First-order Lagrangians and the Hamiltonian Formalism

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ABSTRACT

We consider the problem of constructing a general unconstrained Hamiltonian formalism for a system with a finite number of degrees of freedom, starting from a general first-order Lagrangian. This construction, which uses only elements of linear algebra and the theory of partial differential equations, is given in a rather explicit form. An application of the formalism to the quantization of two-dimensional real self-dual fields is given.

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1 Introduction

This paper is to a large extent inspired by a paper of Floreanini and Jackiw [1] which deals with the quantization of a self-dual (bosonic) field χ in a two-dimensional space-time, i.e. a field that satisfies the equation

$$\left(\frac{\partial}{\partial t} - \frac{\partial}{\partial x}\right)\chi = 0. \quad (1)$$

The model considered by Floreanini and Jackiw is defined by the following Lagrangian (suppressing common time arguments),

$$L = \frac{1}{4} \iint dx dy \chi(x) \epsilon(x-y) \dot{\chi}(y) - \frac{1}{2} \int dx \chi^2(x). \quad (2)$$

The quantization procedure used by Floreanini and Jackiw used as a starting point a commutation relation for the field variable χ , which indeed is a consequence of the canonical structure of the theory described by the Lagrangian (2).

This later gave rise to some discussion in the literature; the commutator given by Floreanini and Jackiw was even said to be unusual in a letter by Costa and Girotti [2], who advocated a formalism with constraints (see also Bernstein and Sonnenschein [3] in this connection). The whole issue depends indeed on the interpretation of the variables in the Lagrangian (2); if one does not identify a minimal set of canonical coordinates and momenta describing the dynamics specified by the Lagrangian (2), one may well be lead to believe that the commutator given by Floreanini and Jackiw is "unusual". However, canonical coordinates and momenta underlying the theory described by the Lagrangian (2) do exist, although these were not exhibited explicitly in the paper by Floreanini and Jackiw. This is true under fairly general circumstances for general Lagrangians which are of first order in "velocities". This whole issue was clarified later in a paper by Faddeev and Jackiw [4], who referred to classical theorems of Darboux (see e.g. Abraham and Marsden [5] for a proof of the relevant theorems) for the existence of the canonical variables.

However, we have not found the existence theorems referred to above very helpful in actually constructing canonical variables in the case of a general first-order Lagrangian, and give therefore in this paper a constructive procedure for obtaining such variables for a general first-order Lagrangian describing a system with a finite number of degrees of freedom, with variables which classically are commuting. The construction in question can easily be generalized to cover the theory described by the Lagrangian (2); in Sec. 5 of this paper we analyse this case in detail and find complete agreement with the treatment of Floreanini and Jackiw.

The issues discussed here have also been taken up by Govaerts [6], who makes a comparison between the unconstrained formalism of Faddeev and Jackiw [4] and a formalism with constraints, which supposedly is more "standard". The work of Govaerts covers the cases of both bosonic and fermionic variables (quantities of even and odd Grassman parity, respectively). However, one should always be cautious about introducing a formalism with constraints if that is not necessitated by the intrinsic properties of the system under consideration. Namely, if one is willing to enlarge the phase space of any dynamical system at will, one can always make it

into a canonical Hamiltonian system (superficially at least) as shown for instance by Arzanykh [7].

In what follows we treat the case of a first-order Lagrangian in detail; the problem of obtaining an explicit canonical formulation in this case is shown to be equivalent to straightforward problems in linear algebra and the theory of partial differential equations.

2 First-order Lagrangians, Equations of Motion and Constraints

We consider a system with a finite number of degrees of freedom, the state of which is described by a time-dependent N -component object $\xi(t) = (\xi_1(t), \dots, \xi_N(t))$ in configuration space. The configuration space is thus taken to be an N -dimensional space, where N is a positive even or odd integer. The dynamics of the system is defined by the following action S ,

$$S = \int dt \left[\sum_{A=1}^N \dot{\xi}_A F_A(\xi) - G(\xi) \right]. \quad (3)$$

The integrand in Eq. (3) defines a first-order Lagrangian $\mathcal{L}^1$, which we write as a Lagrangian one-form,

$$dt \mathcal{L}^1 = \sum_{A=1}^N d\xi_A F_A(\xi) - dt G(\xi). \quad (4)$$

The functions $F_A(\xi)$ and $G(\xi)$ occurring in Eqs. (3) and (4) above are given functions, which are assumed to be smooth enough, for the usual variational procedure connected with the action S to make sense. The variational equations obtained from Eq. (3) are the following

$$\sum_{B=1}^N M_{AB} \dot{\xi}_B - \frac{\partial G}{\partial \xi_A} = 0, \quad (5)$$

where

$$M_{AB}(\xi) := \frac{\partial F_B}{\partial \xi_A} - \frac{\partial F_A}{\partial \xi_B}. \quad (6)$$

In the terminology of classical mechanics [8] the equations of motion (5) are invariantly related to the one-form (4); it should be noted that the quantities $F_A(\xi)$ do not enter directly in the equations of motion (5) but only through the curl $M_{AB}(\xi)$ (6), which can be taken to define a two-form invariantly related to the one-form (4). Rather than using the language of forms, we consider the quantity $M_{AB}(\xi)$ as an antisymmetric tensor quantity; the crucial property of this quantity besides the antisymmetry,

$$M_{AB}(\xi) = -M_{BA}(\xi) \quad (7)$$

is the (Bianchi) identity,

$$\partial_A M_{BC}(\xi) + \partial_B M_{CA}(\xi) + \partial_C M_{AB}(\xi) = 0, \quad (8)$$

which follows straightforwardly from the definition (6).

The main question is now simply whether Eqs. (5) are Hamiltonian equations, albeit in disguise, or whether they contain a subset that is Hamiltonian. As mentioned in the Introduction, this question has been answered by the affirmative under fairly general circumstances by Faddeev and Jackiw [4], who refer to the classical Darboux theorem [5] for the existence of canonical coordinates and momenta related to the system described by the action (3). In this and the following section we simply provide a constructive procedure for obtaining the canonical variables in question, analysing at the same time in more detail the conditions under which the canonical structure actually emerges.

As a first step in this construction it is convenient to consider the so-called normal form of the quantity $M_{AB}(\xi)$ of Eq. (6), now considered as an antisymmetric $N \times N$ matrix. The ξ -dependent quantity $M_{AB}(\xi)$ can be transformed into its normal form at any given fixed point ξ_0 in configuration space, and the resulting form can then be extended by continuity to a suitable region around the chosen point. For definiteness, we choose $\xi_0 = 0$, so that the considerations below can be taken to be valid in a (possibly small, but finite) region around the origin of the coordinates in configuration space.

It is well known [9] that, by making an appropriate basis transformation, any antisymmetric $N \times N$ matrix (M_{AB}) can be transformed into the following normal form:

$$\begin{pmatrix} 0 & \lambda_1 & \dots & \dots & \dots & \dots & \dots & \dots \\ -\lambda_1 & 0 & \dots & \dots & \dots & \dots & \dots & \dots \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \vdots \\ \dots & \dots & \dots & 0 & \lambda_n & \dots & \dots & \dots \\ \dots & \dots & \dots & -\lambda_n & 0 & \dots & \dots & \dots \\ \dots & \dots & \dots & \dots & \dots & 0 & \dots & \dots \\ \dots & \dots & \dots & \dots & \dots & \dots & \ddots & \dots \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots & 0 \end{pmatrix} \quad (9)$$

where the (positive) quantities λ_α , $\alpha = 1, \dots, n$ are the square roots of those characteristic values of the matrix $(-M_{AB}^2)$, which are different from zero. The number n of non-zero characteristic values λ_α^2 is determined by the rank $2n$ of the matrix (M_{AB}).

It is convenient to relate the normal form (9) to the following linear equations (details are given in the Appendix),

$$\sum_{B=1}^N M_{AB} x_{\alpha B} = -\lambda_\alpha y_{\alpha A}, \quad \alpha = 1, \dots, n \quad (10)$$

$$\sum_{B=1}^N M_{AB} y_{\alpha B} = +\lambda_\alpha x_{\alpha A}, \quad \alpha = 1, \dots, n \quad (11)$$

and

$$\sum_{B=1}^N M_{AB} z_{\beta B} = 0, \quad \beta = 1, \dots, N - 2n \quad (12)$$

with the understanding that Eq. (12) is empty if the rank of the matrix (M_{AB}) is N (so that $N = 2n$). In this case the matrix M is *regular*,

$$\det(M_{AB}) \neq 0. \quad (13)$$

If the condition (13) is in force, then the matrix elements M_{AB} can be expressed neatly in terms of the properly orthonormalized solutions $x_{\alpha A}$ and $y_{\alpha A}$ of Eqs. (10), (11) above,

$$M_{AB} = \sum_{\alpha=1}^n \lambda_{\alpha} (x_{\alpha A} y_{\alpha B} - x_{\alpha B} y_{\alpha A}), \quad A, B = 1, \dots, 2n. \quad (14)$$

After these preliminaries we now return to the analysis of the equations of motion (5). First we dispense with the case when Eqs. (12) have at least one non-trivial solution. This is necessarily the case if the dimensionality N of the configuration space is odd, but may also be the case if N is even, in which case Eq. (12) necessarily has an even number of solutions. Contracting the equations of motion (5) with a solution $z_{\beta A}$ of Eq. (12), one obtains

$$\sum_{A=1}^N z_{\beta A}(\xi) \frac{\partial G(\xi)}{\partial \xi_A} = 0, \quad \beta = 1, \dots, N - 2n. \quad (15)$$

But Eqs. (15) are genuine constraints; they cannot be a subset of a (unconstrained) Hamiltonian system, since they contain no derivatives with respect to time. One may of course develop a constrained Hamiltonian formalism, keeping the constraints (15) as they are. However, for (among other things) reasons mentioned in the Introduction, we prefer an explicitly unconstrained formalism. In principle the constraints (15) can be solved at least locally in some appropriate region, which we take to be a region around the origin of coordinates $\xi = 0$. This means that there exists a set of coordinates u_i , say, $i = 1, \dots, 2n$, such that

$$\xi_A = \xi_A(u_1, \dots, u_{2n}), \quad A = 1, \dots, N. \quad (16)$$

Needless to say, the mapping (16) is assumed to be sufficiently smooth, i.e. at least twice continuously differentiable. The proper equations of motion are then obtained from (5) by using the expression

$$\dot{\xi}_B = \sum_{k=1}^{2n} \frac{\partial \xi_B}{\partial u_k} \dot{u}_k \quad (17)$$

and by multiplying Eq. (5) with $\partial \xi_A / \partial u_j$, then by summing over A ,

$$\sum_{k=1}^{2n} m_{jk} \dot{u}_k = \frac{\partial G}{\partial u_j}, \quad (18)$$

where

$$m_{jk} := \sum_{A,B=1}^N \frac{\partial \xi_A}{\partial u_j} M_{AB} \frac{\partial \xi_B}{\partial u_k} = \frac{\partial}{\partial u_j} \left(\sum_{B=1}^N F_B \frac{\partial \xi_B}{\partial u_k} \right) - \frac{\partial}{\partial u_k} \left(\sum_{B=1}^N F_B \frac{\partial \xi_B}{\partial u_j} \right). \quad (19)$$

The original equations of motion (5) including the constraints (15) have thus been replaced by a set of equations (18) of exactly the same form as the original ones, but where the quantity (M_{AB}) is replaced by a quantity (m_{jk}) , which, considered as a $2n \times 2n$ antisymmetric matrix is regular, i.e. of rank $2n$. The quantity (m_{jk}) also satisfies the appropriate Bianchi identity,

$$\frac{\partial}{\partial u_i} m_{jk} + \frac{\partial}{\partial u_j} m_{ki} + \frac{\partial}{\partial u_k} m_{ij} = 0. \quad (20)$$

Thus, by solving the constraints (15) in the manner indicated, one obtains a set of equations of motion without constraints in a reduced configuration space (with the variables u_i as coordinates), which by themselves are invariantly related to a first-order Lagrangian one-form in the reduced configuration space. In the next section we will therefore consider the equations of motion of the form (5) under the assumption that the quantity (M_{AB}) has no eigenvectors corresponding to zero characteristic values, since, as has just been shown, this assumption involves no essential loss of generality.

3 Construction of Canonical Variables

We now return to the equations of motion (5), which we write down once more, with the appropriate range of indices $(A, B, \dots = 1, \dots, 2n)$,

$$\sum_{B=1}^{2n} M_{AB}(\xi) \dot{\xi}_B = \frac{\partial G(\xi)}{\partial \xi_A}, \quad A = 1, \dots, 2n. \quad (21)$$

It is from now on *assumed* that the quantity M_{AB} , considered as an antisymmetric matrix is regular, i.e. of rank $2n$, so that the matrix is invertible,

$$\det(M_{AB}) \neq 0. \quad (22)$$

Then we have

$$\dot{\xi}_B = \sum_{A=1}^{2n} M_{BA}^{-1}(\xi) \frac{\partial G(\xi)}{\partial \xi_A}, \quad B = 1, \dots, 2n. \quad (23)$$

We now analyse the circumstances under which Eqs. (23) are canonical Hamiltonian equations, with the function $G(\xi)$ acting as a Hamiltonian. We start by assuming the existence of canonical variables p_{α} , q_{α} , i.e. a one-to-one correspondence between the variables ξ_A , $A = 1, \dots, 2n$, and pairs of variables p_{α} , q_{α} , $\alpha = 1, \dots, n$:

$$\{\xi_A, A = 1, \dots, 2n\} \longleftrightarrow \{p_{\alpha}, q_{\alpha}, \alpha = 1, \dots, n\}. \quad (24)$$

It goes without saying that the correspondence (24) is always assumed to be smooth enough; the functions involved are always assumed to be at least twice continuously differentiable in the domain of validity of the correspondence (24). The canonical equations corresponding to Eqs. (23) are the following,

$$\dot{q}_{\alpha} = \frac{\partial H(p, q)}{\partial p_{\alpha}}, \quad \dot{p}_{\alpha} = - \frac{\partial H(p, q)}{\partial q_{\alpha}}, \quad (25)$$

where

$$H(p, q) \equiv G(\xi(p, q)). \quad (26)$$

It is now a matter of a simple calculation to show that Eq. (21) and (25) are equivalent if and only if the following conditions hold true:

$$\sum_{B=1}^{2n} \frac{\partial q_\alpha}{\partial \xi_B} (M^{-1})_{BA} = \frac{\partial \xi_A}{\partial p_\alpha} \quad (27)$$

and

$$\sum_{B=1}^{2n} \frac{\partial p_\alpha}{\partial \xi_B} (M^{-1})_{BA} = -\frac{\partial \xi_A}{\partial q_\alpha}. \quad (28)$$

It is likewise simple to show that the relations (27) and (28) hold true if and only if the following relation is in force:

$$M_{AB}^{-1}(\xi) = \sum_{\alpha=1}^n \left(\frac{\partial \xi_A}{\partial q_\alpha} \frac{\partial \xi_B}{\partial p_\alpha} - \frac{\partial \xi_A}{\partial p_\alpha} \frac{\partial \xi_B}{\partial q_\alpha} \right) \equiv \{\xi_A, \xi_B\}_P, \quad (29)$$

where $\{u, v\}_P$ denotes the *Poisson bracket* of any two quantities u, v . We can thus formulate the results so far as follows:

In order that Eqs. (21) may be equivalent to a set of canonical Hamiltonian equations, with the function $G(\xi)$ acting as the Hamiltonian, it is *necessary* that the inverse of the matrix M_{AB} occurring in these equations be identified with the Poisson bracket $\{\xi_A, \xi_B\}_P$ of the configuration space variables ξ in the equations in question.

It is clear that not any (antisymmetric) quantity is acceptable as a Poisson bracket, so it behoves us to analyse the conditions on the quantity M_{AB} implied by the condition (29). For this purpose it is convenient to consider the inverse of the Poisson bracket, i.e. the *Lagrange bracket* $[\cdot, \cdot]_L$. The condition (29) is then equivalent to

$$M_{AB}(\xi) = \sum_{\alpha=1}^n \left(\frac{\partial p_\alpha}{\partial \xi_A} \frac{\partial q_\alpha}{\partial \xi_B} - \frac{\partial q_\alpha}{\partial \xi_A} \frac{\partial p_\alpha}{\partial \xi_B} \right) \equiv [\xi_A, \xi_B]_L. \quad (30)$$

Writing the Lagrange bracket in the equivalent form,

$$[\xi_A, \xi_B]_L = \frac{\partial}{\partial \xi_A} \left(\sum_{\alpha=1}^N p_\alpha \frac{\partial q_\alpha}{\partial \xi_B} \right) - \frac{\partial}{\partial \xi_B} \left(\sum_{\alpha=1}^N p_\alpha \frac{\partial q_\alpha}{\partial \xi_A} \right), \quad (31)$$

one observes that the quantity M_{AB} must be representable as a curl, i.e. that there must exist quantities F_A , $A = 1, \dots, 2n$, such that

$$M_{AB}(\xi) = \frac{\partial}{\partial \xi_A} F_B(\xi) - \frac{\partial}{\partial \xi_B} F_A(\xi). \quad (32)$$

Here we started from Eqs. (21) with a general antisymmetric quantity M_{AB} , and arrived at Eq. (32) as an additional necessary condition on M_{AB} in order that Eqs. (21) be canonical.

Using the Lagrangian one-form (4) as a starting point one need of course not elaborate on the existence of a representation such as (32) for the quantity M_{AB} , since this is granted by the very construction of M_{AB} in that case.

So, the construction of canonical coordinates has finally boiled down to solving Eqs. (30) for the variables p_α and q_α for a given set of functions $F_A(\xi)$. Comparing Eqs. (30), (31) and (32) one observes that this problem is equivalent to the following,

$$\sum_{\alpha=1}^n p_\alpha \frac{\partial q_\alpha}{\partial \xi_A} = F_A(\xi) + \frac{\partial \Phi}{\partial \xi_A} \equiv X_A(\xi), \quad (33)$$

where $\Phi(\xi)$ is an arbitrary function at our disposal. Equations (33) define a straightforward mathematical problem, namely a problem of the type known as Pfaffs problem, for which various methods are available in the literature [10]. Indeed it is inconceivable that the problem of constructing canonical variables using a given Poisson bracket structure as a starting point has not been dealt with before. However, we have not been able to find any such explicit construction in the literature. We therefore continue with our analysis of this problem, which, as shown above, has led to Eqs. (33).

Let us first note that Eqs. (33) imply the conditions (30). Then, defining the quantities W_A by means of the linear equations,

$$\sum_{B=1}^{2n} M_{AB} W_B = X_A, \quad (34)$$

where X_A is the known quantity in Eq. (33), we have,

$$\sum_{\alpha=1}^n p_\alpha \frac{\partial q_\alpha}{\partial \xi_A} = \sum_{B=1}^{2n} \sum_{\alpha=1}^n \left(\frac{\partial p_\alpha}{\partial \xi_A} \frac{\partial q_\alpha}{\partial \xi_B} - \frac{\partial q_\alpha}{\partial \xi_A} \frac{\partial p_\alpha}{\partial \xi_B} \right) W_B. \quad (35)$$

Defining further the quantities Q_α and Π_α as follows,

$$Q_\alpha := - \sum_{B=1}^{2n} W_B \frac{\partial q_\alpha}{\partial \xi_B}, \quad \Pi_\alpha := \sum_{B=1}^{2n} W_B \frac{\partial p_\alpha}{\partial \xi_B} + p_\alpha, \quad (36)$$

one finds that Eqs. (35) are equivalent to the following set of $2n$ linear and homogeneous equations in the $2n$ unknowns Q_α and Π_α ,

$$\sum_{\alpha=1}^n \left(\frac{\partial q_\alpha}{\partial \xi_A} \Pi_\alpha + \frac{\partial p_\alpha}{\partial \xi_A} Q_\alpha \right) = 0, \quad A = 1, \dots, 2n. \quad (37)$$

However the determinant D of Eq. (37) is non-zero, since a straightforward calculation shows that

$$D^2 = \det(M_{AB}) \neq 0. \quad (38)$$

Thus Eq. (37) has only zero solutions Q_α and Π_α so that

$$\sum_{B=1}^{2n} W_B \frac{\partial q_\alpha}{\partial \xi_B} = 0 \quad (39)$$

and

$$\sum_{B=1}^{2n} W_B \frac{\partial p_\alpha}{\partial \xi_B} + p_\alpha = 0. \quad (40)$$

Equations (39) and (40) are independent partial differential equations for the canonical coordinates and momenta, respectively, with known coefficients W_B . However, one should also require that these quantities should have the appropriate Poisson brackets, i.e.

$$\{q_\alpha, q_\beta\}_P = 0, \quad \alpha, \beta = 1, \dots, n \quad (41)$$

$$\{p_\alpha, p_\beta\}_P = 0, \quad \alpha, \beta = 1, \dots, n \quad (42)$$

and

$$\{q_\alpha, p_\beta\}_P = \delta_{\alpha\beta}, \quad \alpha, \beta = 1, \dots, n. \quad (43)$$

Now, from Eq. (29) follows that the Poisson bracket of any two quantities u and v can be expressed as

$$\{u, v\}_P = \sum_{A,B=1}^{2n} M_{AB}^{-1}(\xi) \frac{\partial u}{\partial \xi_A} \frac{\partial v}{\partial \xi_B}. \quad (44)$$

We can thus finally state the complete set of independent equations for the canonical variables. From Eqs. (34) and (39), it follows that the canonical coordinates q_α satisfy the set of n partial differential equations:

$$\sum_{A,B=1}^{2n} X_A M_{AB}^{-1} \frac{\partial q_\alpha}{\partial \xi_B} = 0, \quad \alpha = 1, \dots, n. \quad (45)$$

Furthermore, from the conditions (41) the following set of $\frac{1}{2}n(n-1)$ equations is obtained:

$$\sum_{A,B=1}^{2n} M_{AB}^{-1} \frac{\partial q_\alpha}{\partial \xi_A} \frac{\partial q_\beta}{\partial \xi_B} = 0, \quad \alpha, \beta = 1, \dots, n. \quad (46)$$

The corresponding equations for the canonical momenta p_α are the following, according to Eqs. (34) and (40):

$$\sum_{A,B=1}^{2n} X_A M_{AB}^{-1} \frac{\partial p_\alpha}{\partial \xi_B} = p_\alpha, \quad \alpha = 1, \dots, n. \quad (47)$$

According to the conditions (42), (43) and (44), Eqs. (47) finally have to be completed by the following equations

$$\sum_{A,B=1}^{2n} M_{AB}^{-1} \frac{\partial p_\alpha}{\partial \xi_A} \frac{\partial p_\beta}{\partial \xi_B} = 0, \quad \alpha, \beta = 1, \dots, n \quad (48)$$

and

$$\sum_{A,B=1}^{2n} M_{AB}^{-1} \frac{\partial q_\alpha}{\partial \xi_A} \frac{\partial p_\beta}{\partial \xi_B} = \delta_{\alpha\beta}, \quad \alpha, \beta = 1, \dots, n. \quad (49)$$

The construction of the canonical variables has then finally been reduced to a purely mathematical problem, namely to the solution of the groups of independent partial differential equations

(45), (46) and (47)–(49), respectively. The existence of solutions of these equations is a standard question of regularity conditions for the known functions M_{AB} and F_A (as well as the function Φ , which is at our disposal), and need not be elaborated upon further.

In conclusion we may state our results as follows:

The set of equations (21) are canonical Hamiltonian equations, which can be formulated in terms of a set of canonical coordinates and momenta, respectively, provided the matrix M_{AB} occurring in Eqs. (21) can be taken to define a Poisson bracket structure as given by Eq. (29). This question in turn is equivalent to that of the existence of solutions q_α and p_α to the partial differential equations (45), (46) and (47)–(49), respectively.

In any given problem, it may be as complicated to bother about the canonical structure in the manner discussed above as to solve the problem directly, using whatever means seem appropriate. However, in cases where the details of the canonical structure are of importance, the above method can yield fruitful insights.

We turn to the simple case of constant quantities M_{AB} in the next section. This apparently very simple case is here considered mainly as a preparation for the field theory model to be considered in the following section.

4 Constant Poisson Brackets as a Special Case

In this section we consider the case of *constant* (i.e. ξ -independent) quantities M_{AB} . Needless to say, we still assume the matrix (M_{AB}) to be *regular*. We then use the representation given in Eq. (14) (see also the Appendix),

$$M_{AB} = \sum_{\alpha=1}^n \lambda_\alpha (x_{\alpha A} y_{\alpha B} - x_{\alpha B} y_{\alpha A}). \quad (50)$$

Comparing the Eq. (30) with Eq. (50) above, one may put forward the hypothesis that

$$\frac{\partial p_\alpha}{\partial \xi_A} = C'_\alpha x_{\alpha A} \quad (51)$$

$$\frac{\partial q_\alpha}{\partial \xi_A} = C'_\alpha y_{\alpha A}, \quad (52)$$

where the constants C_α and C'_α , respectively, satisfy the condition

$$C_\alpha C'_\alpha = \lambda_\alpha. \quad (53)$$

We will show below that the quantities p_α and q_α defined by Eqs. (51) and (52) above do indeed satisfy all the equations (45)–(49).

5 Application to Self-Dual Fields in Two Dimensions

We now consider a self-dual field in a two-dimensional space-time, with dynamics specified by the Lagrangian given by Floreanini and Jackiw [1], which was referred to in the Introduction. The action functional is the following,

$$S = \int dt \left\{ \frac{1}{4} \int_{-L}^{+L} \int_{-L}^{+L} dx dy \dot{\chi}(x) \epsilon(x-y) \dot{\chi}(y) - \frac{1}{2} \int_{-L}^{+L} dx \chi^2(x) \right\}, \quad (60)$$

where we follow the notation of Floreanini and Jackiw, with the exception that we initially restrict the (one-dimensional) space to a finite interval $(-L, L)$. The time variable is suppressed in Eq. (60) and in what follows whenever expedient.

The variational equation following from extremizing the action (60) is the following,

$$\frac{1}{2} \int_{-L}^{+L} dy \epsilon(x-y) \dot{\chi}(y) = \chi(x). \quad (61)$$

The boundary conditions for the field χ , which follow from Eq. (61), are as follows:

$$\chi(L) + \chi(-L) = 0. \quad (62)$$

The integrand in Eq. (60) defines a first-order Lagrangian L^1 analogous to the one considered previously. Using the field equation (61) one can immediately identify the analogue of the matrix M_{AB} considered in Sections 3 and 4,

$$M_{AB} \longleftrightarrow M_{xy} = \frac{1}{2} \epsilon(x-y), \quad M_{xy}^{-1} = \delta'(x-y). \quad (63)$$

The field theory model considered here thus corresponds to the case of a *constant* (i.e. χ -independent) antisymmetric quantity M , but still requires a slight generalization of the formalism developed previously; the discrete indices $(A, B, \dots)$ simply have to be replaced by continuous ones $(x, y, \dots)$ and the summation by an integration whenever appropriate. Otherwise most of the considerations in Section 4 can be taken over almost verbatim.

In order to construct the canonical variables corresponding to the system defined by the action (60) we have to solve the following pair of eigenvalue equations, which are analogous to Eqs. (10) and (11) (we change the notation slightly: $x_{\alpha A} \rightarrow f_{\alpha}(x)$, $y_{\alpha A} \rightarrow g_{\alpha}(x)$),

$$\frac{1}{2} \int_{-L}^{+L} dy \epsilon(x-y) f_{\alpha}(y) = -\lambda_{\alpha} g_{\alpha}(x) \quad (64)$$

$$\frac{1}{2} \int_{-L}^{+L} dy \epsilon(x-y) g_{\alpha}(y) = +\lambda_{\alpha} f_{\alpha}(x). \quad (65)$$

The orthonormalized solutions of Eqs. (64) and (65), which are consistent with the boundary conditions (62), are the following,

$$f_{\alpha}(x) = \frac{1}{\sqrt{2L}} (\cos(k_{\alpha}x) + \sin(k_{\alpha}x)) \quad (66)$$

We first note that since the quantities $x_{\alpha A}$ and $y_{\alpha A}$ are constants, Eqs. (51) and (52) immediately lead to the following expressions,

$$p_{\alpha} = \sum_{A=1}^{2n} C_{\alpha} x_{\alpha A} \xi_A \quad (54)$$

$$q_{\alpha} = \sum_{A=1}^{2n} C'_{\alpha} y_{\alpha A} \xi_A. \quad (55)$$

and

In order to verify Eqs. (45)-(49) one now has to evaluate the quantity X_A defined by Eq. (33). Let us first note that an appropriate expression for the quantity F_A , which is related to the quantity M_{AB} by Eq. (32), is the following:

$$F_A = \frac{1}{2} \sum_{B=1}^{2n} \xi_B M_{BA} = \frac{1}{2} \sum_{B=1}^{2n} \sum_{\alpha=1}^n \lambda_{\alpha} \xi_B (x_{\alpha B} y_{\alpha A} - x_{\alpha A} y_{\alpha B}). \quad (56)$$

Choosing then the function Φ in Eq. (33), which is at our disposal, in the following manner,

$$\Phi(\xi) := \frac{1}{4} \sum_{\alpha, A, B} \lambda_{\alpha} (x_{\alpha A} y_{\alpha B} + x_{\alpha B} y_{\alpha A}) \xi_A \xi_B, \quad (57)$$

one finds the following expression for the quantity X_A from Eq. (33),

$$X_A = \sum_{\beta B} \lambda_{\beta} \xi_B p_{\beta B} y_{\beta A}. \quad (58)$$

Using then finally the expression for the inverse of the quantity (50),

$$M_{AB}^{-1} = - \sum_{\alpha=1}^n \lambda_{\alpha}^{-1} (x_{\alpha A} y_{\alpha B} - x_{\alpha B} y_{\alpha A}), \quad (59)$$

as well as the bi-orthogonality properties of the vectors x_{α} and y_{β} (Eqs. (A.7), (A.12) and (A.13)), it is a simple matter to verify that the expressions (54) and (55) indeed do satisfy Eqs. (45)-(49).

We may summarize the discussion as follows:

In the case of *constant* Poisson brackets M_{AB}^{-1} , the canonical momenta and coordinates can be taken to be the expressions (54) and (55), respectively, which are *linear* in the phase-space coordinates ξ_A . The quantities x_{α} and y_{α} occurring in these expressions are the eigenvectors solving the eigenvalue problem (10), (11). The problem of constructing the canonical variables has thus, for the case at hand, been reduced to a straightforward problem in linear algebra.

In the next section we apply (a slight generalization of) the above results, to the case of self-dual fields in a two-dimensional space-time.

and

$$g_\alpha(x) = \frac{1}{\sqrt{2L}} (\cos(k_\alpha x) - \sin(k_\alpha x)) \quad (67)$$

with

$$\lambda_\alpha = k_\alpha^{-1}, \quad k_\alpha = (2\alpha - 1) \frac{\pi}{2L}, \quad \alpha = 1, 2, \dots \quad (68)$$

The canonical momenta p_α and coordinates q_α are simply the following, according to Eqs. (54), (55) and the above results,

$$p_\alpha = \int_{-L}^L dx \sqrt{\lambda_\alpha} f_\alpha(x) \chi(x), \quad q_\alpha = \int_{-L}^L dx \sqrt{\lambda_\alpha} g_\alpha(x) \chi(x). \quad (69)$$

The inverse formula expressing the field χ in terms of the canonical variables is,

$$\chi(x) = \sum_{\alpha=1}^{\infty} \sqrt{k_\alpha} (p_\alpha f_\alpha(x) + q_\alpha g_\alpha(x)). \quad (70)$$

The Hamiltonian G , which immediately can be read off from Eq. (60), is then expressible in terms of the canonical variables,

$$G = \frac{1}{2} \int_{-L}^L dx \chi^2(x) = \sum_{\alpha=1}^{\infty} \frac{k_\alpha}{2} (p_\alpha^2 + q_\alpha^2). \quad (71)$$

The canonical variables obey the Poisson algebra,

$$\{q_\alpha, p_\beta\}_P = \delta_{\alpha\beta}, \quad \{q_\alpha, q_\beta\}_P = 0, \quad \{p_\alpha, p_\beta\}_P = 0. \quad (72)$$

It is a simple matter to obtain, e.g. the Poisson bracket of the field χ at different points in space,

$$\{\chi(x), \chi(y)\}_P = - \sum_{\alpha=1}^{\infty} \lambda_\alpha^{-1} (f_\alpha(x) g_\alpha(y) - f_\alpha(y) g_\alpha(x)) \equiv \delta'(x - y). \quad (73)$$

After all this machinery, it is easy to make a transition to the quantized version of the self-dual field by making the usual replacement of Poisson brackets by $-i$ times commutators (with $\hbar = 1$, of course). Denoting quantum operators by a caret, the quantum version of the Poisson algebra is then the following,

$$[\hat{q}_\alpha, \hat{p}_\beta] = i\delta_{\alpha\beta}, \quad [\hat{q}_\alpha, \hat{q}_\beta] = 0, \quad [\hat{p}_\alpha, \hat{p}_\beta] = 0. \quad (74)$$

Introducing new operators, $\hat{a}_\alpha$ and $\hat{a}_\alpha^\dagger$, by means of the following equations,

$$\hat{p}_\alpha = -\frac{1}{2} [(1+i)\hat{a}_\alpha + (1-i)\hat{a}_\alpha^\dagger], \quad \hat{q}_\alpha = -\frac{i}{2} [(1+i)\hat{a}_\alpha - (1-i)\hat{a}_\alpha^\dagger], \quad (75)$$

one can summarize the commutator algebra (74) as follows,

$$[\hat{a}_\alpha, \hat{a}_\beta^\dagger] = \delta_{\alpha\beta}. \quad (76)$$

The expansion of the field operator $\hat{\chi}$ at a fixed time ($t = 0$) can then be read off from Eq. (70),

$$\hat{\chi}(x) = \frac{-i}{\sqrt{2L}} \sum_{\alpha=1}^{\infty} \sqrt{k_\alpha} (\hat{a}_\alpha \exp(-ik_\alpha x) - \hat{a}_\alpha^\dagger \exp(+ik_\alpha x)). \quad (77)$$

Using the commutators (76) and the expansion (77) above, one immediately obtains the equal-time commutator,

$$[\hat{\chi}(x), \hat{\chi}(y)] = i\delta'(x - y), \quad (78)$$

which is the basic commutator given by Floreanini and Jackiw [1] and which is indeed the most natural commutator imaginable.

From Eq. (71) one finally gets the normal ordered quantum Hamiltonian $\hat{H}$

$$\hat{H} = \sum_{\alpha=1}^{\infty} k_\alpha (\hat{a}_\alpha^\dagger \hat{a}_\alpha) \quad (79)$$

We shall not further discuss the physical interpretation of the operator formalism of the self-dual field; this has been done (in the limit $L \rightarrow \infty$) with admirable clarity in the paper by Floreanini and Jackiw quoted previously.

6 Summary and Conclusions

In this paper we have given a systematic procedure for constructing unconstrained canonical variables for any system that is described by a first-order Lagrangian, which is linear in the (generalized) velocities. The circumstances under which there are constraints in the system are analysed, and it is shown that the constraints can always be eliminated (in principle) in order to obtain an unconstrained system. The conditions under which the unconstrained system is canonical are derived explicitly; using the language of forms, one may state the conditions as follows: the Lagrange one-form leads to an exact two-form, which occurs in the equations of motion, and which is invariantly related to the Lagrangian one-form. The system is canonical if and only if the components of the two-form in question can be identified with a Lagrange bracket of the system.

The Poisson structure of the system is thus defined directly by the form of the Lagrangian; there is no freedom in choosing Poisson brackets for the phase-space variables of the system.

The main part of the analysis has for simplicity been done only for a system with a finite number of degrees of freedom in this paper. In this case, the Lagrange bracket condition referred to above has been shown to lead to a set of independent partial differential equations for the canonical coordinates and momenta, respectively. These equations have coefficients that are determined partly by the Lagrangian and partly by what is essentially a gauge choice. This freedom is in turn related to the possibility of making contact transformations among the canonical variables.

The formalism is finally applied to the case of self-dual fields in a two-dimensional space-time; this requires a slight generalization of the formalism to a system with an infinite number of degrees of freedom. This generalization is straightforward for the system under consideration. The canonical structure of the self-dual field is exhibited explicitly, and then used as a stepping stone for the quantization of the system in question. The results obtained are in agreement with those obtained previously by Floreanini and Jackiw [1].

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Appendix A. Normal form of an antisymmetric matrix

It is well known [9] that any real-valued antisymmetric matrix can be brought into the following normal form, by a suitable basis transformation:

$$\begin{pmatrix} 0 & \lambda_1 & \dots & \dots & \dots & \dots & \dots \\ -\lambda_1 & 0 & \dots & \dots & \dots & \dots & \dots \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots \\ \dots & \dots & \dots & 0 & \lambda_n & \dots & \dots \\ \dots & \dots & \dots & -\lambda_n & 0 & \dots & \dots \\ \dots & \dots & \dots & \dots & \dots & 0 & \dots \\ \dots & \dots & \dots & \dots & \dots & \dots & 0 \end{pmatrix} \quad (\text{A.1})$$

The quantities λ_α in the normal form above are real, and can be taken to be positive without essential loss of generality. We indicate briefly in this Appendix the arguments leading to the normal form above.

Let (M_{AB}) be an arbitrary antisymmetric $N \times N$ matrix, with real-valued matrix elements. Consider the matrix (M_{AB}^2) ,

$$M_{AB}^2 = \sum_{C=1}^N M_{AC} M_{CB}. \quad (\text{A.2})$$

Since the matrix M^2 is self-adjoint (and negative semi-definite), there exists a set of (orthonormal) vectors $\{e_\nu\}_1^N$, which solve the following eigenvalue problem,

$$\sum_B M_{AB}^2 e_{\nu B} = -\lambda_\nu^2 e_{\nu A}, \quad \nu = 1, \dots, N \quad (\text{A.3})$$

with non-negative values for the quantities λ_ν^2 . In fact, the rank of M^2 is even (since the rank of M is even, M being antisymmetric), and therefore there is an even number $(2n)$ of non-zero eigenvalues λ_α^2 ,

$$\lambda_\alpha^2 > 0, \quad \alpha = 1, \dots, 2n. \quad (\text{A.4})$$

If $2n < N$ there are $N - 2n$ remaining eigenvalues that are zero,

$$\lambda_\alpha^2 = 0, \quad \alpha = 2n + 1, \dots, N. \quad (\text{A.5})$$

It can be shown that among the non-zero eigenvalues λ_α^2 ($\alpha = 1, \dots, 2n$) there can be at most n distinct ones. From the set $\{e_\alpha\}_1^{2n}$, we now select a subset of n eigenvectors $\{y_\alpha\}_1^n$, which corresponds to the maximum number of distinct eigenvalues λ_α^2 . Then, renumbering the quantities λ_α^2 appropriately, we have,

$$\sum_B M_{AB}^2 y_{\alpha B} = -\lambda_\alpha^2 y_{\alpha A}, \quad \alpha = 1, \dots, n. \quad (\text{A.6})$$

We take the set $\{y_\alpha\}_1^n$ to be orthonormalized,

$$(y_\alpha, y_\beta) := \sum_{A=1}^N y_{\alpha A} y_{\beta A} = \delta_{\alpha\beta}, \quad \alpha, \beta = 1, \dots, n. \quad (\text{A.7})$$

Without any essential loss of generality we take the quantities λ_α in (A.6) to be *positive*,

$$\lambda_\alpha > 0, \quad \alpha = 1, \dots, n. \quad (\text{A.8})$$

We then define a second set of vectors $\{x_\alpha\}_1^n$ by means of the following equations,

$$\sum_{B=1}^N M_{AB} y_{\alpha B} = +\lambda_\alpha x_{\alpha A}, \quad \alpha = 1, \dots, n. \quad (\text{A.9})$$

From the relations (A.6) and the definitions (A.9) it follows that

$$\sum_{B=1}^N M_{AB} x_{\alpha B} = -\lambda_\alpha y_{\alpha A}, \quad \alpha = 1, \dots, n \quad (\text{A.10})$$

and further that

$$\sum_B M_{AB}^2 x_{\alpha B} = -\lambda_\alpha^2 x_{\alpha A}, \quad \alpha = 1, \dots, n. \quad (\text{A.11})$$

Since M is antisymmetric, Eq. (A.10) implies that the vectors x_α and y_β are orthogonal,

$$(x_\alpha, y_\beta) := \sum_{A=1}^N x_{\alpha A} y_{\beta A} = 0, \quad \alpha, \beta = 1, \dots, n. \quad (\text{A.12})$$

Finally, from the orthonormality condition (A.7) follows that also the vectors x_α defined by Eq. (A.9) are orthonormal,

$$(x_\alpha, x_\beta) := \sum_{A=1}^N x_{\alpha A} x_{\beta A} = \delta_{\alpha\beta}, \quad \alpha, \beta = 1, \dots, n. \quad (\text{A.13})$$

The sets of vectors $\{x_\alpha, y_\alpha\}_1^n$ introduced above thus form a bi-orthogonal system, which (by construction) spans the space of eigenvectors $\{e_\alpha\}_1^{2n}$, corresponding to the non-zero eigenvalues λ_α^2 . The eigenvectors corresponding to vanishing eigenvalues satisfy the following equation:

$$\sum_B M_{AB}^2 e_{\nu B} = 0, \quad \nu = 2n + 1, \dots, N. \quad (\text{A.14})$$

It is convenient to introduce a separate notation for the eigenvectors e_ν satisfying Eq. (A.14),

$$z_\nu := e_{2n+\nu}, \quad \nu = 1, \dots, N - 2n. \quad (\text{A.15})$$

As an immediate consequence of Eqs. (A.14), the eigenvectors z_ν introduced above in Eq. (A.15) also satisfy the following equations,

$$\sum_{B=1}^N M_{AB} z_{\nu B} = 0, \quad \nu = 1, \dots, N - 2n. \quad (\text{A.16})$$

It is now readily verified that the matrix (M_{AB}) has the normal form (A.1) in a basis $\{a_\nu\}_1^N$, which is given as follows,

$$a_{2\nu-1} := x_\nu, \quad a_{2\nu} := y_\nu, \quad \nu = 1, \dots, n \quad (\text{A.17})$$

and

$$a_{2n+\nu} := z_\nu, \quad \nu = 1, \dots, N - 2n. \quad (\text{A.18})$$

In particular, if the matrix M is *regular*, then the normal form (A.1) can be expressed neatly in terms of the (complete set of) vectors x_α, y_α introduced above,

$$M_{AB} = \sum_{\alpha=1}^n \lambda_\alpha (x_{\alpha A} y_{\alpha B} - x_{\alpha B} y_{\alpha A}). \quad (\text{A.19})$$

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