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Abstract

A pendulum using four suspension wires in an X formation is described which can provide an arbitrarily long period in one dimension. Design equations are derived, and tested experimentally. A number of applications are outlined, including a low-frequency vibration isolation system for a laser-interferometric gravity wave observatory.

I. INTRODUCTION

Pendulums of various types have many uses in experimental physics. The restoring force on an object supported by a pendulum is low, and so the effect of small forces is readily measurable. Pendulums are also used as low frequency mechanical oscillators to filter out high frequency vibrations. Examples of this usage include seismometers using a mass suspended by a wire, and the final vibration isolation system for mirrors in a laser interferometer for gravity wave detection as in the ongoing LIGO [1] and VIRGO [2] projects. In these applications, the frequency range is limited by the pendulum frequency. Unfortunately, the frequency scales as the square root of the length so that periods longer than a few seconds require inconveniently tall pendulums. To obtain a longer period in seismometers, various inverted and horizontal pendulums were devised, but they require a pivot or hinge to support the test mass. This is a major limiting factor on performance in precise instruments because the pivot causes noise due to friction and the hinge introduces drift from creep of the material. Moreover, the stiffness of the inevitable auxiliary spring has a much steeper dependence on temperature than its linear dimensions. Therefore, new technologies such as the superspring [3] and other active anti-vibration systems are being developed by many groups.

In this paper we point out the possibility of constructing a pendulum which in its basic form permits an arbitrarily large period in one dimension. We present the principle and the design equations, as well as data from experimental tests. We suggest a number of applications of interest to those in the experimental physics and engineering fields.

II. ANALYSIS OF THE PENDULUM DYNAMICS

The principle of the design can be seen in Fig. 1. Four suspension wires are used. Two wires cross at a shallow angle from the left side of the support to the top plate of the

pendulum on the right side and the other two cross similarly from right to left. In the side view the wires appear to cross in the centre, but of course must actually be staggered in depth so as not to touch. The object to be suspended is connected rigidly to the top plate by a bar or similar structure. It is convenient to analyse the geometry of the suspension in terms of the following parameters: L_1 , the distance between the anchor points at the upper support, L_2 , the similar distance at the mass, l , the length of each wire, and α and α' , the acute angles between the wires and the horizontal at the upper support. When the pendulum is hanging centrally, both α and α' have the value α_0 . Let the origin be midway between the upper anchor points. The left and right anchor points on the mass then have coordinates given by

$$x_L = \frac{L_1}{2} - l \cos \alpha, \quad (1a)$$

$$y_L = -l \sin \alpha, \quad (1b)$$

$$x_R = -\frac{L_1}{2} + l \cos \alpha', \quad (1c)$$

$$y_R = -l \sin \alpha'. \quad (1d)$$

The anchor points on the mass are constrained to be a distance L_2 apart so that

$$L_2^2 = (x_R - x_L)^2 + (y_R - y_L)^2. \quad (2)$$

We need to calculate to at least second order to extract a potential. By symmetry, to first order, we expect $\Delta\alpha' = -\Delta\alpha$. To extend this, let

$$\Delta\alpha' = W\Delta\alpha^2 - \Delta\alpha + O(\Delta^3). \quad (3)$$

Then expanding Eqs. (1) to second order in the angles and using Eq. (2), W becomes

$$W = \frac{2l \cos^2 \alpha_0 - L_2 \cos \alpha_0}{L_2 \sin \alpha_0}. \quad (4)$$

It is then straightforward to calculate the coordinates of a point P on the midline of the pendulum, at a distance H below the midpoint, which we take to be the center of mass of the pendulum:

$$x_P = l \left(\sin \alpha_0 + \frac{2H}{L_1} \cos \alpha_0 \right) \Delta\alpha + \frac{Wl}{2} \left(\sin \alpha_0 + \frac{2H}{L_1} \cos \alpha_0 \right) \Delta\alpha^2, \quad (5a)$$

$$y_P = -l \sin \alpha_0 - H + \frac{l}{2} \left(\sin \alpha_0 - W \cos \alpha_0 + \frac{4lH}{L_1^2} \cos^2 \alpha_0 \right) \Delta\alpha^2. \quad (5b)$$

Eliminating $\Delta\alpha$ between Eqs. (5a) and (5b) then gives an equation for the locus of P:

$$y_P = -l \sin \alpha_0 - H + \frac{x_P^2 \left(\sin \alpha_0 - W \cos \alpha_0 + \frac{4lH}{L_1^2} \cos^2 \alpha_0 \right)}{2l \left(\sin \alpha_0 + \frac{2H}{L_1} \cos \alpha_0 \right)^2}. \quad (6)$$

Figure 1. A side view of the X-pendulum. Each diagonal represents a pair of wires, at different depths in the drawing.

Figure 2. An end view of the X-pendulum, showing the form of the motion when the wire is (a) unstretchable and (b) slightly stretchable. L_3 is representative of the width of the pendulum but is not precisely defined.

For typical dimensions, $\cos \alpha_0 \gg \sin \alpha_0$. Thus for small H , that is, only a small distance below the top, the dominant term in the coefficient of $\Delta \alpha^2$ is $-W \cos \alpha_0$. Since W is typically positive this represents an inverted parabola, and the pendulum will be unstable. Conversely, for large H , the positive term $\frac{4lH}{L_1^2} \cos^2 \alpha_0$ dominates and the pendulum is stable. The critical value of H where the pendulum is on the verge of stability is

$$H_H = \frac{L_1^2 (W \cos \alpha_0 - \sin \alpha_0)}{4l \cos^2 \alpha_0}. \quad (7)$$

Turning now to the dynamics of the system, we consider the Lagrangian

$$L = T - V = \frac{1}{2} M \dot{x}_P^2 + \frac{1}{2} I \dot{\theta}^2 - \frac{1}{2} k x_P^2. \quad (8)$$

Here θ is the tilt of the mass and I is the principal moment of inertia. θ is easily shown to be

$$\theta = \frac{2 \cos \alpha_0 x_P}{L_1 \sin \alpha_0 + 2H \cos \alpha_0}, \quad (9)$$

so that using Lagrange's formalism, the motion is found to be simple harmonic with angular frequency given by

$$\omega^2 = \frac{g}{l} \frac{\sin \alpha_0 - W \cos \alpha_0 + \frac{4lH}{L_1^2} \cos^2 \alpha_0}{\left(\sin \alpha_0 + \frac{2H}{L_1} \cos \alpha_0 \right)^2 + \frac{I}{M} \left(\frac{2 \cos \alpha_0}{L_1} \right)^2}. \quad (10)$$

(where W is given by Eq. (4)).

The moment of inertia, I , will depend on the details of the distribution of the mass of the pendulum. It is clearly impossible to anticipate all designs, so we consider a situation of moderate generality applicable to our test pendulum. Such a pendulum consists of a top plate of mass M_T , a bar of length D and mass M_B , and a weight of mass M_W placed an adjustable distance d from the top. The intrinsic moments of inertia of the parts are I_T , I_B and I_W . The center of mass of the pendulum is then a weighted average of the positions of the parts and the moment of inertia is the second moment of the masses relative to the center of mass, plus the intrinsic moments.

The effective potential is not exactly quadratic of course. In addition there is a quartic term which is positive for all useful values of H . This was determined by solving the geometrical equations numerically and fitting a fourth-order polynomial to the locus so determined. Thus if the pendulum is stable for small θ , it will be stable over its full range of movement. However the frequency will be slightly amplitude dependent, specifically, increasing with amplitude. Similarly if the pendulum is only slightly unstable in the center, there will be two stable points, one on either side of the center.

The two principal sources of energy loss in a well-designed pendulum will be bending of the wires near the anchor points and flexing of the supposedly rigid connection between the top and bottom of the pendulum. The angles at the support are to first order, $\alpha_0 + \Delta \alpha$ and $\alpha_0 - \Delta \alpha$, and the angles at the pendulum are $\alpha_0 + \Delta \alpha - \Delta \theta$ and $\alpha_0 - \Delta \alpha + \Delta \theta$. Putting these values in terms of θ and summing, the net bending of wire for a four wire suspension becomes

$$\Delta\theta_B = 4\Delta\theta. \quad (11)$$

We take the bar and wire materials to have frequency independent intrinsic damping, with fractional imaginary components of elasticity being ϕ_B and ϕ_W [4]. The respective Young's moduli are E_B and E_W , and the moments of cross-sectional area are C_B and C_W . In terms of these parameters, the net Q becomes

$$Q = \frac{\frac{H}{H_H} - 1}{\frac{MgH^3\phi_B}{E_B C_B} + \frac{2\phi_W}{H_H} \left(\frac{E_W C_W}{Mg \sin \alpha_0} \right)^{1/2}}. \quad (12)$$

The numerator is a measure of the extent to which the angular spring constant and hence the stored energy has been cancelled by moving H close to H_H , relative to a simple pendulum (which would have a numerator of 1). Two very small terms representing the elasticity of the wire and the bar have been neglected. Although in principle the cancelling of the other terms could make them relatively large, in practice their effect is indistinguishable from a trivial change in H_H . Since the damping is constant, the Q increases by a corresponding factor. The two terms in the denominator give the damping due to the bar and wire respectively. By suitable choice of materials and dimensions the bar damping can be made small in comparison to that of the wire. In the equivalent expression for a simple pendulum, there is of course no term for the bar, and in the wire term, the factor $\sin \alpha_0$ disappears and the 2 becomes 0.5 [5]. Finally, relative to the simple pendulum, the tension is greater by a factor $1/(4 \sin \alpha_0)$ and so the wire size must be increased appropriately. Since C_W is proportional to the width of the wire but the cube of its thickness, it is advantageous to increase the width preferentially. Using flexure strips is of course an obvious tactic in any case, since only the longitudinal mode is of use.

The temperature response of the pendulum will commonly be important. If the support and the top of the pendulum are taken to have a common thermal expansion coefficient α_T , and the wires have a different coefficient $\alpha_T + \epsilon$, then it can be shown that

$$\frac{d(\omega^2)}{dT} = -\omega^2 \alpha_T + 2\epsilon l \frac{g / \sin^2 \alpha_0 - \omega^2 (l \sin \alpha_0 + 2H)}{\sin \alpha_0 ((l \sin \alpha_0 + 2H)^2 + 4I/M)}. \quad (13)$$

where for simplicity it has been taken that $L_1 = L_2$. In particular if $\epsilon = 0$, the behaviour is that of a simple pendulum. This is one of the merits of the system.

The pendulum has a second mode of much higher frequency, transverse to the first, which although not especially useful, may need to be considered in some applications. In the ideal case when the wires are unstretchable, the amplitude is not too large and the ratio of width to length of the pendulum is not too small, the pendulum moves without rotation as if supported by four short vertical wires of length equal to the gap, as becomes obvious from the end view in Fig. 2(a). The effective length is $l \sin \alpha_0$, so that the frequency is

$$\omega_T^2 = \frac{g}{l \sin \alpha_0}. \quad (14)$$

However the assumption that the wire is unstretchable is very likely to fail. Commonly, wire as thin as possible will be used to minimise energy losses due to bending. Moreover, the elasticity of the wire acts through a very poor mechanical advantage to resist the transverse

mode. When elasticity dominates, there will be an extra degree of freedom and therefore two modes. One of these will be an up-down motion, and the other a back-and-forth rotation centered near the top of the pendulum as in Fig. 2(b). Assuming this mode structure, and ignoring any centripetal forces and nonlinearities, we can derive simple approximations for the frequencies. If the Young's modulus and area of the wire are Y and A_W , the vertical frequency is

$$\omega_{TE}^2 = 4Y A_W \sin^2 \alpha_0 / Ml. \quad (15)$$

Similarly, the transverse frequency becomes

$$\omega_{TE}^2 = \frac{Y A_W L_3^2 \sin^2 \alpha_0 / l + M g H}{M H^2 + I}, \quad (16)$$

where as above I is the principal moment of inertia and H is the position of the center of mass relative to the top. The very small geometrical factor $\sin^2 \alpha_0$ multiplying the Young's modulus tends to produce quite a low frequency, but can be offset by making the width L_3 large.

III. EXPERIMENTAL TESTS

To verify some of the above results, a test pendulum was constructed of aluminium and suspended with 0.2 mm diameter piano wire. It followed the design outlined above, having a top section 3 mm in thickness with $M_T=215$ g, a bar with $M_B=112$ g, and $D=400$ mm, and an adjustable weight with $M_W=686$ g. L_1 and L_2 were both 102 mm, and the gap $l \sin \alpha_0$ was 15.3 mm.

Ten periods of the swing were timed by stopwatch and the frequency was determined for a number of different positions of the weight. Because the restoring force is very small, especially near the critical height, the pendulum needed to be trimmed to bring it precisely into balance. This was done with a small weight at the top (included in the value of M_T above) which could be moved from side to side. There are two symptoms of unbalance. As viewed from the side, the point where the wires appear to cross is displaced to one side when the pendulum is still. Also, very obvious non-simple-harmonic behaviour can be seen when the pendulum is excited; the deceleration at one extreme of the motion is much more abrupt than at the other. However trimming the pendulum until the equilibrium wire crossing is central also causes the pendulum to swing smoothly. The trim was checked and adjusted before each measurement. Because of the slight amplitude dependence, excitations less than 5 mrad peak-to-peak were used.

IV. RESULTS AND ANALYSIS

The results of the measurements are plotted in Fig. 3. The weight positions d appear along the horizontal axis. The period measurements were converted to frequencies in hertz, and appear on the vertical axis. The smooth curve is the theoretical prediction from Eq. (10), and converted to ordinary frequency from rad/s. The theoretical curve reproduces the general behaviour quite well, and correctly predicts both the critical mass position and the

Figure 3. The frequency response of the test pendulum as measured (points) compared with the predicted response (smooth line).

Figure 4. The expected thermal noise for a simple 4 wire pendulum as in the LIGO prototype (the curve lowest at the right), and for the same mass on an X suspension, with comparable dimensions and materials. The two periods were achieved by adjusting $H-H_H$.

maximum frequency. The largest discrepancy is of order 10% near the knee of the curve. The predominant transverse mode was elastic, and the measured frequencies (in the range 1.6 to 2.4 Hz) agreed with Eq. (16) to within 10%, based on a separately measured YA_W product of 6900 N for samples of the wire.

V. DISCUSSION

A laser interferometric gravity wave observatory seeks to detect the effect of a gravitational wave on a laser interferometer in which the end mirrors of each of the two arms are freely suspended and constitute the target masses for the gravity wave. Seismic isolation is one of the key technologies required for a wideband gravitational detector. None of the existing plans have a final version of the isolation system. The proposed pendulum is a clear candidate for a stage in the isolation system due to its simplicity, durability and reliability. The simple pendulum constructed for the purposes of this report easily achieved a period of 30 s with a height budget of less than 30 cm. Periods in excess of 100 s should be readily obtainable in an improved design.

In this application thermal noise is also a major consideration. Such noise is likely to be a limiting factor at midrange frequencies. Each mode of the system contains $k_B T$ of energy (the equipartition theorem) and this energy is smeared over a range of frequencies near the resonance in proportion to the damping (the fluctuation dissipation theorem) [6]. As high a Q as possible is desirable to keep the thermal noise in a narrow band where it can be filtered out in the data analysis. Fig. 4 shows the expected thermal noise from a test version of the LIGO suspension [7] compared with two hypothetical versions with the same suspended mass ($M_W=1.5$ kg) using an X-suspension. We chose $L_1=L_2=0.1$ m, and $l \sin \alpha_0=12$ mm, putting H_H at 0.205 m. We allowed 0.25 kg each for M_T and M_B , so that the overall height budget is about 0.25 m, as before. We assumed wire of the same type (piano wire) as in [7] but with a larger diameter to maintain the same loading relative to the breaking point. We inferred a damping coefficient for piano wire from the highest reported violin mode Q (356000). Two different settings of $H - H_H$ are shown, one giving a period of 10 s, the other of 30 s. Although the factor $H/H_H - 1$ in Eq. (12) has a major effect on the Q itself, this does not translate into extra thermal noise at high frequencies. The only penalty relative to the simple suspension is due to the factor of $2/\sin \alpha_0$ (and the wire thickness which varies with α_0). From the point of view of thermal noise it is therefore optimum to use a wide gap, and to adjust H very close to H_H . However we also note an observation of Saulson [4], that the thermal noise at high frequencies is determined principally by the final stage (even when preceded by stages of modest Q). This is the justification for ignoring the previous stages above, but it also means that an even better strategy may be to terminate the isolation system in a short conventional pendulum using a flexure strip. This reduces the period a little (because the moment of inertia is increased less than the restoring force), but providing the mass ratio is not too small this will give both excellent seismic isolation and excellent thermal noise performance along the key laser axis.

Instead of or as well as the conventional pendulum, a second stage of X-suspension can be introduced to give low-frequency isolation in two dimensions. As with the conventional pendulum, the point of attachment must be below the critical point of the upper stage, and in addition the transverse mode of the top stage must be sufficiently rigid. We expect to

realize such an ultra low frequency anti-vibration system in the near future.

This system can also improve the sensitivity of precise measurements in many other fields. One such application is the measurement of the gravitational constant [8]. The present value is limited mainly by the uncertainty in measurements of associated geometrical quantities due to the use of a torsion pendulum. To make the geometry more precise can only be achieved at the expense of the sensitivity of the balance. This pendulum has the possibility to replace the torsion balance in some extra weak force measurements.

A final application is in validation of the properties of pendulum wire. Although there is some evidence that, as assumed above, frequency independent damping is a common property of materials, this has not been confirmed directly with realistic shapes and stress patterns for the types of materials that may be used in low-frequency ultra-high- Q pendulums because the required observing time is impossibly long. Given a combination of wire material, load and period, an X-suspended pendulum can be constructed which matches all three but has its Q in an experimentally accessible range.

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