B B

- GANIL P 94-14
SCW 94 26

GANIL

CERN LIBRARIES, GENEVA

P00023986

NEW TRENDS IN KINETIC FRAGMENTATION THEORY

ROBERT BOTET

*Laboratoire de Physique des Solides, Bâtiment 510, Université Paris-Sud,
Centre d'Orsay, F-91405 Orsay, France*

and

MAREK PLOSZAJCZAK

*Grand Accélérateur National d'Ions Lourds (GANIL), BP 5027,
F-14021 Caen Cedex, France*

GANIL P 94 14

NEW TRENDS IN KINETIC FRAGMENTATION THEORY

ROBERT BOTET

*Laboratoire de Physique des Solides, Bâtiment 510, Université Paris-Sud,
Centre d'Orsay, F-91405 Orsay, France*

and

MAREK PLOSZAJCZAK

*Grand Accélérateur National d'Ions Lourds (GANIL), BP 5027,
F-14021 Caen Cedex, France*

Abstract

Recent theoretical developments in the kinetic fragmentation theory are presented with a particular emphasis of the newly proposed Fragmentation-Inactivation Binary (FIB) cascading model. We also discuss the origin of scale-invariant, intermittent fluctuations in the cluster fragmentation models such as the percolation model and the binary cascading models.

1. Introduction

Fragmentation occurs in so many fields of the Physics that it is meaningless to try to make an exhaustive list of its applications, moreover we shall see some of them later. Here let us cite only : polymer degradation[1] , rock crushing[2] and breakup of heavy nuclei[3] or liquid droplets[4] , to get some concrete examples in mind.

The dynamics of nuclear multifragmentation in heavy-ion collisions is an example of a situation where a discussion of both quantum effects and complexity due to multiparticle correlations may be essential. In heavy-ion collisions at beam velocities of the order of the average velocity of nucleons in the nucleus at rest, the system created may break up into several nuclei of intermediate mass (see Fig. 1). Whatever the underlying mechanism , its time-scale is of the same order as the typical time-scales of the system. Hence, it is possible that the concepts of thermodynamics cannot be applied and the process has to be analyzed in terms of the off-equilibrium dynamics.

Fragmentation can be defined as the passage between one body characterized by an extensive quantity, say m and called its *size*, and a set of bodies of same nature but with smaller corresponding quantities : $m_1, m_2, \dots, m_n$. In the important example when this quantity is conservative, one must have :

$$m_1 + m_2 + \dots + m_n = m \quad ,$$

and this is the most widely studied case. But if this does not hold, the process is called non-conservative, and one has then the weaker constraint :

$$m_1 + m_2 + \dots + m_n < m \quad . . .$$

Moreover, this fragmentation break-up can be either sequential or instantaneous.

Fig. 1. The collision of a carbon ion of 1144 MeV in a nuclear emulsion. Some 17 fragments with charges between 1 and 5 have been identified [9].

The sequential case occurs when the fragments themselves are allowed to break up. This corresponds to a succession of small jumps in the space of all possible configurations, since the system changes just a little bit at each elementary break-up. On the contrary, instantaneous break-up means that the typical relaxation time of the system is much larger than the overall fragmentation time. This can be represented by a big jump in the configuration space. The distinction between these two cases should be done by investigating the microscopic physics responsible for the fragmentation in each particular example.

In the present review article, we are mainly interested by the sequential conservative fragmentation since it is quite general and, moreover, it is believed to yield *universal* features, i.e. characteristic behaviours which do not depend on the precise mechanism governing the fragmentation. On the other hand, instantaneous break-up is surely very specific of each microscopic physics and so no such universal behaviour is expected.

This idea of universality in the fragmentation processes comes from the observation that many experimental data can be fitted by a single simple law, independently of the sort of the relevant interactions or orders of scale. More precisely, fragmentation data are often fitted by a power-law of the size like :

$$n_s = s^{-\tau} ,$$

and the various values of the exponent τ are remarkably close for any studied

material. As some concrete examples, one can give :

- gabbro fragmentation with lead or steel projectiles [10] : $\tau = 1.48$ or $\tau = 1.57$,
- mid-size meteorites [11] : $\tau = 1.62$,
- crushed quartz, gneiss or granite[12] : $\tau = 1.63, 1.71$ or 1.74 ,
- granite fragmentation in underground chemical or nuclear explosions [13] : $\tau = 1.81$ or $\tau = 1.83$,
- broken coal[14] : $\tau = 1.83$,
- basalt fragmentation with polycarbonate projectiles [15] : $\tau = 1.85$,
- sandy clays[12] : $\tau = 1.87$,
- gravels[12] : $\tau = 1.94$,
- stony meteorites[16] : $\tau = 2.0$,
- asteroids[17] : $\tau = 2.02$.
- nuclear fragments in the heavy-ion collision with $0.5\text{GeV} \leq E_{lab}/A \leq 10\text{GeV}$ [18] : $\tau \simeq 2 + 0.2$.

These "coincidences" recall the history of the phase transitions and of the aggregation processes. Both were known for a long time to show scaling relations very similar for very different systems. For the phase transitions, this remark has given birth to the renormalization group method and the concept of universality classes. For the aggregation phenomena, it has stimulated the study of its relation with the self-organized criticality concept, which is a new and very active subject in statistical physics[19] .

For the fragmentation processes there have been three main theoretical approaches to such a problem. Of course, the precise limits between them are often fuzzy but for the clarity of the exposition, they will be separated. The first one is based on a statistical calculation at equilibrium, the second one relies upon the investigation of some kinetic rate equations, and the last one uses the molecular dynamics approach [20] . The pertinent physical quantities to study will be essentially : the *size*, which can be any extensive variable (e.g. the mass, the momentum, the energy, *etc.*) according to the example considered. In the following, the fragments will be often referred to as clusters of size m_k for the k^{th} fragment. Moreover, if there exists a minimum possible size ¹, any fragment of this minimum size will be called a *monomer*. In this way, one can imagine the fragments of size m as clusters of m monomers, to keep a simple picture in the mind. The physical quantities should be computed under the form of some probability distribution since we are mainly interested by break-up processes leading to a large number of fragments. Also for some applications it will be useful to study some moments of these distributions. Moreover, the time evolution of these probability laws will be investigated when possible.

¹This is not always the case, for instance if the "size" corresponds to the momentum.

To simplify a little bit, in all the following we suppose that either the system is formed with monodisperse clusters, i.e. clusters of the same size, or we work with always the same initial cluster. This is not a very drastic assumption. If this is not the case, all formulas depending on the size of the initial cluster should be averaged over the probability distribution of the initial sizes. This can be done always by a standard integration. But we shall not discuss here the influence of this initial distribution, which is believed to be unimportant if a typical initial size can be defined properly.

This paper presents a status of theoretical studies on fragmentation with a particular emphasis on new developments in the kinetic theories and on the studies of correlations and intermittency in the fragment -size distribution. Different aspects of, in particular, the nuclear fragmentation phenomenon are also discussed in some other recent reviews [5, 6, 7, 8]. In chapter 2 the statistical approach to the fragmentation is discussed through its dominant currents. After discussing basis elements of simple schematic models, such as the percolation model, Mekjian model or the Derrida - Flyvbjerg model, we present statistical sequential decay models and statistical multifragmentation models. In sect. 2.7 we explain the renormalization group ideas applied to the fragmentation process. Dynamical instabilities as the source of fragmentation are discussed in the last section of chapter 2. In chapter 3 we introduce the theory of kinetic rate equations and the newly proposed Fragmentation-Inactivation Binary (FIB) cascading model is discussed in the chapter 4. Fragment-size fluctuations and correlations are discussed in the chapter 5 on example of the Mekjian model, percolation model and some sequential decay models including FIB model. The main ideas and results are summarized in the chapter 6.

2. The statistical approach

2.1 *Introductory models for statistical fragmentation : Partitioning a natural number*

Partitioning a number N into integer summands is yet an old subject in number theory. Maybe the more impressive result is that we know exactly the number $P(N)$ of different partitions of any number, and the formula for large values of N is the Ramanujan-Hardy asymptotics [21] :

$$P(N) \sim \frac{\exp[2\sqrt{\zeta(2)N}]}{4N\sqrt{3}}. \quad (1)$$

Starting from this formula, we can build a simple model which will be called "additive" fragmentation model since a natural number is decomposed into sum of smaller natural numbers. Suppose now that all the partitions of a given number are equally probable. Then, if one makes a large number of experiments, consisting of computing a random partition of N , one finds for the inclusive distribution of the summands of magnitude s [22] :

$$\langle n_s \rangle = \frac{P(N-s) + P(N-2s) + \dots}{P(N)}.$$

The first term of this expansion gives the leading behaviour at large values of s and this means that the size-distribution decreases exponentially with the size [23] :

$$\langle n_s \rangle \sim \exp\left(-s\sqrt{\frac{\zeta(2)}{N}}\right) . \quad (2)$$

The "additive" fragmentation model can also be derived from the principle of minimal information subject to the condition of a total "mass" conservation [23] . This scheme can be generalized in the case where we constrain some partitions to be more probable than others. Every such variant lies within the "statistical approach" of the fragmentation processes of a discrete conservative variable. This way is mainly relevant for reversible fragmentation, when a system undergoes both fragmentation and aggregation, even if one aspect dominates the other.

Another sort of decomposition is the so called *multiplicative* fragmentation. Taking any natural number N , we choose randomly another number, say k , smaller than it and larger than 1 . Then either k is a divisor of N or not. If it is a divisor of N , then one can decompose the number N in a following way :

$$\ln(N) = \ln(k) + \ln(N/k) ,$$

since the fraction N/k corresponds to an integer number. This splitting of "fragments" can be continued furthermore. Of course, only very few numbers k are available for such a break. For example, a prime number cannot be splitted within this rule. The inclusive distribution of the "fragments" of size $s = \ln(k)$ is just given by :

$$\frac{\langle n_s \rangle}{N} = \frac{1}{k} + \frac{1}{k^2} + \frac{1}{k^3} + \dots ,$$

since N/k^m is the number of times one number between 1 and N is a multiple of k^m . We deduce that here again the size-distribution is exponential :

$$\langle n_s \rangle \sim N e^{-s} . \quad (3)$$

But contrary to the "additive" fragmentation, the normalized size-distribution $\langle n_s \rangle / N$ converges to an exponential distribution for large values of N . This behaviour comes from the fact that in this "multiplicative" fragmentation, some "clusters" or "fragments" are inactive, *i.e.* they are unable to break.

2.2 The Kuhn model

This method has been initiated by Kuhn [24] and Flory [25] in the thirties to study theoretically the depolymerization of long chain molecules for which many experimental results and applications are known : *e.g.* the chemical dissolution [26, 27] , the degradation through elongation [28] , nuclear, ultraviolet or ultrasonic irradiation [29] . Consider then some linear clusters of monomers. The size of a cluster is here just the number of monomers which form this cluster. Each pair of neighbouring monomers belonging to the same cluster are linked by bonds (chemical bonds in the case of the depolymerization). At last make the following assumptions

to describe how the break-up process holds for these clusters [30] : at any given time of the evolution, every bond present in the system has the same probability to break. Suppose now that you look at the fragmentation of an initial cluster of size N i.e. composed of N monomers. It contains $N - 1$ bonds, so if $M - 1$ bonds are broken there are :

$$\binom{N - 1}{M - 1}$$

different possible breaks. And all these breaks decompose exactly into

$$\frac{M!}{\prod_j n_j!} \quad (4)$$

different configurations corresponding to :

n_1 clusters of size 1 , n_2 clusters of size 2 , etc.

$$\begin{aligned} \sum_j n_j &= M \\ \sum_j j n_j &= N \end{aligned} \quad (5)$$

Now, the average size-distribution corresponding to a given value of the number $M - 1$ of broken bonds is given by :

$$\left(\binom{N - 1}{M - 1} \right)^{-1} \sum_{\{n_j\}} n_s \frac{M!}{\prod_j n_j!} \quad , \quad (6)$$

where the sum is over all states $\{n_1, n_2, \dots\}$ verifying (5). This summation can be done using the multinomial formula and we get the following simple closed form [31] :

$$\langle n_s \rangle = N \frac{\binom{N - s - 1}{M - 2}}{\binom{N}{M}} \quad , \quad (7)$$

which behaves like $\exp(-s \frac{M}{N})$ when $1 \ll s \ll N$. At last, and this is the key point of this calculation, the values of $M - 1$ are themselves randomly distributed according to a binomial law with a parameter q which is the probability for a given bond to be cut. The average of the size-distribution is finally :

$$\langle n_s \rangle = \begin{cases} q(1 - q)^{s-1} [2 + (N - s - 1)q] & \text{if } s \leq N - 1 \\ (1 - q)^{N-1} & \text{if } s = N \end{cases} \quad (8)$$

For large values of N , this distribution decreases exponentially as :

$$N \frac{q^2}{1 - q} e^{-\alpha s} \quad (9)$$

with the parameter $\alpha = -\ln(1 - q)$. This is the classical Kuhn result[24]. In this approach, the probability q plays the role of the time. This is generally not a direct observable and, at the place of the time or of p , it is often more convenient to use the average reduced molecular weight m_n [32] defined as :

$$m_n = \frac{\langle \sum_j j n_j \rangle}{\langle \sum_j n_j \rangle} = \frac{N}{1 + q(N - 1)} \quad (10)$$

This quantity lies always between 1 and N and decreases as the degree of depolymerization (or equivalently : the time) goes on. m_n is also a natural average size for the system. Note the very simple relation between m_n and q , which is here independent of the hypothesis that the locations of breaks are purely random inside the clusters. This is no more the case for other moments of the size-distribution, like the weight average reduced molecular weight for example which in this model is given by :

$$m_w = \frac{\langle \sum_j j^2 n_j \rangle}{\langle \sum_j j n_j \rangle} = \frac{2 - q}{q} - 2 \frac{1 - q}{N q^2} [1 - (1 - q)^N] \quad (11)$$

These two quantities m_n , m_w are widely studied in the physics of polymers because they occur naturally in some important physical processes, such as the osmotic pressure[33], the vapor pressure, *etc.* for m_n , and the viscosity or the light scattering methods for m_w [30]. But even when we know how to measure the whole size-distribution, these are often very useful tools to extract relevant informations. The magnitude of m_w is always larger than m_n and their ratio characterizes the width of the size-distribution. These two average molecular weights have been written here for the case when we are interested by a mixture of a large amount of fragmenting clusters (initially monodisperse) like the standard polymeric systems. If we work with fragmentation of individual clusters, like in the studies of nuclear fragmentation for example, the relevant average is no more m_n but :

$$m_n' = \langle \frac{\sum_j j n_j}{\sum_j n_j} \rangle = \frac{1}{q} [1 - (1 - q)^N] \quad (12)$$

This q - dependence is calculated easily because it is just the average of the inverse of the multiplicity. In this simple model, the behaviours of m_n and m_n' are quite similar but they can differ noticeably in more complicated cases.

2.3 The Mekjian model

The derivation of the size-distribution in the Kuhn model has been possible because the total number of fragments resulting from one given cluster had a simple probability law (binomial). In any other cases, the calculations are much more difficult. Nevertheless, some alternative models of fragmentation can be built with similar ideas.

At the beginning, let us consider statistical weights :

$$\prod_j \frac{g_j^{n_j}}{n_j!} \quad (13)$$

which are attached to every possible partition $\{n_1, n_2, \dots\}$ of the number N . The parameters g_j measure the preference for a given size to be created by the break-up process. In eq. (13) we have not written multiplicative prefactor which depends on the total size $N = \sum_j j n_j$ as well as on the total multiplicity $M = \sum_j n_j$. These should be calculated by proper normalization of the weights.

A combinatorial theorem on formal polynomials, due to Fine [34] :

$$\prod_{l \geq 1} \left(\sum_{n \geq 0} a_l(n) x^{n_l} y^n \right) = \prod_{l \geq 1} a_l(0) + \sum_{N \geq 1} \sum_{M=1}^N \left(\sum_{\pi(M, N)} \prod_{l \geq 1} a_l(n_l) \right) x^N y^M, \quad (14)$$

in which the last sum is over all possible partitions (5) of N realizing a given value of the multiplicity M , allows to write a simple equation for the average size-distribution for fixed N [35] :

$$\langle n_s \rangle = g_s \frac{Z(N-s)}{Z(N)}, \quad (15)$$

where the generating function for the partition function $Z(N)$ writes :

$$\sum_{N \geq 1} Z(N) u^N = \exp \left(\sum_{l \geq 1} g_l u^l \right) - 1. \quad (16)$$

If one knows a closed form of the right-hand side member of the preceding equation, the value of $\langle n_s \rangle$ is given automatically by the eq. (15). The first choice studied by Mekjian was [36] :

$$g_j = \frac{x}{j}, \quad (17)$$

where x is a free parameter in this model. When replacing g_j by their values in eqs. (15-16), one finds :

$$\langle n_j \rangle = \frac{\Gamma(N+1)}{\Gamma(N+x)} \frac{\Gamma(N+x-s)}{\Gamma(N+1-s)} \frac{x}{s}, \quad (18)$$

where Γ is the Euler gamma function. This is a pure power law for the particular value $x = 1$. Close to this value, the size-distribution shows an exponential correction :

$$\langle n_s \rangle \sim \frac{1}{s} \exp \left[\frac{s}{N} (1-x) \right]. \quad (19)$$

Note here the very different behaviour for x larger than 1 (the distribution decreases more rapidly than a power law for large sizes) and x smaller than 1 (where this distribution increases for large sizes).

Eqs. (15) and (16) can be studied more carefully when averaging over all the configurations of fixed values of both N (the total size) and M (the total multiplicity), instead of keeping just N constant. In this case a strange phenomenon appears. Eq. (15) must now be replaced by [35] :

$$\langle n_s \rangle_M = g_s \frac{Z(N-s, M-1)}{Z(N, M)}, \quad (20)$$

and (16) by :

$$\sum_{N=M}^{\infty} Z(N, M) u^N = \frac{1}{M!} \left(\sum_{l \geq 1} g_l u^l \right)^M . \quad (21)$$

The average size-distribution for a given multiplicity M in the Mekjian model is then [36, 37, 38] :

$$\langle n_s \rangle_M = \frac{1}{s} \frac{\Gamma(N+1)}{\Gamma(N+1-s)} \frac{|S_{N-s}^{(M-1)}|}{|S_N^{(M)}|} , \quad (22)$$

where $S_N^{(M)}$ are the Stirling numbers of the first kind. We remark immediately that this distribution is exactly independent of the free parameter x . This means that the x -dependence of the results (18) and (19) are due to a distribution $P_N(M)$ of the multiplicities which depend strongly on the value of x , namely :

$$P_N(M) = x^M |S_N^{(M)}| \frac{\Gamma(x)}{\Gamma(N+x)} . \quad (23)$$

This also can be seen qualitatively when inspecting the role of x in the definition of the statistical weights (13). This parameter appears just as x^M and so, it acts not on the whole distribution but only on the magnitude of the multiplicity. A small value of x favors small multiplicities and vice-versa. A more detailed argument can be done for this phenomenon. Using the identity :

$$\frac{1}{s} = \frac{1}{a} + \sum_{k=1}^{\infty} \frac{\Gamma(a)}{\Gamma(a+k+1)} (a-s)(a+1-s) \dots (a+k-1-s) ,$$

applied to :

$$a = N + x ,$$

one has the average multiplicity as the simple formula :

$$\sum_{s=1}^N \langle n_s \rangle = \sum_{k=0}^{N-1} \frac{x}{x+k} , \quad (24)$$

where the behaviour with x is clear.

The quantities m_n and m_w introduced in the Kuhn model can be explicitly calculated in the Mekjian model as well. Using the identity :

$$\sum_{s=1}^N \frac{\Gamma(N+x-s)}{\Gamma(N+1-s)} = \frac{N}{x} \frac{\Gamma(N+x)}{\Gamma(N+1)} ,$$

one finds easily :

$$m_n = N \left(\sum_{k=0}^{N-1} \frac{x}{x+k} \right)^{-1} \quad (25)$$

for m_n , and :

$$m_w = \frac{N+x}{1+x} \quad (26)$$

for m_w . Note that these two quantities are smooth functions of x . Like in the previous model, m_n can be used as a simple natural parameter to quantify the degree of fragmentation of the system.

Any other choice of the statistical weight (instead of (17)) defines another model. For example, if the weights of each configuration have a constant value independent of the configuration, one has the "additive" fragmentation model of a natural number as described in the sect. 2.1, and the result (2) holds : i.e. the exponentially decreasing average size distribution . This sort of exponential behaviour seems to be very general for this kind of models, and so the $x = 1$ case of the Mekjian model is quite particular.

Let us solve another case very close to it. Take the statistical weights of the form :

$$g_i = x \quad (27)$$

instead of (17). Then after replacing all the g_i 's by x in (15-16), one finds the average size-distribution[38] :

$$\langle n_s \rangle = x \frac{N}{N-s} \frac{L_{N-s-1}^{(1)}(-x)}{L_{N-1}^{(1)}(-x)} , \quad (28)$$

where $L_N^{(1)}(x)$ is the generalized Laguerre polynomial : $-dL_{N+1}(x)/dx$. A standard asymptotic expansion gives now the large - N behaviour :

$$\langle n_s \rangle \sim \exp\left(-s\sqrt{\frac{x}{N}}\right) , \quad (29)$$

which is still an exponential function of s . There is a great similarity with the "additive" fragmentation (see (2)) , in particular the size scale $\sqrt{N}$ which appears in both cases.

The Kuhn model and the latter case (27) of the Mekjian model are closely related. Since the statistical weights (13) with $g_j = 1$ correspond to (4) without the multiplicity constraint (5), the average of the size -distribution (7) according to a binomial law for $M-1$ gives the Kuhn model and a similar average according to a uniform distribution of $M-1$ gives (28) with $x = 1$.

2.4 The reversible fragmentation models

In many physical applications, fragmentation is in competition with aggregation. If fragmentation is dominant the previous models hold. But if an equilibrium is reached, this implies some precise constraints on the parameters of the model. The number of ways to produce the size-distribution $\{n_1, n_2, \dots\}$, submitted to the conditions :

$$\sum_s n_s = M$$

$$\sum_s s n_s = N, \quad (30)$$

is [39] :

$$N! \prod_s \frac{g_s^{n_s}}{n_s! (s!)^{n_s}}, \quad (31)$$

since this is the number of different ways of partitioning N monomers into n_1 subsets with monomers, n_2 subsets of two monomers, etc. multiplied by the number of ways $g_1^{n_1} g_2^{n_2} \dots$ of assembling the subsets of monomers into polymers. At this stage, one can find the most probable distribution n_s^* by maximizing the statistical weights with the two Lagrange multipliers corresponding to the constraints (30) on both the mass and the multiplicity. The classical result is [39, 40] :

$$n_s^* = \frac{g_s}{s!} e^{\gamma - \beta s}, \quad (32)$$

where the constants β and γ are found by normalization. Suppose now that this distribution at equilibrium has been the result of the aggregation with rate K_{ij} ($\equiv$ probability that two clusters of size i and j merge) and of the fragmentation with the rate F_{ij} ($\equiv$ probability that one cluster of size $i+j$ splits into one fragment of size i and another one of size j)[41]. Then the detailed balance condition writes [42] :

$$K_{ij} n_i^* n_j^* = F_{ij} n_{i+j}^*. \quad (33)$$

This means that the fragmentation rates can be expressed as a function of both the aggregation kernel and the weight coefficients g_i [43] :

$$F_{ij} = \frac{g_i g_j}{g_{i+j}} \frac{(i+j)!}{i! j!} K_{ij} \quad (34)$$

This is one condition for equilibrium to be possible.

2.5 The Derrida - Flyvbjerg model

A quite different statistical fragmentation model has been introduced to study the fragmentation of the phase-space of a disordered medium [44]. Since it is easier to describe this model in a continuous version, we shall consider now the fragmentation of an object of size 1. The break-up scheme is as follows : the initial cluster splits into two fragments, one with size x_1 and the other with size $1 - x_1$. One of them, say the object of size x_1 becomes inactive, i.e. it cannot break up any more. On the other hand, the fragment of size $1 - x_1$ breaks into one piece of size $(1 - x_1)x_2$ and another one of size $(1 - x_1)(1 - x_2)$. The fragment of size $(1 - x_1)x_2$ is then inactive, while the other breaks up, and so on. The size distribution of one event is then :

$$\{x_1, (1 - x_1)x_2, (1 - x_1)(1 - x_2)x_3, \dots\}.$$

The average size-distribution can be calculated by an integral equation. In effect, the averaged sum of the sizes of all s -clusters is just the sum of the size s of the

cluster at the first step of the fragmentation if $x_1 = s$, and of the sum of the sizes of all the clusters of sizes $s/(1-x_1)$ appearing in the sequence of fragmentation of the cluster of size $1-x_1$. If $\rho(x)$ is the probability distribution of the numbers x_j , one has the equation :

$$s < n_s > = s\rho(s) + \int_0^{1-s} \rho(x_1) \frac{s}{1-x_1} < n_{\frac{s}{1-x_1}} > dx_1 . \quad (35)$$

This can be solved for the class of functions :

$$\rho(x) = (\alpha + 1)(1-x)^\alpha ,$$

and $\alpha > -1$. Moreover, this multiplicative process is easy to simulate numerically and any averaged quantities of interest can be calculated straightforwardly. A canonical example is the case $\rho(x) = 1$ which corresponds to a random breaking with a uniform distribution. We obtain in this case the power law distribution :

$$< n_s > = \frac{1}{s} .$$

But the main interest of this model is that one can find analytical informations about quantities which are not commonly studied, like the probability distribution $P_1(s)$ of the largest fragment. To compute this distribution, one can note first that if x_1 is larger than $1/2$, this is certainly the size of the largest fragment. So :

$$P_1(s) = \frac{1}{s} \quad \text{if } s > \frac{1}{2} .$$

If there is a fragment of size s between $1/3$ and $1/2$, this can be the largest one or the second largest one. So :

$$P_1(s) + P_2(s) = \frac{1}{s} \quad \text{if } \frac{1}{3} < s < \frac{1}{2} ,$$

where $P_2(s)$ is the probability distribution that the second largest fragment has size s . And so on :

$$P_1(s) + P_2(s) + \dots + P_n(s) = \frac{1}{s} \quad \text{if } \frac{1}{n+1} < s < \frac{1}{n} .$$

Each distribution $P_n(s)$ can be obtained by solving an integral equation analogous to (35) and knowing all the $P_m(s)$ for $m < n$. One finds for example [44] :

$$\begin{aligned} P_1(s) &= \frac{1}{s} \quad \text{if } s > \frac{1}{2} \\ P_1(s) &= \frac{1}{s} - \frac{1}{s} \ln \left(\frac{1-s}{s} \right) \quad \text{if } \frac{1}{3} < s < \frac{1}{2} \\ \dots &= \dots , \end{aligned} \quad (36)$$

and we see here that the point $s = 1/2$ is singular. The right derivative at this point is equal to -4 when its left derivative is $+4$. There are such singular bumps in $s = 1/n$ for each integer n . They decrease in magnitude with $1/n$. There are

also such singularities for the distribution of the second largest fragment and the distribution of the second moment of the size-distribution : $\sum_s s^2 n_s$.

Such singularities are the consequence of both the conservation of the total size and the multiplicative structure of the process. So they are expected to hold in every fragmentation satisfying these two conditions.

2.6 The percolation model

The Kuhn model can be generalized to the case where the fragmenting initial cluster is more complicated in shape than just a chain without branching. The first step in complexity is to suppose that the cluster is a part of a Bethe lattice of the coordination number z . Each node is then connected to z neighbours and there are no loops. A particular case is $z = 2$, where the Bethe lattice is identical to a linear graph ². We must recover the results of the Kuhn model for this value of the parameter z . Consider then an infinite Bethe lattice of the coordination number z . We will normalize in the following the number of clusters by the total number of sites to get finite quantities in the limit when the total mass goes to infinity. We assume that each node on the Bethe lattice is occupied by one monomer and all the bonds are present. Now cut randomly the bonds with a given probability q . The exact size distribution of the fragments which has been derived by Fisher and Essam, is given by [45] :

$$\langle n_s \rangle = \frac{2}{s[(z-2)s+2]} \binom{(z-1)s}{s-1} (1-q)^{s-1} q^{(z-2)s+1} \quad (37)$$

For $z = 2$, as necessary, we recover the result :

$$(1-q)^{s-1} q^2 \quad , \quad (38)$$

which is identical to eq. (9) when N is infinite. A careful study of the probability for a given site to belong to a finite cluster shows, that an infinite cluster which contains a finite proportion of the total number of monomers, exists if q is larger than a critical value :

$$q_c = \frac{z-2}{z-1} \quad . \quad (39)$$

The passage through this point can be proved to be equivalent to a phase transition, *i.e.* there is a sudden disaggregation of the infinite cluster when the probability q goes from a value slightly larger than q_c to a value slightly smaller than this critical value q_c . This transition can also be seen directly in the size-distribution. For large values of the size s , the behaviour of the average size-distribution can be written in a more pretty form, namely :

$$\langle n_s \rangle \sim s^{-5/2} \exp(-B(q, z)s) \quad , \quad (40)$$

²All the results concerning bond percolation on the Bethe lattice of coordination number z are written here for site clusters of size s . Since such an s -cluster has exactly $s-1$ bonds, one has to replace $s+1$ by s and σ by $z-1$ in ref. [45] , and $s+1$ by s in ref. [46] to recover the results as they are written here.

where

$$B(q, z) = \ln \left[\frac{1-q}{1-q_c} \left(\frac{q}{q_c} \right)^{z-2} \right] , \quad (41)$$

which is a power law with an exponent $-5/2$ corrected by an exponential function of the size. This exponential function disappears when the function $B(q, z)$ vanishes and, as it is easy to verify, this happens when $q = q_c$. So, we obtain a power-law distribution of the sizes only at the critical point[47]. On the other hand, the expression $B(q, z)$ is always positive if q is not equal to q_c . Hence, contrary to the Mekjian's model, one finds here that the size-distribution is decreasing faster than any power-law for every value of q other than q_c . But we have to remark that the eqs. (37-41) are the distributions for the finite clusters only. The largest one must be taken into account when it exists (i.e. for $q > q_c$). The number- and weight- averages of the size-distribution are also known in this model. They are equal respectively :

$$m_n = \frac{z}{z\kappa(q) + 2 - z} , \quad (42)$$

and :

$$m_w = \frac{z - (z-1)\kappa(q)}{2 - z + (z-1)\kappa(q)} , \quad (43)$$

where the function $\kappa(q)$ is given implicitly by the largest root of the equation :

$$(1 - \kappa(q))\kappa^{z-2}(q) = (1 - q)q^{z-2} \quad (44)$$

For $q \geq q_c$, such a solution is trivial :

$$\kappa(q) = q .$$

But for $q < q_c$, $\kappa(q)$ increases monotonically to reach the value 1 for $q = 0$. Moreover, near $q = q_c$, one has :

$$\kappa(q) \simeq q_c + |q - q_c| .$$

The critical behaviour is clear from the eq. (43) which shows a divergency of m_w when q approaches q_c . This quantity remains finite for $q > q_c$ because, in this case, it is calculated for all the clusters except the largest (the infinite) one.

All these features have also been studied on the ordinary lattice at d -dimension, and the behaviours are very similar even if there are usually no more exact analytical results. Anyway, the critical behaviour is easy to handle numerically and it has been shown that it is characterized by critical exponents depending only on the dimension of the space. The most important characteristic here is the behaviour of the average size-distribution, which writes [47, 48] :

$$\langle n_s \rangle \sim s^{-\tau} f\left(\frac{s}{s_\xi}\right) , \quad (45)$$

for q close to q_c . The quantity s_ξ in the above expression is the typical largest cluster size which diverges at the transition like :

$$s_\xi \sim |q - q_c|^{-1/\sigma} \quad (46)$$

The exponents τ and σ depend on the space dimension only and are respectively equal to $5/2$ and $1/2$ for the Bethe lattice (any $z > 2$) and to 2 and 1 for the linear chain ($z = 2$). These are respectively upper and lower bounds for the values of τ and σ in any other cases.

When we are dealing with fragmentation processes, we have often to consider finite systems. In this general case, the characteristic cluster size s_ξ cannot diverge since it is limited by the overall size N of the system, and so, it must be replaced, at the transition point, by some power of N [49] :

$$s_\xi \sim N^{D/d} \quad , \quad (47)$$

where d is the space dimension and D is an exponent characterizing the fractal dimension of the "infinite" cluster. For the Bethe lattice, D equals 4 , whereas for the linear chain D equals 1 . With this finite-size scaling, one finds the modified form of the size-distribution at $q = q_c$:

$$\langle n_s \rangle \sim s^{-\tau} f\left(\frac{s}{N^{D/d}}\right) \quad , \quad (48)$$

from which one may compute the moments of the size-distribution. The moment m_n in this case becomes independent of N for large systems and :

$$m_w \sim N^{\frac{D}{d}(3-\tau)} \quad . \quad (49)$$

for $z \neq 2$. When the mean-field is correct, the dimension d in the above formulas must be replaced by the value of the upper critical dimension d_c , which is defined as the smallest dimension of the space where the mean-field results hold. For the percolation model this happens for $d \geq d_c = 6$. The moments m_n and m_w grow with the size of the system as power-laws with exponents 0 and $1/3$ respectively for the Bethe lattice. These results are consistent with general finite-size scaling arguments which imply that for any singular quantity $A(q, N)$ [50] one finds :

$$A(q, \infty) \sim |q - q_c|^a \Leftrightarrow A(q_c, N) \sim N^{-a \frac{d_c}{d}} \quad , \quad (50)$$

and the space dimension d is replaced by the upper critical dimension for the corresponding mean-field system [51].

2.7 The renormalization group method

In the Introduction, we have remarked that the experimental size-distributions of strongly fragmenting systems are often well described by the power-laws of the sizes of the fragments, at least if natural cut-offs are neglected, the finite size of the whole initial object as the upper cut-off and the minimal size, if any, for the smallest cut-off. Between these two limits, the system does not feel any characteristic size

which should separate two defined regimes. This behaviour is well-known for critical phenomena, where the phase-transition destroys every scale and this is the precise reason of the occurrence of the power-laws in such phenomena. But conversely, to find a power-law is not an absolute proof of some kind of the phase transition and we have to be very careful on this point. In the models presented before, the Mekjian or Derrida-Flyvbjerg models are not critical, contrary to the percolation, though they can show also power-law size-distributions. An astonishing example has been given long time ago by Rayleigh [52] for the fragmentation of solid bodies. Take the case of two spheres impinging directly with a relative velocity equal to v . We assume that the occurrence of fracture depends only on a maximum strain, say e , and not of the duration of the impact. Then, according to the ordinary elastic theory of bodies, this strain, for one of the sphere, is only a function of its radius R , Young modulus E , Poisson ratio σ , its density ρ , and the relative velocity v . But since E/ρ has dimensions of the square of a velocity, and e as well as σ , have no dimensions, we deduce that :

$$e = e(E/\rho v^2, \sigma) \quad (51)$$

We see that this result is independent of the radius of the sphere. So the break-up probability is, in this realistic model, independent of the size of the colliding spheres. This result is closely related to the remark done by Rayleigh more than one century ago : "... the intensity of the pressure between bodies on first impact is independent of the size of the bodies..." .

So, the important point is that a power-law size-distribution is the sign that there is no size-scale in the system or, in other words, that this system is invariant under any change of the size-scale. This sort of self-similarity has been widely studied in the past by powerful tools of the renormalization group method[53]. We shall see now a simple application of this theory to a fragmentation process, as it has been introduced to study the fragility of geological materials[54].

Let us take a bidimensional solid, represented schematically by a square with side length equal to L . We suppose that it can break into four identical fragments similar to the parent (i.e. each of them is a square of side $L/2$). This process occurs with a fragmentation probability p_F , the value of which is not discussed for the moment. We suppose also that this probability is independent of the length scale. This fragmentation process can be continued with each of the fragments, then with the fragments of the fragments, and so on. At each step, a given fragment has the possibility either to break up with the probability p_F or to become inactive with the probability $1 - p_F$. We search now for the average number of active and inactive fragments appearing at each step. These numbers can be easily calculated recurrently by the matrix equation :

$$\begin{pmatrix} N_{act}(m) \\ N_{inact}(m) \end{pmatrix} = \begin{pmatrix} 4p_F N_{act}(m-1) \\ (1 - p_F)N_{act}(m-1) \end{pmatrix} \quad (52)$$

In fact, we are only interested by the distribution of the inactive clusters, since these clusters will give the final size-distribution when the length-scale $L/2^m$ goes

to 0 . When m is large, eq. (52) leads to the result :

$$N_{inact}(m) \sim (1 - p_F) (4p_F)^{m-1} , \quad (53)$$

and this corresponds to the total number of inactive clusters of linear size $L/2^m$. The ratio $N_{inact}(m)/N_{inact}(m-1)$ appears asymptotically as independent of the number m of steps, i.e. of the length-scale. So, the system is found self-similar, and its size-distribution is assumed to behave like a power-law :

$$n_s \sim s^{-\tau} . \quad (54)$$

The exponent τ is defined here, as usually, using the distribution of sizes (here the surfaces). After (53) , the exponent τ is given explicitly by the following function of p_F :

$$\tau = 2 + \frac{\ln p_F}{\ln 4} , \quad (55)$$

which is always smaller than 2 .

Fig. 2. The six topologically different configurations corresponding to the renormalization scheme described in the text. A square of side L is sound if at least two squares of side $L/2$ form a "pillar" which links two opposite sides. The number of configurations is shown below each of them.

The precise value of p_F must be calculated by another reasoning. Take any square fragment of length $L/2^m$. It can break with the probability $p_F(m)$ into four similar fragments of length $L/2^{m+1}$. Here, we have noted p_F depending on the generation number m , but to be consistent with the preceding approach, we shall search for values of this probability independent on m . The fragments formed can themselves be fragile (with probability $p_F(m+1)$) or not. We have to put a "majority rule" telling at which condition one parent is fragile according to the configurations of fragments. One example is the following[55] : if the sound fragments form a "pillar" in the parent cluster, i.e. there is at least one column of sound fragments linking two opposite sides of the parent, then the parent cluster is sound itself. In any other case, it is fragile and breaks up. The six possibilities

are shown in Fig. 2. This leads to the renormalization equation :

$$p_F(m) = 2p_F^2(m+1)[1 - p_F(m+1)]^2 + 4p_F^3(m+1)[1 - p_F(m+1)] + p_F^4(m+1) \quad (56)$$

The standard discussion of this sort of equations is based on the location of the fixed points. Take the eq. (56) as an iterative equation with m . It is schematically drawn in Fig. 3. If one starts from a sufficiently small initial value $p_F(m) > p_F^*$ and a large value for m , the iterative probability fragmentation leads rapidly to a very small value of p_F : the initial system is not fragile. On the contrary, if one starts

Fig. 3. Fragmentation probability at order m versus fragmentation probability at order $m+1$, from the eq. (56). Typical iterations are shown for values of $p_F(m+1)$ smaller, larger, or equal to the unstable fixed point p_F^* .

from a value of $p_F(m)$ larger than p_F^* , the iterative procedure leads to a value 1 of the fragmentation probability. These two trivial solutions ($p_F = 0$ and $p_F = 1$) are solutions of the fixed-point equation :

$$p_F = p_F^2(2 - p_F^2) \quad (57)$$

corresponding to (56). But there exists a non-trivial unstable solution of the equation above, this is :

$$p_F^* = \frac{\sqrt{5} - 1}{2} \quad (58)$$

With this value of the fragmentation probability, the system is scale-invariant since $p_F(m) = p_F^*$ is independent of the length-scale $L/2^m$. For this renormalisation approach (and this majority rule) the value of τ is then :

$$\tau = 2 + \frac{\ln\left(\frac{\sqrt{5}-1}{2}\right)}{\ln 4} = 1.65... \quad (59)$$

This renormalisation group has been extended to the three dimensional case [55] and it leads to a value of τ equal to 1.94. Other majority rules have also been proposed according to their physical relevance in particular processes[54].

2.8 The statistical sequential models

In certain fragmentation processes, such as the nuclear fragmentation in intermediate energy heavy-ion reactions, in addition to the experimental information about fragment sizes and their multiplicities, one has also some informations about the energies, velocities (momenta) and angular distributions of the produced fragments. Hence, one aims at constructing the fragmentation model which would incorporate also certain kinematical features of this heavy-ion process. Complex fragment emission has been observed at all excitation energies and for all kinds of reactions. As an omnipresent decay mechanism it demands a consistent theory describing its salient properties, classifying them in a broad range of excitation energies. At lower excitation energies a compound nucleus formation is assumed at a certain stage of reaction. The compound nucleus in a statistical equilibrium decays then through all available channels, including those of complex fragment emission or binary fission. The equilibration time is much shorter than both the compound nucleus life time and the total fragmentation time. Consequently, one expects that the compound nucleus deexcites through the chain of sequential binary decays which can be either direct or statistical. In statistical models of this type (see e.g. refs. [56, 57, 58]) which consider all possible binary divisions of an initial compound nucleus, one employs a Monte Carlo technique to follow the decay chains of individual compound nuclei until the reaction products are cold, i.e. they are unable to decay anymore.

The decay width for the evaporation of fragments heavier than protons or neutrons are usually calculated using the Hauser - Feshbach formalism [59]. For the emission of a light particle of charge Z_i ($Z_i = 1, 2$), mass A_i and angular momentum J_i from a parent system of charge Z_0 , mass A_0 , excitation energy E_0^* and angular momentum J_0 , which leaves the residual system of charge Z_j , mass A_j and angular momentum J_j , the decay width is given by :

$$\Gamma_{J_j}(Z_i, A_i; Z_j, A_j) = \frac{2J_i + 1}{2\pi\rho_0} \sum_{l=|J_0-J_j|}^{J_0+J_j} \int_0^{E_j^* - B_j - E_{rot}(J_j)} T_l(\varepsilon) \rho_j(U_j, J_j) d\varepsilon. \quad (60)$$

In this equation, ρ_0 is the level density of the initial system and l and ε are the orbital angular momentum and kinetic energy of the emitted particle respectively. $\rho_j(U_j, J_j)$ denotes the level density of a residual system $\{Z_j, A_j, J_j\}$ with the

thermal excitation energy :

$$U_j = E_j^* - B_j - E_{rot}(J_j) - \varepsilon ,$$

where B_j is the binding energy of the nucleus (j) , $E_{rot}(J_j)$ is its rotation plus deformation energy. $T_i(\varepsilon)$ in (60) stands for the transmission coefficients through the Coulomb plus centrifugal barriers.

For the binary division, corresponding to the emission of heavier fragments ($Z_i > 2$) , the decay width is calculated for example using the transition state formalism [60] . One obtains :

$$\Gamma(Z_i, A_i; Z_j, A_j) = \frac{1}{2\pi\rho_0} \int_0^{E^* - E_{sad}(J_0)} \rho_{sad}(U_{sad}, J_0) d\varepsilon , \quad (61)$$

where U_{sad} is the thermal energy :

$$U_{sad} = E^* - E_{sad}(J_0) - \varepsilon \quad (62)$$

of the conditional saddle-point configuration and ε is the translational kinetic energy. ρ_{sad} in (61) stands for the level density of the conditional saddle-point configuration and $E_{sad}(J_0)$ in (62) is the rotation plus deformation energy of the saddle point configuration. Different variants of this decay scheme are obtained for various prescriptions to calculate the energy of the saddle point configuration and the nuclear level density as functions of excitation energy and spin. Usually, the Fermi gas expression :

$$\rho(U, J) = (2J + 1) \left(\frac{\hbar^2}{2\mathcal{J}} \right) \frac{\sqrt{a} \exp(2\sqrt{aU})}{12 U^2} , \quad (63)$$

is applied for all level densities, where $\mathcal{J}$ is the moment of inertia of the residual nucleus or saddle-point configuration and a is the level density parameter. This expression may not be valid at higher excitation energies as it has wrong dependence on mass and energy reducing the probability for the symmetric split of a heavy fragment[61] .

At higher excitation energies, complication arise in this scheme as the temperature affects the surface energy coefficient and nuclear radii , and hence, the potential barriers. Unsuufficient knowledge of these barriers may well lead to significant changes in the decay widths. The secondary products formed in the binary decay of the initial system are allowed to undergo sequential binary decays and it is usually assumed that the time between subsequent fragmentations is long enough to establish the statistical equilibrium in all fragment products. The spin of each residual system at any step in the decay chain is calculated using the Monte Carlo technique from the partial widths $\Gamma_{J_j}(Z_i, A_i; Z_j, A_j)$ and their excitation energy is taken as :

$$E_j^* = U_j - 2t_j + E_{rot}(J_j) \quad (64)$$

where the temperature t_j equals approximately $\sqrt{U_j/a}$. More details can be found in refs. [56, 57] .

2.9 The transition-state formalism

The chain of sequential statistical decays loses its physical meaning if the fragmentation times become comparable or even shorter than the equilibration time. In this case the successive decay steps are correlated and the rates of the binary decays may strongly differ from those obtained for equilibrated statistical system. The deexcitation of such a hot system into several interacting prefragments has been modelled using the transition-state formalism [62, 63]. In this model it is assumed that after creation of a very hot excited system of A nucleons and of the total energy E , the system achieved a transient equilibrium and the statistical considerations can be applied. Let a particular fragmentation be characterized by a vector $\{A_i, \vec{r}_i, \vec{p}_i, \varepsilon_i \mid i = 1, \dots, N\}$ where A_i , m_i , $\vec{r}_i$, $\vec{p}_i$ and ε_i denote the mass number, fragment mass, position, momentum and internal energy of fragment i , respectively. The total width $\Gamma_A^N(E)$ for break-up into any N prefragments can be obtained by summing over various contributing mass partitions :

$$\Gamma_A^N(E) = \frac{1}{N!} \prod_{i=1}^N \sum_{A_i} \delta \left(\sum_{i=1}^N A_i - A \right) \Gamma_{A_1, \dots, A_N}(E) \quad , \quad (65)$$

and the total break-up width is obviously :

$$\Gamma_A^{(tot)}(E) = \sum_N \Gamma_A^N(E) \quad . \quad (66)$$

Let us now define for a given fragmentation α , the disassembly coordinate $\tilde{q}$ and its conjugate momentum $\tilde{p}$ as follows :

$$\begin{aligned} \tilde{q}^2 &= \frac{1}{m_0} \sum_{i=1}^N m_i r_i^2 \\ \tilde{p}^2 &= \frac{1}{\tilde{q}} \sum_{i=1}^N \vec{p}_i \cdot \vec{r}_i \quad , \end{aligned} \quad (67)$$

In these formulas we have omitted the fragmentation index α which should be placed at each disassembly variables below. The disassembly variables correspond to the radial position and momentum in the $3N$ -dimensional space in which the multifragmentation configuration can be embedded. $\tilde{q}$ and $\tilde{p}$ are classical conjugate variables, associated with inertial mass :

$$m_0 = \sum_i m_i \quad ,$$

and the kinetic energy in the disassembly degree of freedom is :

$$K = \frac{1}{2} \tilde{p} \dot{\tilde{q}} = \frac{p^2}{2m_0} \quad .$$

The available energy for statistical excitations is given by :

$$E^* = E - E_{1, \dots, N}^{(0)} - V(\vec{r}_1, \dots, \vec{r}_N) - K \quad , \quad (68)$$

where $E_{1,\dots,N}^{(0)}$ is the sum of the ground-state energies of the given set of fragments and $V(\vec{r}_1, \dots, \vec{r}_N)$ is the potential energy associated with their particular configuration. The available energy is then shared between the random kinetic energy of fragments κ and their internal excitation ε . As we can see, this formalism which provides a generalization of the fission decay, bases on the existence and identification of multidimensional barrier governing the multifragment decay.

The outwards probability current, defined as the number of elementary multifragmentation states that pass by a given value of q per unit time is given by :

$$\nu_{A_1,\dots,A_N}(E) = h^3 \frac{(2\pi m_0)^{-2}}{\Gamma(\frac{3}{2}N - 2)} \prod_{i=1}^N \left[\left(\frac{m_i}{2\pi\hbar^2} \right)^{3/2} \int d\vec{r}_i \right] \int \frac{dp}{h} \frac{p}{m_0} \int d\varepsilon \rho_{1,\dots,N}(\varepsilon) \kappa^{\frac{3}{2}N-3} \delta(\vec{R}_\alpha) \delta(\tilde{q} - q) \quad , \quad (69)$$

where $\vec{R}_\alpha$ denotes the center of mass position of the fragmenting configuration α . Averaging over the constrained positions of fragments one obtains [63] :

$$\nu_{A_1,\dots,A_N}(E) \approx \frac{1}{h} \frac{\sqrt{4\pi}}{\Gamma(\frac{3}{2}N - \frac{3}{2})} < \left(\frac{m_0 q_{1,\dots,N}^2 \tilde{\tau}}{2\hbar^2} \right)^{\frac{3}{2}N-2} \rho_{1,\dots,N}(\varepsilon_{1,\dots,N}) \tilde{\tau} > \quad . \quad (70)$$

In the above expression, $q_{1,\dots,N}$ denotes the value of q for which the current is minimal, the maximal internal energy is $\varepsilon_{1,\dots,N}$ and the corresponding maximal temperature is $\tau_{1,\dots,N}$. $\tilde{\tau}$ denotes the most probable temperature of a given fragmentation configuration. The averaging $< >$ has been constrained by the requirement that the fragmentation has the center of mass at the origin ($\vec{R}_\alpha = 0$) and their r.m.s. extension is fixed at $q = q_{1,\dots,N}$. For a given values of the constrained positions $\{\vec{r}_i\}$, integrand in (69) has a minimum at some value $q_{1,\dots,N}$ as a function of the scaling parameter λ :

$$\vec{r}_i \longrightarrow \lambda \vec{r}_i \quad .$$

This "barrier top" is a generalization of the conditional saddle point of the asymmetric fission.

Then the total rate of an irreversible transition towards fragmentation is approximated by the above current (70) under the assumption that the local value of q is chosen as the transition value $q_{1,\dots,N}$ for which the integrand has a minimum. Consequently, the break-up rate of the system into specified prefragments is given by :

$$\Gamma_{A_1,\dots,A_N}(E) = h \frac{\nu_{A_1,\dots,A_N}(E)}{\rho(A, E)} \approx \frac{1}{\rho(A, E)} \frac{\sqrt{r\pi}}{\Gamma(\frac{3}{2}N - \frac{3}{2})} < \left(\frac{m_0 q_{1,\dots,N}^2 \tilde{\tau}}{2\hbar^2} \right)^{\frac{3}{2}N-2} \rho_{1,\dots,N}(\varepsilon_{1,\dots,N}) \tilde{\tau} > \quad (71)$$

The above formula expresses the disassembly width Γ as the outwards transition current relative to the total number of states. The local transition flux is a product

of a macroscopic factor :

$$\propto \sqrt{4\pi} \left(\frac{m_0 q^2 \tau}{2\hbar^2} \right)^{\frac{3}{2}N-2} \frac{1}{\Gamma(\frac{3}{2}N - \frac{3}{2})} ,$$

associated with the overall motion of prefragments and the microscopic factor :

$$\propto \rho \tau ,$$

which yields the outwards probability current for each such macroscopic state.

The level density $\rho(A, E)$ of the highly excited source is largely unknown because of the many collective modes accessible at high excitations. In ref. [63] a simple expression was used :

$$\rho(A, E) = \frac{4}{3} \pi r_0^3 A \left(\frac{m_0 \tau}{2\pi \hbar^2} \right)^{3/2} \rho_A(\varepsilon_0) , \quad (72)$$

which has the same form as the level density employed for each individual prefragment, i.e. the Fermi gas type expression.

At a second stage, once the prefragment are formed and their configuration and energies are determined, one can study their dynamical evolution [63] into the final state. Collective expansion of the fragmenting equilibrated source has been omitted in the above model. That this may be very important shows the analysis of Friedman in the expanding - emitting source model [64] which yields results close to those of the statistical multifragmentation model.

2.10 The statistical multifragmentation models

If the dynamical relaxation times, associated with the compound nucleus or residuum formation are shorter or comparable with the fragmentation time, an equilibrated source at freeze-out can be formed. Under this assumption, the fragmentation into many fragments can be described as a simultaneous multiple disassembly of the expanded, hot equilibrated system at a fixed freeze-out density [65, 66, 67, 68]. In this case, one may aim at using the thermodynamical language for the description of this process. This approach has been implemented in a variety of models. (For a recent review see [6].) In the nuclear case, where one deals with a finite and small number of fermions, with an important component of the long-range Coulomb force, several precautions should be obeyed.

A key quantity of a statistical description of a many-body system at equilibrium is the partition sum Z , i.e. the number of all accessible quantum states of the system, from which other thermodynamical quantities of interest such as the entropy $S = \ln Z$, or the temperature $T^{-1} = \partial S / \partial E$ can be obtained. Semiclassically, the partition sum is the sum of all different realizations of the system in the phase space. For a heavy nucleus in the excitation energy of interest, it can take "astronomical" values of some 10^{30} or more. Hence, various approximate techniques must be developed to deal with such a fantastic number of configurations. Probably, the most sophisticated approach has been developed by Gross and his

collaborators [6] and uses the Monte - Carlo - Metropolis sampling to calculate the expectation values of quantities of interest such as e.g. the free energy F . In this case, the expectation values : $\langle F \rangle \equiv \text{tr}(F\rho)/\text{tr}(\rho)$, where $\rho(E)$ is a microcanonical statistical operator, are evaluated as the average over a large sample of most important configurations appearing with a frequency proportional to their relative phase-space volume. The selected configurations correspond all to a system at a given fixed excitation energy, momentum, angular momentum, mass, charge, which is an important element of the statistical description of any finite fermion system. This model was very successful in explaining several salient features of the nuclear multifragmentation data including the conditional moments [69], the scaled factorial moments [70] or correlations between intermediate mass fragments [71]. The most essential difference between existing codes of statistical multifragmentation reflect ambiguity in possible properties of the equilibrated source, its size and deformation, treatment of secondary decay of hot fragments, the level density. These differences may be important in view of the fact that results depend strongly on the shape and size of the freeze-out volume.

2.11 The spinodal instabilities

If the relaxation times associated with the compound nucleus formation are much longer than the dynamical time scales involved in breaking up the system into fragments, then the deexcitation process should be considered as the dynamical, off-equilibrium process. This process can be either sequential or not, and the fragment production is governed by fragmentation rates. An example of a model simulating such a process is the Fragmentation - Inactivation Binary (FIB) model discussed in chapter 4. On a more microscopic level, in view of a long mean-free path of nucleons in nuclear medium, the Boltzmann - Langevin type of approaches has been proposed[72] and studied in some details[73, 74].

2.11.1 Volume instabilities

The fragmentation process in heavy-ion collision can be studied from the viewpoint of growth of instabilities in hot expanding nuclear matter. Let us investigate conditions under which density fluctuations may grow during the expansion of a sphere of heated or superdense nuclear matter. We shall consider the Fermi liquid, unstable to density fluctuations, by using Landau theory in hydrodynamic limit[75]. This restricts the validity of the approach to the disturbances for which the wavenumber is much less than $1/\lambda$, where λ is a nucleon mean free path. Let us consider the modes in a uniform static medium at a particular density, neglecting the Coulomb and surface effects.

A convenient way to study the density instabilities is provided by the language of the dispersion relations :

$$\omega^2 = c_s^2 q^2 \quad ,$$

where q is the wavenumber, c_s^2 is the (isothermal) sound speed :

$$c_s^2 = \frac{1}{m} \left(\frac{\partial P}{\partial \rho} \right)_T ,$$

and ρ , T are the fluid density and the temperature respectively. The isothermal spinodal line is given by $c_s^2 = 0$. Inside the spinodal line, bulk matter is unstable to small-amplitude long-wavelength density fluctuations. Small fluctuations can grow to such extent that the matter is disrupted. Between spinodal and coexistence lines, bulk matter is unstable to large, non-linear fluctuations. In this region, the density inhomogeneities lead to nucleation of localized regions. In general, c_s^2 is a function of q related to the bulk modulus :

$$B(q) = \frac{m}{\rho} c_s^2(q) = \left(\frac{\partial^2 F}{\partial \rho_q^2} \right)_T \longrightarrow \frac{m}{\rho} c_s^2 \quad \text{for } q \longrightarrow 0 ,$$

where F is the free energy. For small wave-numbers :

$$m c_s^2(q) = m c_{s0}^2 + \rho \beta q^2 + \dots$$

Consequently :

$$\omega^2 = c_{s0}^2 q^2 + \frac{\rho}{m} \beta q^4 + \dots , \quad (73)$$

where the second and higher terms are responsible for the anomalous dispersion. For unstable modes, the growth rate is :

$$\Gamma = -i\omega ,$$

and hence :

$$\Gamma^2 = -c_{s0}^2 q^2 - \frac{\rho}{m} \beta q^4 . \quad (74)$$

with the maximum :

$$\Gamma_{max}^2 = \left(\frac{\rho \beta}{m} \right)^{1/2} q_m^2 , \quad (75)$$

at :

$$q_m^2 = -\frac{1}{2} \left(\frac{m}{\rho \beta} \right) c_s^2 . \quad (76)$$

The modes become stable for :

$$q_0^2 > - \left(\frac{m}{\rho \beta} \right) c_s^2 . \quad (77)$$

From above, one expects strong suppression of short-wavelength instabilities, an equivalent of the wavelength dependent inactivation term.

To get the idea of the size of fragments produced by an instability of a given wavelength, we calculate the number of particles in a cube of side inbetween $\lambda/2$ and λ ³:

$$\frac{3A}{4\pi\nu^3} < \tilde{A} < \frac{6A}{\pi\nu^3} ,$$

A is the particle number of the fragmenting system, $q = \pi\nu/R$ and R is the radius of a sphere containing A particles. Hence, the size of a typical fragment is :

$$\tilde{A} \sim q^{-3} ,$$

and the growth rate of an unstable mode is :

$$\Gamma \sim -iq \sim -i\tilde{A}^{-1/3} .$$

In the volume $\sim R^3$, there are $A/\tilde{A}$ of such instabilities and, hence the total growth rate is :

$$\Gamma_{eff} \sim \tilde{A}^{-4/3} \quad (78)$$

Assuming that Γ_{eff} is approximately constant during the fragmentation process, one finds that the necessary time to disrupt the system in a given unstable mode is :

$$t_c \sim \Gamma_{eff}^{-1} ,$$

and, consequently, the fragmentation probability of a system A on a fragment of size i is $\sim i^{-4/3}$. Hence, for binary break-up chain, the fragmentation function would be :

$$F_{i,A-i} \sim i^{-4/3} + (A-i)^{-4/3} . \quad (79)$$

The discussion of the spinodal instability of nuclear matter can be done also in the collisionless limit of the Fermi liquid. One finds :

$$\Gamma = \frac{2}{\pi} \frac{1+F_0}{F_0} v_F q \quad , \quad (80)$$

where v_F is the Fermi velocity and F_0 is the spin- and isospin- symmetric Landau interaction function and the fragmentation function for binary break-up chain is again (78).

The above estimates have been done under several restrictions and several remarks should be made :

- The linear approximation has been applied at the edge of unstable region inside the spinodal line. The relations in the non-linear regime are largely unknown and numerical test lead to the contradictory statements with respect to the role of mode coupling in the non-linear regime [73]. In this situation, it may be reasonable to approximate the process by a binary, non-equilibrium chain of break-ups for the additive, homogeneous fragmentation probability function.

³This procedure is analogous to the one used in the physics of gravitational collapse.

- Effects of a finite time, the system stays in the unstable region as well as corrections to the "ideal" dispersion relations could be modelled by the phenomenological inactivation function.
- Effects of a finite mean-free path inbetween two subsequent particle collisions are not considered, but one hopes that this changes the proportionality factor in front of the growth rate Γ but not its wave number dependence $\sim q$ [75].

The fragmentation through spinodal instabilities and its effective description has several similarities with the instabilities of a flow of a classical fluid in the chaotic regime for high Reynolds numbers. In the classical fluid, in this regime, the flow is both unstable and unpredictable, i.e. small perturbation at a certain time t_0 may rapidly lead to a strong distortions of the flow pattern. Hence, even though we have at our disposal the Navier - Stokes eqs. whose validity is not questioned, we are unable to solve them reliably. In other words, we are unable to build up the statistical mechanics of turbulence on the basis of the Navier - Stokes equations. The correct approach is to study the energy transfer in the language of unstable, turbulent structures at all scales, the so called *eddies*, which generate smaller "eddies" by their instabilities [76]. This hierarchy of "eddies" is described by the random cascading (the β -model) with phenomenological fragmentation function $P(\beta)$ [77, 78]. The ultimate relation between singularities of Navier - Stokes eqs. and the exact form of $P(\beta)$ has not yet been given in spite of intense efforts over several decades. This should provide a warning for those who wish to apply the "realistic" kinetic equations in the unstable region. Contrary to the classical fluid, the Boltzmann - Langevin type of approaches provide the approximation to the BBGKY hierarchy of kinetic equations which neglects higher order correlations. It is to be expected that, like in the approximations using Navier - Stokes eqs., the higher order correlations play intricate role in driving lowest order distribution function.

In the hot nuclear matter, spinodal instabilities lead to the picture of unstable "blobs" of matter at different scales, which generate smaller "blobs" and so on until either the nucleon scale or the scale introduced by the time the system stays in the unstable region. These two effects could be modelled by universal phenomenological fragmentation function and a certain inactivation probability function, very much in the same way as found for random cascading model of classical fluid in a turbulent regime.

2.11.2 The Coulomb instabilities The role of a competition between the Coulomb and surface energies in stabilizing the atomic nucleus has been realized long ago. The influence of the rotation in this competition is also a long studied problem of the classical mechanics [79, 80]. In the nuclear physics, Swiatecki and his coworkers have studied in details equilibrium configurations of rotating nuclei in the liquid drop model [81]. The disruptive role of the Coulomb force has been particularly strongly emphasized in the statistical multifragmentation models [65, 66, 67, 68], though it is clear that this force, independently of the disruption mechanism and its details, plays a principal role in the final stage of the nuclear

fragmentation process. Hence, any study of the fragmentation mechanism should subtract its trivial aspects. The eigenfrequencies of a uniformly charged, incompressible and nonviscous spherical liquid drop are :

$$\omega^2 = \frac{\gamma}{\rho R^3} l(l-1) [(l+2) - 4x] \quad , \quad (81)$$

where γ is the surface energy coefficient, R is the radius of the sphere and ρ its density and x is the *fissility parameter* :

$$x = \frac{E_{Coul}}{2E_{surf}}$$

expressed in terms of the Coulomb , E_{Coul} , and surface , E_{surf} , energies of the spherical drop. l in (81) is the order of the spherical harmonic. The value $x = 1$ represents the maximum amount of charge a droplet's surface tension can support, i.e. no stable configurations exist for $x > 1$. At $x = 1$, the frequency ω^2 becomes zero for a quadrupole mode $l = 2$. This marks the onset of the quadrupole instability of the charged drop. With increasing value of the fissility, progressively higher modes attain the instability regime. The growth rate of these instabilities :

$$\Gamma^2 = -\frac{\gamma}{\rho R^3} l(l-1) [(l+2) - 4x] \quad ,$$

attains maximum at :

$$l_m \simeq \text{Int} \left\{ \frac{1}{3} \sqrt{4x-1} [1 + \sqrt{(4x-1)-6}] \right\}$$

higher l - values. The modes become stable again for :

$$l_0 \geq \text{Int}\{4x-2\}$$

For realistic values of the fissility parameter it can be shown that this instability cannot be approximated by the binary chain of fragmentations for the homogeneous fragmentation kernel. This is an important difference with the spinodal instabilities. It is interesting to notice, that it is not always the fission mode ($l = 2$) which is most unstable at high values of the fissility parameter. So the strongly charged classical drop has a tendency for breaking in many small droplets through a fastest growing instability associated with a high multipole mode. The numerical studies of Coulomb instabilities using Boltzmann - Uehling - Uhlenbeck (BUU) formalism [82] suggest that it can be realized experimentally in heavy-ion collisions at bombarding energies $E/A \simeq 50 \text{ MeV}$.

2.11.3 The surface instabilities

Another class of instabilities, that has been discussed in connection with nuclear multifragmentation, are the Rayleigh instabilities [83] which depend solely on the presence of the surface energy term. These instabilities have been suggested based

on the numerical experiment using BUU equation where unstable disks [84], toroids and bubbles [85] have been observed. In a nearly symmetric head-on collision, as a result of side-squeezing of nuclear matter, a disk develops whose diameter increases with the increasing bombarding energy. The sufficiently thin disk breaks up into several fragments of size given by the disk thickness. Importance of this mechanism for the nuclear multifragmentation is an open question as yet.

3. The kinetic rate equation

Fragmentation is basically a phenomenon due to instability. So, if this is the dominant behaviour for a system one expects the relevant physics to be irreversible. The statistical approaches do not hold in these cases, and the most fruitful way to attack this problem has been an extension of the Smoluchowski's works on aggregation[86]. Aggregation and fragmentation are not equivalent by time reversal. The reason is that one cannot exchange the causes and the effects. Nevertheless, under some assumptions which will be detailed below, one can write the following general equation for break up process[87]:

$$\frac{\partial}{\partial t} n_s(t) = -a(s)n_s(t) + \int_s^\infty a(s')b(s|s')n_{s'}(t) ds' \quad , \quad (82)$$

with the initial condition :

$$n_s(0) = \delta(s - N) \quad . \quad (83)$$

As usual, $n_s(t)$ denotes the number of fragments of size s per unit of volume at time t . Then the right-hand side term is the sum of one loss term, which represents the break-up of clusters of size s into smaller ones, and one gain term which is the increase of the number of fragments of size s formed by the breaking of larger size fragments. The function $b(s|s')$ is the rate at which one fragment of size s is formed by fragmentation of one fragment of larger size s' , and $a(s')$ is the rate at which s' -clusters break up. These rates are assumed here to be independent of the time. One can cancel any t -dependence of the form : $a(s, t) = a(s)f(t)$ by a suitable renormalization of the time, if the function $f(t)$ itself does not depend on the size s . The conservation of the mass gives the normalization condition :

$$\int_0^{s'} s b(s|s') ds = s' \quad , \quad (84)$$

and the average number of fragments during such a break is given by :

$$\int_0^{s'} b(s|s') ds \quad . \quad (85)$$

The main assumption used to write this linear integro-differential equation is that fragmentation of the initial cluster of size N consists in a series of independent cascade of break-up, each break-up being an individual process (monomolecular reaction) of the type :

$$(s') \longrightarrow (s_1) + (s_2) + \dots \quad . \quad (86)$$

In other words, the fragmentation of one cluster or fragment is not influenced by the presence of other clusters all around it. Moreover, eq. (82) has been written for the average size-distribution and it provides a mean-field, low-density approximation.

These assumptions can fail in some applications. One case when this could happen is the fragmentation due to collisions between the clusters. The relevant mechanism in this case is now :

$$(s') + (s'') \longrightarrow (s_1) + (s_2) + \dots, \quad (87)$$

what leads to the non-linear equation [88] :

$$\begin{aligned} \frac{\partial}{\partial t} n_s(t) = & - \int_0^\infty a(s, s') n_s(t) n_{s'}(t) ds' \\ & + \int_0^\infty \int_s^\infty a(s', s'') b(s | s', s'') n_{s'}(t) n_{s''}(t) ds' ds'' \quad , \quad (88) \end{aligned}$$

which resembles the original equation for aggregation given by Smoluchowski [86] . But this sort of equation for fragmentation has not yet been investigated in details up to now.

Before giving the properties and solutions of the general equation (82), it is instructive to discuss briefly the microscopic origin of this equation.

3.1 Derivation of the rate equation

Let us take a given configuration, say $\{n_1, n_2, \dots\}$, at a time t . For convenience, we shall write equivalently this configuration as a formal vector $\vec{n}$, with integer components. We ask then how this state is changed during the time-interval dt . There is a fragmentation of one cluster of size s' with a probability $a(s')dt$. The distribution of fragment products is given by the function $b(\vec{k} | s')$, which is the probability to get a set of fragments $\vec{k} = \{k_1, k_2, \dots\}$ from the s' -cluster. Knowing that the initial cluster has size N at time 0, one can derive time-variation of the probability $W_N(\vec{n}, t)$ to have the state $\vec{n}$ at time t . The change of this probability per unit of time is given by the probabilistic equation :

$$\frac{\partial}{\partial t} W_N(\vec{n}, t) = \sum_{s'} a(s') \left(\sum_{\vec{k}} W_N(\vec{n} - \vec{k}, t) n_{s'}(\vec{n} - \vec{k}) b(\vec{k} | s') - W_N(\vec{n}, t) n_{s'}(\vec{n}) \right) \quad , \quad (89)$$

where $n_{s'}(\vec{n})$ means the number of clusters of size s' in the state $\vec{n}$ (or more formally : the coordinate of order s' of the vector $\vec{n}$). The first term in the brackets is the formation of the configuration $\vec{n}$ from the configuration $\vec{n} - \vec{k} \equiv \{n_1 - k_1, n_2 - k_2, \dots\}$ by the break of one cluster of size s' , belonging to the state $\vec{n} - \vec{k}$, into an ensemble of fragments : $\vec{k}$. The second term is the disappearance of a state $\vec{n}$ by break-up of one of its clusters. Note that the function $b(\vec{k} | s')$, normalized by :

$$\sum_{\vec{k}} b(\vec{k} | s') = 1 \quad ,$$

is directly related to the function $b(s | s')$ introduced in eq. (82) by the condition :

$$b(s | s') = \sum_{\vec{k}} k_s(\vec{k}) b(\vec{k} | s') \quad , \quad (90)$$

where the sum runs over all possible ensemble of fragments formed by splitting a s' - cluster. That is also to say, over all vectors $\vec{k}$ satisfying :

$$\sum_s s k_s(\vec{k}) = s' \quad . \quad (91)$$

The eq. (89) can be written in a more compact form using the generating function :

$$G_N(\vec{x}, t) = \sum_{\vec{n}} W_N(\vec{n}, t) \vec{x}^{\vec{n}} \quad , \quad (92)$$

where the sum is over all possible states $\vec{n}$ and $\vec{x}^{\vec{n}}$ means $x_1^{n_1} * x_2^{n_2} * \dots$. Then we find the differential equation for $G_N(\vec{x}, t)$ in the form :

$$\frac{\partial}{\partial t} G_N(\vec{x}, t) = \sum_{s'} a(s') \left(\sum_{\vec{k}} b(\vec{k} | s') \vec{x}^{\vec{k}} - x_{s'} \right) \frac{\partial G}{\partial x_{s'}}(\vec{x}, t) \quad (93)$$

We recover in the brackets the generating function of the probability distribution of the fragments. All the correlation functions can be computed readily from the equation above. In this chapter we will be interested only by the results :

$$\begin{aligned} 1 &= G_N(\vec{1}, t) \\ \langle n_s \rangle_N &= \sum_{\vec{n}} n_s(\vec{n}) G_N(\vec{n}, t) = \frac{\partial G_N}{\partial x_s}(\vec{1}, t) \quad , \end{aligned} \quad (94)$$

with $\vec{1}$ being the vector : $\{1, 1, \dots\}$. We have omitted explicetely the time t in the $\langle n_s \rangle_N$ to simplify the notation. Differentiating eq. (93) with respect to x_s gives :

$$\begin{aligned} \frac{\partial \langle n_s \rangle_N}{\partial t} &= a(s) \frac{\partial^2 G_N}{\partial x_s^2}(\vec{x}, t) \left[\sum_{\vec{k}} b(\vec{k} | s) \vec{x}^{\vec{k}} - x_s \right] \\ &- a(s) \frac{\partial G_N}{\partial x_s}(\vec{x}, t) \\ &+ \sum_{s' > s} a(s') \frac{\partial G_N}{\partial x_{s'}} \left[\sum_{\vec{k}} b(\vec{k} | s') k_s \frac{\vec{x}^{\vec{k}}}{x_s} \right] \quad . \end{aligned} \quad (95)$$

If one wishes to know the time evolution of the average size-distribution, one has to replace vector $\vec{x}$ by vector $\vec{1}$. We find then the so-called master-equation :

$$\frac{\partial \langle n_s \rangle_N}{\partial t} = -a(s) \langle n_s \rangle_N + \sum_{s' > s} a(s') b(s | s') \langle n_{s'} \rangle_N \quad . \quad (96)$$

This is exactly the discrete counter-part of the continuous rate-equation (82). Equation (93) is usually known as the Marcus-equation, because a similar approach has been used by Marcus[89] to study the kinetic aggregation equations.

3.2 The cascade equation

Following ideas of Kolmogorov[76], one can write another equation which must be satisfied by the probability functions W_N . To see this, let us consider the very first fragmentation :

$$(N) \longrightarrow (s_1) + (s_2) + \dots, \quad (97)$$

or more shortly : $\vec{N} \rightarrow \vec{k}$, where $\vec{N}$ is the vector of zero components except at the N -th place where it is equal to 1, and $\vec{k}$ is the vector whose components are the number k_s of fragments of given size s . This state $\vec{k}$ appears with the probability $b(\vec{k} | N)$, and this event occurs itself with the probability $a(N) \exp(-a(N)t') dt'$. Then, we can write that the state $\vec{n}$ at a time t has been formed during the duration $t - t'$ by the superposition of the independent fragmentations starting from $s_1, s_2, \dots$. The corresponding equation is [90] :

$$\begin{aligned} W_N(\vec{n}, t) &= \int_0^t a(N) e^{-a(N)t'} dt' \sum_{\vec{k}} b(\vec{k} | N) \\ &\times \sum_{\vec{n}'_1 + \vec{n}'_2 + \dots = \vec{n} - \vec{k}} \prod_i W_i^{k_i}(\vec{n}'_i, t - t') + \delta(\vec{n} - \vec{N}) e^{-a(N)t} \end{aligned} \quad (98)$$

Note here the important point that the multiplication of the probabilities is allowed because of the assumption of the independence of all the cascades of fragments. The last term must be written here since it can happen that no event occurs before the time t .

As in the Marcus approach, the generating functions for the probability functions are useful to write a simpler equation, namely :

$$G_N(\vec{x}, t) = x_N e^{-a(N)t} + \int_0^t a(N) \sum_{\vec{k}} b(\vec{k} | N) \prod_i G_i^{k_i}(\vec{x}, t - t') e^{-a(N)t'} dt' \quad (99)$$

By time differentiation one get another more familiar form :

$$\frac{\partial G_N}{\partial t}(\vec{x}, t) = -a(N) G_N(\vec{x}, t) + a(N) \sum_{\vec{k}} b(\vec{k} | N) \prod_i G_i^{k_i}(\vec{x}, t) \quad (100)$$

An important remark must be done here : contrary to (93), eq. (100) links the probabilities G_N for different starting configurations (N) . This is still more obvious when deriving the equation for the averages distributions $\langle n_i \rangle_N$ from (100). One finds after differentiating with respect to x_s then putting $\vec{x} = \vec{1}$:

$$\frac{\partial \langle n_s \rangle_N}{\partial t} = -a(N) \langle n_s \rangle_N + a(N) \sum_{\vec{k}} b(\vec{k} | N) \sum_i k_i \langle n_s \rangle_i \quad (101)$$

In terms of the time-Laplace transforms of the $\langle n_s \rangle_N$, the cascade-equation (101) is a forward recurrent equation on N at fixed s , while the master-equation (96) is a backward recurrent equation on s at fixed N . Of course, both equations have the same set of solutions.

These equations are linear in the average size-distributions. This implies in particular that the solution for a given polydisperse initial distribution can be written as a linear superposition of the solutions known for the monodisperse initial distributions, which will be the only cases reported in this review.

3.3 Exact solutions of the continuous rate - equation

Two classes of solutions are known to satisfy the eq. (82) with some particular choice of rate functions $a(s)$ and $b(s | s')$. The first example is the choice [91] :

$$\begin{aligned} a(s) &= s^\lambda \\ b(s | s') &= \frac{\mu}{s'} \left(\frac{s}{s'} \right)^{\mu-2} \end{aligned} \quad , \quad (102)$$

with the exponent $\mu > 0$, in order to ensure the existence of the integral appearing in the mass-conservation (84). But the average number of fragments is found by the eq. (85) to be infinite when $\mu < 1$ and equal to $\mu/(\mu - 1)$ when $\mu > 1$. Since this average number must be larger than 2, the physically admissible values of μ are : $0 < \mu \leq 2$.

Substituting the rate-functions (102) into eq. (82), gives the following equation :

$$\frac{\partial n_s}{\partial t}(t) = -s^\lambda n_s(t) + \mu s^{\mu-2} \int_s^\infty (s')^{\lambda-\mu+1} n_{s'}(t) ds' \quad . \quad (103)$$

Then try one function of the form :

$$n_s(t) = e^{-tN^\lambda} \left(\delta(s - N) + \sum_{l \geq 1} a_l \frac{t^l}{l!} \right) \quad . \quad (104)$$

The solution of eq. (82) in this case becomes [91] :

$$\begin{aligned} n_s(t) = e^{-tN^\lambda} \left\{ \delta(s - N) + \mu t N^{\lambda-1} \left(\frac{s}{N} \right)^{\mu-2} \right. \\ \left. \times {}_1F_1 \left(1 + \frac{\mu}{\lambda}, 2 ; \left[1 - \left(\frac{s}{N} \right)^\lambda \right] t N^\lambda \right) \right\} \end{aligned} \quad (105)$$

and satisfies the initial condition : $n_s(0) = \delta(s - N)$. In this solution ${}_1F_1(a, b; z)$ is the Kummer's confluent hypergeometric function.

The other case which has been solved is for the choice [92] :

$$\begin{aligned} a(s) &= s^\lambda \\ b(s | s') &= \frac{\mu\lambda}{\mu - \lambda} \frac{1}{s'} \left[\left(\frac{s}{s'} \right)^{\lambda-2} - \left(\frac{s}{s'} \right)^{\mu-2} \right] \end{aligned} \quad . \quad (106)$$

The following solution has been conjectured on the basis of an exact scaling solution, then verified to work directly on the rate-equation :

$$n_s(t) = e^{-tN^\lambda} \left(\delta(s - N) + \mu t N^{\lambda-1} \left(\frac{s}{N} \right)^{\mu-2} \int_0^{1-(\frac{s}{N})^\lambda} (1-x)^{-\frac{\mu}{\lambda}} e^{tN^\lambda x} dx \right) \quad (107)$$

The integral which appears in the right-hand side can be expressed in terms of the incomplete Euler Gamma-function. This result holds also for the particular case $\lambda = \mu$ involving a logarithmic term, that is to say for the rate-functions :

$$\begin{aligned} a(s) &= s^\lambda \\ b(s | s') &= \frac{\lambda^2}{s'} \left(\frac{s}{s'} \right)^{\lambda-2} \ln \left(\frac{s'}{s} \right) \end{aligned} \quad (108)$$

and the solution in this case becomes :

$$n_s(t) = e^{-tN^\lambda} \delta(s - N) + \lambda t N^{\lambda-1} \left(\frac{s}{N} \right)^{\lambda-2} [Ei(-tN^\lambda) - Ei(-ts^\lambda)] \quad (109)$$

with Ei the exponential-integral function.

3.4 Scaling solutions of the continuous rate - equation

All these exact solutions have about the same form :

$$n_s(t) = e^{-tN^\lambda} \delta(s - N) + \frac{1}{N} F(tN^\lambda, \frac{s}{N}) \quad (110)$$

After certain time, the initial state is expected to play no role and this is a reason to search for some scaling solutions depending on simple ratios of the variables. To have some possibilities to succeed, one has first to assume homogeneous rate-functions. So, let us put :

$$\begin{aligned} a(s) &= s^\lambda \\ b(s | s') &= \frac{1}{s'} B\left(\frac{s}{s'}\right) \end{aligned} \quad (111)$$

where the scaling function B is such that the mass-conservation condition is satisfied :

$$\int_0^1 u B(u) du = 1 \quad (112)$$

This choice of rate functions is quite interesting also for another reason : it leads to a very simple equation on the moments of the average size-distribution. To see this, let us define the moments $M_n(t)$ by :

$$M_n(t) = \int_0^\infty s^n n_s(t) dt \quad (113)$$

and, similarly, define the moments B_n of the scaling function B by :

$$B_n = \int_0^1 u^n B(u) du \quad (114)$$

Hence, calculating the derivative of M_n by the rate-equation (82), one finds :

$$\frac{dM_n}{dt}(t) = (B_n - 1)M_{n+\lambda}(t) \quad , \quad (115)$$

which can be used to find exact solutions of the rate-equation with the choice (111) for the rate-functions. But in this section, we will use this approach to give exact scaling solutions of the rate-equation[92] .

Let us now define the scaling function[93, 94, 95, 96, 97] :

$$n_s(t) = N\lambda s^{-2}\Phi(ts^\lambda) \quad , \quad (116)$$

which is similar, to the self-preserving function, introduced by Friedlander[98] to study the aggregation processes. The mass-conservation implies :

$$\int_0^\infty \frac{\Phi(u)}{u} du = 1 \quad . \quad (117)$$

In particular, one must verify on the solutions, that : $\Phi(0) = 0$. With this trial solution, the rate-equation (82) becomes :

$$\lambda z^{1-\frac{2}{\lambda}} \left(\frac{d\Phi}{dz}(z) + \Phi(z) \right) = \int_z^\infty u^{-\frac{2}{\lambda}} \Phi(u) B\left(\left(\frac{z}{u}\right)^{\frac{1}{\lambda}}\right) du \quad , \quad (118)$$

which can be written in a more convenient form[99] :

$$\Phi(z) = \int_z^\infty \Phi(u) \bar{B}\left(\left(\frac{z}{u}\right)^{\frac{1}{\lambda}}\right) du \quad , \quad (119)$$

where the function $\bar{B}$ is the cumulative mass-distribution of the fragments :

$$\bar{B}(u) = \int_0^u x B(x) dx \quad . \quad (120)$$

Equivalently, eq.(118) can be written in terms of the moments :

$$M_n(t) = \frac{N}{t^{\frac{n-1}{\lambda}}} \int_0^\infty u^{\frac{n-\lambda-1}{\lambda}} \Phi(u) du \quad , \quad (121)$$

where the moments of the function Φ are defined recursively by the eq. [92] :

$$\begin{aligned} \Phi_r &= \frac{r}{1 - B_{\lambda r+1}} \Phi_{r-1} \\ \Phi_0 &= \frac{1}{-\int_0^1 u \ln(u) B(u) du} \\ \Phi_{-1} &= 1 \quad . \end{aligned} \quad (122)$$

This approach has been used to find an exact scaling solution in the case where [92] :

$$\begin{aligned} a(s) &= s^\lambda \\ b(s | s') &= \frac{1}{s'} \left[\alpha \gamma \left(\frac{s}{s'}\right)^{\gamma-2} + (1 - \alpha) \delta \left(\frac{s}{s'}\right)^{\delta-2} \right] \quad , \end{aligned} \quad (123)$$

and the parameters are submitted to the physical constraints [91] :

$$0 \leq \alpha \leq 1 \quad , \quad \lambda > 0 \quad , \quad \gamma > 0 \quad , \quad \delta > 0 \quad ,$$

$$\frac{\alpha\gamma}{\gamma-1} + \frac{(1-\alpha)\delta}{\delta-1} \geq 2 \quad ,$$

$$\int_0^x x(B(x) - B(1-x))dx \geq 0 \quad \text{for } 0 < x \leq \frac{1}{2} \quad (124)$$

Then the solution has one of the two forms :

$$\Phi(z) = \frac{\Gamma(\frac{(1-\alpha)\gamma+\alpha\delta}{\lambda})}{\Gamma(\frac{\gamma}{\lambda})\Gamma(\frac{\delta}{\lambda})} z^{\frac{\gamma}{\lambda}} e^{-z} U\left((1-\alpha)\frac{\gamma-\delta}{\lambda}, \frac{\gamma-\delta}{\lambda} + 1; z\right) \quad (125)$$

and

$$\Phi(z) = \frac{\Gamma(\frac{(1-\alpha)\gamma+\alpha\delta}{\lambda})}{\Gamma(\frac{\gamma}{\lambda})\Gamma(\frac{\delta}{\lambda})} z^{\frac{\delta}{\lambda}} e^{-z} U\left(\alpha\frac{\gamma-\delta}{\lambda}, \frac{\gamma-\delta}{\lambda} + 1; z\right) \quad , \quad (126)$$

with $U(a, b; z)$ the confluent hypergeometric function. The complete solution satisfying the proper initial condition is not known except for the two cases :

$$\alpha = \begin{cases} \frac{\delta}{\delta-\lambda} \quad , & \gamma = \lambda \\ 1 & \end{cases} \quad (127)$$

which have been discussed in the previous section.

3.5 The large-time behaviour

The recurrent formula (122) for the moments allows to write explicitly all the Φ -moments of integer order as :

$$\Phi_n = \frac{n! \Phi_0}{\prod_{k=1}^n (1 - B_{\lambda k+1})}$$

Since the moment of an exponential is essentially the factorial of the order :

$$\int_0^\infty u^n e^{-u} du = n! \quad ,$$

we see that the function Φ should behave quite naturally as an exponentially decreasing function, at least if the moments of the function B decreases faster than the inverse of its order. Under this condition, the large-time behaviour of the scaling function (116) writes :

$$n_s(t) \propto N t^{\frac{2-\tau}{\lambda}} s^{-\tau} e^{-(ts)^\lambda} \quad . \quad (128)$$

Note that this is very similar to a Weibull dependence[100] :

$$n_s(t) \propto s^{-\tau} e^{-(\frac{s}{\sigma})^{2-\tau}} \quad , \quad (129)$$

which is widely used to fit the experimental results. The short-time behaviour is, unfortunately, much less clear.

3.6 The shattering transition

The scaling solution (116) holds only for an overall fragmentation rate $a(s)$ which is an increasing function of the size s . To see this breakdown of the scaling[101, 102, 88], one can remark that because of the mass-conservation, the first moment of the size-distribution should be a constant :

$$M_1(t) = N \quad , \quad (130)$$

and eq. (115) relating different moments writes in the case of the homogeneous function $a(s) = s^\lambda$:

$$\frac{dM_{1-\lambda}}{dt}(t) = (B_{1-\lambda} - 1)N \quad , \quad (131)$$

where the coefficient $B_{1-\lambda}$:

$$B_{1-\lambda} = \int_0^1 x^{1-\lambda} B(x) dx \quad , \quad (132)$$

is a constant, smaller than :

$$\int_0^1 x B(x) dx = 1 \quad , \quad (133)$$

as long as λ has a negative value. This implies that $M_{1-\lambda}(t)$ must vanish for the finite time :

$$t_c = ((1 - B_{1-\lambda})N^\lambda)^{-1} \quad .$$

This conclusion is obviously in contradiction with standard rate-equation which constrains $n_N(t)$ to be always larger than $\exp(-a(N)t)$ since :

$$\frac{\partial n_s}{\partial t}(t) \geq -a(s) n_s(t) \quad , \quad (134)$$

holds for any size s and any time t . We must conclude that something is wrong in the derivation of the moment equation (115). But even if $\lambda < 0$, the moments $M_{1-\lambda}$ and $B_{1-\lambda}$ are bounded by N and 1 respectively and so, they are well-defined. The only additional assumption which could be wrong is then the mass-conservation (130). To see how, one can use an explicit exact solution of the rate-equation for a negative value of λ . This can be achieved with the choice (102) for which the complete solution is (105). Now the moments are equal to [91, 103] :

$$M_n(t) = N^n e^{-tN^\lambda} {}_1F_1\left(\frac{\mu}{\lambda}, \frac{\mu + n - 1}{\lambda}; tN^\lambda\right) \quad , \quad (135)$$

when $\lambda > 0$ and to :

$$\begin{aligned}
M_n(t) &= N^n e^{-tN^\lambda} \left\{ {}_1F_1 \left(\frac{\mu}{\lambda}, \frac{\mu+n-1}{\lambda}; tN^\lambda \right) \right. \\
&\quad - (tN^\lambda)^{\frac{\lambda-\mu+1-n}{\lambda}} \frac{\Gamma(\frac{\mu+n-1}{\lambda})\Gamma(1-\frac{n-1}{\lambda})}{\Gamma(2-\frac{\mu+n-1}{\lambda})\Gamma(\frac{\mu}{\lambda})} \\
&\quad \times {}_1F_1 \left(1-\frac{n-1}{\lambda}, 2-\frac{\mu+n-1}{\lambda}; tN^\lambda \right) \Big\} \quad (136)
\end{aligned}$$

when $\lambda < 0$. But we do not need all the moments and we shall discuss only the moments of order 1, that is to say :

$$\begin{aligned}
M_1(t) &= N \quad \text{if } \lambda > 0, \\
M_1(t) &= N \frac{\Gamma(1-\frac{\mu}{\lambda}, tN^\lambda)}{\Gamma(1-\frac{\mu}{\lambda})} \quad \text{if } \lambda < 0.
\end{aligned} \quad (137)$$

The important result lies in the last equation which shows that the mass-conservation fails. This is because a part of the mass is lost in zero-size (infinitesimal size) clusters[104]. An analogous behaviour happens for aggregation processes where, under some conditions on the aggregation rates, a finite proportion of the total mass is lost by the very rapid formation of an infinite-size cluster. This phenomenon is called the gelation transition[105]. Here, this behaviour can be also studied as a transition phenomenon between two phases : a *normal phase* and a *shattered phase*[91] the latter being defined by the condition :

$$\int_0^\infty s n_s(t) ds < N \quad (138)$$

The integral is not sensible to zero-measure singularities, when they are not strong, and so, the infinite number of zero-size clusters is not taken into account in this left-hand side definition of the total mass. The difference between the two sides of the preceding equation is an order parameter for the shattering transition. Note also that the total number of clusters (M_0) does not sign this transition in the sense that it can be infinite in the "normal" phase.

This strange behaviour is a direct consequence of the form of the rate-function $a(s)$. This function increases when the size decreases (i.e. when λ is negative) and smaller are the clusters the faster they break up. This accelerating cascading process explains why there is so rapid formation of the shattered phase, as soon as the fragmentation process begins.

Let us remark that the exact size-distribution (105) leads to a size all-size distribution :

$$n_s(t) \sim s^{-\tau},$$

with an exponent :

$$\tau = 2 + \lambda.$$

In this case, the transition can be determined as well by its characteristic value : $\tau = 2$.

3.7 The discrete binary fragmentation case

An important particular case is when the splitting is binary, *i.e.* the overall fragmentation is a sequential series of elementary break-ups where one cluster gives birth to exactly two fragments at each time. This is not exactly similar to put the average number of fragments equal to 2 in the discussion of preceding sections since, from a mathematical point of view, the rate-equation can be solved even for an average number of fragments which is smaller than 2 . For example, in the case (111) discussed before, to find the scaling solutions a simple change in the variables :

$$\begin{aligned} s &\longrightarrow s^\alpha \\ n_s &\longrightarrow n_s s^\beta \end{aligned}$$

with the parameters α and β given by the conditions :

$$\int_0^1 u^{\alpha+\beta-1} B(u) du = 2 \quad , \quad (139)$$

and

$$\int_0^1 u^{2\alpha+\beta-1} B(u) du = 1 \quad , \quad (140)$$

maps exactly the problem with any average number of fragments onto a problem with an average number of fragments equal to 2 .

The model we wish to discuss here is then the case where each elementary fragmentation is a binary splitting. Usually, the rate-functions $a(s')$ and $b(s | s')$ are expressed in the form :

$$\begin{aligned} a(s') &= \int_0^{s'} F_{s,s'-s} ds \\ b(s | s') &= \frac{2F_{s,s'-s}}{\int_0^{s'} F_{s,s'-s} ds} \quad , \end{aligned} \quad (141)$$

where now the symmetric function F_{s_1,s_2} gives the rate that a cluster of size $(s_1 + s_2)$ breaks into a fragment of size s_1 , and another of size s_2 . With this new function, the rate-equation writes[106] :

$$\frac{\partial n_s}{\partial t}(t) = -n_s(t) \int_0^s F_{s',s-s'} ds' + 2 \int_s^\infty n_{s'}(t) F_{s,s'-s} ds' \quad , \quad (142)$$

in its continuous version, and[107, 108] :

$$\frac{dn_s}{dt}(t) = -n_s(t) \sum_{s'=1}^{s-1} F_{s',s-s'} + 2 \sum_{s'=s+1}^N n_{s'}(t) F_{s,s'-s} \quad , \quad (143)$$

with discrete variables.

When the fragmentation kernel F_{s_1, s_2} depends only on the sum $s_1 + s_2$ [109], this means that the break-up is independent of the fragment sizes. In this case, all the couples of fragments have the same probability to be created from a given cluster. A solution is known in the generic case :

$$F_{s_1, s_2} = (s_1 + s_2)^{\lambda-1} \quad (144)$$

which corresponds exactly to the choice (102) with $\mu = 2$.

More interesting are the discrete equations (143) which can be solved exactly. First, one can note that these equations can be put as a set of N linear coupled equations. This means that they can be solved by standard matrix algebra [110]. This way has been investigated to get numerical solutions for comparison with experiments [110]. If A is the matrix of kinetic coefficients :

$$\begin{pmatrix} 0 & F_{1,2} + F_{1,2} & F_{1,3} + F_{2,3} & \dots & F_{1,N} + F_{N-1,N} \\ 0 & -F_{1,2} & F_{1,3} + F_{2,3} & \dots & F_{2,N} + F_{N-2,N} \\ 0 & 0 & -\sum_{s=1}^2 F_{s,3} & \dots & F_{3,N} + F_{N-3,N} \\ \vdots & \vdots & \vdots & \dots & \vdots \\ 0 & 0 & 0 & \dots & -\sum_{s=1}^{N-1} F_{s,N} \end{pmatrix}, \quad (145)$$

where the generic off-diagonal term is $F_{i,j} + F_{j-i,j}$, then the configuration $\vec{n} = \{n_1, n_2, \dots\}$ satisfies the following equation :

$$\frac{d\vec{n}}{dt}(t) = A \vec{n}(t) \quad (146)$$

The matrix A is triangular due to the constraint that no recombination is allowed between fragments. So, the eigenvalues of A are just its diagonal terms, say $\{\lambda_1, \lambda_2, \dots\}$, and the matrix U of eigenvectors is also triangular. And if the eigenvalues are not degenerated, the components of the matrix U are given recursively by :

$$\begin{aligned} U_{j,j} &= 1 \\ U_{i-1,j} &= \frac{A_{i-1,j}}{A_{j,j} - A_{i-1,i-1}} U_{i,j} \end{aligned}, \quad (147)$$

Therefore, the complete solution is given by the expression :

$$\vec{n}(t) = U \Lambda(t) U^{-1} \vec{n}_0, \quad (148)$$

and the diagonal matrix Λ is just :

$$\Lambda_{i,i}(t) = \lambda_i e^{\lambda_i t} \quad (149)$$

The case where the eigenvalues are degenerated is not very important here, but anyway, it can be solved by a slight modification of this procedure and has been discussed by Ballauff and Wolf [111]. By this way, the discrete rate-equation can

be solved for finite N and any given fragmentation kernel. This has been done[27, 112] in particular to compare the random scission[113, 114, 115] :

$$F_{i,j} = 1 \quad ,$$

the Gaussian scission :

$$F_{i,j} = \exp(-(j - i/2)^2 / 2\sigma_i^2) \quad ,$$

and central scission :

$$F_{i,j} = \begin{cases} 1 & \text{if } j = (i - 1)/2 \\ 1/2 & \text{if } j = i/2 \text{ or } j = i/2 - 1 \\ 0 & \text{otherwise} \end{cases}$$

Some exact solutions are analytically known for particular choice of kernels. We shall see some of them now.

The case $F_{i,j} = 1$: One can show [116] that this problem is exactly equivalent to the Kuhn model. The solution for any initial conditions writes then :

$$n_s(t) = (1 - q)^{s-1} \left(n_s(0) + \sum_{s'=s+1}^{\infty} n_{s'}(0) [2q + q^2(s' - s - 1)] \right) \quad . \quad (150)$$

The following solutions will be all given for the monodisperse initial condition : $n_s(0) = \delta(s - N)$, and written under the form :

Fragmentation kernel : $F_{i,j}$

Solution : $n_s(t)$

Here after are these solutions :

The first one[116] is, by far, the simplest example after the Kuhn model :

$$\begin{aligned} F_{i,j} &= i + j \\ n_s(t) &= \frac{N}{s} e^{-s^2 t} (e^{st} - e^{-st}) \quad \text{if } 1 \leq s < N \\ n_N(t) &= e^{-N(N-1)t} \end{aligned} \quad (151)$$

Then come two other cases for which the fragmentation kernel looks very similar, but the respective solutions are of different complexities[117, 108] :

$$\begin{aligned} F_{i,j} &= \frac{1}{i + j + 1} \\ n_s(t) &= \sum_{s'=s}^N (-1)^{s'-s} \binom{s'}{s} \binom{N}{s'} e^{-t \frac{s'-1}{s'+1}} \end{aligned} \quad (152)$$

and

$$\begin{aligned}
F_{i,j} &= \frac{1}{i+j-1} \\
n_s(t) &= e^{-t} \sum_{s'=1}^{\infty} \frac{(2t)^{s'}}{s'!} C_{s,s'}^{(N)} \quad \text{if } 1 \leq s < N \\
n_N(t) &= e^{-t}
\end{aligned} \tag{153}$$

The coefficients $C_{s,s'}^{(N)}$ are given by :

$$\begin{aligned}
C_{s,1}^{(N)} &= \frac{1}{N-1} \\
C_{s,j}^{(N)} &= \frac{1}{N-1} \sum_{s_j=s+1}^{N-1} \frac{1}{s_j-1} \sum_{s_{j-1}=s_j+1}^{N-1} \frac{1}{s_{j-1}-1} \cdots \sum_{s_2=s_3+1}^{N-1} \frac{1}{s_2-1} \tag{154}
\end{aligned}$$

Contrary to the two last cases, which were at the shattering transition, the following two cases are inside the shattered phase [101] :

$$\begin{aligned}
F_{i,j} &= \frac{1}{(i+j)^2-1} \\
n_s(t) &= \frac{N}{s} [(N-s+1)e^{-\frac{t}{N+1}} \\
&\quad + 2(N-s)e^{-\frac{t}{N}} + (N-s-1)e^{-\frac{t}{N-1}}] \quad \text{if } 1 < s < N \\
n_1(t) &= N - \frac{N^2}{2}(N-1)e^{-\frac{t}{N+1}} \\
&\quad + N(N-1)(N-2)e^{-\frac{t}{N}} - \frac{N}{2}(N-2)(N-3)e^{-\frac{t}{N-1}} \\
n_N(t) &= e^{-\frac{t}{N+1}}
\end{aligned} \tag{155}$$

and

$$\begin{aligned}
F_{i,j} &= \frac{1}{(i+j)[(i+j)^2-1]} \\
n_s(t) &= e^{-\frac{t}{N(N+1)}} - e^{-\frac{t}{N(N-1)}} \quad \text{if } 1 < s < N \\
n_1(t) &= N - \frac{(N+2)(N-1)}{2} e^{-\frac{t}{N(N+1)}} + \frac{(N+1)(N-2)}{2} e^{-\frac{t}{N(N-1)}} \\
n_N(t) &= e^{-\frac{t}{N(N+1)}}
\end{aligned} \tag{156}$$

3.8 Particular fragmentation kernels

In some cases, precise informations can be obtained on the binary fragmentation kernel associated with the break-up models. Most of these results have been found for percolation models as we shall see below.

We have remarked in sect. (2.6) that the average size-distribution of clusters formed by random breaking of bonds on a Bethe lattice is exactly known. This problem has also been attacked by a modified rate-equation with a source sink [118]. In effect, for any lattice, the formulation of the percolation break-up problem can be written as :

$$\frac{\partial n_s}{\partial t} = -a(s)n_s + \sum_{s' \geq s+1} a(s')b(s' | s)n_{s'} + a(\infty)b(s | \infty) \quad (157)$$

The notations have here the same meaning as for the rate equation (82). The corresponding rate probability functions $a(\infty)$ and $b(s | \infty)$ are respectively the rate at which the infinite cluster breaks up and the rate at which the s -cluster is formed by fragmentation of the infinite cluster. Of course, these quantities are strictly equal to 0 when $q < q_c$ since there is no infinite cluster in this case. Cutting one bond cannot produce more than 2 fragments, so this is a pure binary process and (157) should be written as :

$$\frac{dn_s}{dt} = -n_s \sum_{s'=1}^{s-1} F_{s',s-s'} + 2 \sum_{s'=s+1}^{\infty} F_{s,s'-s} n_{s'} + F_{s,\infty} \quad (158)$$

Now, the kinetic coefficients $F_{s,s'}$ are directly related to the different possible breaks of the "red bonds", i.e. these particular bonds which break the cluster into two fragments when cut. Note that for the Bethe lattice, all the bonds are "red bonds" since there are no loops in the geometric structure.

When on a Bethe lattice, and if the sizes s and s' are finite, the fragmentation probability $F_{s,s'}$ is just proportional to the total number of ways of building a cluster of size $s + s'$ containing the bond to be cut, with one cluster of size s at a side of this bond and another cluster, of size s' , at the opposite side. So :

$$F_{s,s'} \propto W_s W_{s'} \quad ,$$

where W_s denotes the number of distinct configurations of s sites containing one given site. This number is exactly known [46], and equals to :

$$W_s = \frac{1}{(z-2)s+1} \binom{(z-1)s}{s} \quad (159)$$

For the large values of both s and s' , one finds the remarkable homogeneous behaviour [119] :

$$F_{s,s'} \propto (ss')^{-3/2} \quad , \quad (160)$$

which is valid for any $z > 2$. This fragmentation kernel is independent of s and s' if $z = 2$. The factor of proportionality in the last equation is a function of the size of the parent : $s + s'$. It can be calculated explicitly since in this bond-percolation process, the overall rate probability of fragmenting $a(s)$ of an s -cluster must be proportional to the number $s-1$ of its bonds. The eq. (141) linking $a(s)$ to the $F_{s,s'}$ writes :

$$\sum_{s'=1}^{s-1} F_{s',s-s'} = c(s-1) \quad , \quad (161)$$

where the coefficient c is not important ⁴. One concludes that the precise form of the fragmentation kernel must be :

$$F_{s,s'} \propto (s + s')^{5/2} (ss')^{-3/2} , \quad (162)$$

where now, the proportionality factor is independent of the sizes s and s' . It can be set equal to 1 by a suitable choice of the time scale.

This kernel (162) does not depend on the value of the proportion q of cut bonds. In eq. (158) this parameter appears only in the value of $F_{s,\infty}$, which must vanish for $q < q_c$. If needed, this quantity can be found [46] by replacing in eq. (158) the q -dependent size-distribution by its exact value (37).

On the basis of scaling relations on the average number of "red bonds", it has been conjectured [119] that the fragmentation kernel of the bond-percolation model at the transition threshold $q = q_c$, should be of the form :

$$F_{s,s'} = (s + s')^{\sigma-1} g\left(\frac{s'}{s + s'}\right) \quad (163)$$

where g is a scaling function and the critical exponent σ characterizes how the size of the infinite cluster diverges at the transition :

$$s_\infty \sim (q_c - q)^{-\frac{1}{\sigma}} . \quad (164)$$

For the bond-percolation on a Bethe lattice with any coordination number $z > 2$, one has : $\sigma = 1/2$ and (163) is verified with :

$$g(x) = (x(1-x))^{-3/2} .$$

For the Kuhn model, $\sigma = 1$ and $g(x) = 1$. The two-dimensional bond-percolation model on the square lattice has also been investigated numerically [119]. The results are consistent with (163), and the homogeneity relation :

$$F_{\lambda s, \lambda s'} = \lambda^{\sigma-1} F_{s,s'} \quad (165)$$

with $\sigma = 36/91$ is well verified asymptotically.

4. The Fragmentation-Inactivation Binary (FIB) model [121, 122, 123]

Even if the two approaches to the fragmentation process given above can be quite useful for some applications, nevertheless they have evident conceptual drawbacks : the statistical models suppose quasi-static evolution whereas the kinetic rate-equation cannot reach any final state. In fact, most of the fragmenting systems deal with strong off-equilibrium processes which stop after a certain time due to some dissipative phenomenon. To take these features into account, one generalization of the rate-equation has been introduced (see also [118] for another

⁴The coefficient c is just a scale for the inverse of the time.

generalization with inflow and removal), where a dissipative process stops randomly the sequential fragmentation.

The FIB model is defined as follows. At any stage of the process, one given cluster can be either in an active state, *i.e.* it can break up, or in an inactive state, *i.e.* it cannot break up. Moreover, total irreversibility is assumed, *i.e.* once inactive, the cluster is not allowed to become active at a later time. At the initial time $t = 0$, the system consists of just one active cluster of size N . Every active cluster, say of size s , either becomes inactive or at any time it can break up to give two fragments of respective sizes s' and $s - s'$. Similarly to the kinetic rate-equation, we suppose that the fragmentation occurs with a probability $F_{s', s-s'}$ per unit of time, and the inactivation with a probability I_s per unit of time. By this way, the initial state and the two rate-functions define completely the system and its evolution. Note that the standard kinetic rate-equation must be recovered if one sets $I_s = 0$ in the FIB model.

In the following, we suppose purely binary process : one fragmenting cluster gives birth to (exactly) two fragments. This assumption can be removed and one can consider as simply also more general cases like for example the statistical distribution of fragments at each fragmentation. But there is no need for the moment for such refinements and the binary case will be considered as the *standard model* of this fragmentation-inactivation model which itself can be modified and extended if necessary.

4.1 The master- and cascade- equation

The derivation of the sect. 3.1 and 3.2 for the basic equations of the kinetic rate-equation model can be generalized to find the Marcus and cascade-equations for the FIB model. Let us consider the state of the system at a given time t . It contains, say, m_1 active clusters of size 1, m_2 active clusters of size 2, *etc.*, as well as n_1 inactive clusters of size 1, n_2 inactive clusters of size 2, *etc.* The conservation of the total mass leads to the constraint :

$$\sum_{k=1}^N k (m_k + n_k) = N \quad . \quad (166)$$

We can write the change, per unit of time, of the probability to get this state :

$$\begin{aligned} \frac{\partial W_N(\{\dots\}; t)}{\partial t} = & \sum_{i,j} F_{i,j} [(m_{i+j} + 1) \\ & \times W_N(\{\dots, m_i - 1, \dots, m_j - 1, \dots, m_{i+j} + 1, \dots\}; t) \\ & - m_{i+j} W_N(\{\dots\}; t)] \\ & + \sum_k I_k [(m_k + 1) W_N(\{\dots, m_k + 1, \dots, n_k - 1, \dots\}; t) \\ & - m_k W_N(\{\dots\}; t)] \quad , \quad (167) \end{aligned}$$

where dots in the brackets $\{\dots\}$ denote the state under consideration : $\{m_1, m_2, \dots; n_1, n_2, \dots\}$. Only the changes from this state have been written

explicitely. The meaning of the different terms of this equation is clear : at the right-hand side, the first sum corresponds to the binary fragmentation of a $(i+j)$ - cluster into one cluster of size i and another of size j . The second sum corresponds to the inactivation of a k -cluster. Of course, all the states written in eq. (167) must verify the same constraint (166).

This equation could be used for very small systems where all the configurations can be explicitly written down. But it is practically much more convenient to rewrite this differential equation with the following generating function of the probabilities W :

$$G_N(\vec{x}, \vec{y}; t) = \sum_{\{m_1, \dots, n_N\}} W_N(\{m_1, \dots, n_N\}; t) x_1^{m_1} \dots x_N^{m_N} y_1^{n_1} \dots y_N^{n_N} \quad , \quad (168)$$

where the summation occurs over all the states satisfying to (166). This leads to the following equation, which is a generalization of (93) :

$$\frac{\partial G_N}{\partial t}(\vec{x}, \vec{y}; t) = \sum_k \left[\sum_{i=1}^{k-1} F_{i, k-i} (x_i x_{k-i} - x_k) + I_k (y_k - x_k) \right] \frac{\partial G_N}{\partial x_k}(\vec{x}, \vec{y}; t) \quad , \quad (169)$$

and the boundary conditions of this equation are the normalization condition :

$$G_N(1, 1; t) = 1 \quad ,$$

and the initial condition :

$$G_N(x, y; 0) = x_N \quad .$$

All the correlations can be calculated from this generating function. In this section we shall use the identities :

$$\begin{aligned} \langle m_k(t) \rangle_N &= \frac{\partial G_N}{\partial x_k}(\vec{1}, \vec{1}; t) \\ \langle n_k(t) \rangle_N &= \frac{\partial G_N}{\partial y_k}(\vec{1}, \vec{1}; t) \quad . \end{aligned} \quad (170)$$

From the Marcus equation (169), we deduce the linear master equations for the average size-distributions :

$$\begin{aligned} \frac{d \langle m_k(t) \rangle_N}{dt} + \Phi_k \langle m_k(t) \rangle_N &= 2 \sum_{i=1}^{N-k} F_{i, k} \langle m_{i+k}(t) \rangle_N \\ \frac{d \langle n_k(t) \rangle_N}{dt} &= I_k \langle m_k(t) \rangle_N \quad , \end{aligned} \quad (171)$$

where we used the notation :

$$\Phi_k = I_k + \sum_{i=1}^{k-1} F_{i, k-i} \quad . \quad (172)$$

The initial conditions are :

$$\langle m_k \rangle_N = \delta(k - N)$$

and

$$\langle n_k \rangle_N = 0 .$$

The cascade equation (see sect. 3.2) is found by noticing that the very first event is either a fragmentation with the probability $F_{i,N-i}dt$ or an inactivation with the probability $I_N dt$. Then, the probability to obtain the state $\{m, n\}$ at time t can be expressed as the product of the probabilities to get the state $\{\mu, \nu\}$ from an initial cluster of size i , and the state $\{m - \mu, n - \nu\}$ from an initial cluster of size $N - i$. More precisely :

$$\begin{aligned} W_N(\{\vec{m}, \vec{n}\}; t) &= \sum_{i=1}^{N-1} \int_0^t \sum_{\{\vec{\mu}, \vec{\nu}\}} W_i(\{\vec{\mu}, \vec{\nu}\}; t - t') \\ &\times W_{N-i}(\{\vec{m} - \vec{\mu}, \vec{n} - \vec{\nu}\}; t - t') e^{-\Phi_N t'} F_{i,N-i} dt' \\ &+ \delta(\vec{m} - \vec{N}) \delta(\vec{n}) e^{-\Phi_N t} \\ &+ \delta(\vec{m}) \delta(\vec{n} - \vec{N}) \int_0^t e^{-\Phi_N t'} I_N dt' \end{aligned} \quad (173)$$

, where we have used the property that $\exp(-\Phi_N t)$ is the probability that no event occurs in the time-interval $[0, t]$.

We see on example of this equation that the FIB model is a non-linear process, since the probability distribution W_N is a linear function of products like $W_i * W_{N-i}$.

As for the Marcus equation, the cascade-equation is simpler when written in terms of the generating function (168) :

$$\begin{aligned} \frac{\partial G_N(\vec{x}, \vec{y}; t)}{\partial t} + \Phi_N G_N(\vec{x}, \vec{y}; t) &= \sum_{i=1}^{N-1} F_{i,N-i} G_i(\vec{x}, \vec{y}; t) G_{N-i}(\vec{x}, \vec{y}; t) \\ &+ I_N y_N . \end{aligned} \quad (174)$$

An important remark must be done now : while the linear Marcus equation (169) works with the probabilities G_N corresponding to the same initial state (N), the non-linear cascade-equation (174) deals with the probabilities G_i for different initial conditions. There are the same informations in both equations, but they are not written in the same way. This is still more evident when we derive the equations for the average size-distributions from the cascade-equation :

$$\frac{d \langle n_k(t) \rangle_N}{dt} + \Phi_k \langle n_k(t) \rangle_N = 2 \sum_{i=1}^{N-1} F_{i,N-i} \langle n_k(t) \rangle_i + I_k \delta(k - N) . \quad (175)$$

We have to keep the memory of the initial state for every quantity, and this is achieved with the notation $\langle \dots(t) \rangle_N$ which means the average value depending on the time t with a total system size equal to N .

4.2 A soluble case : $F_{i,j} = 1$

We wish to solve in this section an important case : the final ($t = \infty$) size-distribution for any inactivation kernel I_N and when $F_{i,j}$ does not depend on the sizes of the fragments. First, the general final size-distribution is given by the eqs. (171) and (175) where the time is set equal to infinite, namely :

$$\Phi_k < n_k >_N = 2I_k \sum_{i=k+1}^N \frac{< n_i >_N}{I_i} + I_k \delta(k - N) \quad , \quad (176)$$

which corresponds to the master-equation, and :

$$\Phi_k < n_k >_N = 2 \sum_{i=1}^{N-1} < n_k >_i + I_k \delta(k - N) \quad , \quad (177)$$

which corresponds to the cascade-equation. The first equation is a backward recurrent equation on the size k for a fixed initial condition N , while the other (177) is an upward recurrent equation on the initial condition N for a fixed size k . Obviously, these two equations must have the same set of solutions

These equations are invariant when replacing :

$$\{F_{i,j}, I_{i+j}\}$$

by :

$$\left\{ \frac{F_{i,j}}{f(i+j)}, \frac{I_{i+j}}{f(i+j)} \right\}$$

So, if the fragmentation kernel $F_{i,j}$ depends only on the sum $i+j$, it can be set equal to 1 by a proper renormalization of the inactivation kernel. We will consider then the case : $F_{i,j} = 1$: The eqs. (176) or (177) can be solved by considering them for two consecutive values of the parameters. The resulting size-distribution can be written as a finite product :

$$\begin{aligned} < n_k >_N &= \frac{2}{I_N + N - 1} \frac{I_k}{I_k + k + 1} \prod_{j=k}^{N-1} \left(1 + \frac{2}{I_j + j - 1} \right) \quad \text{for } k < N \\ < n_N >_N &= \frac{I_N}{I_N + N - 1} \end{aligned} \quad (178)$$

If the inactivation kernel is such that $I_k = o(k)$ then the product behaves like $(N/k)^2$ and the size-distribution is :

$$< n_k >_N \sim \frac{NI_k}{k^3} \quad (179)$$

An important particular case occurs when the inactivation kernel is a power-law :

$$I_k = I_1 k^\beta \quad .$$

In this homogeneous kernel, the positive coefficient I_1 is called the strength of the inactivation kernel and β its exponent. Then, if β is smaller than 1, we obtain naturally a power-law size-distribution :

$$\langle n_k \rangle_N \sim k^{-\tau} \quad ,$$

and $\tau = 3 - \beta$. This value is here independent of the strength I_1 : it is always larger than 2 and tends to 2 when β approaches the value 1. We must compare this result with what happens if one takes $\beta = 1$. In this latter case, the finite product in (178) behaves still as a power-law with N/k , but with an exponent $2/(I_1 + 1)$ instead of 2. This shows that the average size-distribution is a power-law with the exponent $\tau = 2/(I_1 + 1)$. This value of τ depends now of the strength I_1 and is always smaller than 2. Then, there is a jump in the value of τ when β crosses the value $\beta = 1$ for a given strength. For β larger than 1, there is no more power-law behaviour of the size-distribution.

This different behaviour for $\beta < 1$ and $\beta > 1$ in the case of the constant fragmentation kernel, can be examined in more details with the help of other quantities. The first one we will study now is the so-called *dust-mass*, and is defined as :

$$\nu(N; t) = \frac{\langle n_1(t) \rangle_N}{N} \quad , \quad (180)$$

where N is the total size of the system. This quantity characterizes the total mass contained in the smallest-size clusters. In a continuous framework, this should correspond to the total mass in the zero-mass phase as defined in sect. (3.6). The cascade equation (175) allows to write the following equation for the dust-mass when $F_{i,j} = 1$:

$$\frac{d\nu}{dt}(N; t) + (I_N + N - 1) \nu(N; t) = \frac{2}{N} \sum_{i=1}^{N-1} i \nu(i; t) \quad , \quad (181)$$

with the initial condition :

$$\nu(1; t) = 1 - e^{-I_1 t} \quad . \quad (182)$$

In terms of the Laplace transform $\nu(N; p)$ of the dust-mass, this writes as a simple recurrent equation :

$$N(\Phi_N + p) \nu(N; p) = 2 \sum_{i=1}^{N-1} i \nu(i; p) \quad , \quad (183)$$

which for the initial condition :

$$\nu(N; p) = \frac{I_1}{p(I_1 + p)} \quad ,$$

solves as :

$$p \nu(N; p) = \frac{1}{N(p + \Phi_N)} \frac{2I_1}{p + I_1} \prod_{j=2}^{N-1} \left(1 + \frac{2}{p + \Phi_j} \right) \quad . \quad (184)$$

Let us tend with N to infinity in the last expression. We see that the behaviour of the dust-mass will be governed by the convergence of the sum :

$$\sum_n \frac{1}{\Phi_n} .$$

If this sum is convergent then the dust-mass will be asymptotically zero, while it can be finite when the sum diverges. For the moment we will discuss only the final state

Fig. 4. Behaviour of the order parameter $1 - \nu(N)$ at the limit $N = \infty$ in the case $\alpha = 0$. This quantity is equal 1 for all values of β larger than 1 [121].

($t = \infty$) and not the whole time evolution. In this case, the final dust-mass is given by :

$$\nu(N) = \frac{2}{N(I_N + N - 1)} \exp \left[\sum_{j=2}^{N-1} \ln \left(1 + \frac{2}{I_j + j - 1} \right) \right] , \quad (185)$$

and because of the inequality :

$$\frac{1}{I_j + j - 1} \geq \frac{1}{j} - \frac{I_j}{j^2} , \quad (186)$$

we deduce a lower-bound for the asymptotic dust-mass :

$$\lim_{N \rightarrow \infty} \nu(N) \geq 0.858 \exp \left[-2 \sum_{j=2}^{\infty} \frac{I_j}{j^2} \right] . \quad (187)$$

It is then clear that if the inactivation kernel I_N is $o(N)$, the asymptotic dust-mass is finite.

When the inactivation kernel is a homogeneous function of the size, the infinite system experiences a phase transition when the exponent β crosses the value 1 . For $\beta > 1$, the asymptotic dust-size is zero, and when $\beta < 1$ a finite part of the total mass is contained in the monomers. We recover here the so-called *shattering transition* as described in sect. 3.6 for the rate-equation model. Fig. 4 shows the behaviour of the asymptotic dust-mass, which plays the role of the phase parameter, when the inactivation kernel is a power-law with exponent β . For β smaller but close to 1, the dust-mass behaves like :

$$\exp \left(-\frac{2I_1}{1-\beta} \right) .$$

In the terms of the phase transition, this corresponds to an essential singularity.

4.3 More particular cases

In the preceding section, we have discussed the choice $F_{i,j} = 1$ and any inactivation kernel. For special cases of the function I_k , one can find simple expression for the size-distribution. These explicit examples can be useful to understand certain features of more general cases.

(i) $F_{i,j} = 1$ and $I_k = I_1/k$:

The eq. (178) gives :

$$\begin{aligned} \langle n_k \rangle_N &= \frac{2I_1 N}{(k^2 + I_1)^2 - k^2} \quad \text{for } k < N \\ \langle n_N \rangle_N &= \frac{I_1}{N^2 - N + I_1} , \end{aligned} \quad (188)$$

and we easily verify that $\tau = 4$ according to the formula $\tau = 3 - \beta$. Moreover, the value of the asymptotic dust-mass is :

$$\lim_{N \rightarrow \infty} \nu(N) = \frac{2}{I_1 + 2} , \quad (189)$$

so that the system is inside the shattered phase. Moreover, we can compute for example the moment of order 0 (the total number of clusters in the final state) :

$$\sum_{k=1}^N \langle n_k \rangle_N = N \sum_{k=1}^{N-1} \frac{1}{k^2 + k + I_1} + 1 , \quad (190)$$

from which one define a typical cluster size :

$$S = \left(\sum_{k=1}^{\infty} \frac{1}{k^2 + k + I_1} \right)^{-1}, \quad (191)$$

which is finite in this case.

(ii) $F_{i,j} = 1$ and $I_k = I_1$:

The exponent τ must be equal 3 since $\beta = 0$. This is obvious since the eq. (178) writes here as :

$$\begin{aligned} \langle n_k \rangle_N &= \frac{2I_1(N + I_1)}{(k + I_1 - 1)(k + I_1)(k + I_1 + 1)} \quad \text{for } k < N \\ \langle n_N \rangle_N &= \frac{I_1}{N + I_1 - 1} \end{aligned} \quad (192)$$

The system is still in the shattered phase since :

$$\lim_{N \rightarrow \infty} \nu(N) = \frac{2}{(I_1 + 1)(I_1 + 2)} \quad (193)$$

is positive.

Like in the previous example, the total number of clusters is almost proportional to the total size :

$$\sum_{k=1}^N \langle n_k \rangle_N = \frac{N + I_1}{1 + I_1}, \quad (194)$$

and the typical size of the clusters is finite :

$$S = 1 + I_1 \quad (195)$$

(iii) $F_{i,j} = 1$ and $I_k = k + a$:

This is a very important case indeed, since it is exactly located at the shattering transition. The eq. (178) gives :

$$\begin{aligned} \langle n_k \rangle_N &= \frac{2(k + a)}{(2k + a)^2 - 1} \quad \text{for } k < N \\ \langle n_N \rangle_N &= \frac{N + a}{2N + a - 1} \end{aligned} \quad (196)$$

The value of τ ($= 1$) is consistent with the formula $\tau = 2/(I_1 + 1)$ since $I_1 = 1$. But the dust-mass behaves now for large values of N :

$$\nu(N) \sim \frac{2}{N(a + 3)}, \quad (197)$$

and its asymptotic value is obviously 0.

The total number of clusters is given by :

$$\sum_{k=1}^N \langle n_k \rangle_N = 1 + \sum_{k=2}^N \frac{1}{2k + a - 1} \quad , \quad (198)$$

which shows that the typical size of the clusters diverges logarithmically as :

$$S \sim \frac{2N}{\ln(N)} \quad . \quad (199)$$

This is the sign that the large clusters dominate in this system.

(iv) $F_{i,j} = 1$ and $I_k = (k+1)^2$:

No power-law is expected in this case since the system is no more in the shattered phase. More precisely the formula (178) writes :

$$\begin{aligned} \langle n_k \rangle_N &= \frac{2}{(N+2)(N+3)} \frac{k+1}{k} \quad \text{for } k < N \\ \langle n_N \rangle_N &= \frac{(N+1)^2}{N(N+3)} \quad . \end{aligned} \quad (200)$$

In principle, if one does not take into account the largest mass $k = N$, the size-distribution could be described as a particular power-law with an exponent $\tau = 0$. But this is not very important since almost all the total mass is contained in one cluster of size N . Of course, the asymptotic dust-mass is zero :

$$\nu(N) \sim \frac{4}{N^3} \quad , \quad (201)$$

the average total number of clusters is finite :

$$\sum_{k=1}^N \langle n_k \rangle_N = \frac{N^2 + 6N + 3 + 2 \sum_{k=1}^N k^{-1}}{(N+2)(N+3)} \quad , \quad (202)$$

and hence, the typical size of the clusters behaves like N .

In fact all these exact formulas are particular example of a more general case. To derive this case, one seeks size-distributions under the factorized form :

$$\langle n_k \rangle_N = c(N)f(k) \quad . \quad (203)$$

Then, the cascade equation (177) can be written as :

$$I_N = \frac{2}{c(N)} \sum_{i=k+1}^{N-1} F_{i,N-i} c(i) - \sum_{i=1}^{N-1} F_{i,N-i} + \frac{2F_{k,N-k}}{c(N)f(k)} \langle n_k \rangle_k \quad . \quad (204)$$

The function I_N must be independent on k such that, writing the preceding equation for k and for $k-1$ one obtains a recurrent equation for the values of $\langle n_k \rangle_k$ which solves according to :

$$\langle n_k \rangle_k = \frac{f(k)}{F_{k,N-k}} \left[\sum_{i=2}^k F_{i,N-i} c(i) + \frac{F_{1,N-1}}{f(1)} \right] \quad . \quad (205)$$

Since the left-hand side quantity does not depend on the value of N , one concludes that the fragmentation kernel must be a constant. We will take $F_{i,j} = 1$. Moreover, (205) must be coherent with the total size-conservation :

$$\langle n_k \rangle_k = 1 - \frac{c(k)}{k} \sum_{i=1}^{k-1} i f(i) . \quad (206)$$

This implies that the size-distribution is exactly given by (203) with :

$$c(N) = N \left(\sum_{i=1}^{N-1} i f(i) \right)^{-1} \left[1 - f(N) \left(\sum_{i=2}^N c(i) + \frac{1}{f(1)} \right) \right] , \quad (207)$$

provided the kernels are such that :

$$\begin{aligned} F_{i,j} &= 1 , \\ I_N &= \sum_{i=1}^{N-1} \left(\frac{2c(i)}{c(N)} - 1 \right) + \frac{2}{c(N)} \left(\frac{1}{f(1)} - c(1) \right) . \end{aligned} \quad (208)$$

In this way, this class of exact solutions is defined by the value of the function $f(k)$ and $c(1)$.

4.4 The power-law solutions

We shall discuss in this section a more general case where the fragmentation and inactivation kernels are given respectively by the homogeneous functions of the sizes :

$$\begin{aligned} F_{i,j} &= (ij)^\alpha \\ I_k &= I_1 k^\beta . \end{aligned} \quad (209)$$

We know two recurrent equations to compute the final average size-distribution , the master equation :

$$\Phi_k \langle n_k \rangle_N = 2I_k \sum_{i=k+1}^N F_{k,i-k} \frac{\langle n_i \rangle_N}{I_i} + I_N \delta(k - N) , \quad (210)$$

and the cascade equation :

$$\Phi_N \langle n_k \rangle_N = 2 \sum_{i=1}^{N-1} F_{i,N-i} \langle n_k \rangle_i + I_N \delta(k - N) . \quad (211)$$

We search for approximate solutions of the form :

$$\langle n_k \rangle_N \sim c(N) k^{-\tau} , \quad (212)$$

with the positive function $c(N)$ depending only of the total mass N , and this equation should be valid for $1 \ll k < N$.

If we suppose that $\tau + \beta - \alpha > 1$, the eq. (210) writes :

$$I_1 k^\beta = 2k^{\tau+\beta+\alpha} \sum_{i=1}^{\infty} \frac{i^\alpha}{(i+k)^{\tau+\beta}} - \sum_{i=1}^{k-1} i^\alpha (k-i)^\alpha \quad (213)$$

The two sums appearing in this equation can be replaced by their expansion :

$$\begin{aligned} \sum_{i=1}^{N-1} i^a (N-i)^b &= \frac{\Gamma(a+1)\Gamma(b+1)}{\Gamma(a+b+2)} N^{a+b+1} \\ &+ \sum_{n=0}^{\infty} \frac{(-1)^n}{n!} \left[\frac{\Gamma(a+1)}{\Gamma(a+1-n)} \zeta(-b-n) N^{a-n} \right. \\ &\left. + \frac{\Gamma(b+1)}{\Gamma(b+1-n)} \zeta(-a-n) N^{b-n} \right] \quad , \quad (214) \end{aligned}$$

and :

$$\begin{aligned} \sum_{i=1}^{\infty} \frac{i^a}{(i+N)^b} &= \frac{\Gamma(a+1)\Gamma(b-a-1)}{\Gamma(b)} N^{a-b+1} \\ &+ \sum_{n=0}^{\infty} \frac{(-1)^n}{n!} \frac{\Gamma(b+n)}{\Gamma(b)} \zeta(-a-n) N^{-b-n} \quad , \quad (215) \end{aligned}$$

at least if the parameters a and b are not equal to $-1, -2, \dots$. Replacing in (213) and taking the leading terms when k is large implies that :

$$\tau = 2\alpha + 3 - \beta \quad \text{if} \quad \alpha > -2 \quad \text{and} \quad \beta < 2\alpha + 1 \quad (216)$$

$$\frac{\Gamma(\tau + \alpha)}{\Gamma(\tau + 2\alpha + 1)} = \frac{\Gamma(\alpha + 2)}{\Gamma(2\alpha + 3)} + \frac{I_1}{2\Gamma(\alpha + 1)} \quad \text{if} \quad \alpha > -2, \alpha \neq -1 \quad \text{and} \quad \beta = 2\alpha + 1 \quad (217)$$

$$\psi(\tau - 1) = \psi(1) - \frac{I_1}{2} \quad \text{if} \quad \alpha = \beta = -1 \quad (218)$$

In the above expression ψ denotes the Euler digamma function. Unfortunately, when α is smaller than -2 , the assumption $\tau + \beta - \alpha > 1$ can break down. An analysis of the leading singularities of the generating function of the size-distribution :

$$\sum_{k=1}^N \left[\frac{I_k + \sum_{i=1}^{k-1} i^\alpha (k-i)^\alpha}{k^\alpha} - 2 \sum_{j=1}^{k-1} j^\alpha e^{-jx} \right] \frac{<n_k>_N}{I_k} e^{kx} = \frac{e^{Nx}}{N^\alpha} \quad , \quad (219)$$

derived from the eq. (210) leads to the values of τ :

$$\tau = \alpha - \beta \quad \text{if} \quad \alpha < -2 \quad \text{and} \quad \beta < \alpha - 1 \quad (220)$$

$$\tau = 1 - \frac{I_1}{2\zeta(-\alpha - 1)} \quad \text{if} \quad \alpha < -2 \quad \text{and} \quad \beta = \alpha - 1 \quad (221)$$

Even if, qualitatively, the behaviour of the system is similar for α smaller or larger than 2 (there is one critical value of β , say β_c , for which the shattered phase exists when $\beta < \beta_c$), there are many important quantitative differences. For example, τ is always larger than 2 if $\alpha > -2$ but is larger than 1 if $\alpha < -2$. On the transition line, the value of τ verifies the double inequality :

$$-\alpha < \tau < 2 \quad \text{if} \quad \alpha > -2$$

and

$$\tau < 1 \quad \text{if} \quad \alpha < -2 \quad .$$

The line $\alpha = -2$ appears here as very particular but its meaning is not well understood. The values found here for the general case applies indeed for the particular choice $\alpha = 0$ detailed in the sects. (4.2) and (4.3), namely :

$$\tau = 3 - \beta \quad \text{if} \quad \beta < 1$$

and

$$\tau = \frac{2}{1 + I_1} \quad \text{if} \quad \beta = 1 \quad .$$

The same analysis can be done for another homogeneous choice :

$$\begin{aligned} F_{i,j} &= \frac{1}{2}(i^{2\alpha} + j^{2\alpha}) \\ I_k &= I_1 k^\beta \quad . \end{aligned} \tag{222}$$

which could be relevant for the binary approximation to the spinodal instabilities of the Fermi liquid (see sect. 2.10). The results are of course very similar to the previous multiplicative case. They write :

$$\tau = 2\alpha + 3 - \beta \quad \text{if} \quad 2\alpha > -2 \quad \text{and} \quad \beta < 2\alpha + 1 \tag{223}$$

$$\begin{aligned} \frac{\Gamma(\tau)}{\Gamma(\tau + 2\alpha + 1)} &= \frac{\tau - 1}{(\tau + 2\alpha)\Gamma(2\alpha + 2)} + \frac{I_1}{\Gamma(2\alpha + 1)} \\ &\text{if} \quad 2\alpha > -2, 2\alpha \neq -1 \quad \text{and} \quad \beta = 2\alpha + 1 \end{aligned} \tag{224}$$

$$\psi(\tau - 1) = \psi(1) - I_1 \quad \text{if} \quad 2\alpha = -1 \quad \text{and} \quad \beta = 0 \tag{225}$$

and for $\alpha < -2$ the formulas must be changed according to :

$$\tau = -\beta \quad \text{if} \quad 2\alpha < -2 \quad \text{and} \quad \beta < -1$$

$$\tau = 1 - \frac{I_1}{\zeta(-2\alpha - 1)} \quad \text{if} \quad 2\alpha < -2 \quad \text{and} \quad \beta = -1 \quad .$$

These results are quite important. In particular, because they show clearly that there is no need for any critical phenomenon to get power-law size distributions. This explains, at least in principle, why so many different physical fragmentation processes lead experimentally to such distributions. It would be strange to try to explain systematically all these processes by some kind of a critical behaviour. Nevertheless, we will see in the next section that a true phase transition occurs in the FIB model according to the qualitative change in the size-distribution.

4.5 The shattering transition

We have seen in the sect. (4.2) that when the inactivation kernel is a sufficiently fast decreasing function of the size, then a finite part of the total mass is found in the monomers. This shattered phase is defined by the non-vanishing dust-mass parameter :

$$\lim_{N \rightarrow \infty} \frac{\langle n_1 \rangle_N}{N} \quad . \quad (226)$$

For any general fragmentation and inactivation kernels, the cascade equation verified by the dust-mass writes :

$$\frac{\langle n_1 \rangle_N}{N} = \frac{2}{N\Phi_N} \sum_{i=1}^{N-1} i F_{i,N-i} \frac{\langle n_1 \rangle_i}{i} \quad . \quad (227)$$

Let us suppose now that the following large- N expansion holds :

$$\frac{\langle n_1 \rangle_N}{N} = a_0 + \frac{a_1}{N^{b_1}} + \dots \quad , \quad (228)$$

with a_0 a non-negative coefficient, a_1 a non-zero coefficient and the b_j 's are increasing and all positive. Replacing in the eq. (227) leads to the equality :

$$a_0 N I_N + a_1 N^{1-b_1} I_N + \dots = a_1 \sum_{i=1}^{N-1} (2i^{1-b_1} - N^{1-b_1}) F_{i,N-i} + \dots \quad , \quad (229)$$

where we have used the symmetry of the function $F_{i,j}$ to cancel terms of the type :

$$\sum_{j=1}^{N-1} (N-2j) F_{j,N-j} \quad .$$

For the homogeneous multiplicative kernels (209), the expansion (229) leads to the condition :

$$b_1 = \text{Max}\{2\alpha + 2, \alpha\} - \beta - 1 \quad \text{if} \quad a_0 > 0 \quad .$$

Since this exponent must be positive, we deduce that the shattered phase ($a_0 > 0$) lies inside the region :

$$\beta < 2\alpha + 1 \quad \text{if} \quad \alpha > -2$$

$$\beta < \alpha - 1 \quad \text{if} \quad \alpha < -2 \quad .$$

Considering the case, $a_0 = 0$, one can show similarly that this is also a sufficient condition. We remark that this phase corresponds to the region of the phase-diagram where the size-distribution behaves like a power-law (after sect. 4.4), and we will see another characterization in the following section. This is the same for the additive fragmentation kernel. The Fig. 5 gives a sketch of the phase diagram of the FIB model.

Fig. 5. Phase diagram of the FIB model for the homogeneous multiplicative kernels .

An important alternative characterization of this transition can be given now. In the FIB model, each individual event is either a fragmentation or an inactivation. One can define the probability attached to this choice, namely the probability p_F that the event is a fragmentation without specifying the sizes of the daughter fragments :

$$p_F(k) = \frac{\sum_{i=1}^{k-1} F_{i,k-i}}{I_k + \sum_{i=1}^{k-1} F_{i,k-i}} \quad , \quad (230)$$

and the probability that the event is an inactivation : $p_I = 1 - p_F$. One characterization of the transition is that these probabilities p_F or p_I are independent of

the size of the object which is under consideration. Moreover, when the instability of the larger clusters is more important than the instability of the smaller ones, p_F is an increasing function of the size and the system is in the shattered phase, and conversely. This criterium can be very useful to define from some informations about the microscopic processes leading to the fragmentation, in which phase the system is situated.

4.6 The infinite-cluster behaviour

The average size of the largest cluster is not so easy to handle analytically, even if this is a simple quantity to compute by numerical simulations. Nevertheless, a recurrent inequality can be written by the following cascade reasoning.

Let us call $\langle s_{max} \rangle_N$ the average final largest size in the system. The very first event is either an inactivation of the cluster of size N (and then $s_{max} = N$) or its fragmentation into one (i) -cluster and one $(N-i)$ -cluster. Then, the cascade

Fig. 6. The reduced average maximum size versus the total size N of the system for $\alpha = -2.5$ and two values of β : $\beta = -3.5$ (on the transition line) and $\beta = -4.0$ (in the shattered phase) [123] .

beginning with the cluster of size i leads to a largest cluster of size $s_{max}(i)$ with

average $\langle s_{max} \rangle_i$, and similarly with the cascade beginning with the $(N - i)$ -cluster. This means that :

$$\langle s_{max} \rangle_N \geq \sum_{i=1}^{N-1} \text{Max}(\langle s_{max} \rangle_i, \langle s_{max} \rangle_{N-i}) \frac{F_{i,N-i}}{\Phi_N} + N \frac{I_N}{\Phi_N} \quad , \quad (231)$$

with the initial condition :

$$\langle s_{max} \rangle_1 = 1 \quad .$$

Suppose now that the ratio : $\langle s_{max} \rangle_N / N$ behaves like a positive number σ when N tends to infinity. Replacing $\langle s_{max} \rangle_N$ by σN in (231) leads to the lower bound :

$$\sigma \geq \left(1 + 2 \frac{\sum_{i=1}^{N/2} i F_{i,N-i}}{N I_N} \right)^{-1} \quad . \quad (232)$$

It is easy to show that when the system is not in the shattered phase, the right-hand side of the preceding inequality tends to 1 . This implies that $\sigma = 1$ outside of the shattered phase. This means also that almost all the mass is concentrated into one big cluster. For this reason, we shall call this phase : *the infinite-cluster phase*.

A typical example is shown in Fig. 6, where we have simulated the FB process 1000 times with $\alpha = -2.5$, $I_1 = 1$ and two different values of β ($\beta = -3.5$ is on the transition line, and $\beta = -4.0$ which is in the shattered phase). At the transition point, the value of σ remains finite while $\langle s_{max} \rangle_N / N$ behaves like a power-law in the shattered phase ($N^{-0.5}$ for $\beta = -4.0$). This parameter could be used as another order-parameter to define the different phases : it is equal to 1 in the infinite-cluster phase and is 0 in the shattered phase. On the transition line, its value is between 0 and 1 .

4.7 The time evolution

Up to now, we have discussed the properties and characterizations of the final state ($t = \infty$), and we have seen that different features of the size-distribution depend on some mixing of the fragmentation and inactivation kernels. For example, the value of the exponent τ of the final size-distribution is given by $2\alpha + 1 - \beta$ when the system is inside the shattered phase and $\alpha > -2$. This means that the knowledge of the value of τ alone is not enough to determine the unknown rate functions $F_{i,j}$ and I_k . This degeneracy is believed to be broken if one has the whole time evolution of the distribution. This is because the number and the lengths of the cascades should depend strongly and by a very non-trivial way on the fragmentation and inactivation kernels, even if the final average state is the same.

An analytical approach may be useful here to clarify this point. The master-equation (171) can be rewritten under the form :

$$\frac{d \langle n_k(t) \rangle_N}{dt} + \Phi_k \langle n_k(t) \rangle_N = 2I_k \sum_{i=k+1}^N \frac{F_{k,i-k}}{I_i} \langle n_i(t) \rangle_N + I_N \delta(k-N) \quad , \quad (233)$$

where the initial condition is :

$$\langle n_k(0) \rangle_N = 0 \quad .$$

The formal set of solutions of these equations is :

$$\langle n_k(t) \rangle_N = \sum_{j=k}^N I_k b_{j,k} (1 - e^{-\Phi_j t}) \quad , \quad (234)$$

where the coefficients $b_{j,k}$ which depend also on the value of the total mass N , are given by the following recurrences :

$$\begin{aligned} b_{N,N} &= \frac{1}{\Phi_N} \\ b_{j,k} &= \frac{2}{\Phi_k - \Phi_j} \sum_{i=k+1}^j F_{k,i-k} b_{j,i} \quad \text{for } k < N \quad \text{and } j > k \\ b_{k,k} &= \frac{2}{\Phi_k} \sum_{i=k+1}^N F_{k,i-k} \sum_{j=1}^N b_{j,i} - \sum_{j=k+1}^N b_{j,k} \quad \text{for } k < N \quad . \end{aligned} \quad (235)$$

So, we see that the approach to equilibrium occurs via a function which is the sum of N exponentials of the time. For very small systems, this approach should be mainly exponential, but for large systems this can be quite different. In principle, the knowledge of the time evolution of the average size-distribution can lead to the calculation of all the values of Φ_k by the formula (234).

4.8 The Monte-Carlo numerical simulations

We have seen that many features of the FIB model can be handled analytically more conveniently than for the standard rate-equation model. This is because we are often interested by the final state, and take the limit $t \rightarrow \infty$ in all equations. This leads to great simplifications. Of course, if we have to find the whole time evolution, the difficulties are quite similar for the two models. Anyway, the FIB model is very easy to simulate on a computer, and this is one important point. The scheme is as follows :

0. Take one cluster of size N
1. Choose randomly one active cluster
2. Calculate the probability p_F that this cluster break up
3. Choose one random number x uniformly distributed between 0 and 1
4. If x is larger than p_F , the cluster becomes inactive, then go to 1
- 4'. If x is smaller than p_F , the cluster breaks up : goto 5
5. Calculate the probabilities of all the possible fragmentations of the cluster
6. Choose randomly one of these fragmentations according to these probabilities
7. Split the cluster into the two active fragments corresponding to this fragmentation, then goto 1

The loop 'go to 1' takes place as long as at least one active cluster is present in the system.

Under this form, the time is not computed during the simulation. We only take care of the cascades. In particular, the step 1 can be replaced by any other choice. For example, choose always the largest cluster. as well, since all the cascades are independent. If one wishes to have the physical time evolution, one has to compute at which time every event occurs. This is achieved noting that the probability that one event occurs for a given cluster of size k at a time t after its formation, is just :

$$1 - e^{-\Phi_k t}$$

This gives readily the probability distribution of this time.

5. Fluctuations in some fragmentation models

Relevance of the non-statistical fluctuations for understanding the fragmentation mechanism and cluster formation has been first emphasized in ref. [125]. In this section, we shall discuss some features of the two-size correlation functions and related to it fluctuations in the multiplicity distribution. With some increasing analytical difficulties, one may generalize these results to more general n - size correlation functions, but we expect that higher order correlations do not introduce different qualitative phenomenon. An extensive discussion of the fluctuations in multiplicity distributions and methods to analyze them can be found in a recent review by Božek et al [8] . Here we give only some basic expressions.

From the production probabilities, one can construct the moments of the multiplicity distribution P_n . For example, the i -th moment is :

$$\langle n^i \rangle = \sum_{n=1}^{\infty} n^i P_n . \quad (236)$$

Similarly, the scaled moment of the multiplicity distribution is :

$$C_i = \frac{\langle n^i \rangle}{\langle n \rangle^i} . \quad (237)$$

As we shall see below, the scaled factorial moments are a better tool to study the multiplicity distributions, because they are not contaminated by the statistical noise. Actually, they deconvolute the observed multiplicity distributions from the Poisson distribution, which is a natural *ansatz* for statistical noise superimposed on top of the "physical" distribution. The factorial moment is defined as :

$$\langle n(n-1) \dots (n-i+1) \rangle = \sum_{n=1}^{\infty} n(n-1) \dots (n-i+1) P_n , \quad (238)$$

and the scaled factorial moment as :

$$F_i = \frac{\langle n(n-1) \dots (n-i+1) \rangle}{\langle n \rangle^i} .$$

Let us write the discrete multiplicity distribution P_n in form of a Poisson transform :

$$P_n = \int_0^\infty f(x) \frac{(x\bar{n})^n e^{-x\bar{n}}}{n!} dx ,$$

where $\bar{n} = \langle n \rangle$ and $f(x)$ fulfills the normalization conditions :

$$\begin{aligned} \int_0^\infty f(x) dx &= 1 \\ \int_0^\infty x f(x) dx &= 1 . \end{aligned}$$

The Poisson transform means a convolution of the statistical poissonian noise of mean $\bar{n}$ with the "physical" distribution $f(x)$. The scaled factorial moments of the discrete distribution P_n are related to the moments of the function $f(x)$:

$$F_i = \int_0^\infty x^i f(x) dx . \quad (239)$$

If some correlations are present, the formation of m clusters enhances the formation probability of the $(m+1)$ -st cluster. As a result, the multiplicity distributions are broader than the Poisson distribution. This gives information on integrated correlation functions and not on the correlations between clusters.

The factorial moments of the multiplicity distribution can be expressed also as integrals of the n -particle distribution over a domain Ω of the studied distribution, for example the cluster-size distribution. This gives :

$$\begin{aligned} \langle n \rangle_\Omega &= \int_\Omega dy \rho_1(y) , \\ \dots &= \dots , \\ \langle n(n-1)\dots(n-i+1) \rangle_\Omega &= \int_\Omega dy_1 \dots \int_\Omega dy_i \rho_i(y_1, \dots, y_i) , \end{aligned} \quad (240)$$

and correspondingly the scaled factorial moment :

$$F_i = \frac{\int_\Omega dy_1 \dots \int_\Omega dy_i \rho_i(y_1, \dots, y_i)}{\left(\int_\Omega dy \rho_1(y) \right)^i} . \quad (241)$$

For details concerning scaled factorial moments and related quantities, careful reader should consult ref. [8] and references quoted therein.

5.1 The Mekjian model

The two-size correlation functions for the Mekjian model with the statistical weights (17), can be calculated easily [126, 37] :

$$\begin{aligned} \langle n_j(n_k - \delta_{j,k}) \rangle_N &= \frac{\Gamma(1+N)}{\Gamma(x+N)} \frac{x^2}{jk} \frac{\Gamma(x+N-j-k)}{\Gamma(1+N-j-k)} \quad \text{if } j+k \leq N \\ &= 0 \quad \text{if } j+k > N \end{aligned} \quad (242)$$

These results are obtained from the generating function of the partition probabilities (16). Using the average distribution (18), one may write the two-size correlation function in a form :

$$\begin{aligned}
\langle n_j(n_k - \delta_{j,k}) \rangle_N &= \langle n_j \rangle_N \langle n_k \rangle_N \\
&\times \frac{\Gamma(x+N) \Gamma(1+N-j)}{\Gamma(1+N) \Gamma(x+N-j)} \\
&\times \frac{\Gamma(1+N-k) \Gamma(x+N-j-k)}{\Gamma(x+N-k) \Gamma(1+N-j-k)} \\
\langle n_j(n_k - \delta_{j,k}) \rangle_N &= 0 \quad \text{if } j+k > N,
\end{aligned} \tag{243}$$

which clearly shows that the difference between these results and those obtained for a purely poissonian distribution is small and comes mainly from the mass conservation (second equation in (242)). This effect must disappear rapidly when the total mass increases and hence, no important and non-trivial correlations are expected in this model.

This can also be seen in the analysis using the scaled factorial moments [124] which allow to study the possible non-poissonian part of the fluctuations of a signal. This is obtained by calculating the scaled moments of the signal in given bins, deconvoluted with the uncorrelated poissonian signal of the same average value in these bins. These moments are then studied as a function of the width of the bins. More precisely, one defines usually the second vertical scaled factorial moment as :

$$F_{2,V}(\delta s) \equiv \sum_{j=1}^{N/\delta s} F_{2,V}(j; \delta s) = \sum_{j=1}^{N/\delta s} \frac{\langle m_j(m_j - 1) \rangle}{\langle m_j \rangle^2}, \tag{244}$$

where now m_j is the number of fragments in the j -th bin of width δs and N is the "size" of the total studied interval, e.g. the total mass of the fragmenting system in the case of the cluster-size distribution. The dependence of this moment on the bin size is the sign of non-poissonian fluctuations. Moreover, the magnitude of this moment gives information about the fluctuations of the moment of order 0 (the total number of fragments). One can also define the corresponding second horizontal scaled factorial moment as :

$$F_{2,H}(\delta s) = \frac{\sum_{j=1}^{N/\delta s} \langle m_j(m_j - 1) \rangle}{\sum_{j=1}^{N/\delta s} \langle m_j \rangle^2}, \tag{245}$$

where the sum runs over all the $N/\delta s$ bins, and this moment is studied versus the width δs of the bins. In the case of a fragmenting system, the number of fragments in the j^{th} bin of width δs is given by :

$$m_j = \sum_{k=1}^{\delta s} n_{k+\delta s(j-1)}, \tag{246}$$

and the vertical moment is then :

$$F_{2,V}(j, \delta s) = \frac{\sum_{k,l=1}^{\delta s} \langle n_{k+\delta s(j-1)} (n_{l+\delta s(j-1)} + \delta_{k,l}) \rangle_N}{\sum_{k,l=1}^{\delta s} \langle n_{k+\delta s(j-1)} \rangle_N \langle n_{l+\delta s(j-1)} \rangle_N} \quad (247)$$

The knowledge of the two-size correlation functions (242) allows to calculate these second moments. More precisely in the case where x is equal to 1, one finds explicitly [37, 126] :

$$F_{2,H}(\delta s) \sim \exp \left[-\frac{2}{N} \frac{\delta s}{\ln^2(\delta s)} \right] \quad (248)$$

We recover that there is no important variation of this moment with the width bin δs , and these variations are just due to the mass-conservation [126]. A numerical investigation of (247), or (245) with (246), justifies the conclusion that the non-poissonian fluctuations are irrelevant in this model [37, 126].

5.2 Simple sequential decay models

The fluctuations in simple sequential binary decay models have been studied by Richert and Wagner [129]. They have approximated the set of coupled rate equations for production yields :

$$\frac{dP_i}{dt} = \sum_{j>i} w_{i,j-i}(\{x\}, t) P_j(t) - \frac{1}{2} P_i(t) \sum_{l>i} w_{l,i-l}(\{x\}, t) \quad , \quad (249)$$

by a simplified set of uncoupled equations :

$$\frac{dP_i}{dt} = w_i P_i(t) \quad (i = 1, \dots, A) \quad , \quad (250)$$

where

$$w_i = \langle w_i \rangle + w_{fl} \quad .$$

In the above equation, w_{fl} is taken to be a random gaussian number and the average $\langle \rangle$ is taken over all internal states of fragment i . P_i ($i = 1, \dots, A$) in (249) are the production yields for fragments i , $w_{i,j-i}$ are the transition rates for the binary decay $j \longrightarrow i + (j-i)$ and $\{x\}$ stands for other variables such as the excitation energies of fragments, kinetic energies of fragments, etc. It is well known [127, 128], that these equations exhibit a *temporal intermittency*. If the time interval $[0, t]$ is cut into N intervals $t = N\tau$, then the normalized moments of order k of the probability distribution :

$$\frac{\langle P_i^k(t) \rangle}{\langle P_i(t) \rangle^k} \sim \exp \left(N\tau^2 \sigma^2 \frac{k(k-1)}{2} \right) \quad , \quad (251)$$

where σ^2 is the variance of the gaussian distribution of w_{fl} , increase exponentially in time in any interval i , with an exponent (*intermittency exponent*) of the k -th

moment growing like $k^2 - k$. This result is a consequence of the multiplicative

Fig. 7. The scaled factorial moments as a function of the bin size δs . The averaged transition rates corresponds either to the Weisskopf prescription [58] (the top figure) or to the transition state theory prescription [130] and the fluctuating component of rates is uniformly distributed around zero. The abrupt change in the slope is due to both the finite-size effect and insufficient number of events. The selected events correspond to the mass fragmentation of ^{197}Au . (From ref. [131].)

temporal evolution of the probability distribution P_i :

$$P_i[(N+1)\tau] \simeq \exp[(\langle w_i \rangle + w_{fi})] P_i[N\tau] \quad , \quad (252)$$

where τ is supposed to be small. By construction, there are no correlations between probabilities $P_i(t)$ for different intervals i in (250). Hence, even though for each interval i separately, fluctuations in the time-distribution are strong and grow exponentially, the size distribution does not show any manifestation of the fragment-size correlations. Consequently, the factorial moment for the mass range divided into M intervals of equal length $\delta s = A/M$:

$$\ln \frac{\sum_{l=1}^M \langle P_l^k(t) \rangle}{\sum_{l=1}^M \langle P_l(t) \rangle^k} \sim N\tau^2 \sigma^2 \frac{k(k-1)}{2} = \text{const}(M) \quad , \quad (253)$$

is a constant independent of M for each given instant of time $t \equiv N\tau$. This simplifying hypothesis (250) seems to be well satisfied in some realistic sequential decay models as is shown in Fig. 7.

The simplest model where the self-similar density fluctuations occur is the self-similar cascade model. Those type of models were first introduced in the description of the energy dissipation in the fully developed turbulence [132, 133]. The main ideas behind this model are the independent evolution of different branches of the cascade with increasing resolution and the random multiplicative nature of the density fluctuations [134]. As a model of multiparticle production in high energy collisions, this model (α -model) was introduced by Białas and Peschanski [124].

At each step of the cascade, all the subintervals of the density distribution are subdivided into λ parts. In Fig. 8, seven steps of the cascade are shown for $\lambda = 2$. At each subdivision, the density in the resulting size interval is multiplied by a random variable W . So that after n steps the density ρ_m in an interval m of length $\delta s = \Delta S/\lambda^n$ is given by a product of n independent random variables :

$$\rho_m = W_1 \dots W_n \quad . \quad (254)$$

Due to the self-similarity of the cascade all these random variables have the same probability distribution $P(W)$:

$$\langle W \rangle = \int P(W) W dW = 1 \quad . \quad (255)$$

It is easy to see that values of the density in the neighbouring bins are strongly correlated and depend on the multiplicative "history" (254) of the cascade. The moments of the density ρ_m in the m -th bin have a power-law dependence on the resolution l :

$$\langle \rho_m^i \rangle \sim \left(\frac{\Delta S}{\delta s} \right)^{\nu_i} \quad , \quad (256)$$

where ν_i are the intermittency exponents :

$$\nu_i = \frac{\ln \langle W^i \rangle}{\ln \lambda} \quad . \quad (257)$$

This leads to a power-law behaviour of the scaled factorial moments of the multiplicity distributions obtained by a superposition of the poissonian noise on top of the fluctuations of ρ . The formula (257) is valid in the case of the weak intermittency [136] $\nu_i < i - 1$. Otherwise, one obtains the limiting exponents of the strong intermittency $\nu_i = i - 1$.

Fig. 8. An example of the random cascading in the α model for the branching ratio $\lambda = 2$.

The α - model shows global density fluctuations in addition to the local density fluctuations. One may however define a simpler version of it for $\lambda = 2$, the so called p - model [137], which exhibits only local density fluctuations. In this model, one imposes constraint on the probability distribution :

$$P(W) = \frac{1}{2}\delta(W - p) + \frac{1}{2}\delta(W + 2 - p) \quad , \quad (258)$$

that the two bins created in the division of a given bin, in a given step of the cascade, have complementary density factors : p and $2 - p$.

5.3 The percolation model

Contrary to the Mekjian model, and to the 1 -d bond-percolation model, a

3 - d percolation system can exhibit a critical behaviour. At a singular point the fluctuations are known to be quite important, but some of their aspects are still unclear. The horizontal scaled factorial moments (245) have been calculated for the 3 - d bond-percolation model on a finite cubic lattice [125, 135] . The results are shown in Fig. 9 for three values of the bond-probability : $q = 0.21$ (below the transition), $q = 0.23$ (at the transition), $q = 0.28$ (above the transition), and a number of sites $N = 6^3 = 216$. It is clear that there is no relevant bin-size dependence except at the transition point. At this point, the scaled factorial moment behaves like a power-law of the width of the bins.

By studying for the 1 - d percolation the properties of the two fragment size-correlations one can identify three main contributions to the correlations [126] :

- the upper mass bound $\langle n_j n_k \rangle = 0$ for $j + k > A$,
- the repulsive correlations arising from the fact that the second fragment is chosen in a smaller system $(A - j)$,
- the attractive correlations arising from the configurations where the two fragments have one or more common inactive links in their perimeters.

The first two types of correlations are numerically not important. All these contributions to the two fragment size-correlations are also present in the percolation in higher dimensions. For simplicity, let us consider the correlations between one large fragment of mass s and one fragment of mass $s' = 1$ ($\equiv c_{s,1}$) in $d \geq 2$. The mean number of clusters of mass n_s is [48] :

$$\langle n_s \rangle = A \sum_u g_{su} q^s (1 - q)^u , \quad (259)$$

where g_{su} is the number of clusters with size s and perimeter u per lattice site. Taking into account only the configurations where the two clusters have at most one common link in the perimeter one obtains for the two fragment cumulant correlations :

$$c_{s,1} \equiv \frac{\langle n_s n_1 \rangle - \langle n_s \rangle \langle n_1 \rangle}{\langle n_s \rangle \langle n_1 \rangle} \sim \frac{1}{A} \left(\frac{u}{1 - q} - s - u \right) , \quad (260)$$

where n_s is the number of fragments of size s . The perimeter u at the percolating point takes the form [120] :

$$u = \frac{s(1 - q)}{q} + \text{const } s^\zeta \quad \text{as } (s \rightarrow \infty) , \quad (261)$$

where the second term is not present below q_c . From (260) and (261) one finds :

$$c_{s,1} \sim \frac{\text{const } q}{A(1 - q)} s^\zeta , \quad (262)$$

the *attractive correlations* whose appearance is directly linked with the appearance of the excess perimeter $\sim \text{const } s^\zeta$ at the percolating point. In other words, the geometrical properties of the clusters at the percolation point are a principal source

of the attractive, finite-size correlations in the fragment size distribution. If the size s of a cluster is identified with its mass, then the perimeter u is the quantity

Fig. 9. The dependence of scaled factorial moments of rank $q = 2 - 5$ on the bin size in the 3-d bond percolation model with 6^3 sites, for different values of the bond activation probabilities [125].

which defines the structure of this mass. At first sight one would expect that :

$$u \propto s^{(1-1/d)} ,$$

as in ordinary geometry for which :

$$surface \propto volume^{(1-1/d)} .$$

However, the perimeter is not a surface in the usual sense, because above the critical point the infinite cluster is not compact containing in its interior many inactive bonds. Only the second term in eq. (261) may be interpreted as coming from the usual surface contribution, since for $q > q_c$ one has $\zeta = 1 - 1/d$ [48]. Interestingly, near the percolation point, in analogy to the usual cluster scaling (45), one expects also that the perimeter distribution function obeys the scaling behaviour [120]:

$$n'_u(\tau', \sigma') \sim u^{-\tau'} f(\varepsilon u^{\sigma'}), \quad (263)$$

where τ' and σ' are the scaling exponents from which all other scaling exponents can be found. They can also be related to those of the cluster-size scaling (45) [120].

For $q > q_c$ the number of fragments of finite size decreases and, consequently, the importance of these correlation effects diminishes as well. This is not only true for the fragments of size s and $s' = 1$ but for any two fragments of finite size.

5.4 The non-critical FIB model

Let us suppose that the system is governed by a FIB model out of the critical line. The fluctuations of the size-distribution are then expected to be small around their average values. We choose a trial probability function:

$$W_N(\{\vec{m}, \vec{n}\}; t) = e^{f(\vec{m}, \vec{n}; t)} \prod_k \left(\frac{\langle m_k \rangle^{m_k} \langle n_k \rangle^{n_k}}{m_k! n_k!} \right), \quad (264)$$

where the function f is still unknown. Using the formulas (171) to express the time derivatives of $\langle m_k \rangle$ and $\langle n_k \rangle$, the Marcus equation (167) writes as:

$$\frac{\partial f}{\partial t}(\vec{m}, \vec{n}; t) = \sum_k \sum_{i+j=k} F_{i,j} \left(\frac{m_i m_j}{\langle m_i \rangle \langle m_j \rangle} - \frac{m_i}{\langle m_i \rangle} - \frac{m_j}{\langle m_j \rangle} \right) \quad (265)$$

Vectors $\vec{m}$ and $\vec{n}$ are not independent⁵, and therefore, it would be wrong to conclude that the function f does not depend on the inactive set of clusters $\{n_1, n_2, \dots\}$. Nevertheless, if the fluctuations are small, one can replace all the values of m_k in eq. (265) by their average values. As a first-order approximation, this equation shows that the function f is a function of the time only, namely:

$$\frac{1}{f} \frac{df}{dt} = - \sum_k \langle m_k \rangle \sum_{i+j=k} F_{i,j} \quad (266)$$

When the fluctuations of the size-distribution are small, the process is then essentially poissonian. No anomalous fluctuations are expected in this case.

An analytical result can be derived in one such case for the scaled factorial moments. First, we have to derive the equations allowing the explicit derivation of the two-size correlation functions. These can be found using the eqs. (169) and

⁵Vectors $\vec{m}$ and $\vec{n}$ are linked by the mass-conservation.

(174) written in terms of the generating functions of the probability functions W_N .
Because of the relation :

$$\frac{\partial^2 G_N}{\partial x_j \partial x_k}(\vec{1}, \vec{1}; t) = \langle m_j(m_k - \delta_{j,k}) \rangle_N \quad , \quad (267)$$

we deduce :

$$\begin{aligned} \frac{d}{dt} \langle m_k(m_l - \delta_{k,l}) \rangle_N &= 2 \sum_{i=k+1}^N F_{k,i-k} \langle m_i(m_l - \delta_{i,l}) \rangle_N \\ &+ 2 \sum_{i=l+1}^N F_{l,i-l} \langle m_i(m_k - \delta_{i,k}) \rangle_N \\ &- (\Phi_k + \Phi_l) \langle m_k(m_l - \delta_{k,l}) \rangle_N \\ &+ 2F_{k,l} \langle m_{k+l} \rangle_N \end{aligned} \quad (268)$$

$$\begin{aligned} \frac{d}{dt} \langle m_k n_l \rangle_N &= 2 \sum_{i=k+l}^N F_{k,i-k} \langle m_i n_l \rangle_N - \Phi_k \langle m_k n_l \rangle_N \\ &+ I_k \langle m_k(m_l - \delta_{k,l}) \rangle_N \end{aligned} \quad (269)$$

$$\frac{d}{dt} \langle n_k(n_l - \delta_{k,l}) \rangle_N = I_k \langle m_k n_l \rangle_N + I_l \langle n_k m_l \rangle_N \quad (270)$$

from the Marcus equation and :

$$\begin{aligned} \frac{d}{dt} \langle m_k(m_l - \delta_{k,l}) \rangle_N &+ 2\Phi_N \langle m_k(m_l - \delta_{k,l}) \rangle_N = \\ &2 \sum_{i=1}^{N-1} F_{i,N-i} [\langle m_k(m_l - \delta_{k,l}) \rangle_i \\ &+ \langle m_k \rangle_i \langle m_l \rangle_{N-i}] \end{aligned} \quad (271)$$

$$\begin{aligned} \frac{d}{dt} \langle m_k n_l \rangle_N &+ 2\Phi_N \langle m_k n_l \rangle_N = \\ &2 \sum_{i=1}^{N-1} F_{i,N-i} (\langle m_k n_l \rangle_i + \langle m_k \rangle_i \langle n_l \rangle_{N-i}) \end{aligned} \quad (272)$$

$$\begin{aligned} \frac{d}{dt} \langle n_k(n_l - \delta_{k,l}) \rangle_N &+ 2\Phi_N \langle n_k(n_l - \delta_{k,l}) \rangle_N = \\ &2 \sum_{i=1}^{N-1} F_{i,N-i} [\langle n_k(n_l - \delta_{k,l}) \rangle_i \\ &+ \langle n_k \rangle_i \langle n_l \rangle_{N-i}] \quad , \end{aligned} \quad (273)$$

from the cascade equation.

Once the correlation functions have been found using eqs. (268)-(270) or (271)-(273), one may calculate the scaled factorial moments :

$$F_{2,H}(\delta s) = \frac{\sum_{j=1}^{N/\delta s} \sum_{k,l=1}^{\delta s} \langle n_{k+\delta s(j-1)} (n_{l+\delta s(j-1)} + \delta_{k,l}) \rangle_N}{\left(\sum_{k=1}^{\delta s} \langle n_k \rangle_N \right)^2} \quad (274)$$

written here for the second horizontal scaled factorial moment. The parameter $N/\delta s$ denotes the total number of bins. The rest of the calculation is straightforward.

To solve the eqs. (268)-(273) we need explicitly the average size-distributions. So , to obtain exact results, we are constrained to use only the examples given in sect. (4.3). The simplest case is the choice for which both fragmentation and inactivation kernels are constant, independent of the sizes. In this case the final ($t = \infty$) solution writes :

$$\begin{aligned} \langle n_k (n_l - \delta_{k,l}) \rangle_N &= \frac{\langle n_k \rangle_N \langle n_l \rangle_N}{(N + I_1)} \left\{ \sum_{i=k+l+2}^N \frac{(i + 2I_1 - 1)(i + 2I_1)}{(i + I_1 - 1)(i + I_1)} \right. \\ &\quad + \frac{(k + I_1)(k + I_1 + 1)(l + I_1)(l + I_1 + 1)}{(k + l + I_1)(k + l + I_1 + 1)} \\ &\quad \times \left. \left(\frac{1}{k + l + I_1 - 1} + \frac{1}{k + I_1} + \frac{1}{l + I_1} \right) \right\} \\ &\quad \text{if } k + l < N \end{aligned} \quad (275)$$

$$\langle n_k (n_l - \delta_{k,l}) \rangle_{k+l} = \frac{2 \langle n_k \rangle_N \langle n_l \rangle_N}{k + l + I_1 - 1} \quad (276)$$

With this exact form, we can use the double inequality :

$$\begin{aligned} \frac{\langle n_k \rangle_N \langle n_l \rangle_N}{N + I_1} (N - k - l) &\leq \langle n_k (n_l - \delta_{k,l}) \rangle_N \\ &\leq \frac{\langle n_k \rangle_N \langle n_l \rangle_N}{N + I_1} \left(N + \frac{1}{2}(k + l) + 2I_1 \ln N \right) \end{aligned} \quad (277)$$

to conclude that the term $j = 1$, in the numerator of formula (274), is of order N^2 while all other terms are of order $(N/\delta s)^2$. This means that if $\delta s \gg 1$, then only the first bin gives an important contribution to the numerator of (274). This is a quite general effect when the average size-distribution is singular (diverges) for the zero-sizes. This shows also that the horizontal factorial moment is essentially identical to the vertical factorial moment computed for the first bin. One has then an approximation of the factorial moment :

$$F_{2,H}(\delta s) \simeq \frac{\sum_{k,l=1}^{\delta s} \langle n_k (n_l - \delta_{k,l}) \rangle_N}{\left(\sum_{k=1}^{\delta s} \langle n_k \rangle_N \right)^2} = F_{2,V}(1, \delta s) \quad , \quad (278)$$

which is easier to handle. The leading term of (278) is easy to find :

$$F_{2,H}(\delta s) \sim \frac{N}{N + I_1} \quad , \quad (279)$$

and we see that it is close to 1 for large values of N . This is consistent with ordinary fluctuations in the number of clusters. The following term has been investigated numerically from (274), and it does not show any relevant bin-width dependence. All these results confirm that this case has mainly poissonian fluctuations.

5.4 The critical FIB model

The problem is very different if the system is on the transition line. The fluctuations are believed to be important and their structure may be complicated. The case where the fragmentation kernel is equal to 1, and the inactivation kernel is : $I_k = k + a$ can be derived analytically. The solutions of (263)-(270) or (271)-(273) are given in this case by :

$$\begin{aligned} \langle n_k(n_l - \delta_{k,l}) \rangle_N &= \langle n_k \rangle_N \langle n_l \rangle_N \left\{ \sum_{i=k+l+2}^N \frac{1}{i + \frac{a-1}{2}} \right. \\ &\quad \left. + \frac{4k^2 + 4l^2 + 12kl + 4(2a+1)(k+l) + (a+1)(3a-1)}{(2k+2l+a)^2 - 1} \right\} \\ &\quad \text{if } k+l < N \end{aligned} \quad (280)$$

$$\langle n_k(n_l - \delta_{k,l}) \rangle_{k+l} = \frac{2 \langle n_k \rangle_k \langle n_l \rangle_l}{2(k+l) + a - 1} \quad , \quad (281)$$

and the double inequality :

$$\begin{aligned} \langle n_k \rangle_N \langle n_l \rangle_N &\leq \langle n_k(n_l - \delta_{k,l}) \rangle_N \\ &\leq \langle n_k \rangle_N \langle n_l \rangle_N [\ln(N) + 2a + 4] \end{aligned} \quad (282)$$

holds. This shows that, like in the previous case (275 - 277), only the first term of the sum in the numerator of (274) is important when $\delta s \gg 1$. Another, more precise double inequality can now be used to further simplify the expression for the factorial moment. This is :

$$\ln(N) - \ln(k+l) - \frac{2}{k+l} < \{ \} < \ln(N) - \ln(k+l) + 2a + 4 \quad , \quad (283)$$

where $\{ \}$ means the expression inside the brackets in (280). With this result, one finds that $F_{2,H}$ is :

$$F_{2,H}(\delta s) \simeq \ln(N) - \frac{\sum_{k,l=1}^{\delta s} \langle n_k \rangle_N \langle n_l \rangle_N \ln(k+l)}{\sum_{k,l=1}^{\delta s} \langle n_k \rangle_N \langle n_l \rangle_N} \quad , \quad (284)$$

where we have neglected terms of order 1 . The two sums are then easy to compute, and the final leading behaviour of the scaled factorial moment is :

$$F_{2,H}(\delta s) \sim \ln(N) - \frac{2}{3} \ln \delta s \quad . \quad (285)$$

One can also be interested by the value of this moment at the point $\delta s = 1$, since it gives some informations about the multiplicity fluctuations. A direct calculation gives :

$$F_{2,H}(1) = \left\{ 1 + \left(8 \sum_{k=1}^{\infty} \frac{(k+a)^2}{[(2k+a)^2 - 1]^2} \right)^{-1} \right\} \ln(N) \quad . \quad (286)$$

Let us now discuss these results. The leading behaviour (285) can be written as :

$$\ln F_{2,H}(\delta s) = \ln(\ln N) + \frac{2}{3} \frac{(-\ln \delta s)}{\ln N} \quad , \quad (287)$$

and this shows that the scaled factorial moment of the size-distribution behaves as a power-law of the bin-size, with the exponent equal $2/(3 \ln(N))$. Notice also large value ($\sim \ln(N)$) of this moment for all values of δs .

A following general feature of this analytical result should be noticed : when intermittency of the fluctuations occurs, the exponent of the dependence of the factorial moment on the bin-size is a decreasing function of the total size N of the system [138, 126] . In order to be able to compare different sizes of the system, it seems useful to consider an idea put forward in the context of the fully developed turbulence. The good variable in this case is the ratio between the logarithm of the length of the system and the logarithm of the local Reynolds number, which is proportional to the typical length of the dissipative structures [133] . An equivalent parameter in our case is :

$$\mu = - \frac{\ln N}{\ln \delta s} \quad (288)$$

If the factorial moment is normalized by $\ln(N)$, the result (287) becomes :

$$\frac{F_{2,H}(\delta s)}{\ln(N)} \sim 1 + \frac{2}{3\mu} \quad , \quad (289)$$

independent of the total size N . So, this parameter (288) could be a good parameter to study the intermittent behaviour of the fluctuations in such systems.

Independent information about the correlations is provided by the direct studies of the two fragment correlations. In the previous section, we have discussed them in the percolation for $d \geq 2$. In FIB model, the factorial cumulant correlations $c_{s',s}$ between fragments of size s and s' can be calculated analytically. In particular, for $s' = 1$ at the transition line : $\alpha = 0$, $\beta = 1$, $I_1 = 1$ one obtains :

$$c_{s,1} \sim \ln \frac{N}{s} \quad , \quad (290)$$

in the limit $s \gg 1$. The correlations are attractive and increase with the size of the fragmenting system. On the contrary, in the shattered phase : $\alpha = 0$, $\beta = 0$, $I_1 = 1$, the factorial cumulant :

$$c_{s,1} \sim -\frac{s}{N} \quad (291)$$

is repulsive.

6. Conclusions

In the Introduction, we have seen that fragmentation is a very common process in Nature, and despite both the diversity of systems and their parameters, the experimental cluster-size distributions show remarkable similarities. Certainly, the most constant characteristic is the power-law behaviour of these distributions. This shows that in many fragmenting systems, there is apparently no typical size-scale other than the total size of the fragmenting bodies. This resembles the characteristic features of critical phenomena, though it is hard to believe that the inverse is true, i.e. that the power-law of the size-distributions exhibited in so many experimental systems of fragmentation, implies some critical behaviour. Moreover, if this scale-invariance is found in many systems, the measured values of the exponent τ of the power-laws are sufficiently different one from another, to conclude that all these systems surely cannot be grouped in a few classes by the values of this parameter, as it is the case for the phase transitions. This strongly reminds the situation in aggregation phenomena, where universality classes have been precisely defined by their geometrical structures, but not by their size-distributions, though they are qualitatively similar in a given class. There is here a striking resemblance with one kinetic fragmentation model, namely the FIB model, which has been developed in details in this review. This model splits the set of all the sequentially fragmenting systems in three classes. Two of them exhibit power-law size-distribution, but the possible values of the exponent τ is smaller than 2 in one class and is larger than 2 in the other one. Despite this qualitative resemblance and weak characterization, these two classes are of fundamentally different nature. For example, the cluster-size fluctuations in both classes are different. Whereas in the first class, strong non-statistical (non-poissonian) fluctuations dominate at fine resolutions exhibiting a power-law behaviour, in the second class the fluctuations are generally weak and no particular scale dependence has been noticed, at least in large systems. The third class cannot be described naturally in terms of scale-invariance. Nevertheless, this FIB model as it is described here, is by far too schematic to be able to conclude definitively about the existence, number and quality of such classes, even if a global structure of universality begins to appear for the first time in the sequential fragmentation processes. In principle, we have now to show how the other existing models can enter in a FIB framework. Let us give just one example : qualitatively, the percolation fragmentation is evidently a particular case of a FIB model since when one cuts randomly, irreversibly and sequentially the bonds of a lattice, a cut will sometimes lead to the fragmentation of one big cluster into two smaller ones. If the total number of cut is limited, a final state will be reached. But the parameters of the corresponding FIB model are only partially known. For

example, we know the fragmentation kernel for the infinite cluster at the percolation threshold [119, 139] but this is not enough to describe the whole evolution of the system. To achieve this goal, it is necessary to have precise informations about the fragmentation kernels for all the sizes of clusters and all the values of the bond probabilities.

In this review, we have presented the method of scaled factorial and discussed briefly its application for the analysis of the multifragment densities. This method opens the possibility to study quantitatively the non-poissonian fluctuations at small scales and in this way it provides a new source of informations about the aggregates undergoing the transition to final clusters. The scaled factorial moments appear as new observables which characterize the fragment distributions in a more detailed way than the inclusive observables, such as the fragment-size distribution. Complete understanding of the rising of scaled factorial moments at small scales is still missing, but the examples provided by the percolation and FIB models suggest their association with the second order phase-transition. At the moment we encounter difficulty in separating various finite-size correlation effects from the scale-invariant correlations due to the finite system phase transition. To clarify the situation, one needs more experimental data on systems of different size. This is particularly true in the domain of low energy nuclear collisions where the high quality exclusive data, event by event, are virtually non-existent.

We have said that aggregation universality classes were properly defined by the geometrical properties of the clusters while the kinetic ones give only a weak description of these classes. But one important point to note is that these two approaches are decoupled, at least to a large extent. One can study, for example, aggregation geometry without precise information about the kinetics, as it is done in the "hierarchical" model of Botet, Jullien and Kolb [140] and, conversely, one can study the kinetics without precise informations about the geometry of the clusters, like in the Smoluchowski approach [86]. In the former case, we need only to know if the average size-distribution is unimodal or bimodal and, in the latter case, we need the form of the aggregation kernels which are weakly depending on the geometry. This seems quite miraculous but has been explained. The key point in the explanation is the fact that the upper critical dimensionality of the Smoluchowski approach is very small and equals to 2 [141]. In other words, the mean-field kinetic Smoluchowski equations are valid as soon as the space dimension is larger than 2. In the ordinary 3-d space, the fluctuations of concentrations, which are very dependent on the shapes of the clusters, can be neglected.

This remarkable feature has been the main motivation of the kinetic approaches for fragmentation. If one trusts some qualitative analogies between aggregation and fragmentation, one can imagine that these sort of mean-field equations will remain valid to describe real systems without taking care of the, yet unknown, space fluctuations. This assumption is also conformed by the remark that a number of very different, more or less simple models are able to fit some precise experimental data of fragmenting nuclear systems [122, 142]. The complete description of such fragmentation processes should take into account both geometry and the kinetics. But very few geometrical approaches are available up to now. Percolation and reversible aggregation [143] are surely pertinent for some applications but not

easy to generalize. This should be a very fruitful road to investigate in the future.

Acknowledgements

During our activity on this subject we have profited from many discussions and collaborations with P. Božek , X. Campi , D.H.E. Gross , J. Richert and A. Tucholski and we thank them very warmly.

We thank J. Richert for providing the figure with his published results.

This work was possible due to the financial support of the Commissariat à l'Énergie Atomique and Centre National de Recherches Scientifiques.

References

- [1] W.R. Johnson and C.C. Price, *J. Polym. Sci.* **45** , 217 (1960);
E.W. Merrill, H.S. Mickley and A. Ram, *J. Polym. Sci.* **62** , 109 (1962).
- [2] H.J. Moore and D.E. Gault, *Ann. Prog. Rep. (Astr. Stud.) U.S. Geol. Survey Part B* , 127 (1965);
J.J. Gilvarry, *J. Appl. Phys.* **32**, 391 (1961);
P.R. Halmos, *Ann. Math. Stat.* **15** , 182 (1944);
L. Austin, K. Shoji, V. Bhatia K. Savage and R. Klimpel, *Ind. Eng. Chem. Process Des. Dev.* **15** , 187 (1976).
- [3] J.E. Finn et al., *Phys. Rev. Lett.* **49** , 1321 (1982).
- [4] R. Shinnar, *J. Fluid Mech.* **10** , 259 (1961);
B.L. Holian and D.E. Grady, *Phys. Rev. Lett.* **60** , 1355 (1988).
- [5] J. Hüfner, *Phys. Rep.* **125** , 129 (1985);
J. Aichelin, *Phys. Rep.* **202** , 233 (1991);
L.I. Csernai and J.I. Kapusta, *Phys. Rep.* **131** , 233 (1986);
L.G. Moretto and G.J. Wozniak, *Prog. Part. and Nucl. Phys.* **21** , 401 (1988).
- [6] D.H.E. Gross, *Rep. Progr. Phys.* **53** , 605 (1990).
- [7] L.G. Moretto and G.J. Wozniak, "Multifragmentation in Heavy-Ion Process", LBL Preprint LBL-34176 (1993).
- [8] P. Bozek, M. Ploszajczak and R. Botet, "Two and Many Particle Correlations in Nuclear and High Energy Physics", GANIL Preprint P 94/13 (1994).
- [9] B. Jakobsson, G. Jonsson, B. Lindkvist and A. Oskarsson, Cosmic and Subatomic Physics Report LUIP 8207 (1982); *Z. Phys.* **A307** , 293 (1982).
- [10] M.A. Lange, T.J. Ahrens and M.B. Baslough, *Icarus* **58** , 383 (1984).
- [11] R.E. McCrosky, *Spec. Rep.* **280**, Smithsonian Astrophys. Observ., Washington D.C. (1968).
- [12] D.E. Grady and M.E. Kipp, *J. Appl. Phys.* **58** , 1210 (1985);
W.K. Hartmann, *Icarus* **10** , 201 (1969).
- [13] V.E. Schoutens, *Nuclear Geophysics Source Book* **55** Part. 2.4; Report DNA 65 01H-4.2, Bethesda, MD, (1979).

- [14] J.G. Bennett, *J. Instr. Fuel* **10** , 22 (1936).
- [15] A. Fujiwara, G. Kamimoto and A. Tsukamoto, *Icarus* **31** , 277 (1977).
- [16] G.S. Hawkins, *Annu. Rev. Astron. Astrophys.* **2** , 149 (1964);
W.K. Hartmann, *Sci. Am.* , 84 (1977) (January).
- [17] J.S. Dohnanyi, *J. Geophys. Res.* **75** , 3468 (1970);
J.R. Donnison and R.A. Sugden, *Mon. Not. R. Astron. Soc.* **210** , 673 (1984).
- [18] W. Trautmann, U. Milkau, U. Lynen and J. Pochodzalla, *Z. Phys.* **A344** , 447 (1993).
- [19] *Spontaneous Formation of Space-Time Structures and Criticality*, T. Riste and D. Sherrington eds., NATO ASI Series, Series C, Vol. 349, Kluwer Academic Publ., (1991).
- [20] F.F. Abraham, S.W. Koch and R.C. Desai, *Phys. Rev. Lett.* **49** , 923 (1982);
J.A. Blink and W.G. Hoover, *Phys. Rev.* **A32** , 1027 (1985);
B.L. Holian and D.E. Grady, *Phys. Rev. Lett.* **60** , 1355 (1988);
V. Latora, M. Belkacem and A. Bonasera, "Dynamics of Instabilities and Intermittency", LNS Preprint (1993).
- [21] G.H. Hardy, *Ramanujan, Twelve Lectures on Subjects Suggested by His Life and His Work*, Chelsea, N.Y., (1959);
A. Abramowitz and I.A. Stegun, *Handbook of Mathematical Functions* , Dover Inc., N.Y., (1965)
- [22] L.G. Sobotka and L.G. Moretto, *Phys. Rev.* **C31** , 668 (1985).
- [23] J. Aichelin, J. Hüfner and R. Ibarra, *Phys. Rev.* **C30** , 107 (1984);
J. Aichelin and J. Hüfner, *Phys. Lett.* **B136** , 15 (1984).
- [24] W. Kuhn, *Ber. Chem. Dtsch. Ges.* **63** , 1502 (1930).
- [25] P.J. Flory, *J. Am. Chem. Soc.* **58** , 1877 (1936).
- [26] A.R. Kerstein and A.L.R. Bug, *Phys. Rev.* **B34** , 1754 (1986);
M. Sahimi and T.T. Tsotsis, *Phys. Rev. Lett.* **59** , 888 (1987).
- [27] A.M. Basedow, K.H. Ebert and H.J. Ederer, *Macromolecules* **11** , 774 (1978).
- [28] A. Keller, J.A. Odell and H.H. Wills, *Colloid Polym. Sci.* **263** , 181 (1952).
- [29] S. Egusa, K. Ishigure and Y. Tabata, *Macromolecules* **12** , 939 (1979).
- [30] E.W. Montroll and R. Simha, *J. Chem. Phys.* **8** , 721 (1940).
- [31] H. Mark and R. Simha, *Trans. Faraday Soc.* **35** , 611 (1940).
- [32] A. Charlesby, *Proc. Roy. Soc. (London)* **A224** , 120 (1954).
- [33] E.O. Kraemer and W.D. Lansing, *J. Phys. Chem.* **39** , 153 (1935).
- [34] N.J. Fine, Report of the Institute in the Theory of Numbers, Univ. of Colorado, Boulders, Colorado (1959) p. 86.
- [35] R. Botet, "Formation d'Aggregats Fractals par Collage d'Amas en Diffusion : Simulations Numeriques et Theorie", These d'Etat, Universite Paris-Sud, 1985.
- [36] A.Z. Mekjian, *Phys. Rev. Lett.* **64** , 2125 (1990); *Phys. Rev.* **C41** , 5103 (1990).
- [37] A.Z. Mekjian and S.J. Lee, *Phys. Rev.* **A44** , 6294 (1991);
S.J. Lee and A.Z. Mekjian, *Phys. Rev.* **C45** , 1284 (1992); *ibid.* **C47** , 2266 (1993).
- [38] D.H.E. Gross, A. Ecker and A.R. DeAngelis, *Ann. Physik* **1** , 340 (1992).

- [39] W.H. Stockmayer, *J. Chem. Phys.* **11** , 45 (1943);
P.J. Flory, *Principles of Polymer Chemistry*, Cornell Univ. Press, Ithaca, N.Y., 1953.
- [40] P.J. Blatz and A.V. Tobolsky, *J. Phys. Chem.* **49** , 77 (1945);
R.J. Cohen and G.B. Benedek, *J. Phys. Chem.* **86** , 3696 (1982).
- [41] R.K. Bajpai, D. Ramkrishna and A. Prokop, *Chem. Eng. Sci.* **31** , 13 (1976).
- [42] H. Hayakawa, *Modern Phys. Lett.* **B6** , 773 (1988).
- [43] P.G.J. van Dongen and M.H. Ernst, *J. Stat. Phys.* **37** , 301 (1984);
P.G.J. van Dongen and M.H. Ernst, *Phys. Rev. Lett.* **54** , 1396 (1985).
- [44] B. Derrida and H. Flyvbjerg, *J. Phys.* **A20** , 5273 (1987).
- [45] M.E. Fisher and J.W. Essam, *J. Math. Phys.* **2** , 609 (1961).
- [46] A.R. Kerstein, *J. Phys.* **A22** , 3371 (1989).
- [47] D. Stauffer, *Phys. Rep.* **54** , 1 (1979).
- [48] D. Stauffer, *Introduction to Percolation Theory* (Taylor and Francis, London and Philadelphia, Penn., 1985).
- [49] H.J. Herrmann, *Phys. Rep.* **136** , 153 (1986).
- [50] M.E. Fisher and M.N. Barber, *Phys. Rev. Lett.* **28** , 1516 (1972);
M.P. Nightingale, *Physica* **A83** , 561 (1976); *Phys. Lett.* **A59** , 485 (1977).
- [51] E. Brezin, *J. de Physique* **43** , 15 (1982);
R. Botet, R. Jullien and P. Pfeuty, *Phys. Rev. Lett.* **49** , 478 (1982).
- [52] Lord Rayleigh, *Nature* **XCIII** , 188 (1914).
- [53] K.G. Wilson and J. Kogut, *Phys. Rep.* **12C** , 75 (1974).
- [54] D.L. Turcotte, *J. of Geophys. Research* **91** , 1921 (1986).
- [55] C.J. Allegre, J.L. Le Mouel and A. Provost, *Nature* **297** , 47 (1982).
- [56] R.J. Charity et al., *Nucl. Phys.* **A511** , 59 (1990).
- [57] R.J. Charity et al., *Nucl. Phys.* **A476** , 476 (1988).
- [58] C. Barbagallo, J. Richert and P. Wagner, *Z. Phys.* **A324** , 97 (1986).
- [59] H. Hauser and H. Feshbach, *Phys. Rev.* **87** , 366 (1952).
- [60] L.G. Moretto, *Nucl. Phys.* **A247** , 211 (1975).
- [61] D.H.E. Gross and R. Heck, *Phys. Lett.* **B318** , 405 (1993).
- [62] J. Randrup and S.E. Koonin, *Nucl. Phys.* **A356** , 223 (1981).
- [63] J.A. Lopez and J. Randrup, *Nucl. Phys.* **A503** , 183 (1989); *ibid.* **A512** , 345 (1990).
- [64] W.A. Friedman, *Phys. Rev.* **C42** , 667 (1990); *Phys. Rev. Lett.* **60** , 2125 (1988); AIP Conf. Proceedings **250** Nikko, Japan (1992) 442.
- [65] D.H.E. Gross, L. Satpathy, Meng Ta-Chung and M. Satpathy, *Z. Physik* **A309** , 41 (1982);
D.H.E. Gross, *Phys. Scripta* **T5** , 213 (1983);
D.H.E. Gross and B.H. Sa, *Nucl. Phys.* **A437** , 643 (1985);
D.H.E. Gross and X.Z. Zhang, *Phys. Lett.* **161B** , 47 (1985).
- [66] J.P. Bondorf, *Nucl. Phys.* **A387** , 22c (1982);
J.P. Bondorf et al., *Phys. Lett.* **B150** , 57 (1985);
J.P. Bondorf et al., *Nucl. Phys.* **A443** , 321 (1985); *ibid.* **A444** , 460 (1986).

- [67] H. Barz et al., *Phys. Lett.* **B191** , 232 (1987); *ibid.* **B211** , 25 (1988); *ibid.* **B244** , 161 (1990);
H.W. Barz et al., *Nucl. Phys.* **A448** , 753 (1986); *ibid.* **A545** , 215c (1992).
- [68] K. Sneppen and R. Donangelo, *Phys. Rev.* **C39** , 263 (1989);
K. Sneppen, *Nucl. Phys.* **A470** , 213 (1987);
K. Sneppen and L. Vinet, *Nucl. Phys.* **A480** , 342 (1988).
- [69] H.R. Jaqaman G. Papp and D.H.E. Gross, *Nucl. Phys.* **A514** , 327 (1990) ;
ibid. **A524** , 321 (1991).
- [70] A.R. DeAngelis, D.H.E. Gross and R. Heck, *Nucl. Phys.* **A537** , 606 (1992).
- [71] O. Schapiro, A.R. DeAngelis and D.H.E. Gross, *Nucl. Phys.* **A568** , 333 (1994).
- [72] S. Ayik and C. Gregoire, *Phys. Lett.* **B212** , 269 (1988);
A. Ayik and C. Gregoire, *Nucl. Phys.* **513** , 187 (1990);
P.G. Reinhard, E. Suraud and S. Ayik, *Ann. Phys. (N.Y.)* **213** , 204 (1992).
- [73] J. Randrup and B. Remaud, *Nucl. Phys.* **A514** , 339 (1990);
P. Chomaz , G.F. Burgio and J. Randrup, *Phys. Lett.* **B254** , 340 (1991);
Nucl. Phys. **A529** , 157 (1991); *ibid.* **A567** , 637 (1994);
G.F. Burgio, P. Chomaz and J. Randrup, *Phys. Rev. Lett.* **69** , 885 (1992);
G.F. Burgio, M. Baldo and A. Rapisarda, *Phys. Lett.* **B321** , 307 (1994).
- [74] Bao-An Li and D.H.E. Gross, *Nucl. Phys.* **A554** , 257 (1993).
- [75] C.J. Pethick and D.G. Ravenhall, *Ann. Phys. (N.Y.)* **183** , 131 (1988); *Nucl. Phys.* **A471** , 19c (1987);
H. Heiselberg, C.J. Pethick and D.G. Ravenhall, *Phys. Rev. Lett.* **61** , 818 (1988).
- [76] A. Kolmogorov, *Math. Ann.* **104** , 415 (1931).
- [77] G. Paladin and A. Vulpiani, *Phys. Rep.* **156** , 147 (1987).
- [78] L.F. Richardson, *Proc. Roy. Soc. (London)* **A110** , 709 (1926);
A.N. Kolmogorov, *Dokl. Akad. Nauk SSSR* **30** , 299 (1941).
- [79] S. Chandrasekhar, *Proc. Roy. Soc. (London)* **A286** , 1 (1964).
- [80] G.A. Pik-Pichak, *Soviet Phys. JETP* **34** , 238 (1958); *ibid.* **15** , 897 (1962);
ibid. **16** , 1201 (1963);
R. Behringer and W.J. Knox, *Phys. Rev.* **121** , 1195 (1961);
D. Sperber, *Phys. Rev.* **130** , 468 (1963);
C.E. Rosenkilde, *J. Math. Phys.* **8** , 98 (1967);
W.J. Swiatecki, *Proc. Second Int. Conf. on Peaceful Uses of Atomic Energy*, Geneva, 1958, Pergamon Press, London, Paper P/651;
S. Cohen, F. Plasil and W.J. Swiatecki, *Proc. Third Conf. on Reactions Between Complex Nuclei*, eds. A. Ghiorso, R.M. Diamond and H.E. Conzett, p. 325, University of California Press, 1963.
- [81] S. Cohen, F. Plasil and W.J. Swiatecki, *Ann. Phys. (N.Y.)* **82** , 557 (1974).
- [82] B. Borderie, M.F. Rivet, B. Remaud, *Phys. Lett.* **B302** , 15 (1992).
- [83] Lord Rayleigh, *Proc. London Math. Soc.* **X** , 4 (1879).
- [84] L.G. Moretto et al., *Phys. Rev. Lett.* **69** , 1884 (1992).
- [85] C.Y. Wong, *Phys. Rev. Lett.* **55** , 1973 (1985);
W. Bauer, G.F. Bertsch and H. Schulz, *Phys. Rev. Lett.* **69** , 1888 (1992);
D.H.E. Gross, Bao-An Li, A.R. DeAngelis, *Ann. Phys.* **1** , 467 (1992).

- [86] M.V. Smoluchowski, *Z. Phys. Chem.* **92** , 129 (1917);
R. Jullien and R. Botet, *Aggregates and Fractal Aggregates*, World Scientific, Singapore (1987) and references therein.
- [87] T.W. Peterson, M.V. Scotto and A.F. Sarofim, *Powder Technol.* **45** , 87 (1985).
- [88] Z. Cheng and S. Redner, *J. Phys.* **A23** , 1233 (1990).
- [89] A.H. Marcus, *Technometrics* **10** , 133 (1968);
A.A. Lushnikov, *J. Colloid Interf. Sci.* **65** , 276 (1978).
- [90] S.E. Esipov, L.P. Gor'kov and T.J. Newman , *J. Phys.* **A26** , 787 (1993).
- [91] E.D. McGrady and R.M. Ziff, *Phys. Rev. Lett.* **58** , 892 (1987).
- [92] R.M. Ziff, *J. Phys.* **A24** , 2821 (1991).
- [93] S.L. Goren, *Can. J. of Chem. Eng.* **46** , 185 (1968);
T.W. Peterson, *Aero. Sci. Tech.* **5** , 93 (1986).
- [94] P.C. Kapur, *Chem. Eng. Sci.* **25** , 899 (1970).
- [95] M.M.R. Williams, *Aero. Sci. Tech.* **12** , 538 (1990).
- [96] F. Family, P. Meakin and J.M. Deutsch, *Phys. Rev. Lett.* **57** , 727 (1986).
- [97] T. Vicsek and F. Family, *Phys. Rev. Lett.* **52** , 1669 (1984).
- [98] S.K. Friedlander and C.S. Wang, *J. Colloid Interf. Sci.* **22** , 126 (1966).
- [99] D. Ramkrishna, *Chem. Eng. Sci.* **29** , 987 (1974).
- [100] P. Rosin and E. Rammler, *J. Inst. Fuel* **7** , 29 (1933);
W.K. Brown, R.R. Karpp and D.E. Grady, *Astrophys. Space Sc.* **94** , 401 (1983).
- [101] R.M. Ziff and E.D. McGrady, *Macromolecules* **19** , 2513 (1986).
- [102] Z. Cheng and S. Redner, *Phys. Rev. Lett.* **60** , 2450 (1988).
- [103] M.H. Ernst and G. Szamel, *J. Phys.* **A26** , 6085 (1993).
- [104] A.F. Filippov, *Theory Prob. Its Appl.* **6** , 275 (1961).
- [105] J.L. Spouge, *J. Phys.* **A16** , 767 (1983);
E.M. Hendriks and J.L. Spouge, *Z. Phys.* **B58** , 219 (1985);
M. Kolb and H.J. Hermann, *J. Phys.* **A18** , L435 (1985).
- [106] O. Saito, *J. Phys. Soc. Japan*, **13** , 198 (1958).
- [107] H.H.G. Jellinek and G. White, *J. Polym. Sci.* **6** , 745 (1951).
- [108] R. Simha, *J. Appl. Phys.* **12** , 569 (1941).
- [109] G.M. Bristow, *J. Polym. Sci.* **A1** , 2261 (1963).
- [110] M.A.K. Mostafa, *J. Polym. Sci.* **22** , 535 (1956).
- [111] M. Ballauff and B.A. Wolf, *Macromolecules* **14** , 654 (1981).
- [112] M. Ballauff and B.A. Wolf, *Macromolecules* **17** , 209 (1984).
- [113] E. Montroll, *J. Am. Chem. Soc.* **63** , 1215 (1940).
- [114] A.M. Kotliar, *J. Polym. Sci.* **45** , 541 (1960).
- [115] M. Inokuti, *J. Chem. Phys.* **38** , 1174 (1963);
M. Inokuti and M. Dole, *J. Polym. Sci.* **A1** , 3289 (1963).
- [116] R.M. Ziff and E.D. McGrady, *J. Phys.* **A18** , 3027 (1985).
- [117] R.M. Ziff, *J. Phys.* **A25** , 2569 (1992).
- [118] E.D. McGrady and R.M. Ziff, *AIChE J.* **34** , 2073 (1988).
- [119] M.F. Gyure and B.F. Edwards, *Phys. Rev. Lett.* **68** , 2692 (1992);
B.F. Edwards, M.F. Gyure and M. Ferer, *Phys. Rev.* **A46** , 6252 (1992).

- [120] R.M. Ziff, *Phys. Rev. Lett.* **56** , 545 (1986).
- [121] R. Botet and Ploszajczak, *Intermittency Patterns of Fluctuations in Disaggregating Systems*, in *Complex Fluids*, p. 303, ed. L. Garrido, Springer Verlag (Berlin), 1993.
- [122] R. Botet and M. Ploszajczak, *Phys. Rev. Lett.* **69** , 3696 (1992);
R. Botet and M. Ploszajczak, *Proc. Workshop on Fragmentation Phenomena*, Ecole de Physique Theorique, Centre de Physique des Houches, 12 - 17 April 1993, GANIL Preprint P 93/14;
R. Botet and M. Ploszajczak, *Phys. Lett.* **B312** , 30 (1993).
- [123] R. Botet and M. Ploszajczak, *Acta Phys. Pol.* **B25** , 353 (1994).
- [124] A. Bialas and R. Peschanski, *Nucl. Phys.* **B273** , 703 (1986); ib d. **B308** , 857 (1988).
- [125] M. Ploszajczak and A. Tucholski, *Phys. Rev. Lett.* **65** , 1539 (1990).
- [126] P. Bozek, M. Ploszajczak and A. Tucholski, *Nucl. Phys.* **A539** , 693 (1992);
M. Ploszajczak, P. Bozek and A. Tucholski, in *Proc. XXX Int. Winter Meeting on Nuclear Physics*, ed. I. Iori, Bormio, Italy (1992), Proc. Univ. degli Studi di Milano, Suppl. 91 (1992).
- [127] Ya.B. Zeldovich, A.A. Ruzmaikin and D.D. Sokoloff, *World Scientific Lecture Notes in Physics* **20** (1990).
- [128] A.S. Mikhailov and I.V. Uporov, *Usp. Fiz. Nauk* **144** , 79 (1984)
Ya.B. Zeldovich, *Dokl. Akad. Nauk SSSR* **270** , 1369 (1983);
Ya.B. Zeldovich et al., *Zh. Eksp. Teor. Fiz.* **89** , 2061 (1985); *Usp. Fiz. Nauk* **152** , 3 (1987).
- [129] B. Elattari, J. Richert and P. Wagner, *Phys. Rev. Lett.* **69** , 45 (1992).
- [130] W.J. Swiatecki, *Aus. J. Phys.* **36** , 641 (1983).
- [131] B. Elattari, J. Richert and P. Wagner, *Nucl. Phys.* **A538** , 403c (1992).
- [132] B. Mandelbrot, *J. Fluid Mech.* **62** , 719 (1974);
U. Frisch, P. Sulem and M. Nelkin, *J. Fluid Mech.* **87** , 719 (1978).
- [133] G. Parisi and U. Frisch, in *Turbulence and Predictability in Geophysical Fluid Dynamics and Climate Dynamics*, *Proc. Int. School of Physics E. Fermi*, Varenna, Italy, eds. M. Ghil, R. Benzi and G. Parisi, Course LXXXVIII (North-Holland, Amsterdam) 1985, p. 84;
U. Frisch and M. Vergassola, *Europhys. Lett.* **14** , 439 (1991);
Y. Gagne and B. Castaing, *C. R. Acad. Sci.* **312** , 441 (1991).
- [134] S.E. Esipov, "Fluctuations of the Instantaneous Space Averages in Fragmentation with Kolmogorov Time Scales", Univ. of Chicago Preprint (1994).
- [135] M. Ploszajczak and A. Tucholski, *Nucl. Phys.* **A523** , 651 (1991).
- [136] K. Zalewski and A. Bialas, *Phys. Lett.* **B228** , 155 (1989).
- [137] C. Meneveau and K.R. Srenivasan, *Phys. Rev. Lett.* **59** , 1424 (1987);
P. Lipa and B. Buschbeck, *Phys. Lett.* **B223** , 465 (1989).
- [138] M. Ploszajczak, A. Tucholski and P. Bozek, *Phys. Lett.* **B262** , 183 (1991).
- [139] J.-F. Gouyet, *Phys. Rev.* **B47** , 5446 (1993).
- [140] R. Botet, R. Jullien and M. Kolb, *J. Phys.* **A17** , L75 (1984).
- [141] K. Kang and S. Redner, *Phys. Rev.* **A30** , 2833 (1984).

- [142] P. Kreutz et al., *Nucl. Phys.* **A556** , 672 (1993);
 C.A. Ogilvie et al., *Nucl. Phys.* **A553** , 271c (1993);
 H.W. Bartz et al., Niels Bohr Institute Preprint NBI-92-88 (1992);
 Bao-An Li, A.R. DeAngelis and D.H.E. Gross, *Phys. Lett.* **B303** , 225 (1993).
- [143] R. Botet and R. Jullien, *Phys. Rev. Lett.* **55** , 1943 (1985);
 P. Meakin, *J. Chem. Phys.* **83** , 3645 (1985);
 M. Kolb, *J. Phys.* **A19** , L263 (1986).