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INSTITUTE OF THEORETICAL PHYSICS

50-204 Wrocław, Pl. Maxa Born 9, tel. 222363, fax 201467, telex 715623 UWRPL

Twisting Poincaré Algebras⁺)Jerzy Lukierski^{*)1),4),5)}, Anatol Nowicki^{**)2),5)},
Henri Ruegg^{*)4)} and Valeriy N. Tolstoy^{*)3),4)}^{*)}Dépt. de Physique Théorique, Université de Genève,
24, quai Ernest-Ansermet, 1211 Genève 4, Switzerland^{**)}Physikalisches Inst., Universität Bonn, Nussallee 12, 53115 Bonn, Germany

Abstract. We consider the twisting of the Hopf structure for the classical enveloping algebra $U(\hat{g})$, where $\hat{g}$ is an inhomogeneous rotation algebra, with explicit formulae given for $D = 4$ Poincaré algebra ($\hat{g} = \mathcal{P}_4$). We show that the quantum deformation of the Poincaré algebra recently proposed by Chaichian and Demiczev is a twisted classical Poincaré algebra. The interpretation of the twisted Poincaré algebra as describing relativistic symmetries with clustered 2-particle states is proposed.

- Presented by J. Lukierski.

¹⁾ On leave of absence from the Institute for Theoretical Physics, University of Wrocław, pl. Maxa Born 9, 50-204 Wrocław, Poland.

²⁾ On leave of absence from the Institute of Physics, Pedagogical University, Pl. Śłowiański 6, 65-029 Zielona Góra, Poland.

³⁾ On leave of absence from Institute of Nuclear Physics, Moscow State University, 119899 Moscow, Russia.

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1. Introduction

Let us consider Poincaré algebra $\mathcal{P}_4$ with the generators $\hat{g} = (P_\mu, M_{\mu\nu})$ as a classical Hopf algebra. We supplement the well-known algebraic relations

$$\begin{aligned} [M_{\mu\nu}, M_{\rho\tau}] &= i(g_{\mu\tau}M_{\nu\rho} - g_{\nu\tau}M_{\mu\rho} + g_{\nu\rho}M_{\mu\tau} - g_{\mu\rho}M_{\nu\tau}) , \\ [M_{\mu\nu}, P_\rho] &= i(g_{\nu\rho}P_\mu - g_{\mu\rho}P_\nu) , \quad [P_\mu, P_\nu] = 0 \end{aligned} \quad (1.1)$$

by the “primitive” coproduct relations

$$\Delta_0(M_{\mu\nu}) = M_{\mu\nu} \otimes 1 + 1 \otimes M_{\mu\nu} , \quad \Delta_0(P_\mu) = P_\mu \otimes 1 + 1 \otimes P_\mu \quad (1.2)$$

and the antipode $S_0(\hat{g}) = -\hat{g}$ ($\hat{g} \in \mathcal{P}_4$). The relations (1.1) lead to (1.1), (1.2) preserving the Hopf algebra structure (for general framework see e.g.[3,4]). In this paper we would like to consider the mildest quantum deformations of (1.1), (1.2) obtained by the twisting procedure [5-9]. Following Drinfeld [5] two Hopf algebras $\mathcal{A} = (A, \Delta, S, \varepsilon)$ and $\mathcal{A}^F = (A, \Delta^F, S^F, \varepsilon)$ are related by twisting if there exists an invertible function $F = \sum_i f_i \otimes f^i \in \mathcal{A} \otimes \mathcal{A}^{(1)}$ satisfying the “cocycle” condition [5,7,8]

$$F_{23}(1 \otimes \Delta)F = F_{12}(\Delta \otimes 1)F \quad (1.3)$$

and $(\varepsilon \otimes 1)F = (1 \otimes \varepsilon)F = 1$. In such a case Δ^F and Δ are related as follows $(a \otimes b \cdot c \otimes d = ac \otimes bd)$

$$\Delta^F(a) = F \cdot \Delta(a) \cdot F^{-1} \quad (1.4a)$$

Introducing $U = \sum_i f_i \cdot S(f^i)$ one obtains also that

$$S^F(a) = US(a)U^{-1} \quad (1.4b)$$

If $\mathcal{A}$ is the quasitriangular Hopf algebra and the relations (1.3) are replaced by [5,6]

$$(\Delta \otimes 1)F = F_{13}F_{23} , \quad (1 \otimes \Delta)F = F_{13}F_{12} \quad (1.5)$$

the universal R -matrices for $\mathcal{A}$ and $\mathcal{A}^F$ are related by the formulae ($\tilde{F} = \sigma \cdot F = \sum_i f^i \otimes f_i$)

$$R^F = F^{-1} \cdot R \cdot \tilde{F} \quad (1.6)$$

For the complex simple Lie algebras $\hat{g}$ there were considered twistings described by

$$F = \exp f \quad f \in \hat{c} \otimes \hat{c} \quad (1.7)$$

¹Strictly speaking we consider below F belonging to an extension of $\mathcal{A} \otimes \mathcal{A}$.

where $\hat{c}$ is the commutative subalgebra of $\hat{g}$ (Cartan subalgebra in [6], Borel subalgebra in [8]). Indeed it is easy to check that if $f \in \hat{c} \otimes \hat{c}$, and $\hat{c}$ is abelian, the conditions (1.5) are valid.

In this lecture we shall consider the twisting of physically important case of the D=4 Poincaré algebra $\hat{g} = O(3,1) \ni T_4$. In such nonsimple algebras one can select the commutative subalgebra C_m in several ways, e.g.

- a) Cartan subalgebra $(h_1, \dots, h_n)$ ($n = \frac{D}{2}$ for D even, $n = \frac{D-1}{2}$ for D odd)
- b) Translation generators $(P_1 \dots P_D)$
- c) “Mixed” Cartan-translation algebra C_k $k \leq \frac{N}{2}$

$$C_k = (h_1 \dots h_k, P_{2k+1} \dots P_D) \quad (1.8)$$

We shall consider in explicit way the twisted D=4 Poincaré algebras $U^F(\mathcal{P}_4)$ with the choice of the algebra $\hat{c}$ (see (1.7)) described by the formula (1.8) with $k = 0, 1, 2$, and as an example we present twisted Poincaré algebra the quantum Poincaré algebra considered recently by Chaichian and Demiczev [10]. Further we shall discuss the elements of the representation theory of twisted Poincaré algebras, and present an outlook: some generalizations as well unsolved problems.

2. Twisting of the classical Poincaré algebra.

2.1 General scheme.

Let us denote the basis of the commutative algebra $\hat{c}$ ($F \in \hat{c} \otimes \hat{c}$) by $(c_1 \dots c_n)$. We define

$$F = F_+ F_- \quad F_\pm = \exp f_\pm \quad (2.1)$$

where $f_\pm = \pm \sigma \cdot f$ (σ is the exchange map: $\sigma(c_i \otimes c_j) = c_j \otimes c_i$), and

$$f^{(\pm)} = \frac{1}{2} \alpha_{ij}^{(\pm)} (c_i \otimes c_j \pm c_j \otimes c_i) \quad (2.2)$$

i.e. one can assume that $\alpha_{\pm ij} = \pm \alpha_{\mp ji}$.

If we twist the coproducts of classical Lie algebra we obtain from the commutativity of $\hat{c}$ that

$$U = \sum f_i \otimes S(f^i) = \exp(-\alpha_{+ij} c_i c_j) \quad (2.3)$$

and after using (1.6) the R -matrix takes the particular form:

$$R = \exp(-2f_-) = (F_-)^{-2} \quad (2.4)$$

The formulae for the coproduct Δ^F depend on the particular choice of the algebra $\hat{c}$. We shall further specify our algebra for the case of classical Poincaré algebra (1.1). One can consider the following four types of the twist function:
(i) $\hat{c} = (M_3 = M_{12}, N_3 = M_{30})$. We postulate

$$f_+ = \alpha_+ M_3 \otimes M_3 + \beta_+ (M_3 \otimes N_3 + N_3 \otimes M_3) + \gamma_+ N_3 \otimes N_3 ,$$

$$f_- = \beta_- (M_3 \otimes N_3 + N_3 \otimes M_3) \quad (2.5)$$

(ii) $\hat{c} = (M_3 = M_{12}, P_3, P_0)$ We assume that $(r, s = 3, 0)$

$$f_+ = \alpha_+ M_3 \otimes M_3 + \delta_+^r (M_3 \otimes P_r + P_r \otimes M_3) + \rho_+^{rs} P_r \otimes P_s ,$$

$$f_- = \delta_-^r (M_3 \otimes P_r + P_r \otimes M_3) . \quad (2.6)$$

(iii) $\hat{c} = (P_1, P_2, N_3 = M_{30})$. Putting $(a, b=1, 2)$

$$f_+ = \rho_+^{ab} P_a \otimes P_a + \xi_+^a (N_3 \otimes P_a + P_a \otimes N_3) + \gamma_+ N_3 \otimes N_3 ,$$

$$f_- = \xi_-^a (N_3 \otimes P_a + P_a \otimes N_3) \quad (2.7)$$

where $a, b=1, 2$. (iv) $\hat{c} = (P_1, P_2, P_3, P_0)$ One can write $(\rho_{\pm}^{\mu\nu} = \pm \rho_{\pm}^{\nu\mu})$

$$f_+ = \rho_+^{\mu\nu} (P_\mu \otimes P_\nu + P_\nu \otimes P_\mu) , \quad f_- = \rho_-^{\mu\nu} (P_\mu \otimes P_\nu - P_\nu \otimes P_\mu) . \quad (2.8)$$

Because the split Casimir

$$C_2^{\text{split}} \equiv \Delta(P_\mu P^\mu) - P_\mu P^\mu \otimes 1 - 1 \otimes P_\mu P^\mu = 2P_\mu \otimes P^\mu \quad (2.9)$$

commutes with $\Delta(\hat{a})$ for any $\hat{a} \in U(\mathcal{P}_4)$, one can assume further that $\rho_+^{\mu\nu} \eta_{\mu\nu} = \rho_-^{\mu\nu} \eta_{\mu\nu} = 0$ ($\eta_{\mu\nu} = \text{diag}(1, -1, -1, -1)$).

In general we assume that the Poincaré algebra is the complex one, and the twist function parameters are also complex. The reality condition imposed on the Poincaré generators imply the reality conditions for the coefficient in the formulae (2.5)-(2.8). For simplicity we shall consider the last example of the twist function, given by (2.8). It is known that if the real structure is an antihomomorphism in the algebra sector, still one can impose on the generators of twisted Poincaré algebra two types of reality conditions [10, 14]:

a) Standard one, denoted in [14] by $+$. In the case one obtains

$$(\Delta^F(M_{\mu\nu}))^+ = \Delta^F(M_{\mu\nu}) \implies \alpha^{\rho\tau} \text{ real} \quad (2.10a)$$

b) nonstandard one, used e.g. in [15], and denoted in [10] by $\oplus$. In such a case

$$(\Delta^F(M_{\mu\nu}))^\oplus = \Delta^F(M_{\mu\nu}) \implies \alpha^{\rho\tau} = (\alpha^{\tau\rho})^* \quad (2.10b)$$

i.e. the matrix $\alpha \equiv (\alpha^{\rho\tau})$ is Hermitean.

In this Section we considered classical twisted Poincaré algebras, parametrized by multiparameter twist functions. These Hopf algebras by duality relations determine multiparameter deformations of the functions of the Poincaré group. Using the duality relation between multiplication and comultiplication

$$\langle a \cdot b, c \rangle = \langle a \otimes b, \Delta(c) \rangle \quad (2.11)$$

one sees easily that all the antisymmetric contributions to the twisted coproducts lead to noncommutativity of the generators of the corresponding dual quantum Poincaré group.

It is an interesting exercise to classify the quantum Poincaré groups dual to the classical twisted Poincaré algebras.

2.2 An example: Chaichian-Demichev quantum Poincaré algebra

We shall show that the example of q -Poincaré algebra given in [10] is isomorphic as a Hopf algebra to twisted classical Poincaré algebra. We shall describe firstly the complexified classical Lorentz algebra $SO(4; C) = SO(3; C) \oplus SO(3; C)$ as follows:

$$[e_i, e_{-j}] = \delta_{ij} , \quad h_i[h_i, h_j] = 0 , \quad [h_i, e_{\pm j}] = \pm 2\delta_{ij} e_{\pm j} , \quad (2.12)$$

where (e_1, e_{-1}, h_1) and (e_2, e_{-2}, h_2) describe two $O(3; C)$ sectors.

We introduce

$$L_1 = e_{-1} , \quad L_2 = e_{-2} , \quad L_5 = \frac{1}{2}(h_1 + h_2) ,$$

$$L_3 = e_{+2} , \quad L_4 = e_{+1} , \quad L_6 = \frac{1}{2}(h_2 - h_1) \quad (2.13)$$

and perform the twist of the “primitive” coproduct by means of the twist function

$$F = q^{h_2 \otimes h_1} = q^{(L_5 + L_6) \otimes (L_5 - L_6)} \quad (2.14)$$

One gets $(\Delta^F(L_a) = F \cdot \Delta(L_a) \cdot F^{-1})$

$$\Delta^F(L_1) = L_1 \otimes I + q^{-2(L_5 + L_6)} \otimes L_1 .$$

$$\begin{aligned}\Delta^F(L_2) &= I \otimes L_2 + L_2 \otimes q^{-2(L_5-L_6)} , \\ \Delta^F(L_3) &= I \otimes L_3 + L_3 \otimes q^{2(L_5-L_6)} ,\end{aligned}\quad (2.15)$$

$$\Delta^F(L_5) = \Delta(L_5) , \quad \Delta^F(L_6) = \Delta(L_6) .$$

Introducing

$$\begin{aligned}\tilde{L}_1 &= L_1 , & \tilde{L}_2 &= q^{-2} L_2 q^{-2(L_5-L_6)} , & \tilde{L}_3 &= q^{-2} L_3 q^{2(L_5-L_6)} \\ \tilde{L}_4 &= L_4 , & \tilde{L}_5 &= L_5 , & \tilde{L}_6 &= L_6\end{aligned}\quad (2.16)$$

one can identify the transformed classical Lorentz algebra with the q -deformed Lorentz algebra proposed in [10], in algebra as well as coalgebra sector. Modifying the standard fourmomentum operators as follows

$$\begin{aligned}\tilde{P}_1 &= q^{L_5-L_6} P_1 , & \tilde{P}_2 &= q^{L_5-L_6} P_2 \\ \tilde{P}_3 &= q^{L_6-L_5} P_3 , & \tilde{P}_4 &= q^{L_6-L_5} P_4\end{aligned}\quad (2.17)$$

we obtain the relations

$$\begin{aligned}[\tilde{P}_1, \tilde{P}_2]_{q^2} &= [\tilde{P}_4, \tilde{P}_1]_{q^2} = [\tilde{P}_2, \tilde{P}_3]_{q^2} = [\tilde{P}_3, \tilde{P}_4]_{q^2} = 0 , \\ [\tilde{P}_1, \tilde{P}_3] &= [\tilde{P}_2, \tilde{P}_4] = 0\end{aligned}\quad (2.18)$$

and the coproducts

$$\begin{aligned}\Delta^F(\tilde{P}_1) &= \tilde{P}_1 \otimes 1 + q^{-2L_6} \otimes \tilde{P}_1 , & \Delta^F(\tilde{P}_2) &= \tilde{P}_2 \otimes 1 + q^{2L_5} \otimes \tilde{P}_2 , \\ \Delta^F(\tilde{P}_3) &= \tilde{P}_3 \otimes 1 + q^{2L_6} \otimes \tilde{P}_3 , & \Delta^F(\tilde{P}_4) &= \tilde{P}_4 \otimes 1 + q^{-2L_5} \otimes \tilde{P}_4 .\end{aligned}\quad (2.19)$$

The relations (2.18) and (2.19) describe the translation sector of Chaichian–Demichev quantum algebra.

Let us recall that recently the quantum Lorentz groups have been classified by Woronowicz and Zakrzewski [11], where besides the Drinfeld–Jimbo parameter q a new parameter t has been introduced. It can be shown that the quantum deformation, proposed by Chaichian and Demichev corresponds to $q = 1$. This condition as the necessary requirement for the existence of nontrivial quantum deformation of Poincaré algebra, with the Lorentz part as the Hopf subalgebra, has been obtained in [12] (see also [13]).

It should be stressed that in [11] there were given also other examples of the quantum deformations of the Lorentz group, which satisfy the condition $q = 1$ and can be extended to the quantum deformations of the Poincaré algebra without supplementing an eleventh dilatation generator. It would be interesting to prove the conjecture that all quantum deformations of Poincaré algebra which do have the deformed Lorentz algebra as its Hopf subalgebra are classical twisted Poincaré algebras.

3. Discussion

i) Twisted Poincaré algebra from the contraction of $U_q(O(4, 2))$.

In a recent paper [15] two of the present authors proposed the contraction of $U_q(O(4, 2))$ to quantum Poincaré algebra. It can be shown that the result of the contraction is a twisted Poincaré algebra with the twist function depending on the fourmomenta and one central charge Z , obtained from the contraction of the dilatation generator in the conformal algebra.

ii) Nonabelian choice of twist functions.

It is interesting to consider more general classes of twisting functions, with F spanned by nonabelian sectors of the algebra. In particular such a twisting function is provided by the universal R -matrix, which interchanges two noncocommutative coproducts Δ and $\Delta' = \sigma \cdot \Delta$ of a quantum algebra. It is known that for Drinfeld–Jimbo deformations $U_q(\hat{g})$ of simple Lie algebras the universal R -matrix can be decomposed into the product [16,17]

$$R = \prod_{\alpha \in \Delta^{(+)}} R_\alpha \cdot K \quad (3.1)$$

where

$$R_\alpha = \exp q_\alpha(a_\alpha(q)e_\alpha \otimes e_{-\alpha}) \quad (3.2)$$

and K depends only on the Cartan generators. It appears that any component (3.2) of the product (3.1) can be used as a twist function F [18]. It is interesting to find nontrivial twist functions for the quantum κ -Poincaré algebra, proposed in [19,20]. Because the universal R -matrix for the κ -Poincaré algebra is not known, the type of twisting proposed in [18] cannot be applied.

iii) Finally we would like to mention that the expanded version of this conference report is presented in [21].

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