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Local Gauge Fields based on a Yang-Mills Theory in Loop Space

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Abstract

We construct a Yang-Mills theory in loop space (the space of all loops in Minkowski space) with the Kac-Moody gauge group in such a way that the theory possesses reparametrization invariance. On the basis of the Yang-Mills theory, we derive the usual Yang-Mills theory and a non-abelian Stueckelberg formalism extended to local antisymmetric and symmetric tensor fields of the second rank. The local Yang-Mills field and the second-rank tensor fields are regarded as components of a Yang-Mills field on the loop space.

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1 Introduction

Nowadays, gauge theories are established as the fundamental theories in physics. All the fundamental interactions in nature are described by the gauge theories. It is an attractive property of the gauge theories that local gauge invariance requires gauge fields and uniquely determines interactions between the gauge fields and matter fields and self-interactions of the gauge fields. Among the gauge theories, Yang-Mills theories are known to be renormalizable by virtue of their desirable interaction terms. The gauge theories are well understood in terms of differential geometry. The gauge fields are interpreted as connections on fiber bundles.

In order to incorporate matter fields into the gauge fields, various extensions of gauge theories have been considered [1-6]. Supersymmetric and higher-dimensional gauge theories are examples of such extended gauge theories. The advantages of the gauge theories are inherited by the extended gauge theories: interactions among matter fields are uniquely determined from gauge invariance and the matter fields are geometrically characterized as connections. In the supersymmetric gauge theories [1], spinor and scalar fields as well as usual gauge fields can be treated as components of gauge fields on superspace. In the higher-dimensional gauge theories [2], components of gauge fields along continuous or discrete extra dimensions are identified with scalar fields such as Higgs fields. We remark that the main idea in the extensions is to enlarge spacetime appropriately and to introduce gauge fields into the enlarged spacetime.

How can we incorporate tensor fields into the gauge fields? An answer to this question has been given by Freund and Nepomechie [3], who showed that an abelian antisymmetric tensor field of the second rank is a constrained $U(1)$ gauge field on loop space, the space of all loops in spacetime. The present authors have elegantly obtained the same in a gauge-invariant manner, without taking the small-loop limit [4]. In addition it was found that besides the second-rank antisymmetric tensor field, abelian vector and abelian third-rank tensor fields governed by the Stueckelberg formalism are also constrained $U(1)$ gauge fields on the loop space, or components of a $U(1)$ gauge field on the loop space [4,5]. Since the $U(1)$ gauge field is a functional field on spacetime, it contains an infinite number of local abelian tensor fields of higher rank. Consequently, enlarging spacetime up to the loop space, we can deal with the abelian tensor fields collectively as a $U(1)$ gauge field.

The purpose of the present paper is to generalize the above to the non-abelian case: we try to derive non-abelian gauge theories of local tensor fields from a Yang-Mills theory defined in the loop space. We then expect that interactions among the tensor fields are uniquely determined. A non-abelian generalization has already been studied by Nepomechie in order to construct a non-abelian gauge theory of second-rank antisymmetric tensor fields [6]. He, however, concluded that interaction terms of the tensor fields vanish in the small-loop limit and hence his generalization reduces to the abelian theory.

In the present paper we consider a different non-abelian generalization, whose essential

idea is to take the Kac-Moody group as the gauge group of the theory. The Kac-Moody group plays an important role in deriving local interactions among tensor fields. From the Yang-Mills theory in loop space, we actually obtain the usual Yang-Mills theory and a non-abelian gauge theory of second-rank antisymmetric and symmetric tensor fields interacting with the local Yang-Mills field. The interaction terms are uniquely determined as we expected. The gauge theory of the second-rank tensor fields is a non-abelian Stueckelberg formalism extended to the tensor fields. We conclude that the second-rank tensor fields as well as the local Yang-Mills field are components of a Yang-Mills field on the loop space.

This paper is organized as follows. In section 2 we construct a Yang-Mills theory in loop space with the Kac-Moody gauge group in such a way that the theory possesses reparametrization invariance. In section 3 we derive the usual Yang-Mills theory from the Yang-Mills theory in loop space. In section 4, the gauge group is restricted to the $U(1)$ Kac-Moody group. In this case, it is easy to see how a Yang-Mills field on the loop space is expressed in terms of local tensor fields. We show that the Yang-Mills theory in loop space yields a massive abelian gauge theory of local antisymmetric and symmetric tensor fields of the second rank. In section 5, referring to sections 3 and 4, we try to derive a non-abelian version of the abelian gauge theory. After modifying the Yang-Mills theory in loop space, we actually obtain a massive non-abelian gauge theory of the second-rank tensor fields. The modification is justified in section 6 by means of the formalism of nonlinear realization. Section 7 contains some remaining discussions. The appendix is devoted to construction of a metric that is necessary to incorporate reparametrization invariance into the Yang-Mills theory in loop space.

2 A Yang-Mills theory in loop space

We define a loop space ΩM^D as the set of all loops (closed paths) in D -dimensional Minkowski space M^D . An arbitrary point in ΩM^D corresponds to a loop in M^D having a parametrization $x^\mu = x^\mu(\sigma)$ ($0 \leq \sigma \leq 2\pi$) with $x^\mu(0) = x^\mu(2\pi)$, and hence we hereafter denote coordinates of a point in ΩM^D by $(x^{\mu\sigma})$ ¹⁾ with $x^{\mu\sigma} \equiv x^\mu(\sigma)$. Instead of $(x^{\mu\sigma})$, we can use coefficients $(x^{\mu n})$ ($n \in \mathbf{Z}$) of the Fourier expansion $x^\mu(\sigma) = \sum_{n=-\infty}^{\infty} x^{\mu n} e^{in\sigma}$ as coordinates of a point in ΩM^D . The loop space ΩM^D is isomorphic to the product space $M^D \times \tilde{\Omega} M^D$ with coordinates $(x^{\mu 0}, x^{\mu n})_{n \neq 0}$. Under an infinitesimal reparametrization

$$\sigma \longrightarrow \bar{\sigma} = \sigma - \varepsilon(\sigma), \quad (2.1)$$

the coordinates $(x^{\mu\sigma})$ transform as

$$x^{\mu\sigma} \longrightarrow \bar{x}^{\mu\sigma} = x^{\mu\sigma} + \varepsilon(\sigma) x'^\mu(\sigma), \quad (2.2)$$

since $x^\mu(\sigma)$ is a reparametrization scalar function. Here $x'^\mu(\sigma) \equiv dx^\mu(\sigma)/d\sigma$, and $\varepsilon(\sigma)$ is an infinitesimal function of σ with $\varepsilon(0) = \varepsilon(2\pi)$.

Next we introduce a Kac-Moody group $\hat{G}$ [7], the Kac-Moody extension of a compact Lie group G , whose generators $T_a(\sigma)$ satisfy the Kac-Moody algebra without the central extension,

$$[T_a(\rho), T_b(\sigma)] = if_{ab}{}^c \delta(\rho - \sigma) T_c(\rho), \quad (2.3)$$

and the hermiticity conditions $T_a(\sigma)^\dagger = T_a(\sigma)$, where $f_{ab}{}^c$ are the structure constants of G ²⁾. In terms of $T_{an} \equiv \int_0^{2\pi} \frac{d\sigma}{2\pi} T_a(\sigma) e^{in\sigma}$, (2.3) is written as $[T_{am}, T_{bn}] = if_{ab}{}^c T_{cm+n}$. From (2.3), we see that the generators $T_a(\sigma)$ transform as

$$T_a(\sigma) \longrightarrow \bar{T}_a(\bar{\sigma}) = \frac{d\sigma}{d\bar{\sigma}} T_a(\sigma) \quad (2.4)$$

under a reparametrization $\sigma \rightarrow \bar{\sigma}(\sigma)$. Then the transformation rule of T_{an} under (2.1) is determined to be $\delta_r T_{an} (\equiv \bar{T}_{an} - T_{an}) = -i\varepsilon^p (\ell_p)_n{}^m T_{am}$ ³⁾, where $(\ell_p)_n{}^m \equiv n\delta_{p+n}{}^m$ and ε^p are Fourier coefficients of $\varepsilon(\sigma)$. This rule is consistent with the commutation relations $[L_m, T_{an}] = -nT_{am+n}$, where L_n are the generators of $\text{Diff} S^1$ satisfying the Virasoro algebra $[L_m, L_n] = (m-n)L_{m+n}$ and $L_n^\dagger = L_{-n}$. The generators T_{an} and L_n constitute the Lie algebra of the semidirect product group $\hat{G} \odot \text{Diff} S^1$, where $\odot$ indicates the semidirect product. An arbitrary element U of the Lie algebra $\hat{\mathcal{G}}$ of $\hat{G}$ can be expanded in terms of $T_a(\sigma)$ or T_{an} : $U = \int_0^{2\pi} \frac{d\sigma}{2\pi} U^{a\sigma} T_a(\sigma) = U^{an} T_{an}$. The components U^{an} transform, under (2.1), as $\delta_r U^{an} = iU^{am} \varepsilon^p (\ell_p)_m{}^n$ so that U does not vary.

¹⁾ In the present paper, the indices $\iota, \kappa, \lambda, \mu, \nu$ and ξ take the values 0, 1, 2, ..., $D-1$, while the indices ρ, σ, χ and ω take continuous values from 0 to 2π .

²⁾ The indices a, b, c, d and e take the values 1, 2, 3, ..., $\dim G$.

³⁾ The indices m, n, p and q take all integers.

Let us consider a Yang-Mills theory in the loop space ΩM^D . We assume that the gauge group is the Kac-Moody group $\hat{G}$. Similar to the usual Yang-Mills theory, the infinitesimal gauge transformation of a Yang-Mills field $\mathcal{A}_{\mu\sigma}[x]$ on ΩM^D is given by

$$\delta \mathcal{A}_{\mu\sigma}[x] = \mathcal{D}_{\mu\sigma} \Lambda[x] \equiv \partial_{\mu\sigma} \Lambda[x] + i[\mathcal{A}_{\mu\sigma}[x], \Lambda[x]], \quad (2.5)$$

where $\partial_{\mu\sigma} \equiv \partial/\partial x^{\mu\sigma}$, and Λ is an infinitesimal scalar function on ΩM^D taking values in $\hat{G}$: Λ is written as

$$\Lambda[x] = \int_0^{2\pi} \frac{d\sigma}{2\pi} \Lambda^{a\sigma}[x] T_a(\sigma) \quad (2.6)$$

with infinitesimal scalar functions $\Lambda^{a\sigma}$ on ΩM^D . Since the gauge symmetry has no relation with reparametrizations, Λ has to be reparametrization invariant. As seen from (2.4), $d\sigma T_a(\sigma)$ is a reparametrization scalar; consequently, $\Lambda^{a\sigma}$ have to be reparametrization invariant: $\Lambda^{a\sigma}[\bar{x}] = \Lambda^{a\sigma}[x]$, from which, substituting (2.1) and (2.2), we have

$$x'^\mu(\sigma) \partial_{\mu\sigma} \Lambda[x] = \frac{\partial \Lambda^{a\sigma}[x]}{\partial \sigma} T_a(\sigma). \quad (2.7)$$

This is different from the condition $x'^\mu(\sigma) \partial_{\mu\sigma} \Lambda[x] = 0$ discussed in ref.[4]. The difference is due to the fact that the Λ in the present case is a linear combination of $T_a(\sigma)$ with the transformation rule (2.4). As a result, $x'^\mu(\sigma) \mathcal{A}_{\mu\sigma}[x] = 0$ does not hold in the present case.

We now give an action integral for $\mathcal{A}_{\mu\sigma}$. As done in ref.[4], we insert the damping factor $\exp(-L/l^2)$ with $L \equiv -\int_0^{2\pi} \frac{d\sigma}{2\pi} \eta_{\mu\nu} x'^\mu(\sigma) x'^\nu(\sigma)$ into the action integral, where $l (> 0)$ is a constant with dimensions of length, and $\eta_{\mu\nu}$, $\text{diag} \eta_{\mu\nu} = (1, -1, -1, \dots, -1)$, is the metric tensor on M^D . The action integral for $\mathcal{A}_{\mu\sigma}$ with the damping factor is given by

$$S_R = \frac{k}{V_R} \int [dx] \mathcal{L} \exp(-\frac{L}{l^2}) \quad (2.8)$$

with the Lagrangian

$$\mathcal{L} = -\frac{1}{4} \mathcal{G}^{\kappa\rho, \lambda\sigma} \mathcal{G}^{\mu\chi, \nu\omega} \text{TR}[\mathcal{F}_{\kappa\rho, \mu\chi} \mathcal{F}_{\lambda\sigma, \nu\omega}] .^4) \quad (2.9)$$

Here $[dx] \equiv \prod_{\mu=0}^{D-1} \prod_{n=-\infty}^{\infty} dx^{\mu n}$, $V_R \equiv \int \{dx\} \exp(-L/l^2)$ with $\{dx\} \equiv \prod_{\mu=0}^{D-1} \prod_{n=-\infty}^{\infty, n \neq 0} dx^{\mu n}$, k is a constant and $\mathcal{F}_{\mu\rho, \nu\sigma}$ is the field strength of $\mathcal{A}_{\mu\sigma}$:

$$\mathcal{F}_{\mu\rho, \nu\sigma} \equiv \partial_{\mu\rho} \mathcal{A}_{\nu\sigma} - \partial_{\nu\sigma} \mathcal{A}_{\mu\rho} + i[\mathcal{A}_{\mu\rho}, \mathcal{A}_{\nu\sigma}], \quad (2.10)$$

which transforms, under (2.5), as

$$\delta \mathcal{F}_{\mu\rho, \nu\sigma} = i[\mathcal{F}_{\mu\rho, \nu\sigma}, \Lambda]. \quad (2.11)$$

⁴⁾ We employ Einstein's convention for indices in the loop space ΩM^D ; for example, $U^{\mu\sigma} V_{\mu\sigma} = \sum_{\mu=0}^{D-1} \int_0^{2\pi} \frac{d\sigma}{2\pi} U^{\mu\sigma} V_{\mu\sigma}$.

The (inverse) metric tensor $\mathcal{G}^{\mu\rho,\nu\sigma}$ on ΩM^D is induced from the (inverse) metric tensor $\eta^{\mu\nu}$ on M^D as follows:

$$\mathcal{G}^{\mu\rho,\nu\sigma} = \eta^{\mu\nu} G^{\rho\sigma} , \quad (2.12)$$

where $G^{\rho\sigma}$ is a metric tensor yielding inner products that are reparametrization invariant. The metric $G^{\rho\sigma}$ has actually been constructed in ref.[4] by utilizing the coset manifold $\text{Diff}S^1/S^1$. In the present paper, however, we use $\delta^{\rho\sigma} = \delta(\rho - \sigma)$ instead of $G^{\rho\sigma}$; we can regard $\delta^{\rho\sigma}$ as a form of $G^{\rho\sigma}$ in a particular gauge of the reparametrizations.

The symbol “TR” denotes an inner product of two elements of $\hat{\mathcal{G}}$. In the usual (quantum) Yang-Mills theory, the inner product in the Lie algebra of a gauge group is defined with the Killing metric of the gauge group. The Killing metric of $\hat{G}$ is given by

$$K_{a\rho,b\sigma} = \kappa_{ab} \delta_{\rho\sigma} , \quad (2.13)$$

where $\delta_{\rho\sigma} = \delta(\rho - \sigma)$, and κ_{ab} is the Killing metric of the Lie group G . Let U and V be arbitrary elements of $\hat{\mathcal{G}}$ having the gauge transformation rule (2.11). Then the inner product

$$\text{Tr}[UV] \equiv K_{a\rho,b\sigma} U^{a\rho} V^{b\sigma} \quad (2.14)$$

is, of course, gauge invariant; however, it is not invariant under reparametrizations other than rotations. To maintain both reparametrization invariance and gauge invariance in the Lagrangian (2.9), we have to use an inner product that is reparametrization invariant as well as gauge invariant. As is discussed in appendix, we can indeed define such an inner product, which we denote by TR, by introducing a new metric. However, we use, instead of TR, the inner product Tr for concrete calculations in sections 3, 4 and 5; Tr is regarded as a form of TR in a certain gauge, which fact is also discussed in appendix.

The damping factor $\exp(-L/l^2)$ restricts contribution of large loops in the integrals with respect to $x^{\mu n}$; as a result, the integrals are well defined. (The integrations with respect to x^{0n} are carried out after the Wick rotation $x^{0n} \rightarrow -ix^{0n}$.) Similar to the metric $\delta^{\rho\sigma}$ and the inner product Tr, we can regard $\exp(-L/l^2)$ as a form of a certain damping factor that is reparametrization invariant [4]; the action integral (2.8) can be replaced by a fully reparametrization-invariant form. (Note here that the measures $[dx]$ and $\{dx\}$ are reparametrization invariant.)

⁵⁾ We employ Einstein's convention $U_{a\sigma} V^{a\sigma} = \sum_{a=1}^{\dim G} \int_0^{2\pi} \frac{d\sigma}{2\pi} U_{a\sigma} V^{a\sigma}$.

3 Local Yang-Mills Field

Now we consider the simplest solution of (2.7):

$$\Lambda^{(0)}[x] = \int_0^{2\pi} \frac{d\sigma}{2\pi} q_0 \lambda^a(x(\sigma)) T_a(\sigma), \quad (3.1)$$

where q_0 is a dimensionless constant and λ^a are infinitesimal scalar functions on M^D . Corresponding to (3.1), we restrict $\mathcal{A}_{\mu\sigma}$ to

$$\mathcal{A}_{\mu\sigma}^{(0)}[x] = q_0 A_\mu^a(x(\sigma)) T_a(\sigma), \quad (3.2)$$

where A_μ^a are local vector fields on M^D . Note that under the reparametrizations, the right-hand side of (3.2) transforms in the same manner as $\mathcal{A}_{\mu\sigma}$. With the aid of (2.3), the substitution of (3.1) and (3.2) into (2.5) leads to the usual gauge transformation of local Yang-Mills fields:

$$\delta A_\mu^a(x) = D_\mu \lambda^a(x) \equiv \partial_\mu \lambda^a(x) - q_0 A_\mu^b(x) \lambda^c(x) f_{bc}^a. \quad (3.3)$$

Substituting (3.2) into (2.10) and using (2.3), we obtain

$$\mathcal{F}_{\mu\rho,\nu\sigma}^{(0)}[x] = q_0 \delta(\rho - \sigma) F_{\mu\nu}^a(x(\sigma)) T_a(\sigma) \quad (3.4)$$

with

$$F_{\mu\nu}^a(x) \equiv \partial_\mu A_\nu^a(x) - \partial_\nu A_\mu^a(x) - q_0 A_\mu^b(x) A_\nu^c(x) f_{bc}^a. \quad (3.5)$$

The field strength of $\mathcal{A}_{\mu\sigma}^{(0)}$ is thus written in terms of that of A_μ^a . The Lagrangian (2.9) with (3.4) becomes

$$\mathcal{L}^{(0)} = -\frac{1}{4} q_0^2 \delta(0)^2 \int_0^{2\pi} \frac{d\sigma}{2\pi} F_{\mu\nu}^a(x(\sigma)) F^{\mu\nu a}(x(\sigma)), \quad (3.6)$$

where lowering of the index a has been done with κ_{ab} .

We next carry out the Taylor expansion of $F_{\mu\nu}^a(x(\sigma)) F^{\mu\nu a}(x(\sigma))$ around $(x^{\mu 0})$:

$$\begin{aligned} F_{\mu\nu}^a(x(\sigma)) F^{\mu\nu a}(x(\sigma)) &= F_{\mu\nu}^a(x^0) F^{\mu\nu a}(x^0) \\ &+ \sum_{k=1}^{\infty} \frac{1}{k!} \left[\left(\prod_{j=1}^k \sum_{\mu_j=0}^{D-1} \sum_{n_j \neq 0} x^{\mu_j n_j} e^{i n_j \sigma} \partial_{\mu_j} \right) (F_{\mu\nu}^a F^{\mu\nu a}) \right]_{x^\mu = x^{\mu 0}}, \end{aligned} \quad (3.7)$$

where x^0 stands for $(x^{\mu 0})$. Since $-\infty < x^{\mu 0} < \infty$, all the differential coefficients at $(x^{\mu 0})$ in the Taylor series (3.7) vanish after integration with respect to $x^{\mu 0}$; by virtue of this fact, the action integral (2.8) with (3.7) reduces to

$$S_R^{(0)} = -\frac{1}{4} \int d^D x F_{\mu\nu}^a(x) F^{\mu\nu a}(x) \quad (3.8)$$

with $d^D x \equiv \prod_{\mu=0}^{D-1} dx^\mu$, after we set the normalization condition $k q_0^2 \delta(0)^2 = 1$. Here $x^{\mu 0}$ have been replaced by x^μ . We thus arrive at the action integral for local Yang-Mills fields.

4 An abelian gauge theory of local second-rank tensor fields

The next simplest solution of (2.7) is

$$\Lambda^{(1)}[x] = \int_0^{2\pi} \frac{d\sigma}{2\pi} q_1 \frac{x'^\mu(\sigma)}{\sqrt{-x'(\sigma)^2}} \lambda_\mu^a(x(\sigma)) T_a(\sigma), \quad (4.1)$$

where $x'(\sigma)^2 \equiv x'_\mu(\sigma)x'^\mu(\sigma)$, q_1 is a dimensionless constant, and λ_μ^a are infinitesimal vector functions on M^D .

Before studying the non-abelian case, we focus our attention to the abelian case, that is, $G = [\text{U}(1)]^N$. In this case we can easily determine from (2.5) what local fields on M^D have to be introduced and how the Yang-Mills field $\mathcal{A}_{\mu\sigma}$ is expressed in terms of the local fields. Taking into account

$$\begin{aligned} \partial_{\mu\sigma} \Lambda^{(1)}[x] &= q_1 Q^\nu(\sigma) \{ \partial_\mu \lambda_\nu^a(x(\sigma)) - \partial_\nu \lambda_\mu^a(x(\sigma)) \} T_a(\sigma) \\ &\quad - q_1 Q_\mu(\sigma) Q^\nu(\sigma) Q^\kappa(\sigma) \partial_\nu \lambda_\kappa^a(x(\sigma)) T_a(\sigma) \\ &\quad - q_1 \lambda_\nu^a(x(\sigma)) \frac{d}{d\sigma} \left(\Pi_\mu^\nu(\sigma) \frac{T_a(\sigma)}{\sqrt{-x'(\sigma)^2}} \right) \end{aligned} \quad (4.2)$$

with

$$\Pi_{\mu\nu}(\sigma) \equiv \frac{1}{-x'(\sigma)^2} (x'_\mu(\sigma)x'_\nu(\sigma) - \eta_{\mu\nu} x'(\sigma)^2), \quad (4.3)$$

$$Q_\mu(\sigma) \equiv \frac{x'_\mu(\sigma)}{\sqrt{-x'(\sigma)^2}}, \quad (4.4)$$

we restrict $\mathcal{A}_{\mu\sigma}$ to the following form:

$$\begin{aligned} \mathcal{A}_{\mu\sigma}^{(1)}[x] &= q_1 Q^\nu(\sigma) B_{\mu\nu}^a(x(\sigma)) T_a(\sigma) - \frac{1}{2} q_1 Q_\mu(\sigma) Q^\nu(\sigma) Q^\kappa(\sigma) C_{\nu\kappa}^a(x(\sigma)) T_a(\sigma) \\ &\quad - e_1 \phi_\nu^a(x(\sigma)) \frac{d}{d\sigma} \left(\Pi_\mu^\nu(\sigma) \frac{T_a(\sigma)}{\sqrt{-x'(\sigma)^2}} \right), \end{aligned} \quad (4.5)$$

where e_1 is a constant with dimensions of length, $B_{\mu\nu}^a (= -B_{\nu\mu}^a)$ and $C_{\mu\nu}^a (= C_{\nu\mu}^a)$ are local antisymmetric and symmetric tensor fields on M^D , respectively, and ϕ_μ^a are local vector fields on M^D . Then, (2.5) in the abelian case yields the gauge transformation of the local fields:

$$\delta B_{\mu\nu}^a(x) = \partial_\mu \lambda_\nu^a(x) - \partial_\nu \lambda_\mu^a(x), \quad (4.6a)$$

$$\delta C_{\mu\nu}^a(x) = \partial_\mu \lambda_\nu^a(x) + \partial_\nu \lambda_\mu^a(x), \quad (4.6b)$$

$$\delta \phi_\mu^a(x) = m_1 \lambda_\mu^a(x), \quad m_1 \equiv \frac{q_1}{e_1}. \quad (4.6c)$$

We now substitute (4.5) into (2.10) in the abelian case. After some tedious calculation, we obtain

$$\begin{aligned}
\mathcal{F}_{\mu\rho,\nu\sigma}^{(1)}[x] = & \frac{1}{2}q_1\delta(\rho-\sigma)[Q^\lambda(\rho)H_{\lambda\mu\nu}{}^a(x(\rho))T_a(\rho) + \tilde{B}_{\mu\nu}{}^a(x(\rho))\frac{d}{d\rho}\left(\frac{T_a(\rho)}{\sqrt{-x'(\rho)^2}}\right) \\
& - \frac{1}{2}I_{\lambda[\mu}{}^a(x(\rho))\frac{d}{d\rho}(Q_{\nu]}(\rho)Q^\lambda(\rho)\frac{T_a(\rho)}{\sqrt{-x'(\rho)^2}}) \\
& - \frac{1}{2}Q^\kappa(\rho)Q^\lambda(\rho)Q_{[\mu}(\rho)J_{\nu]\kappa\lambda}{}^a(x(\rho))T_a(\rho) \\
& + Q^\lambda(\sigma)H_{\lambda\mu\nu}{}^a(x(\sigma))T_a(\sigma) + \tilde{B}_{\mu\nu}{}^a(x(\sigma))\frac{d}{d\sigma}\left(\frac{T_a(\sigma)}{\sqrt{-x'(\sigma)^2}}\right) \\
& - \frac{1}{2}I_{\lambda[\mu}{}^a(x(\sigma))\frac{d}{d\sigma}(Q_{\nu]}(\sigma)Q^\lambda(\sigma)\frac{T_a(\sigma)}{\sqrt{-x'(\sigma)^2}}) \\
& - \frac{1}{2}Q^\kappa(\sigma)Q^\lambda(\sigma)Q_{[\mu}(\sigma)J_{\nu]\kappa\lambda}{}^a(x(\sigma))T_a(\sigma)] \\
& - \frac{1}{2}q_1\delta'(\rho-\sigma)[\{Q^\lambda(\rho)Q_{(\mu}(\rho)I_{\nu)\lambda}{}^a(x(\rho)) \\
& - (3Q_\mu(\rho)Q_\nu(\rho) + \eta_{\mu\nu})Q^\kappa(\rho)Q^\lambda(\rho)\tilde{C}_{\kappa\lambda}{}^a(x(\rho))\}\frac{T_a(\rho)}{\sqrt{-x'(\rho)^2}} \\
& + \{Q^\lambda(\sigma)Q_{(\mu}(\sigma)I_{\nu)\lambda}{}^a(x(\sigma)) \\
& - (3Q_\mu(\sigma)Q_\nu(\sigma) + \eta_{\mu\nu})Q^\kappa(\sigma)Q^\lambda(\sigma)\tilde{C}_{\kappa\lambda}{}^a(x(\sigma))\}\frac{T_a(\sigma)}{\sqrt{-x'(\sigma)^2}}] ,^6) \quad (4.7)
\end{aligned}$$

where

$$\begin{aligned}
H_{\lambda\mu\nu}{}^a(x) & \equiv \partial_\lambda B_{\mu\nu}{}^a(x) + \partial_\mu B_{\nu\lambda}{}^a(x) + \partial_\nu B_{\lambda\mu}{}^a(x) \\
& = \partial_\lambda \tilde{B}_{\mu\nu}{}^a(x) + \partial_\mu \tilde{B}_{\nu\lambda}{}^a(x) + \partial_\nu \tilde{B}_{\lambda\mu}{}^a(x) , \quad (4.8)
\end{aligned}$$

$$\tilde{B}_{\mu\nu}{}^a(x) \equiv B_{\mu\nu}{}^a(x) - \frac{1}{m_1}(\partial_\mu \phi_\nu{}^a(x) - \partial_\nu \phi_\mu{}^a(x)) , \quad (4.9)$$

$$\tilde{C}_{\mu\nu}{}^a(x) \equiv C_{\mu\nu}{}^a(x) - \frac{1}{m_1}(\partial_\mu \phi_\nu{}^a(x) + \partial_\nu \phi_\mu{}^a(x)) , \quad (4.10)$$

and

$$I_{\mu\nu}{}^a(x) \equiv \tilde{B}_{\mu\nu}{}^a(x) - \tilde{C}_{\mu\nu}{}^a(x) , \quad (4.11)$$

$$J_{\lambda\mu\nu}{}^a(x) \equiv \partial_\mu \tilde{B}_{\lambda\nu}{}^a(x) + \partial_\mu \tilde{C}_{\lambda\nu}{}^a(x) - \partial_\lambda \tilde{C}_{\mu\nu}{}^a(x) . \quad (4.12)$$

The field strength $\mathcal{F}_{\mu\rho,\nu\sigma}^{(1)}$ is invariant under the gauge transformation (4.6), since $H_{\lambda\mu\nu}{}^a$, $\tilde{B}_{\mu\nu}{}^a$ and $\tilde{C}_{\mu\nu}{}^a$ are invariant. We notice that the terms proportional to $\delta(\rho-\sigma)$ are antisymmetric under the exchange of μ and ν , while the terms proportional to $\delta'(\rho-\sigma)$ are symmetric under

⁶⁾ $X_{[\mu}Y_{\nu]} \equiv X_\mu Y_\nu - X_\nu Y_\mu$, $X_{(\mu}Y_{\nu)} \equiv X_\mu Y_\nu + X_\nu Y_\mu$.

the exchange of μ and ν ; because of this, $\mathcal{F}_{\mu\rho,\nu\sigma}^{(1)} = -\mathcal{F}_{\nu\sigma,\mu\rho}^{(1)}$ is evident. The substitution of (4.7) into (2.9) leads to the following Lagrangian:

$$\begin{aligned}
\mathcal{L}^{(1)} = & -\frac{1}{4}q_1^2 \int_0^{2\pi} \frac{d\sigma}{2\pi} [\delta(0)^2 Q^\kappa(\sigma) Q^\lambda(\sigma) H_{\kappa\mu\nu a}(x(\sigma)) H_\lambda{}^{\mu\nu a}(x(\sigma)) \\
& -\frac{1}{2}\delta(0)^2 Q^\kappa(\sigma) Q^\lambda(\sigma) Q^\mu(\sigma) Q^\nu(\sigma) J_{\xi\kappa\lambda a}(x(\sigma)) J^\xi{}_{\mu\nu}{}^a(x(\sigma)) \\
& +2\delta(0)^2 Q^\lambda(\sigma) H_{\lambda\mu\nu a}(x(\sigma)) \tilde{B}^{\mu\nu a}(x(\sigma)) \frac{d}{d\sigma} \left(\frac{1}{\sqrt{-x'(\sigma)^2}} \right) \\
& +2\delta(0)^2 \frac{1}{\sqrt{-x'(\sigma)^2}} Q^\kappa(\sigma) Q^\lambda(\sigma) Q'^\mu(\sigma) H_{\kappa\mu\nu a}(x(\sigma)) I_\lambda{}^{\nu a}(x(\sigma)) \\
& -2\delta(0)^2 Q^\kappa(\sigma) Q^\lambda(\sigma) Q^\mu(\sigma) J_{\nu\kappa\lambda a}(x(\sigma)) \tilde{B}_\mu{}^{\nu a}(x(\sigma)) \frac{d}{d\sigma} \left(\frac{1}{\sqrt{-x'(\sigma)^2}} \right) \\
& +\delta(0)^2 Q^\kappa(\sigma) Q^\lambda(\sigma) J_{\nu\kappa\lambda a}(x(\sigma)) I_\mu{}^{\nu a}(x(\sigma)) \frac{d}{d\sigma} \left(\frac{Q^\mu(\sigma)}{\sqrt{-x'(\sigma)^2}} \right) \\
& -\delta(0)^2 \frac{1}{\sqrt{-x'(\sigma)^2}} Q^\kappa(\sigma) Q^\lambda(\sigma) Q^\mu(\sigma) Q^\nu(\sigma) Q'^\xi(\sigma) J_{\xi\kappa\lambda a}(x(\sigma)) \tilde{C}_{\mu\nu}{}^a(x(\sigma)) \\
& +R_\kappa{}^{\kappa'}{}_\lambda{}^\lambda(\sigma) \tilde{B}_{\mu\nu a}(x(\sigma)) \tilde{B}^{\mu\nu a}(x(\sigma)) - 2R_\kappa{}^{\kappa'\lambda\mu}(\sigma) I_{\lambda\nu a}(x(\sigma)) \tilde{B}_\mu{}^{\nu a}(x(\sigma)) \\
& +\frac{1}{2}R^{\kappa\nu,\lambda}{}_\nu(\sigma) I_{\kappa\mu a}(x(\sigma)) I_\lambda{}^{\mu a}(x(\sigma)) - \frac{1}{2}R^{\kappa\lambda,\mu\nu}(\sigma) I_{\kappa\mu a}(x(\sigma)) I_{\nu\lambda}{}^a(x(\sigma)) \\
& -\frac{3}{2}\delta''(0)\delta(0) \frac{1}{x'(\sigma)^2} Q^\lambda(\sigma) Q^\mu(\sigma) I_{\nu\lambda a}(x(\sigma)) I^\nu{}_\mu{}^a(x(\sigma)) \\
& +\frac{3}{4}(D-3)\delta''(0)\delta(0) \frac{1}{x'(\sigma)^2} Q^\kappa(\sigma) Q^\lambda(\sigma) Q^\mu(\sigma) Q^\nu(\sigma) \tilde{C}_{\kappa\lambda a}(x(\sigma)) \tilde{C}_{\mu\nu}{}^a(x(\sigma))] ,
\end{aligned} \tag{4.13}$$

where

$$\begin{aligned}
R^{\kappa\lambda,\mu\nu}(\sigma) \equiv & \frac{d}{d\sigma} \left(\frac{1}{\sqrt{-x'(\sigma)^2}} Q^\kappa(\sigma) Q^\lambda(\sigma) \right) \frac{d}{d\sigma} \left(\frac{1}{\sqrt{-x'(\sigma)^2}} Q^\mu(\sigma) Q^\nu(\sigma) \right) \delta(0)^2 \\
& + \frac{1}{x'(\sigma)^2} Q^\kappa(\sigma) Q^\lambda(\sigma) Q^\mu(\sigma) Q^\nu(\sigma) \delta''(0)\delta(0) .
\end{aligned} \tag{4.14}$$

In deriving (4.13) we have used $\text{Tr}[T_a(\rho)T_b(\sigma)] = K_{a\rho,b\sigma}$ and the integral formulas

$$\int_0^{2\pi} \frac{d\rho}{2\pi} f(\rho) \delta'(\rho - \sigma) \delta'(\rho - \sigma) = -\delta''(0)f(\sigma) , \tag{4.15a}$$

$$\int_0^{2\pi} \frac{d\rho}{2\pi} f(\rho) \delta(\rho - \sigma) \delta(\rho - \sigma) \delta'(\rho - \sigma) = 0 , \tag{4.15b}$$

$$\int_0^{2\pi} \frac{d\rho}{2\pi} f(\rho) \delta(\rho - \sigma) \delta(\rho - \sigma) \delta''(\rho - \sigma) = \delta''(0)\delta(0)f(\sigma) , \tag{4.15c}$$

$$\int_0^{2\pi} \frac{d\rho}{2\pi} f(\rho) \delta(\rho - \sigma) \delta'(\rho - \sigma) \delta'(\rho - \sigma) = -\frac{1}{2}\delta''(0)\delta(0)f(\sigma) , \tag{4.15d}$$

where $f(\sigma)$ is an arbitrary continuous function of σ . These formulas are established by certain regularizations of the δ functions.

As done in section 3, we expand the functions of $x^\mu(\sigma)$ in (4.13) around $(x^{\mu 0})$. Then, in the action integral (2.8) with (4.13) (which we denote by $S_R^{(1)}$), all the differential coefficients at $(x^{\mu 0})$ in the each Taylor series vanish after integration with respect to $x^{\mu 0}$; as a result, we have $S_R^{(1)}$ in which the arguments $x^\mu(\sigma)$ of the functions are replaced by $x^{\mu 0}$. By using the formulas

$$\begin{aligned} & \int \{dx\} \int_0^{2\pi} \frac{d\sigma}{2\pi} g(x'(\sigma)^2) Q^\kappa(\sigma) Q^\lambda(\sigma) Q^\mu(\sigma) Q^\nu(\sigma) \exp(-\frac{L}{l^2}) \\ &= \frac{1}{D(D+2)} (\eta^{\kappa\lambda} \eta^{\mu\nu} + \eta^{\kappa\mu} \eta^{\lambda\nu} + \eta^{\kappa\nu} \eta^{\lambda\mu}) \int \{dx\} \int_0^{2\pi} \frac{d\sigma}{2\pi} g(x'(\sigma)^2) \exp(-\frac{L}{l^2}), \quad (4.16a) \end{aligned}$$

$$\begin{aligned} & \int \{dx\} \int_0^{2\pi} \frac{d\sigma}{2\pi} g(x'(\sigma)^2) Q^\kappa(\sigma) Q^\lambda(\sigma) Q'^\mu(\sigma) Q'^\nu(\sigma) \exp(-\frac{L}{l^2}) \\ &= \frac{-1}{D(D-1)(D+2)} \{ (D+1) \eta^{\kappa\lambda} \eta^{\mu\nu} - (\eta^{\kappa\mu} \eta^{\lambda\nu} + \eta^{\kappa\nu} \eta^{\lambda\mu}) \} \\ & \times \int \{dx\} \int_0^{2\pi} \frac{d\sigma}{2\pi} g(x'(\sigma)^2) Q'_\mu(\sigma) Q'^\mu(\sigma) \exp(-\frac{L}{l^2}) \quad (4.16b) \end{aligned}$$

satisfied with any function g of $x'(\sigma)^2$, the action integral $S_R^{(1)}$ becomes

$$\begin{aligned} S_R^{(1)} = \int d^D x & \left[\frac{1}{12} H_{\lambda\mu\nu a}(x) H^{\lambda\mu\nu a}(x) + 4v \partial^\lambda \tilde{B}_{\lambda\nu a}(x) \partial_\mu \tilde{B}^{\mu\nu a}(x) \right. \\ & - 8v \partial^\lambda \tilde{B}_{\lambda\nu a}(x) \partial_\mu \tilde{C}^{\mu\nu a}(x) + 3v \partial_\lambda \tilde{C}_{\mu\nu a}(x) \partial^\lambda \tilde{C}^{\mu\nu a}(x) \\ & - 2v \partial^\lambda \tilde{C}_{\lambda\nu a}(x) \partial_\mu \tilde{C}^{\mu\nu a}(x) - 2v \partial^\lambda \tilde{C}_{\lambda\nu a}(x) \partial^\nu \tilde{C}_\mu{}^{\mu a}(x) \\ & + v \partial_\lambda \tilde{C}_\mu{}^\mu{}_a(x) \partial^\lambda \tilde{C}_\nu{}^{\nu a}(x) - \frac{1}{4} m_1^2 \tilde{B}_{\mu\nu a}(x) \tilde{B}^{\mu\nu a}(x) \\ & \left. - 3v m_2^2 \tilde{C}_{\mu\nu a}(x) \tilde{C}^{\mu\nu a}(x) - 3v m_3^2 \tilde{C}_\mu{}^\mu{}_a(x) \tilde{C}_\nu{}^{\nu a}(x) \right] \quad (4.17) \end{aligned}$$

after the integration over the subspace $\tilde{\Omega} M^D$. Here we have set the normalization conditions

$$k q_1^2 \delta(0)^2 \frac{6D+13}{2D(D+2)} = 1, \quad (4.18a)$$

$$\frac{k e_1^2}{l^2} \left\{ \left(\frac{2D-3}{2D} K_1 + \frac{D-2}{D(D-1)} K_2 \right) \delta(0)^2 - K_3 \delta''(0) \delta(0) \right\} = 1, \quad (4.18b)$$

and have defined v , m_2^2 and m_3^2 by $v \equiv 1/4(6D+13)$,

$$m_2^2 \equiv \frac{k q_1^2}{12v l^2} \left\{ \frac{1}{D+2} \left(\frac{1}{2} K_1 + \frac{D}{D-1} K_2 \right) \delta(0)^2 - \frac{7D-3}{2D(D+2)} K_3 \delta''(0) \delta(0) \right\}, \quad (4.19a)$$

$$m_3^2 \equiv \frac{k q_1^2}{12v l^2} \left\{ \frac{-1}{D+2} \left(\frac{1}{2D} K_1 + \frac{1}{D-1} K_2 \right) \delta(0)^2 - \frac{3D-11}{4D(D+2)} K_3 \delta''(0) \delta(0) \right\}, \quad (4.19b)$$

where

$$K_1 \equiv \frac{l^2}{V_R} \int \{dx\} \int_0^{2\pi} \frac{d\sigma}{2\pi} \left\{ \frac{d}{d\sigma} \left(\frac{1}{\sqrt{-x'(\sigma)^2}} \right) \right\}^2 \exp(-\frac{L}{l^2}) , \quad (4.20a)$$

$$K_2 \equiv \frac{l^2}{V_R} \int \{dx\} \int_0^{2\pi} \frac{d\sigma}{2\pi} \frac{1}{x'(\sigma)^2} Q'_\mu(\sigma) Q'^\mu(\sigma) \exp(-\frac{L}{l^2}) , \quad (4.20b)$$

$$K_3 \equiv -\frac{l^2}{V_R} \int \{dx\} \int_0^{2\pi} \frac{d\sigma}{2\pi} \frac{1}{x'(\sigma)^2} \exp(-\frac{L}{l^2}) . \quad (4.20c)$$

Note that K_1 , K_2 and K_3 are positive and dimensionless.

Let us now consider the local field theory defined by (4.17). The Euler-Lagrange equations for $\tilde{B}_{\mu\nu}^a$ and $\tilde{C}_{\mu\nu}^a$ are, from (4.17),

$$\partial^\lambda H_{\lambda\mu\nu}^a - 8v\partial^\lambda \partial_{[\mu} (\tilde{B}_{\nu]\lambda}^a + \tilde{C}_{\nu]\lambda}^a) + m_1^2 \tilde{B}_{\mu\nu}^a = 0 , \quad (4.21)$$

$$\begin{aligned} \square \tilde{C}_{\mu\nu}^a + \frac{1}{3} \partial^\lambda (2\partial_{(\mu} \tilde{B}_{\nu)\lambda}^a - \partial_{(\mu} \tilde{C}_{\nu)\lambda}^a) - \frac{1}{3} \eta_{\mu\nu} \partial^\kappa \partial^\lambda \tilde{C}_{\kappa\lambda}^a \\ + \frac{1}{3} (\eta_{\mu\nu} \square - \partial_\mu \partial_\nu) \tilde{C}^a + m_2^2 \tilde{C}_{\mu\nu}^a + m_3^2 \eta_{\mu\nu} \tilde{C}^a = 0 , \end{aligned} \quad (4.22)$$

where $\tilde{C}^a \equiv \tilde{C}_\mu{}^{\mu a}$ and $\square \equiv \partial_\mu \partial^\mu$. Comparing the divergence of (4.21),

$$\square (\tilde{B}_\mu{}^a - \partial^\nu \tilde{C}_{\nu\mu}^a) + \partial_\mu \partial^\lambda \partial^\nu \tilde{C}_{\lambda\nu}^a + \frac{1}{8v} m_1^2 \tilde{B}_\mu{}^a = 0 , \quad (4.23)$$

with that of (4.22), we have

$$m_1^2 \tilde{B}_\mu{}^a + 12vm_2^2 \tilde{C}_\mu{}^a = 0 , \quad (4.24)$$

where $\tilde{B}_\mu{}^a \equiv \partial^\nu \tilde{B}_{\nu\mu}^a$ and $\tilde{C}_\mu{}^a \equiv \partial^\nu \tilde{C}_{\nu\mu}^a + (m_3/m_2)^2 \partial_\mu \tilde{C}^a$. Evidently $\tilde{B}_\mu{}^a$ and $\tilde{C}_\mu{}^a$ satisfy the divergenceless conditions $\partial^\mu \tilde{B}_\mu{}^a = \partial^\mu \tilde{C}_\mu{}^a = 0$. Using (4.24), we can eliminate $\partial^\nu \tilde{C}_{\nu\mu}^a$ in (4.23) and arrive at the Klein-Gordon equation

$$(\square + m_4^2) \tilde{B}_\mu{}^a = 0 , \quad m_4^2 \equiv \frac{3m_1^2 m_2^2}{2(m_1^2 + 12vm_2^2)} . \quad (4.25)$$

Obviously $\tilde{C}_\mu{}^a$ also satisfy (4.25). By virtue of (4.24) and (4.25), eq. (4.21) reduces to the Klein-Gordon equation

$$(\square + m_1^2) \tilde{B}_{\mu\nu}^{*a} = 0 \quad (4.26)$$

for the divergenceless parts $\tilde{B}_{\mu\nu}^{*a} \equiv \tilde{B}_{\mu\nu}^a + (1/m_4^2) \partial_{[\mu} \tilde{B}_{\nu]}^a$, satisfying $\partial^\mu \tilde{B}_{\mu\nu}^{*a} = 0$.

Contracting μ and ν in (4.22) and using $\partial^\mu \tilde{C}_\mu{}^a = 0$, we obtain

$$(\square + m_5^2) \tilde{C}^a = 0 , \quad m_5^2 \equiv \frac{3m_2^2(m_2^2 + Dm_3^2)}{(D+2)(m_2^2 + m_3^2)} . \quad (4.27)$$

By virtue of (4.24), (4.25) and (4.27), eq. (4.22) also gives

$$(\square + m_2^2)\tilde{C}_{\mu\nu}^{*a} = 0 \quad (4.28)$$

with

$$\begin{aligned} \tilde{C}_{\mu\nu}^{*a} \equiv & \tilde{C}_{\mu\nu}^a - \frac{1}{D}\eta_{\mu\nu}\tilde{C}^a + \frac{1}{m_4^2}\partial_{(\mu}\tilde{C}_{\nu)}^a \\ & - \frac{(D+2)(m_2^2 + m_3^2)}{3(D-1)m_2^4}(\partial_\mu\partial_\nu - \frac{1}{D}\eta_{\mu\nu}\square)\tilde{C}^a. \end{aligned} \quad (4.29)$$

Notice that $\tilde{C}_{\mu\nu}^{*a}$ are traceless and divergenceless, namely, $\tilde{C}_\mu^{*\mu a} = \partial^\mu\tilde{C}_{\mu\nu}^{*a} = 0$.

Each of the independent fields $\tilde{B}_{\mu\nu}^{*a}$, $\tilde{B}_\mu^a$ (or $\tilde{C}_\mu^a$), $\tilde{C}_{\mu\nu}^{*a}$ and $\tilde{C}^a$ is irreducible in the sense that it is traceless and divergenceless. We have found that these fields are massive and satisfy Klein-Gordon equations. The action integral (4.17) thus describes massive fields without spoiling the gauge invariance, with the aid of the Nambu-Goldstone-like fields ϕ_μ^a . The fields $\tilde{B}_{\mu\nu}^a$ and $\tilde{C}_{\mu\nu}^a$ are nothing but $B_{\mu\nu}^a$ and $C_{\mu\nu}^a$ in the unitary gauge, respectively. From (4.18) and (4.19), we see that the masses m_1 , m_2 , m_4 and m_5 are real and inversely proportional to the size l of the loops under suitable regularizations of the divergent quantities. This is similar to the results in refs. [4] and [5]. The massless limit, that is, $l \rightarrow \infty$ can not be taken in the action integral (4.17); after rewriting (4.17) by means of additional auxiliary fields, we can take the massless limit.

5 A non-abelian gauge theory of the second-rank tensor fields

In this section we derive a non-abelian version of the gauge theory discussed in the previous section, on the basis of the Yang-Mills theory in loop space.

Let $\Lambda^{(p)}[x]$ be a solution of (2.7) consisting of infinitesimal p th-rank symmetric tensor functions $\lambda_{\mu_1\mu_2\cdots\mu_p}^a(x)$ on M^D :

$$\Lambda^{(p)}[x] = \int_0^{2\pi} \frac{d\sigma}{2\pi} q_p Q^{\mu_1}(\sigma) Q^{\mu_2}(\sigma) \cdots Q^{\mu_p}(\sigma) \lambda_{\mu_1\mu_2\cdots\mu_p}^a(x(\sigma)) T_a(\sigma), \quad (5.1)$$

where q_p is a dimensionless constant. (The solutions $\Lambda^{(0)}$ and $\Lambda^{(1)}$ were given in sections 3 and 4, respectively.) In a similar fashion to sections 3 and 4, we consider a Yang-Mills field $\mathcal{A}_{\mu\sigma}^{(p)}[x]$ on ΩM^D associated with $\Lambda^{(p)}$. From the abelian gauge transformation $\delta\mathcal{A}_{\mu\sigma}^{(p)} = \partial_{\mu\sigma}\Lambda^{(p)}$, we find a concrete form of $\mathcal{A}_{\mu\sigma}^{(p)}$ written in terms of local $(p+1)$ th-rank and p th-rank tensor fields on M^D . (In the $p=0$ case, the vector fields A_μ^a alone were introduced.) Any general solution of (2.7) is a linear combination of the fundamental solutions $\Lambda^{(p)}$ ($p=0, 1, 2, \dots$). Absorbing coefficients in the linear combination into the constants q_p , we can write the general solution as

$$\Lambda[x] = \sum_{p=0}^{\infty} \Lambda^{(p)}[x]. \quad (5.2)$$

The Yang-Mills field associated with (5.2) is given by

$$\mathcal{A}_{\mu\sigma}[x] = \sum_{p=0}^{\infty} \mathcal{A}_{\mu\sigma}^{(p)}[x]. \quad (5.3)$$

Substituting (5.2) and (5.3) into (2.5) and comparing both sides of the resulting equation, we conclude that

$$\delta\mathcal{A}_{\mu\sigma}^{(p)} = \partial_{\mu\sigma}\Lambda^{(p)} + i \sum_{k=0}^p [\mathcal{A}_{\mu\sigma}^{(k)}, \Lambda^{(p-k)}]. \quad (5.4)$$

The gauge transformation of $\mathcal{A}_{\mu\sigma}^{(0)}$ has already been discussed in section 3. Here we restrict our attention to the gauge transformation of $\mathcal{A}_{\mu\sigma}^{(1)}$.

The result of the substitution of (3.2) and (4.1) into $\partial_{\mu\sigma}\Lambda^{(1)} + i[\mathcal{A}_{\mu\sigma}^{(0)}, \Lambda^{(1)}]$ shows that in the non-abelian case, (4.5) should be modified to

$$\begin{aligned} \mathcal{A}_{\mu\sigma}^{(1)}[x] = & q_1 Q^\nu(\sigma) B_{\mu\nu}^a(x(\sigma)) T_a(\sigma) - \frac{1}{2} q_1 Q_\mu(\sigma) Q^\nu(\sigma) Q^\kappa(\sigma) C_{\nu\kappa}^a(x(\sigma)) T_a(\sigma) \\ & - e_1 \phi_\nu^a(x(\sigma)) D_\sigma(\Pi_\mu^\nu(\sigma) \frac{T_a(\sigma)}{\sqrt{-x'(\sigma)^2}}), \end{aligned} \quad (5.5)$$

where D_σ is defined by

$$D_\sigma U_a(\sigma) \equiv \frac{dU_a(\sigma)}{d\sigma} + q_0 x'^\mu(\sigma) A_\mu^b(x(\sigma)) f_{ba}^c U_c(\sigma) \quad (5.6)$$

with any function U_a of σ . After the modification, (5.4) actually yields the non-abelian version of (4.6):

$$\delta B_{\mu\nu}^a(x) = D_\mu \lambda_\nu^a(x) - D_\nu \lambda_\mu^a(x) - q_0 B_{\mu\nu}^b(x) \lambda^c(x) f_{bc}^a, \quad (5.7a)$$

$$\delta C_{\mu\nu}^a(x) = D_\mu \lambda_\nu^a(x) + D_\nu \lambda_\mu^a(x) - q_0 C_{\mu\nu}^b(x) \lambda^c(x) f_{bc}^a, \quad (5.7b)$$

$$\delta \phi_\mu^a(x) = m_1 \lambda_\mu^a(x) - q_0 \phi_\mu^b(x) \lambda^c(x) f_{bc}^a, \quad (5.7c)$$

where (3.3) has been used. The transformation rule (5.7a) (and its extension to higher-rank antisymmetric tensor fields) have been considered in topological field theories [8].

We now substitute (5.3) into (2.10). Taking into account the ranks of the local tensor fields contained in $\mathcal{F}_{\mu\rho,\nu\sigma}$, we decompose $\mathcal{F}_{\mu\rho,\nu\sigma}$ as follows:

$$\mathcal{F}_{\mu\rho,\nu\sigma} = \sum_{p=0}^{\infty} \mathcal{F}_{\mu\rho,\nu\sigma}^{(p)} \quad (5.8)$$

with

$$\mathcal{F}_{\mu\rho,\nu\sigma}^{(p)} \equiv \partial_{\mu\rho} \mathcal{A}_{\nu\sigma}^{(p)} - \partial_{\nu\sigma} \mathcal{A}_{\mu\rho}^{(p)} + i \sum_{k=0}^p [\mathcal{A}_{\mu\rho}^{(k)}, \mathcal{A}_{\nu\sigma}^{(p-k)}]. \quad (5.9)$$

As we have seen in section 3, $\mathcal{F}_{\mu\rho,\nu\sigma}^{(0)}$ is given by (3.4). Under the gauge transformation (5.4), $\mathcal{F}_{\mu\rho,\nu\sigma}^{(p)}$ transforms as

$$\delta \mathcal{F}_{\mu\rho,\nu\sigma}^{(p)} = i \sum_{k=0}^p [\mathcal{F}_{\mu\rho,\nu\sigma}^{(k)}, \Lambda^{(p-k)}], \quad (5.10)$$

which is compatible with (2.11). For $p \geq 1$, (5.10) is not a homogeneous transformation in the sense that its right hand side is not a single commutator of $\mathcal{F}_{\mu\rho,\nu\sigma}^{(p)}$ with an element of $\hat{\mathcal{G}}$. For this reason we can not construct, from $\mathcal{F}_{\mu\rho,\nu\sigma}^{(p)}$, an action integral for $\mathcal{A}_{\mu\sigma}^{(p)}$ in a gauge invariant form. The only exception is $\mathcal{F}_{\mu\rho,\nu\sigma}^{(0)}$, which transforms homogeneously, and hence we were able to construct an action integral from $\mathcal{F}_{\mu\rho,\nu\sigma}^{(0)}$ without spoiling its gauge invariance.

Let us confine our discussion to the $p = 1$ case. In order to construct an appropriate action integral for $\mathcal{A}_{\mu\sigma}^{(1)}$, we introduce a $\hat{\mathcal{G}}$ -valued scalar field $\Phi^{(1)}[x]$ on ΩM^D obeying the gauge transformation rule

$$\delta \Phi^{(1)} = \Lambda^{(1)} + i[\Phi^{(1)}, \Lambda^{(0)}]. \quad (5.11)$$

From (5.10) and (5.11), we find that

$$\hat{\mathcal{F}}_{\mu\rho,\nu\sigma}^{(1)} \equiv \mathcal{F}_{\mu\rho,\nu\sigma}^{(1)} - i[\mathcal{F}_{\mu\rho,\nu\sigma}^{(0)}, \Phi^{(1)}] \quad (5.12)$$

transforms homogeneously as

$$\delta \hat{\mathcal{F}}_{\mu\rho,\nu\sigma}^{(1)} = i[\hat{\mathcal{F}}_{\mu\rho,\nu\sigma}^{(1)}, \Lambda^{(0)}] . \quad (5.13)$$

By replacing $\mathcal{F}_{\mu\rho,\nu\sigma}$ in (2.9) by $\hat{\mathcal{F}}_{\mu\rho,\nu\sigma}^{(1)}$, we define an action integral $\hat{S}_R^{(1)}$ for $\mathcal{A}_{\mu\sigma}^{(1)}$. Evidently the gauge transformation (5.13) leaves $\hat{S}_R^{(1)}$ invariant. In the abelian case, $\hat{\mathcal{F}}_{\mu\rho,\nu\sigma}^{(1)}$ agrees with $\mathcal{F}_{\mu\rho,\nu\sigma}^{(1)}$. In terms of

$$\hat{\mathcal{A}}_{\mu\sigma}^{(1)} \equiv \mathcal{A}_{\mu\sigma}^{(1)} - (\partial_{\mu\sigma} \Phi^{(1)} + i[\mathcal{A}_{\mu\sigma}^{(0)}, \Phi^{(1)}]) , \quad (5.14)$$

$\hat{\mathcal{F}}_{\mu\rho,\nu\sigma}^{(1)}$ is written as

$$\hat{\mathcal{F}}_{\mu\rho,\nu\sigma}^{(1)} = \partial_{\mu\rho} \hat{\mathcal{A}}_{\nu\sigma}^{(1)} - \partial_{\nu\sigma} \hat{\mathcal{A}}_{\mu\rho}^{(1)} + i[\mathcal{A}_{\mu\rho}^{(0)}, \hat{\mathcal{A}}_{\nu\sigma}^{(1)}] + i[\hat{\mathcal{A}}_{\mu\rho}^{(1)}, \mathcal{A}_{\nu\sigma}^{(0)}] . \quad (5.15)$$

As will be discussed in the next section, the field $\Phi^{(1)}$ and its gauge transformation rule can be derived in a systematic way.

Consider now how $\Phi^{(1)}$ is expressed in terms of local fields on M^D . Since (5.11) has no relation with the reparametrizations and $\Lambda^{(p)}$'s are reparametrization invariant, $\Phi^{(1)}$ also has to be reparametrization invariant. Taking this into account, we express $\Phi^{(1)}$ as

$$\Phi^{(1)}[x] = \int_0^{2\pi} \frac{d\sigma}{2\pi} e_1 Q^\mu(\sigma) \phi_\mu^a(x(\sigma)) T_a(\sigma) \quad (5.16)$$

by utilizing the constant e_1 and the vector fields ϕ_μ^a . Then (5.11) actually reduces to (5.7c), and (5.14) becomes

$$\begin{aligned} \hat{\mathcal{A}}_{\mu\sigma}^{(1)}[x] &= q_1 Q^\nu(\sigma) \hat{B}_{\mu\nu}^a(x(\sigma)) T_a(\sigma) \\ &\quad - \frac{1}{2} q_1 Q_\mu(\sigma) Q^\nu(\sigma) Q^\kappa(\sigma) \hat{C}_{\nu\kappa}^a(x(\sigma)) T_a(\sigma) \end{aligned} \quad (5.17)$$

with

$$\hat{B}_{\mu\nu}^a(x) \equiv B_{\mu\nu}^a(x) - \frac{1}{m_1} (D_\mu \phi_\nu^a(x) - D_\nu \phi_\mu^a(x)) , \quad (5.18)$$

$$\hat{C}_{\mu\nu}^a(x) \equiv C_{\mu\nu}^a(x) - \frac{1}{m_1} (D_\mu \phi_\nu^a(x) + D_\nu \phi_\mu^a(x)) . \quad (5.19)$$

Under the gauge transformation (5.7), $\hat{B}_{\mu\nu}^a$ and $\hat{C}_{\mu\nu}^a$ transform homogeneously as

$$\delta \hat{B}_{\mu\nu}^a = -q_0 \hat{B}_{\mu\nu}^b \lambda^c f_{bc}^a , \quad \delta \hat{C}_{\mu\nu}^a = -q_0 \hat{C}_{\mu\nu}^b \lambda^c f_{bc}^a . \quad (5.20)$$

Substituting (3.2) and (5.17) into (5.15), we obtain an expression that is given by the following replacement in (4.7):

$$\begin{aligned} H_{\lambda\mu\nu}^a &\rightarrow \hat{H}_{\lambda\mu\nu}^a , & \tilde{B}_{\mu\nu}^a &\rightarrow \hat{B}_{\mu\nu}^a , & \tilde{C}_{\mu\nu}^a &\rightarrow \hat{C}_{\mu\nu}^a , \\ I_{\mu\nu}^a &\rightarrow \hat{I}_{\mu\nu}^a , & J_{\lambda\mu\nu}^a &\rightarrow \hat{J}_{\lambda\mu\nu}^a , \end{aligned} \quad (5.21)$$

and $\frac{d}{d\rho} \rightarrow D_\rho$, $\frac{d}{d\sigma} \rightarrow D_\sigma$. Here

$$\widehat{H}_{\lambda\mu\nu}{}^a(x) \equiv D_\lambda \widehat{B}_{\mu\nu}{}^a(x) + D_\mu \widehat{B}_{\nu\lambda}{}^a(x) + D_\nu \widehat{B}_{\lambda\mu}{}^a(x), \quad (5.22)$$

$$\widehat{I}_{\mu\nu}{}^a(x) \equiv \widehat{B}_{\mu\nu}{}^a(x) - \widehat{C}_{\mu\nu}{}^a(x), \quad (5.23)$$

$$\widehat{J}_{\lambda\mu\nu}{}^a(x) \equiv D_\mu \widehat{B}_{\lambda\nu}{}^a(x) + D_\mu \widehat{C}_{\lambda\nu}{}^a(x) - D_\lambda \widehat{C}_{\mu\nu}{}^a(x). \quad (5.24)$$

As is expected, the same expression is also obtained by substituting (3.4), (5.5) and (5.16) into (5.12). For this expression we can directly check the transformation rule (5.13). Let $\widehat{\mathcal{L}}^{(1)}$ be the Lagrangian defined by replacing $\mathcal{F}_{\mu\rho,\nu\sigma}$ in (2.9) by $\widehat{\mathcal{F}}_{\mu\rho,\nu\sigma}^{(1)}$. After being substituted the expression, $\widehat{\mathcal{L}}^{(1)}$ becomes

$$\widehat{\mathcal{L}}^{(1)} = \widehat{\mathcal{L}}_1^{(1)} + \widehat{\mathcal{L}}_2^{(1)}, \quad (5.25)$$

where $\widehat{\mathcal{L}}_1^{(1)}$ is given from (4.13) by the replacement (5.21), and

$$\begin{aligned} \widehat{\mathcal{L}}_2^{(1)} \equiv & -\frac{1}{4}q_1^2\delta(0)^2 \int_0^{2\pi} \frac{d\sigma}{2\pi} [-2q_0 Q^\kappa(\sigma) Q^\lambda(\sigma) f_{abc} A_\kappa^a(x(\sigma)) \widehat{H}_{\lambda\mu\nu}{}^b(x(\sigma)) \widehat{B}^{\mu\nu c}(x(\sigma)) \\ & - q_0 Q^\kappa(\sigma) Q^\lambda(\sigma) Q^\mu(\sigma) Q^\nu(\sigma) f_{abc} A_\kappa^a(x(\sigma)) \widehat{J}_{\xi\lambda\mu}{}^b(x(\sigma)) \widehat{I}^{\xi\nu c}(x(\sigma)) \\ & - 2q_0 \frac{Q^\kappa(\sigma)}{\sqrt{-x'(\sigma)^2}} f_{abc} A_\kappa^a(x(\sigma)) \widehat{I}_\lambda{}^{\nu b}(x(\sigma)) \widehat{B}_{\mu\nu}{}^c(x(\sigma)) \frac{d}{d\sigma} (Q^\lambda(\sigma) Q^\mu(\sigma)) \\ & + q_0 Q^\kappa(\sigma) Q^\lambda(\sigma) f_{abc} A_\lambda^a(x(\sigma)) \widehat{I}_{\mu\nu}{}^b(x(\sigma)) \widehat{I}_\kappa{}^{\nu c}(x(\sigma)) \frac{d}{d\sigma} \left(\frac{Q^\mu(\sigma)}{\sqrt{-x'(\sigma)^2}} \right) \\ & + q_0 Q^\kappa(\sigma) Q^\lambda(\sigma) Q^\mu(\sigma) f_{abc} A_\mu^a(x(\sigma)) \widehat{I}_{\nu\lambda}{}^b(x(\sigma)) \widehat{I}_{\kappa\xi}{}^c(x(\sigma)) \\ & \times \frac{d}{d\sigma} \left(\frac{1}{\sqrt{-x'(\sigma)^2}} Q^\nu(\sigma) Q^\xi(\sigma) \right) \\ & + S_{ab}^{\kappa\lambda}{}_{\kappa\lambda}(\sigma) \widehat{B}_{\mu\nu}{}^a(x(\sigma)) \widehat{B}^{\mu\nu b}(x(\sigma)) - \frac{1}{2} S_{ab}^{\kappa\lambda\mu\nu}(\sigma) \widehat{C}_{\kappa\lambda}{}^a(x(\sigma)) \widehat{C}_{\mu\nu}{}^b(x(\sigma)) \\ & + S_{ab}^{\kappa\lambda}{}_{\mu}(\sigma) \widehat{I}_{\kappa\nu}{}^a(x(\sigma)) \{-2\widehat{B}_\lambda{}^{\nu b}(x(\sigma)) + \frac{1}{2} \widehat{I}_\lambda{}^{\nu b}(x(\sigma))\}] \end{aligned} \quad (5.26)$$

with

$$\begin{aligned} S_{ab}^{\kappa\lambda\mu\nu}(\sigma) \equiv & q_0^2 Q^\kappa(\sigma) Q^\lambda(\sigma) Q^\mu(\sigma) Q^\nu(\sigma) Q'(\sigma) Q^\xi(\sigma) \\ & \times f_{ac}{}^e f_{bde} A_e^c(x(\sigma)) A_\xi^d(x(\sigma)). \end{aligned} \quad (5.27)$$

The action integral with the Lagrangian (5.25) consists of two parts:

$$\widehat{S}_R^{(1)} = \widehat{S}_{R1}^{(1)} + \widehat{S}_{R2}^{(1)}, \quad (5.28)$$

where $\widehat{S}_{R1}^{(1)}$ and $\widehat{S}_{R2}^{(1)}$ are defined from $\widehat{\mathcal{L}}_1^{(1)}$ and $\widehat{\mathcal{L}}_2^{(1)}$, respectively. Following the procedure for deriving (4.17), we have

$$\widehat{S}_{R1}^{(1)} = \int d^D x \left[\frac{1}{12} \widehat{H}_{\lambda\mu\nu a}(x) \widehat{H}^{\lambda\mu\nu a}(x) + 4v D^\lambda \widehat{B}_{\lambda\nu a}(x) D_\mu \widehat{B}^{\mu\nu a}(x) \right]$$

$$\begin{aligned}
& -8vD^\lambda \hat{B}_{\lambda\nu a}(x)D_\mu \hat{C}^{\mu\nu a}(x) + 3vD_\lambda \hat{C}_{\mu\nu a}(x)D^\lambda \hat{C}^{\mu\nu a}(x) \\
& -2vD^\lambda \hat{C}_{\lambda\nu a}(x)D_\mu \hat{C}^{\mu\nu a}(x) - 2vD^\lambda \hat{C}_{\lambda\nu a}(x)D^\nu \hat{C}_\mu^{\mu a}(x) \\
& + vD_\lambda \hat{C}_\mu^{\mu a}(x)D^\lambda \hat{C}_\nu^{\nu a}(x) - \frac{1}{4}m_1^2 \hat{B}_{\mu\nu a}(x)\hat{B}^{\mu\nu a}(x) \\
& - 3vm_2^2 \hat{C}_{\mu\nu a}(x)\hat{C}^{\mu\nu a}(x) - 3vm_3^2 \hat{C}_\mu^{\mu a}(x)\hat{C}_\nu^{\nu a}(x) \Big], \tag{5.29}
\end{aligned}$$

which is nothing but the non-abelian version of (4.17). In the same way as we derived (3.8) and (4.17), we can show that $\hat{S}_{R2}^{(1)}$ becomes

$$\begin{aligned}
\hat{S}_{R2}^{(1)} = & -\frac{1}{4}kq_1^2\delta(0)^2 \int d^D x \Big[\frac{1}{D}q_0 f_{abc}A_\kappa^a(x)\{2\hat{H}^\kappa_{\mu\nu}{}^b(x)\hat{B}^{\mu\nu c}(x) \\
& - \frac{1}{D+2}(\hat{J}^{\mu(\nu\kappa)b}(x)\hat{I}_{\mu\nu}{}^c(x) + \hat{J}_{\mu\nu}{}^{\nu b}(x)\hat{I}^{\mu\kappa c}(x))\} \\
& + \frac{1}{2D(D+2)(D+4)}q_0^2 f_{ac}{}^e f_{bde} \\
& \times (A_\kappa^a(x)A^{\kappa b}(x)\{-(2D+1)(D+4)\hat{B}_{\mu\nu}{}^c(x)\hat{B}^{\mu\nu d}(x) \\
& - (D+2)\hat{C}_{\mu\nu}{}^c(x)\hat{C}^{\mu\nu d}(x) + \hat{C}^c(x)\hat{C}^d(x)\} \\
& + A^{(\kappa a}(x)A^{\lambda)b}(x)\{2\hat{C}^c(x)\hat{C}_{\kappa\lambda}{}^d(x) - D\hat{C}_{\kappa\mu}{}^c(x)\hat{C}_\lambda^{\mu d}(x) \\
& + (D+4)\hat{B}_{\kappa\mu}{}^c(x)(3\hat{B}_\lambda^{\mu d}(x) - 2\hat{C}_\lambda^{\mu d}(x))\}\Big], \tag{5.30}
\end{aligned}$$

where $\hat{C}^a \equiv \hat{C}_\mu^{\mu a}$. Since $\hat{B}_{\mu\nu}{}^a$, $\hat{C}_{\mu\nu}{}^a$ and their covariant derivatives obey the homogeneous gauge transformation rule, (5.29) is gauge invariant, whereas (5.30) is not gauge invariant. This indicates that the gauge invariance which $\hat{S}_R^{(1)}$ originally possesses in the loop space is not preserved in the derivation of the local field theory. Taking the position that gauge invariance should be maintained in the theory, we regularize the $\delta(0)$ in (5.30) so that $kq_1^2\delta(0)^2 = 0$ is satisfied in a certain limit; the $\delta(0)$ in (4.18a) and that in (5.30) are regularized in a different manner. As a result, obtaining $\hat{S}_R^{(1)} = \hat{S}_{R1}^{(1)}$, we can restore gauge invariance of the action integral.

The action integral (4.17) is the quadratic part of (5.29) and can be employed to govern the asymptotic fields of $\hat{B}_{\mu\nu}{}^a$ and $\hat{C}_{\mu\nu}{}^a$, apart from possible renormalization. The abelian fields $\tilde{B}_{\mu\nu}{}^a$ and $\tilde{C}_{\mu\nu}{}^a$ are then regarded as renormalized asymptotic fields of $\hat{B}_{\mu\nu}{}^a$ and $\hat{C}_{\mu\nu}{}^a$, respectively.

Although (5.29) describes “massive” non-abelian fields, it is invariant under the gauge transformation (5.7); the vector fields ϕ_μ^a play the role of Nambu-Goldstone fields. We can hence regard (5.29) as an extension of the non-abelian Stueckelberg formalism proposed by Kunimasa and Gotō [9].

6 Application of a nonlinear realization to the Yang-Mills Theory in loop space

In the previous section we introduced the field $\Phi^{(1)}$ and its gauge transformation rule somewhat artificially. We now show that these can be derived naturally using the formalism of nonlinear realization [10].

Let $\mathfrak{g}^{(p)}$ be the linear subspace of $\hat{\mathcal{G}}$ whose all elements have the following form:

$$\Xi^{(p)}[x] = \int_0^{2\pi} \frac{d\sigma}{2\pi} Q^{\mu_1}(\sigma) Q^{\mu_2}(\sigma) \cdots Q^{\mu_p}(\sigma) \xi_{\mu_1 \mu_2 \cdots \mu_p}^a(x(\sigma)) T_a(\sigma), \quad (6.1)$$

where $\xi_{\mu_1 \mu_2 \cdots \mu_p}^a$ are p th-rank symmetric tensor functions on M^D . The infinitesimal function $\Lambda^{(p)}$ in (5.1) is an element of $\mathfrak{g}^{(p)}$. Using (2.3), we see that $[\Xi^{(p)}, \Xi'^{(q)}] \in \mathfrak{g}^{(p+q)}$ for every $\Xi^{(p)} \in \mathfrak{g}^{(p)}$ and every $\Xi'^{(q)} \in \mathfrak{g}^{(q)}$, which shows that $\mathfrak{g}^{(0)}$ is a Lie algebra, while $\mathfrak{g}^{(p)}$ ($p = 1, 2, 3 \dots$) are not Lie algebras. The direct sum $\mathfrak{g} \equiv \bigoplus_{p=0}^{\infty} \mathfrak{g}^{(p)}$ is obviously a Lie algebra.

Let us consider the Lie groups $\mathbf{G}$ and $\mathbf{G}^{(0)}$ that are associated with $\mathfrak{g}$ and $\mathfrak{g}^{(0)}$, respectively. Any elements $\mathcal{X} \in \mathbf{G}$ and $\mathcal{Y} \in \mathbf{G}^{(0)}$ can be represented, for instance, as $\mathcal{X} = \exp(-i \sum_{p=0}^{\infty} \Xi^{(p)})$ and $\mathcal{Y} = \exp(-i \Xi'^{(0)})$. Both $\mathbf{G}$ and $\mathbf{G}^{(0)}$ are Lie subgroups of the Kac-Moody group $\hat{\mathbf{G}}$. Now, since $\mathbf{G}^{(0)}$ is a Lie subgroup of $\mathbf{G}$, we can define the coset manifold $\mathbf{G}/\mathbf{G}^{(0)}$, which is “reductive” because $[\Xi^{(p)}, \Xi'^{(0)}] \in \mathfrak{g}^{(p)}$ [11] (cf. appendix). From each left-coset in $\mathbf{G}/\mathbf{G}^{(0)}$ we choose a coset representative

$$\mathcal{V}(\Phi) = \exp(-i \sum_{p=1}^{\infty} \Phi^{(p)}[x]). \quad (6.2)$$

Here $\Phi^{(p)}$ ($p \geq 1$) is an element of $\mathfrak{g}^{(p)}$ and we regard it as a scalar field on ΩM^D . We can use $\Phi^{(p)}$'s as coordinates of a point in $\mathbf{G}/\mathbf{G}^{(0)}$. Under the left action of $\mathcal{X}$, the coset representatives $\mathcal{V}(\Phi)$ transform as

$$\mathcal{V}(\Phi) \longrightarrow \mathcal{V}(\bar{\Phi}) = \mathcal{X} \mathcal{V}(\Phi) \mathcal{Y}^{-1}(\Phi, \mathcal{X}), \quad (6.3)$$

where $\mathcal{Y} \in \mathbf{G}^{(0)}$ is now a function of $\Phi^{(p)}$ and $\mathcal{X}$. The transformation rule of $\Phi^{(p)}$ is determined from (6.3). For example, setting $\Xi^{(p)} = \Lambda^{(p)}$ and $\Xi'^{(0)} = \Lambda^{(0)}$ in (6.3), we obtain the infinitesimal transformation

$$\delta \Phi^{(1)} (\equiv \bar{\Phi}^{(1)} - \Phi^{(1)}) = \Lambda^{(1)} + i[\Phi^{(1)}, \Lambda^{(0)}], \quad (6.4)$$

which is nothing but the transformation rule (5.11). Thus the scalar field $\Phi^{(1)}$ and its transformation rule can be derived from the nonlinear realization of symmetry.

From $\mathcal{A}_{\mu\sigma}$ in (5.3) and $\mathcal{V}(\Phi)$, we define the vector field

$$\hat{\mathcal{A}}_{\mu\sigma}[x] \equiv \mathcal{V}^{-1} \mathcal{A}_{\mu\sigma}[x] \mathcal{V} - i \mathcal{V}^{-1} \partial_{\mu\sigma} \mathcal{V}. \quad (6.5)$$

Corresponding to (5.3), we decompose $\hat{\mathcal{A}}_{\mu\sigma}$ as follows:

$$\hat{\mathcal{A}}_{\mu\sigma}[x] = \sum_{p=0}^{\infty} \hat{\mathcal{A}}_{\mu\sigma}^{(p)}[x]. \quad (6.6)$$

The substitution of (5.3), (6.2) and (6.6) into (6.5) then leads to

$$\hat{\mathcal{A}}_{\mu\sigma}^{(0)} = \mathcal{A}_{\mu\sigma}^{(0)}, \quad (6.7a)$$

$$\hat{\mathcal{A}}_{\mu\sigma}^{(1)} = \mathcal{A}_{\mu\sigma}^{(1)} - (\partial_{\mu\sigma}\Phi^{(1)} + i[\mathcal{A}_{\mu\sigma}^{(0)}, \Phi^{(1)}]), \quad (6.7b)$$

and so on. Notice that (6.7b) agrees with (5.14).

We now consider the finite gauge transformation of $\mathcal{A}_{\mu\sigma}$:

$$\mathcal{A}_{\mu\sigma} \longrightarrow \bar{\mathcal{A}}_{\mu\sigma} = \mathcal{X}\mathcal{A}_{\mu\sigma}\mathcal{X}^{-1} - i\mathcal{X}\partial_{\mu\sigma}\mathcal{X}^{-1}, \quad (6.8)$$

which indeed reduces to (5.4) if $\Xi^{(p)} = \Lambda^{(p)}$. From (6.3) and (6.8) it follows that $\hat{\mathcal{A}}_{\mu\sigma}$ transforms as

$$\hat{\mathcal{A}}_{\mu\sigma} \longrightarrow \bar{\hat{\mathcal{A}}}_{\mu\sigma} = \mathcal{Y}\hat{\mathcal{A}}_{\mu\sigma}\mathcal{Y}^{-1} - i\mathcal{Y}\partial_{\mu\sigma}\mathcal{Y}^{-1}. \quad (6.9)$$

Since $\mathcal{Y}\partial_{\mu\sigma}\mathcal{Y}^{-1} \in \mathfrak{g}^{(0)}$, we find that $\hat{\mathcal{A}}_{\mu\sigma}^{(p)}$ other than $\hat{\mathcal{A}}_{\mu\sigma}^{(0)}$ transform homogeneously. From (6.9), the gauge transformation rule of the field strength $\hat{\mathcal{F}}_{\mu\rho,\nu\sigma}$ of $\hat{\mathcal{A}}_{\mu\sigma}$ is determined to be $\hat{\mathcal{F}}_{\mu\rho,\nu\sigma} \rightarrow \bar{\hat{\mathcal{F}}}_{\mu\rho,\nu\sigma} = \mathcal{Y}\hat{\mathcal{F}}_{\mu\rho,\nu\sigma}\mathcal{Y}^{-1}$, which gives the infinitesimal transformation $\delta\hat{\mathcal{F}}_{\mu\rho,\nu\sigma} = i[\hat{\mathcal{F}}_{\mu\rho,\nu\sigma}, \Lambda^{(0)}]$. For all $\hat{\mathcal{F}}_{\mu\rho,\nu\sigma}^{(p)}$ in $\hat{\mathcal{F}}_{\mu\rho,\nu\sigma} = \sum_{p=0}^{\infty} \hat{\mathcal{F}}_{\mu\rho,\nu\sigma}^{(p)}$, we thus obtain

$$\delta\hat{\mathcal{F}}_{\mu\rho,\nu\sigma}^{(p)} = [\hat{\mathcal{F}}_{\mu\rho,\nu\sigma}^{(p)}, \Lambda^{(0)}], \quad (6.10)$$

where each $\hat{\mathcal{F}}_{\mu\rho,\nu\sigma}^{(p)}$ is defined by replacing $\mathcal{A}_{\mu\sigma}^{(p)}$ in (5.9) with $\hat{\mathcal{A}}_{\mu\sigma}^{(p)}$. Since $\hat{\mathcal{F}}_{\mu\rho,\nu\sigma}^{(p)}$, unlike $\mathcal{F}_{\mu\rho,\nu\sigma}^{(p)}$ ($p \geq 1$), transforms homogeneously, we can construct an appropriate action integral from $\hat{\mathcal{F}}_{\mu\rho,\nu\sigma}^{(p)}$.

Let us define the inner product $\widehat{\text{TR}}$ by replacing the metric $\mathcal{K}_{am,bn}(y)$ in (A.6) by

$$\hat{\mathcal{K}}_{am,bn}(y) \equiv D_{am}^{cp}(\hat{w}(y))D_{bn}^{dq}(\hat{w}(y))K_{cp,dq}. \quad (6.11)$$

(See appendix.) Here $\hat{w}(y)$ are coset representatives associated with the coset manifold $\widehat{\mathcal{N}} = \mathbf{G}^{(0)} \odot \text{Diff } S^1 / G \otimes S^1$ given by replacing $\hat{G}$ in (A.1) with $\mathbf{G}^{(0)}$. In the particular gauge $y^{an} = y^n = 0$ of the reparametrizations, $\hat{\mathcal{K}}_{am,bn}(y)$ reduces to the Killing metric $K_{am,bn}$. Utilizing $\widehat{\text{TR}}$, we construct the Lagrangian

$$\hat{\mathcal{L}}_I = \sum_{p=0}^{\infty} \sum_{q=0}^{\infty} \hat{\mathcal{L}}^{(p,q)} \quad (6.12)$$

with

$$\hat{\mathcal{L}}^{(p,q)} \equiv -\frac{1}{4} \mathcal{G}^{\kappa\rho,\lambda\sigma} \mathcal{G}^{\mu\chi,\nu\omega} \widehat{\text{TR}}[\hat{\mathcal{F}}_{\kappa\rho,\mu\chi}^{(p)} \hat{\mathcal{F}}_{\lambda\sigma,\nu\omega}^{(q)}]. \quad (6.13)$$

By virtue of $\widehat{\text{TR}}$, each $\widehat{\mathcal{L}}^{(p,q)}$ is reparametrization invariant as well as gauge invariant. The Lagrangians $\mathcal{L}^{(0)} = \widehat{\mathcal{L}}^{(0,0)}$ and $\widehat{\mathcal{L}}^{(1)} = \widehat{\mathcal{L}}^{(1,1)}$ in the particular gauge $y^{an} = y^n = 0$ have actually been examined in sections 3 and 5, respectively. Using the formalism of nonlinear realization, we can thus define the Lagrangian $\widehat{\mathcal{L}}^{(1)}$ in a systematic way. The integral $(1/V_R) \int [dx] \widehat{\mathcal{L}}^{(0,1)} \exp(-L/l^2)$ vanishes at the integration over ΩM^D .

We see that the mass term

$$\widehat{\mathcal{L}}_{\text{II}} = \frac{1}{2} \sum_{p=1}^{\infty} \sum_{q=1}^{\infty} (M^{(p,q)})^2 \mathcal{G}^{\mu\rho, \nu\sigma} \widehat{\text{TR}}[\widehat{\mathcal{A}}_{\mu\rho}^{(p)} \widehat{\mathcal{A}}_{\nu\sigma}^{(q)}] \quad (6.14)$$

is also reparametrization and gauge invariant. Here $M^{(p,q)}$ are constants with dimensions of mass. The Lagrangian

$$\widehat{\mathcal{L}} = \widehat{\mathcal{L}}_{\text{I}} + \widehat{\mathcal{L}}_{\text{II}} \quad (6.15)$$

defines an extended Stueckelberg formalism describing a massive Yang-Mills theory in loop space. In deriving the local gauge theory of the second-rank tensor fields, the mass term (6.14) works only for rising masses of the massive tensor fields $\widehat{B}_{\mu\nu}^a$ and $\widehat{C}_{\mu\nu}^a$.

From (6.12) (or (6.15)) we can derive Lagrangians defining non-abelian gauge theories of the local higher-rank tensor fields. The tensor fields interact among themselves, obeying interaction terms of the Lagrangians.

7 Discussion

From the Yang-Mills theory in loop space with the Kac-Moody gauge group, we have derived the usual Yang-Mills theory and a non-abelian gauge theory of the local fields $B_{\mu\nu}^a$, $C_{\mu\nu}^a$ and ϕ_μ^a coupled to the local Yang-Mills field A_μ^a . As seen from (5.3), (3.2) and (5.5), A_μ^a , $B_{\mu\nu}^a$, $C_{\mu\nu}^a$ and ϕ_μ^a are components of the Yang-Mills field $\mathcal{A}_{\mu\sigma}$ on the loop space. Besides these local fields, $\mathcal{A}_{\mu\sigma}$ contains an infinite number of local higher-rank tensor fields. It has been found that A_μ^a are massless, while $\hat{B}_{\mu\nu}^a$ and $\hat{C}_{\mu\nu}^a$, $B_{\mu\nu}^a$ and $C_{\mu\nu}^a$ in the unitary gauge, have mass which is inversely proportional to the size of loops. The Yang-Mills theory in loop space can deal with the massless and massive fields on an equal footing.

Massive non-abelian antisymmetric tensor fields have been studied in topologically massive gauge theory [12], where generalized Chern-Simons terms make antisymmetric tensor fields massive. On the one hand, in our theory, the second-rank tensor fields become massive owing to the Nambu-Goldstone-like fields ϕ_μ^a ; the part of our theory containing $\hat{B}_{\mu\nu}^a$ alone is considered to be a non-abelian analog of the massive gauge theory discussed by Kalb and Ramond [13] and by Nambu [14]. (Kalb and Ramond have argued that massive antisymmetric tensor fields of the second rank arise from the interaction among open strings.) It is an interesting subject to study the relation between topologically massive gauge theory and our theory.

To derive the non-abelian gauge theory of the second-rank tensor fields it was necessary to modify the Yang-Mills theory in loop space and to regularize $\delta(0)$'s suitably. The modification was carried out by introducing the scalar field $\Phi^{(1)}$ on the loop space. We constructed the modified Lagrangian (6.12) by utilizing the formalism of nonlinear realization. The non-abelian gauge theory of the second-rank tensor fields was not obtained from (2.9), but from (6.12) – similarly, non-abelian gauge theories of the higher-rank tensor fields are also obtained from (6.12). Hence we should take, instead of (2.9), the Lagrangian (6.12) (or (6.15)) to define the Yang-Mills theory in loop space.

As for the regularization of $\delta(0)$'s, which was made in an ad hoc way to preserve gauge invariance, we have to justify it in the future. Alternatively we might be able to derive, without the regularization, local field theories possessing gauge invariance with further modification of the Yang-Mills theory in loop space. Anyway, these are unsolved problems in our theory.

It is interesting to compare the Yang-Mills theory in loop space with the U(1) gauge theory in loop space. In refs. [4] and [5], the present authors have shown that the U(1) gauge field on loop space consists of a local massless abelian field $B_{\mu\nu}(= -B_{\nu\mu})$ and local massive abelian fields $\tilde{A}_\mu$, $\tilde{A}_{\lambda\mu\nu}(= \tilde{A}_{\lambda\nu\mu})$ and so on. For instance, the antisymmetric tensor field $B_{\mu\nu}$ relates to the U(1) gauge field $\mathcal{A}_{\mu\sigma}$ as follows:

$$\mathcal{A}_{\mu\sigma}[x] = qx'^\nu(\sigma)B_{\mu\nu}(x(\sigma)) , \quad (7.1)$$

where q is a constant with dimensions of $[\text{length}]^{-1}$. Comparing this with (3.2), we see that $B_{\mu\nu}$ and $x^\nu(\sigma)$ correspond to A_μ^a and $T_a(\sigma)$, respectively. Similarly $\tilde{A}_{\lambda\mu\nu}$ corresponds to $\hat{B}_{\mu\nu}^a$ and $\hat{C}_{\mu\nu}^a$. In general, a local abelian tensor field of the $(p+1)$ th rank corresponds to local non-abelian tensor fields of the p th rank. (As an exception $\tilde{A}_\mu$ does not have corresponding non-abelian fields.) Noticing this correspondence, we could deal with the abelian and non-abelian tensor fields collectively within the framework of the Yang-Mills theory in loop space. To do so, it would be important to enlarge the Kac-Moody gauge group so as to incorporate $x^\nu(\sigma)$ into the generators.

In this paper we have concentrated our discussion on the classical field theories. The quantum theories of the abelian fields $B_{\mu\nu}$ and $\tilde{A}_\mu$ were actually studied on the basis of the BRST formalism for the U(1) gauge theory in loop space [4]. In a similar fashion, we will be able to consider quantum theories of the non-abelian fields on the basis of the BRST formalism for the Yang-Mills in loop space. We hope to discuss this subject in the future.

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Appendix

In this appendix we define an inner product so as to maintain reparametrization invariance and gauge invariance in the Lagrangian (2.9). For this purpose, we make use of a coset manifold, following the construction of the metric $G^{\rho\sigma}$ [4].

For the components $\mathcal{F}^{an} \equiv \frac{1}{2}\mathcal{F}_{\mu\rho,\nu\sigma}{}^{an}dx^{\mu\rho}\wedge dx^{\nu\sigma}$ of the field strength two-form, the gauge transformation (2.11) is written as $\delta\mathcal{F}^{an} = i\mathcal{F}^{bm}\Lambda^{cp}(t_{cp})_{bm}{}^{an}$ with $(t_{cp})_{bm}{}^{an} \equiv -if_{cb}{}^a\delta_{p+m}{}^n$. Also, $\mathcal{F}^{an}$ transform, under the reparametrization (2.1), as $\delta_r\mathcal{F}^{an} = i\mathcal{F}^{bm}\varepsilon^p(\ell_p)_{bm}{}^{an}$ with $(\ell_p)_{bm}{}^{an} \equiv m\delta_b{}^a\delta_{p+m}{}^n$. We notice that t_{an} and ℓ_n constitute a matrix representation of the Lie algebra of the semidirect product group $\hat{G} \odot \text{Diff} S^1$, namely, $[t_{am}, t_{bn}] = if_{ab}{}^c t_{cm+n}$, $[\ell_m, t_{an}] = -nt_{am+n}$ and $[\ell_m, \ell_n] = (m-n)\ell_{m+n}$. Notice also that the Killing metric $K_{am,bn}(\equiv \int_0^{2\pi} \frac{d\rho}{2\pi} \int_0^{2\pi} \frac{d\sigma}{2\pi} K_{a\rho,b\sigma} e^{i(m\rho+n\sigma)}) = \delta_{mn}\kappa_{ab}$ of $\hat{G}$ is invariant under the action of the subgroup $G \otimes S^1 (\subset \hat{G} \odot \text{Diff} S^1)$ generated by T_{a0} and L_0 . These facts indicate that a coset manifold suitable for our purpose is

$$\mathcal{N} = \hat{G} \odot \text{Diff} S^1 / G \otimes S^1. \quad (\text{A.1})$$

This is a “reductive” coset manifold [11]; the generators of $G \otimes S^1$ generate rotations in the tangent space at the origin of $\mathcal{N}$. Now we choose a coset representative $w(y) = \exp\{-i\sum_{n \neq 0}(y^{an}T_{an} + y^n L_n)\}$ from each left-coset, where y^{an} ($n \neq 0$) and y^n ($n \neq 0$) are complex parameters satisfying $y^{an*} = y^{a-n}$ and $y^{n*} = y^{-n}$, and are used as coordinates of a point in $\mathcal{N}$. Under the left action of $g \in \hat{G} \odot \text{Diff} S^1$, the coset representatives $w(y)$ transform as

$$w(y) \longrightarrow w(\bar{y}) = gw(y)h(y,g)^{-1}, \quad h(y,g) \in G \otimes S^1. \quad (\text{A.2})$$

With $g = \exp(-i\alpha^{an}T_{an} - i\alpha^n L_n)$, we define the representation D of $\hat{G} \odot \text{Diff} S^1$ by $D(g) = \exp(-i\alpha^{an}t_{an} - i\alpha^n \ell_n)$, where α^{an} and α^n are complex parameters satisfying $\alpha^{an*} = \alpha^{a-n}$ and $\alpha^{n*} = \alpha^{-n}$. Then the finite transformation of $\mathcal{F}^{an}$ is given by $\mathcal{F}^{an} \rightarrow \bar{\mathcal{F}}^{an} = \mathcal{F}^{bm}D_{bm}{}^{an}(g^{-1})$. Using (A.2) and the invariance of the Killing metric $K_{am,bn}$:

$$D_{am}{}^{cp}(h^{-1})D_{bn}{}^{dq}(h^{-1})K_{cp,dq} = K_{am,bn}, \quad (\text{A.3})$$

we can show that the metric

$$\mathcal{K}_{am,bn}(y) \equiv D_{am}{}^{cp}(w(y))D_{bn}{}^{dq}(w(y))K_{cp,dq} \quad (\text{A.4})$$

transforms as

$$\mathcal{K}_{am,bn}(y) \longrightarrow \mathcal{K}_{am,bn}(\bar{y}) = D_{am}{}^{cp}(g)D_{bn}{}^{dq}(g)\mathcal{K}_{cp,dq}(y). \quad (\text{A.5})$$

We define the inner product “TR” by

$$\text{TR}[UV] = \mathcal{K}_{am,bn}(y)U^{am}V^{bn}, \quad U, V \in \hat{\mathcal{G}}. \quad (\text{A.6})$$

Then the Lagrangian (2.9) is evidently invariant under reparametrizations and under the gauge transformation (2.5). Since $\mathcal{K}_{am,bn}(0) = K_{am,bn}$, we can regard $K_{am,bn}$ as a form of $\mathcal{K}_{am,bn}(y)$ in the particular gauge $y^{an} = y^n = 0$. In sections 3, 4 and 5, we carry out concrete calculations after taking the specific gauge.

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