
$$

integration over $t$ leads to

$$
\Psi[B] = \Psi[B_0] \exp \left( \int \frac{i}{\hbar} \langle g^{-1} \partial g, B_0 \rangle dx^1 \right), \quad g \in \mathcal{G}.
$$

An alternative derivation of this exponentiated form of the Gauss law constraint is provided by a Fourier transformation of the gauge invariance property of the physical wave functionals in the connection representation [19]. By construction the wave functions calculated in the previous sections are the general solutions of eq. (55) for gauge transformations connected to the unity. To quantize the $PSL(2, \mathbb{R})$ theory we may rewrite $\mathcal{G}$ as the semidirect product of $\mathcal{G}_e$ (the component connected to the unity) and the zero-th homotopy group $\Pi_0(\mathcal{G}) = \mathbb{Z}$:

$$
\mathcal{G} = \mathcal{G}_e \times \mathbb{Z}
$$

The most general incorporation of the second factor $\mathbb{Z}$ into the quantum theory will be to require the wave functionals to transform according to a unitary representation $D_\theta$ of $\mathbb{Z}$ characterized by an angle $\theta$:

$$
D_\theta(n) = \exp \left( \frac{i2\pi n \theta}{\hbar} \right).
$$

Taking together (55) and (57) we are thus lead to

$$
\Psi[B] \equiv \Psi[gB_0g^{-1}] = \Psi[B_0] \exp \left[ \frac{i}{\hbar} \left( \int \langle g^{-1} \partial g, B_0 \rangle dx^1 + 2\pi n \theta \right) \right]
$$

for $g$ in the $n$-th component of $\mathcal{G}$.

It is easy to verify that (58) is compatible with group multiplication, i.e. if it holds for two gauge transformations $g_1, g_2$, it will also hold for the product $g_1g_2$. The group $\mathbb{Z}$ is generated by one element which may be represented by $g_{(1)}$ as defined in (51). For this reason the wave functions obtained in the sections 2 and 3 will solve (58) for all $g \in \mathcal{G}$, if this identity holds for $g_{(1)}$. Furthermore
(55) will hold for all loops \( B \) which may be written as \( B = g^{-1}B_0g \) for some \( g \in G \), if it holds for the loop \( B_0 \). It is now obvious that in the hyperbolic sector \((Q > 0)\) the large gauge transformations simply relate the \( \Psi (Q, n_Q) \), \((n_Q \equiv n)\), to each other for fixed value of \( Q \) and different values of \( n \). In the elliptic sector \((Q < 0)\) we may apply (58) to the constant loop \( B_0 = B^{\ell II} (B^{\ell II} \text{ as defined in (50)}) \). Noting that \( g(1) \) commutes with \( B^{\ell II} \), we find

\[
\frac{1}{\hbar} \left( \frac{1}{2\pi} \int (g^{-1}_1 \partial g_1, B^{\ell II} ) dx^1 + \theta \right) = \theta - \frac{2c_2}{\hbar} = \theta - sgn(B_1)\sqrt{-Q} \in \mathbb{Z} \quad (59)
\]

in which the signum function can be expressed also in terms of the quantum number \( m_Q \) of eq. (38) via \( sgn(c_2) \equiv sgn(B_1) = 2m_Q - 1 \). Thus in the elliptic sector the support of physical wavefunctions \( \Psi (Q, m_Q) \) is restricted to \( \sqrt{-Q} = (1 - 2m_Q)(i\hbar + \theta), l \in \mathbb{Z} \). The quantum theories we obtain for different choices of \( \theta \) are obviously unitarily inequivalent, as they generate different spectra of \( Q \). This is precisely the result we expected.

At this point we want to mention that these results also hold for the \( PSL(2, R) \) Yang-Mills theory. In this case \( Q \) plays the role of the Hamiltonian.

Via the identification (31) the action (43) together with (44) may also be regarded as the one of a gravity theory (Jackiw-Teitelboim model). In this case the symmetry content of the model is given by diffeomorphisms and local Lorentz transformations (gravitational symmetries). This group consists of a finite number of components not smoothly connected to each other. They differ by \( x^0 \)- and \( x^1 \)-reflection on the space time manifold and by parity transformation and time reversal in the Lorentz bundle. So up to these transformations the symmetry content of the gravity theory seems to coincide with the one of the \( SL(2, R) \) gauge theory. So let us see, how the infinitesimal generators of the gravitational symmetries are identified with the Gauss law constraints (2) generating the \( sl(2, R) \) algebra. To this end we may calculate the Hamiltonian density \( \mathcal{H} \) \(( \mathcal{H} = \frac{1}{\hbar} \mathcal{H} dx^1 \) of the theory as the generator of diffeomorphisms in \( x^0 \)-direction. We find \( \mathcal{H} = -A^i_k G_i \), with \( G_i = \partial B_i + \varepsilon_{ij} A^j B_k \). Analogously the generator of diffeomorphisms in \( x^1 \)-direction is obviously given by \( A^i_k G_i \). Noting that \( G_3 \) precisely generates local Lorentz transformations one concludes that identification of the \( sl(2, R) \) generators and the infinitesimal generators of the gravitational symmetries crucially depends on the condition \( \det e \neq 0 \). Let us investigate the consequences of this observation for the RPS of the gravity theory. With the identifications (31) the solutions we used above to parametrize the RPS of the \( SL(2, R) \) gauge theory correspond to space time manifolds with \( \det g = 0 \). To any of these solutions, however, it is possible to find a gauge transformation yielding a solution corresponding to a nondegenerate space time metric. More precisely, this can be done in an infinite number of gravitationally inequivalent ways. E.g., in the elliptic sector, we might apply one of the
following gauge transformations to $A_{[k]}$:

$$
\begin{align*}
    g_{[k]} &= \begin{pmatrix}
        \cos \chi_k & \sin \chi_k \\
        -\sin \chi_k & \cos \chi_k
    \end{pmatrix}
    \begin{pmatrix}
        1 & b_k \\
        0 & 1
    \end{pmatrix}, \\
    \chi_k &= [\exp(x^0) + 2|\varphi|] \sin(kx^1),
\end{align*}
$$

$$
\begin{align*}
    b_k &= [\exp(x^0) + 2|\varphi|] \cos(kx^1) \quad k \in \mathbb{N}. (60)
\end{align*}
$$

We obtain

$$
A_{[k]} = \begin{pmatrix}
    b_k (\partial dx^1 + d\chi_k) \\
    -b_k (\partial dx^1 + d\chi_k)
\end{pmatrix} \\
    (1 + b_k^2)(\partial dx^1 + d\chi_k) + db_k.
$$

The gauge transformations (60) are smoothly connected to the unity for arbitrary value of $k$ as the $\chi_k$ are periodic functions in $x^1$. Nevertheless the solutions $A_{[k]}$ are gravitationally inequivalent for different values of $k$. To prove this let us again choose a loop $C$ running around the cylinder once. Under the restriction $\det g = \det e^0 \neq 0$ the components of the zweibein $(e^0, e^1)$ induce a map $C : S^1 \to R^2 \setminus \{0\}$ characterized by a winding number (not depending on the choice of $C$). Solutions with different winding numbers cannot be transformed into each other by gravitational symmetries, since they are separated by solutions with $\det e = 0$. (Also the discrete gravitational symmetry transformations mentioned above do not change the winding number). For different values of $k$ the solutions (61) have different winding numbers, which proves our assertion.

This result generalizes to the other sectors of the theory: Solutions which are gauge equivalent in the $SL(2, \mathbb{R})$ gauge theory are not equivalent in the gravity theory, if they have different winding number.

The winding number defined above is related to the kink number as defined in [20] by means of 'turn arounds' of the light cone along non contractible loops. More precisely, winding number $k$ corresponds to kink number $2k$. (Odd kink numbers [20] characterize solutions which are not time orientable. Such solutions are not considered here).

The physical relevance of solutions with nontrivial winding number is not quite clear. They necessarily contain closed lightlike curves. There are, however, also solutions with trivial winding number containing closed lightlike curves. As outlined, in a conventional Hamiltonian treatment of the action (43) the constraints will generate infinitesimal gauge transformations rather than gravitational symmetry transformations. Thus on the Hamiltonian level the kink number will not appear in the parametrization of the reduced phase space, while, however, not all solutions with closed timelike curves can be excluded in this way. A similar situation occurs also when treating other models of two dimensional gravity contained in (32). It would be interesting to see, if the equivalence up to $\det g = 0$ of the Hamiltonian and Lagrangian formulation of four dimensional gravity leads to similarly inequivalent factoring spaces.
5 A Model for Quantum Gravity

In the previous sections we found that a large class of Hamiltonian systems, including gravitational ones, can be reduced to quantum systems of finitely many topological degrees of freedom. The question arises: Can such models serve as toy models for a quantum theory of four dimensional gravity? Indeed even in the absence of local degrees of freedom an illustrative treatment of some conceptual questions of quantum gravity is possible. Most prominent among these is the so called 'problem of time' [11], which we shall take up in this section for the example of $R^2$-gravity with Minkowski signature coupled to $SU(2)$ Yang Mills.

The Lagrangian of this system is

$$S = \int_{S^1 \times R} \left[ \frac{1}{8 \beta^2} R_{ab} \wedge * R^{ab} + \frac{1}{4 \gamma^2} tr(F \wedge * F) \right]$$  \hspace{1cm} (62)

where the Hodge dual operation is, in contrast to (1), performed with the dynamical metric used to define also the torsionless curvature two-form $R_{ab}$ and the trace is taken, e.g., in the fundamental representation of $su(2)$ (the generators $T_i$, fulfilling (44) with $\kappa_{ij} = \delta_{ij}$, are then represented by $T_i = -i \sigma_i/2$, which yields $\kappa_{ij} = -2i \epsilon_{i j} T_j$ now). Rewriting (62) by means of Cartan variables $(\omega^a_b \equiv -\epsilon^a b \omega, \epsilon^a)$ in a Hamiltonian first order form, it becomes

$$S_H = \int_{S^1 \times R} B_a De^a + B_3 d \omega + tr(EF) + [-\beta^2 (B_3)^2 + \gamma^2 tr(E^2)] \epsilon$$  \hspace{1cm} (63)

where we have chosen $E = E^2 T_3$ to denote the 'electric fields' conjugate to the $SU(2)$-connection one-components $A_1$, and the $B$'s are the conjugates to the spin connection $\omega_1$ and the zweibein one-components $e_l^a \equiv (e_l^-, e_1^+).$ (Our conventions are $\epsilon^\pm = (\epsilon^2 \pm \epsilon^1)/\sqrt{2}$, yielding a light cone frame metric $\kappa_{+ -} = 1$, whereas $\epsilon = \epsilon^1 \wedge \epsilon^2 = \epsilon^- \wedge \epsilon^+ \,$ so that $\epsilon^{a b} = \epsilon_{a b} = 1$). Obviously $S_H$ is the sum of an $SU(2)$-$E$-theory (43) (up to a factor $-2$) and an action $S_G$ (32) with $w_3 = -\beta^2 (B_3)^2 + \gamma^2 tr(E^2)$. In explicit terms the constraints following (naturally) from $S_H$ are

$$G_a = \partial B_a + \epsilon^a_l B_l \omega_1 \mp \kappa_{ab} [\beta^2 (B_3)^2 + \gamma^2 tr(E^2)] e_1^b,$$  \hspace{1cm} (64)

$$G_3 = \partial B_3 + \epsilon^3_l B_l e_1^b,$$  \hspace{1cm} (65)

beside the unmodified $SU(2)$ Gauss law $G \equiv 0.$ We will not attempt to reformulate these constraints so as to possibly cure the global deficiencies of them with respect to diffeomorphisms noted at the end of the previous section. Instead we proceed with a straightforward quantization.

There are two independent Dirac observables as functions of the momenta $(q(a) \equiv \int Q(a) dx^1/2\pi)$

$$q_{(1)} = \frac{-1}{\pi} \int tr(E^2) dx^1 \equiv \frac{1}{2\pi} \int E_i E_i dx^1$$

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\[ q_{(2)} = \frac{1}{2\pi} \int [(B)^2 - \frac{2}{3}\beta^2(B_3)^2 + 2\gamma^2 tr(E^2)B_0]dx^1. \]

The corresponding level surfaces have topology \( S^2 \times R^2 \) for \( q_{(1)} \neq 0 \) and \( R^2 \) for \( q_{(1)} = 0 \). This gives rise to the quantization condition (cf. end of sec. 3):

\[ q_{(1)} = n^2/4, n \in N_0. \]

Expanding the physical wave functionals in terms of eigenfunctions of \( q_{(1)} \), the corresponding coefficients are

\[ \exp \left( \frac{i}{\hbar} \int (E_3 \partial \varphi \pm \ln B_\pi \partial B_3 dx^1) \right) \tilde{\Psi}_n(q_{(2)}), \quad n = 2\sqrt{q_{(1)}} \in N_0, \quad q_{(2)} \in R, \]

where we have written the phase factor in some local target space coordinates with \( \tan \varphi \equiv (E_2/E_1) \). The inner product with respect to \( q_{(2)} \) is determined by the hermiticity requirement on

\[ p_{(2)} = -\frac{1}{2} \int \frac{e_1^+}{B_\pi} dx^1, \]

the Dirac observable conjugate to \( q_{(2)} \): as noted already in section 3, \( p_{(2)} \) acts as the usual derivative operator on \( \tilde{\Psi}_n \), thus leading to the ordinary Lebesgue measure \( dq_{(2)} \).

We end up with the Hilbert space \( \mathcal{H} \) of an effective two-point particle system with nontrivial phase space topology. As a basic set of operators acting in \( \mathcal{H} \) we could use \( q_{(2)}, p_{(2)}, q_{(1)}, \) and \( tr[\mathcal{P} \exp(\int A_{\parallel} dx^1)] \). From the latter one may construct a ladder operator \( l : n \rightarrow n + 1 \).

All operators acting in \( \mathcal{H} \) are thus found to be expressible in terms of \( q_{(2)}, p_{(2)}, \) and the number and ladder operators. However, we do not have an operator such as \( g_{\mu\nu}(x^\lambda) \). Following furthermore, any textbook on elementary quantum mechanics, the next step in the quantization procedure would be to introduce an evolution parameter 'time', which we will call \( \tau \), and to require the wave functions to evolve in this parameter according to the Schrödinger equation. In the present case, however, the Hamiltonian following from (63) is a combination of the constraints,

\[ H = -\frac{\hbar}{2} \int \left[ \epsilon_{0}^6 G_3 + \omega_{0} G_3 + tr(A_{\parallel} G) \right], \]

so that the naive Schrödinger equation becomes meaningless.

Both of these items, the nonexistence of space-time dependent quantum operators as well as the apparent lack of dynamics, are correlated and they are not just a feature of the topological theory (62). Also in four dimensional gravity the quantum observables are some (not explicitly space-time dependent)

\footnote{Within the latter level surface the origin is an integral surface by itself. We will in the following disregard this small complication. As suggested already through the chosen notation we assume \( \beta^2 \) and \( \gamma^2 \) to be non negative.}
holonomy equivalence classes and the Hamiltonian vanishes when acting on physical wave functions [21]. Diffeomorphisms are part of the symmetries of any gravity theory; as a consequence the Lie derivative into any ’spatial’ direction can be found to equal the Hamiltonian vector field of some linear combination of the constraints (in our case \( \mathcal{L}_1 = \epsilon_1 i^a G_a + \omega_1 G_3 + t \alpha_1 A_1 \)), whereas, on shell, \( x^i \)-diffeomorphisms will be generated by the Hamiltonian \( H \). Thus, although 4D gravity has local degrees of freedom, any of its (uncountably many) Dirac observables will be also space-time independent.

To orientate ourselves as of how to introduce quantum dynamics within such a system, let us have recourse to the simple case of a nonrelativistic particle (NRP). As is well known, any Hamiltonian system can be reformulated in time reparametrization invariant terms. In the case of the NRP,

\[
\int \left( p \frac{dq}{dt} - \frac{p^2}{2} \right) dt = \int \left( p \dot{q} - \frac{p^2}{2} \right) d\tau,
\]

the equivalent system has canonical coordinates \((q, t; p, p_t)\) and the ’extended’ Hamiltonian is proportional (via a Lagrange multiplier) to the constraint \( C = p^2/2 + p_t \approx 0 \). Quantizing this system, e.g., in the coordinate representation, we observe that the implementation of the constraint \( C\psi(q, t) = 0 \) is equivalent to the Schrödinger equation of the original formulation, if one reinterprets the canonical variable \( t \) as evolution parameter \( \tau \). Therefore, given this formulation of the NRP or similarly of any other system, the postulate of a Schrödinger equation within the transition from the classical to the quantum system becomes superfluous; rather it is already included within the Dirac quantization procedure in terms of a constraint equation.

The identification \( t = \tau \) above can be looked upon also as a gauge condition with gauge parameter \( \tau \). This interpretation is helpful for the quantization of the parametrization invariant NRP in the momentum representation, in which case the space of physical wave functions is isomorphic to the space of functions of the Dirac observable \( p \). The gauge condition \( \bar{C} \equiv t - \tau = 0 \) provides a perfect cross section for the flow of \( C \). Thus it is possible to determine any phase space variable in terms of the Dirac observables \( p, Q = q - pt \), as well as the gauge fixing parameter \( \bar{C} \). Interpreting \( \tau \) as a dynamical flow parameter ’time’, the obtained evolution equations for \( p \) and \( q \), transferred to the quantum level as \( \bar{q}(\tau) = i\hbar d/dp + \tau p, p(\tau) = p \), become equivalent to the Heisenberg evolution equations of the parametrized NRP.

The operator \( \bar{q}(\tau) \) above corresponds to a measuring device that determines the place of the particle at time \( \tau \). A measuring device that determines the time \( t \) at which the particle is at a given point \( q = q_0 \), on the other hand, corresponds to the alternative gauge condition \( \bar{C} \equiv q - q_0 = 0 \). \( \bar{C} \) provides a good cross section only for \( p \neq 0 \). Ignoring this subtlety, e.g., by regarding only wave functions with support at \( p \neq 0 \), the (hermitian) quantum operator for such an experiment is \( \bar{t}(q_0) = -i\hbar [(1/p)d/dp - (1/2p^2)] + q_0/p \). In this second experimental setting Heisenberg’s ’fourth uncertainty relation’ between
time $t$ and energy $p^2/2 \sim -p_t$, usually motivated only heuristically, becomes a strict mathematical equation. We learn that different experimental settings are realized by means of different gauge conditions, and, at least in principle, vice versa.

The wave functions of (63) are basically functions of the Dirac observables, although part of the latter became discretized in the quantum theory. Transferring the ideas above to the gravity system, we should condition gauge conditions to the constraints (64, 65). (It will not be necessary to gauge fix also $G$. As such we will choose

$$\partial B_k = 0, \quad B_3 + \tau B_4 = 0, \quad e_1 \tau = 1.$$  \hspace{2cm} (70)

It is somewhat cumbersome to convince oneself that this is indeed a good gauge condition. However, for $q_{(1)} \neq 0$ it provides even a globally well-defined cross section. The gauge conditions together with the constraints allow to express all gravity phase space variables in terms of Dirac observables. In this way one obtains evolution equations such as

$$B_{-}(\tau) = -\frac{1}{2\pi} p_{(2)} q_{(2)} - \frac{\gamma^2}{2} q_{(1)} \tau - \frac{\beta^2 \pi^2}{3(p_{(2)})^2} \tau^3, \quad B_{+}(\tau) = \frac{\pi}{p_{(2)}},$$  \hspace{2cm} (71)

Antisymmetrizing this with respect to $q_{(2)}$ and $p_{(2)}$, (71) can be taken as an operator in the Hilbert space $\mathcal{H}$ defined above. Similarly one finds $g_{11}(x^b) = 2\epsilon_1 \epsilon_2(x^b) = -p_{(2)} B_{-}(x^b)/\pi$, $(x^b \equiv \tau)$, which now, up to operator ambiguities, becomes a well defined operator in our small quantum gravity theory, too.

Requiring that the $\tau$-dependence of (70) is generated by the Hamiltonian $H$, the gauge conditions determine also the zero components of the zweibein and the spin connection. Actually, one zero mode of these Lagrange multipliers field remains arbitrary as a result of the linear dependence of the constraints $G_i$ (cf. also [13]). Requiring this zero mode to vanish as a further gauge condition, one finds $e_0 = 1$ and $e_i \tau = \omega_i \equiv 0$. In other gauges the Lagrange multipliers can become also non trivial quantum operators. Furthermore, it is a special feature of the chosen gauge that the obtained operators are $x^1$-independent. (The existence of this gauge shows that $B_3 = \text{const}$ is an isometry or Killing direction of the metric). Again different choices of gauge conditions are interpreted as corresponding to different types of questions or measuring devices.

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8One possibility to check the obtainability of (70) is to carefully analyse the Faddeev matrix, taking into account that due to $\int \delta Q dx^3 \equiv 0$ and (11, 37) the gravity constraints are not completely linearly independent. This (infinite dimensional) matrix turns out to be nondegenerate, iff $B_+ \int \delta_1 \tau dx^3 \neq 0$. For $q_{(1)} \neq 0$ any gauge orbit in the loop space contains a representative fulfilling this condition, which suffices to prove the assertion since the space of gauge orbits is connected in the case under study (no quantum number $n_0$).

9The elementary procedure above coincides with the use of Dirac brackets for $\tau$-dependent systems (in which case one extends the symplectic form by $dx \wedge dp_{(2)}$); this explains also $B_{-}$ and $B_{+}$ do not commute anymore.
The alternative, at least for the parametrization invariant NRP equivalent procedure to reintroduce time within the quantum theory was the direct implementation of the gauge in the wave functions. For this it was decisive that the initially chosen polarization of the wave functions contained the phase space variable subject to the gauge. To implement (70) analogously within the gravity theory under consideration, we Fourier transform (66), multiplied by $\delta[\partial Q(2)]$, with respect to $B_\perp(x^1)$. The result is

$$\exp\left(\frac{i}{\hbar}\int E_\partial \partial \varphi + \frac{\partial B_\perp B_\parallel + \frac{[\beta_\perp^2 (B_\parallel)^3 - \gamma^2 \text{tr}(E^2) B_\parallel]}{B_\parallel}}{\text{d}x}^1\right)\Pi_x (\text{const}) \hat{\Psi}_n(p(2)),$$  

(72)

in which $\hat{\Psi}_n$ is the Fourier transform of the ordinary function $\hat{\Psi}_n$. Eq. (72) certainly is in agreement with the general solution of the quantum constraints in a $(B_\perp, B_\parallel, c_1^-, E)$ representation. In the gauge (70) the quantum wave functions take the form

$$\sum_n \exp\left[-\frac{i}{\hbar}\left(\frac{\gamma^2 n^2}{8} \tau + \frac{\beta^2 \pi^2}{\delta p(2)} \tau^3 \right)\right] c_n(p(2))|p\rangle,$$  

(73)

where $|n\rangle$ denotes the eigenfunctions of $q_{(1)}$ (inclusive $\exp[(i/\hbar) \int E_\partial \partial \varphi \text{d}x^1]$) and we have reabsorbed the divergent factor of (72), being a function of $p(2)$, into $c_0(p(2))$.

At this point the case $\beta = 0$ is of special interest: for it $S_H$ is seen to describe a Yang Mills theory coupled to a flat metric. Thus in some sense it is the parametrization (i.e. diffeomorphism) invariant formulation of the usual Yang Mills theory on the cylinder (with rigid Minkowski background metric). If we ignore the $p(2)$ dependence of $c_0$ for a moment, (73) with $\beta = 0$ indeed coincides with the time evolution generated by the (nonvanishing) Yang Mills Hamiltonian $-\gamma^2 \frac{1}{2} \text{tr} E^2 \text{d}x^1 \equiv \gamma^2 \pi q_{(1)}$. This agreement gives support to the method used to derive (73).

The reason for the $p(2)$-dependence of $c_0$ is due to the fact that in the formulation (63) with $\beta = 0$ the metric induced circumference of the cylinder became a dynamical variable (on shell one has $p(2) \propto \frac{1}{B_\perp \text{const}} \sqrt{\gamma^2} \text{d}x^1$).

Within (70) one finds $-\frac{1}{2} G_\perp \sim H$ to effectively implement the Schrödinger equation corresponding to (73). The effective Hamiltonian acting on $c_n(p)$ is $-(\gamma^2/2) \frac{1}{2} \text{tr} E^2 \text{d}x^1 - \beta^2 \pi^2 |p(2)|$. Thus generically the above procedure yields time dependent Hamiltonians (cf. also [13]).

The strategies developed at the example of a NRP to resolve the 'issue of time' within a quantum theory of gravity produced sensible results for the toy model (62). They, however, relied heavily on either the knowledge of all Dirac observables or on some specifically chosen polarization. To cope with the considerable technical difficulties of a quantum theory of four dimensional gravity,
it might be worthwhile to extend the applicability of the method. One way to do so within our model is to allow for equivalence classes of wave functions coinciding at \( \partial Q(2) = 0 \), the latter condition being enforced within the inner product [13]. In this way one can, e.g., implement the gauge condition \( \partial e_1^- = 0 \) as an operator condition in the \( B \) polarization of the wave functions as well, whereas a straightforward implementation of \( \frac{1}{2} e_1^- = \text{const} \) seems again inadmissible.

References


