Analytic Bethe Ansatz
for
Fundamental Representations of Yangians*

by

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Abstract

We study the analytic Bethe ansatz in solvable vertex models associated with the Yangian $Y(X_r)$ or its quantum affine analogue $U_q(X^{(1)}_r)$ for $X_r = B_r, C_r$ and $D_r$. Eigenvalue formulas are proposed for the transfer matrices related to all the fundamental representations of $Y(X_r)$. Under the Bethe ansatz equation, we explicitly prove that they are pole-free, a crucial property in the ansatz. Conjectures are also given on higher representation cases by applying the $T$-system, the transfer matrix functional relations proposed recently. The eigenvalues are neatly described in terms of Yangian analogues of the semi-standard Young tableaux.

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1. Introduction

1.1 General remarks. Among many studies on solvable lattice models, the Bethe ansatz is one of the most successful and widely applied machinery. It was invented at very dawn of the field [1] and is still providing rich insights. Meanwhile, original Bethe’s idea has evolved into several versions of Bethe ansätze called with the adjectives as ‘thermodynamic’ [2], ‘algebraic’ [3], ‘analytic’ [4,5], ‘functional’ [6] and so forth. These are all powerful techniques that involve some deep aspects. We are yet to understand their full contents, a challenge raised on Feynman’s ‘last blackboard’ [7].

In this paper we step forward to it by developing our recent works [8,9,10,11] further. We shall propose eigenvalue formulas for several transfer matrices in the models with the Yangian symmetry [12] or its quantum affine analogue [13,14,15]. An interesting interplay will thereby be exposed between the representation theory of these algebras and the analytic Bethe ansatz. Let us explain our basic setting of the problem.

1.2 Yang-Baxter equation and transfer matrices. Consider the quantum affine algebra $U_q(X_r^{(1)})$ [13,14] associated with any classical simple Lie algebra $X_r$ of rank $r$. Throughout the paper we assume that $q$ is generic. Let $W_m^{(a)} (1 \leq a \leq r, m \in \mathbb{Z}_{\geq 1})$ be the irreducible finite dimensional $U_q(X_r^{(1)})$-module as sketched in section 2.1. See also [16] and [8]. For $W, W' \in \{W_m^{(a)} \mid 1 \leq a \leq r, m \in \mathbb{Z}_{\geq 1}\}$, let $R_{W, W'} (u) \in \text{End}(W \otimes W')$ denote the quantum $R$-matrix satisfying the Yang-Baxter equation [17]

$$R_{W, W'} (u)R_{W, W'} (u + v)R_{W', W'} (v) = R_{W', W'} (v)R_{W, W'} (u + v)R_{W, W'} (u). \quad (1.1)$$

Here, $u, v \in \mathbb{C}$ denote the spectral parameters and $R_{W, W'} (u)$ is supposed to act as identity on $W^u$, etc. As is well known, one has a solvable vertex model on planar square lattice by regarding the matrix elements of the $R$-matrix as local Boltzmann weights. For $R_{W, W'} (u)$, the vertices take dim$W$-states (resp. dim$W'$-states) on, say, horizontal (resp. vertical) edges. The row-to-row transfer matrix under the periodic boundary condition is defined by

$$T_m^{(a)} (u) = \text{Tr}_{W_m^{(a)}} (R_{W_m^{(a)}, W_p} (u - w_1) \cdots R_{W_m^{(a)}, W_p} (u - w_N)) \quad (1.2)$$

up to an overall scalar multiple. Here $N$ is the system size, $w_1, \ldots, w_N$ are complex parameters representing the inhomogeneity, $1 \leq a, p \leq r$ and $m, s \in \mathbb{Z}_{\geq 1}$. Following the QISM terminology [3], we say that (1.2) is the row-to-row transfer matrix with the auxiliary space $W_m^{(a)}$ that acts on the quantum space $(W_p^{(a)})^{\otimes N}$. (More precisely, $W_m^{(a)} (u)$ and $\otimes_{j=1}^N W_p^{(a)}(w_j)$, respectively. See section 2.1.) Note that in (1.2) we have suppressed the quantum space dependence on the lhs. Thanks to the Yang-Baxter equation (1.1), the transfer matrices form a commuting family

$$[T_m^{(a)} (u), T_m^{(a')} (u')] = 0. \quad (1.3)$$

They can be simultaneously diagonalized and we shall write their eigenvalues as $\Lambda_m^{(a)} (u)$, which is also dependent on $p$ and $s$. Our aim is to find an explicit formula for them. So far, the full answer is known only for $X_r = A_r$ [18,19] and $X_r = C_r$ [10]. In this paper we extend the results in [20,21] for $X_r = B_r, C_r$ and $D_r$ further by combining the two basic
ingredients, the analytic Bethe ansatz [5] and the transfer matrix functional relations (T-system) [8,9]. Our approach renders a new insight into the base structure of the module $W_m^{(a)}$ and leads to several conjectures on $\Lambda_m^{(a)}(u)$. Below we shall illustrate our idea along an exposition of the analytic Bethe ansatz (section 1.3) and the T-system (section 1.4) for the simplest example $X_r = sl(2)$.

1.3 Analytic Bethe ansatz. Let us concentrate on the $X_r = sl(2)$ case in this and the next subsections. We write $T_m(u)$ for $T_m^{(1)}(u)$, etc. since the rank of $sl(2)$ is 1. Then $W_m$ denotes the $(m + 1)$-dimensional irreducible representation of $U_q(sl(2))$. For simplicity, we assume that $s = 1$ in (1.2). Then $T_1(u)$ is just the 6-vertex model transfer matrix acting on the vectors labeled by length $N$ sequences of $+$ or $-$ states. We take the local vertex Boltzmann weights as $R_u(\pm, \pm, \pm, \pm) = [2 + u]$, $R_u(\pm, \mp, \mp, \pm) = [u]$ and $R_u(\pm, \mp, \mp, \mp) = [2]$, where the local states $+$ or $-$ are ordered anti-clockwise from the left edge of the vertex. The function $[u]$ is defined by

$$[u] = \frac{q^u - q^{-u}}{q - q^{-1}}.$$  

The eigenvalue $\Lambda_1(u)$ is well known and given by

$$\Lambda_1(u) = \frac{Q(u - 1)}{Q(u + 1)} \phi(u + 2) + \frac{Q(u + 3)}{Q(u + 1)} \phi(u),$$  

$$Q(u) = \prod_{j=1}^{n} [u - iu_j], \quad \phi(u) = \prod_{j=1}^{N} [u - w_j].$$  

Here, $0 \leq n \leq N/2$ is the number of the $-$ states in the eigenvector, which is preserved under the action of $T_1(u)$. $w_j \in \mathbb{C}$ are any solution of the Bethe ansatz equation (BAE)

$$\frac{-\phi(iu_k + 1)}{\phi(iu_k - 1)} = \frac{Q(iu_k + 2)}{Q(iu_k - 2)}.$$  

On the result (1.4-5), one makes a few observations.

(i) The eigenvalue has the “dressed vacuum form (DVF)”, which means the following. The “vacuum vector” $++, +, \ldots, +$ is the obvious eigenvector with the vacuum eigenvalue

$$\prod_{j=1}^{N} R_{u-w_j}(++, +, +, +) + \prod_{j=1}^{N} R_{u+w_j}(-, +, -, +) = \phi(u + 2) + \phi(u).$$  

(ii) The BAE (1.5) ensures that the eigenvalues are free of poles for finite $u$. The apparent pole at $u = iu_k - 1$ in (1.4a) is spurious as the residues from the two terms cancel due to (1.5). The eigenvalues must actually be pole-free because the local Boltzmann weight, hence the matrix elements of $T_1(u)$ are so.
(iii) Properties inherited from the asymptotic behavior in $|u| \to \infty$ and the first/second inversion relations of the $R$-matrix (vertex Boltzmann weights). For example, one has $\Lambda_1(u) = (-)^N \Lambda_1(-2-u)|_{w_j \to -w_j, u \to -u}$ form the last property. See also the remark after (2.12).

The analytic Bethe ansatz is the hypothesis that the postulates (i)-(iii) essentially determine a function of $u$ uniquely and that the so obtained is the actual transfer matrix eigenvalue. As the input data, it only uses the BAE and the $R$-matrix (or the vacuum eigenvalue (1.6)) which should be normalized to be an entire function of $u$. It was formulated in [5] by extracting the idea from Baxter’s solution of the 8-vertex model [4]. See [10,11,20,21] for other applications. In section 2.4, we will introduce a few more conditions than (i)-(iii) above.

1.4 Transfer matrix functional relations. The transfer matrix (1.2) obeys various functional relations. For $X_r = sl(2)$ and $s = 1$ in (1.2), it is known that [18,22]

$$T_m(u + 1)T_m(u - 1) = T_{m+1}(u)T_{m-1}(u) + g_m(u)\text{Id},$$

$$g_m(u) = \prod_{k=0}^{m-1} \phi(u + 2k - m)\phi(u + 4 + 2k - m),$$

where $m \geq 0$ and $T_0(u) = \text{Id}$. Since $T_m(u)$’s can be simultaneously diagonalized, (1.7) may be regarded as an equation for the eigenvalues $\Lambda_m(u)$. By using (1.4a) and $\Lambda_0(u) = 1$ as the initial condition, it is easy to solve the recursion (1.7) to find

$$\Lambda_m(u) = \left( \prod_{k=1}^{m-1} \phi(u + m + 1 - 2k) \right) \sum_{j=0}^{m} \frac{Q(u - m)Q(u + m + 2)\phi(u + m + 1 - 2j)}{Q(u + m - 2j)Q(u + m + 2 - 2j)},$$

in agreement with [18]. To observe a representation theoretical content, we now set

$$[1] = \frac{Q(u - 1)}{Q(u + 1)}\phi(u + 2), \quad [2] = \frac{Q(u + 3)}{Q(u + 1)}\phi(u),$$

where we assume on the lhs that the spectral parameter $u$ is implicitly attached to the single box as well. In this notation (1.4a) reads as $\Lambda_1(u) = [1] + [2]$. Moreover, the result (1.8) for general $m$ can be expressed as follows.

$$\Lambda_m(u) = \sum_{j=0}^{m} \left[ \underbrace{[1] \cdots [1]}_{m-j} \underbrace{[2] \cdots [2]}_{j} \right].$$

Here we interpret the tableau as the product of the $m$ functions (1.9) with the spectral parameter $u$ shifted to $u - m + 1, u - m + 3, \ldots, u + m - 1$ from the left to the right. Notice that the tableaux appearing in (1.10) are exactly the semi-standard ones that label the weight vectors in the $(m+1)$-dimensional irreducible representation $W_m$ of $U_q(\hat{sl}(2))$ (plainly, the spin $\frac{m}{2}$ representation of $sl(2)$). In this sense the eigenvalues $\Lambda_m(u)$ are analogues (“Yang-Baxterizations”) of the characters of the auxiliary space $W_m$, which
may be natural from (1.2). The functional relation (1.7) for $\Lambda_m(u)$ thereby plays the role of a character identity.

1.5 General $X_r$ case. Having seen the $sl(2)$ example, an immediate question then would be, how the “tableau construction” of the eigenvalues as (1.10) can be generalized to the other algebra cases. For $X_r = A_r$, the answer has been given in [19] for the RSOS models [23], which essentially includes (1.10) for $r = 1$. In this case, the $U_q(A_r)\text{-module} W_m^{(a)}$ (the auxiliary space) is a $q$-analogue of the $sl(r + 1)$-module corresponding to the $a \times m$ rectangular Young diagram representation. The eigenvalue $\Lambda_m^{(a)}(u)$ is constructed as in (1.10) from the set of the usual semi-standard tableaux labeling the weight vectors.

An interesting feature emerges for $X_r \neq A_r$ where $U_q(X_r)\text{-module} W_m^{(a)}$ is a $q$-analogue of a reducible $X_r$-module in general. Evaluation of $\Lambda_m^{(a)}(u)$ amounts to finding the tableau-like objects that label the base of such $W_m^{(a)}$. This can actually be done by postulating the $T$-system, the transfer matrix functional relations, proposed in [8]. It is a generalization of (1.7) into arbitrary $X_r$ case and can be solved for $\Lambda_m^{(a)}(u)$ in terms of $\Lambda_1^{(a)}(u + \text{shift})(1 \leq a \leq r)$ and $\Lambda_0^{(a)}(u) = 1$. Thus one can play the following game.

Step 1. Find $\Lambda_1^{(1)}(u), \ldots, \Lambda_1^{(r)}(u)$ by the analytic Bethe ansatz.

Step 2. Solve the $T$-system for $\Lambda_m^{(a)}(u)$ recursively by taking the step 1 result as the initial condition.

Step 3. Find such “tableaux” that the step 2 result is expressed, in an analogous sense to (1.10), as

$$\Lambda_m^{(a)}(u) = \sum \text{tableau}(u).$$

We shall execute the above program in a number of cases for $X_r = B_r, C_r$ and $D_r$. The resulting tableau label for the base of $W_m^{(a)}$ exhibits an interesting contrast with those for the crystal base [24,25] concerning the irreducible $X_r$-modules. We find in several cases that the DVF (1.11), hence the base of $W_m^{(a)}$, can also be labeled by semi-standard-like tableaux obeying remarkably simple rules.

1.6 Plan of the paper. In the next section, we begin by fixing our notations and recall the family of the modules $W_m^{(a)}$, the $T$-system [8] and the BAE [26,21] for models with $U_q(X_r^{(1)})$ symmetry. The Yangian case $Y(X_r)$ corresponds to a smooth rational limit $q \to 1$ of them. Then we discuss the analytic Bethe ansatz and propose a few more hypotheses, “dress universality”, “top term” and “coupling rule”. They supplement (i)-(iii) in section 1.3 and work efficiently for models with general $U_q(X_r^{(1)})$ symmetry. Sections 3, 4 and 5 are devoted to the cases $X_r = C_r, B_r$ and $D_r$, respectively. A peculiarity for the latter two algebras is the presence of the spin representations, whose $U_q(X_r^{(1)})$-analogues are certainly the members of the family \{ $W_m^{(a)} | 1 \leq a \leq r, m \in \mathbb{Z}_{\geq 1}$ \}. ($W^{(r)}_1$ for $B_r$ and $W^{(r-1)}_1, W^{(r)}_1$ for $D_r$.) For these algebras, we introduce two kinds of elementary boxes corresponding to the bases of the vector and the spin representations. We clarify their relation reflecting the fact that the former representation is contained in a tensor product of the latter. These features are quite similar for $B_r$ and $D_r$ cases, hence we shall omit many details for the latter. Section 6 gives the summary and discussion.
2. T-system, BAE and analytic Bethe ansatz

2.1 Modules $W_m^{(a)}$. Let us fix our notations for the data from the simple Lie algebras $X_r$. Let $\alpha_a, \omega_a (1 \leq a \leq r)$ and $(\cdot \mid \cdot)$ denote the simple roots, the fundamental weights and the invariant bilinear form on $X_r$. We identify the Cartan subalgebra and its dual via $(\cdot \mid \cdot)$ and normalize it as $(\alpha \mid \alpha) = 2$ for $\alpha$ = long root. Put

$$t_a = \frac{2}{(\alpha_a \mid \alpha_a)}, \quad 1 \leq a \leq r,$$

$$g = \text{dual Coxeter number of } X_r.$$

By the definition $t_a = 1, 2$ or 3 and $(\omega_a \mid \omega_b) = \delta_{ab}/t_a$. Enumeration of the nodes $1 \leq a \leq r$ on the Dynkin diagram is same with Table 1 in [8]. For $X_r = B_r (r \geq 2), C_r (r \geq 2)$ and $D_r (r \geq 4)$, (2.1) reads explicitly as

$$t_1 = \cdots = t_{r-1} = 1, t_r = 2 \text{ for } B_r,$$

$$t_1 = \cdots = t_{r-1} = 2, t_r = 1 \text{ for } C_r,$$

$$\forall t_a = 1 \text{ for } D_r,$$

$$g = \begin{cases} 
2r - 1 & \text{for } B_r \\
2r & \text{for } C_r \\
2r - 2 & \text{for } D_r 
\end{cases}$$

(2.2)

Now we recall the family of modules $\{W_m^{(a)} \mid 1 \leq a \leq r, m \in \mathbb{Z}_{\geq 1}\}$ first introduced in [16] for the Yangian $Y(X_r)$ extending the earlier examples [26]. Precisely speaking, Yangian modules carry a spectral parameter hence the auxiliary and the quantum spaces in (1.2) are to be understood as $W_m^{(a)}(u)$ and $\otimes_{j=1}^N W_n^{(p)}(w_j)$, respectively. See [27,28] and section 3.2 in [8]. Then $W_m^{(a)}(u)$ has a characterization by the Drinfel’d polynomials [27,28] $\{P_a(v) \mid 1 \leq a \leq r\}$ as

$$P_b(v) = \begin{cases} 
(v - u + \frac{m-1}{t_a})(v - u + \frac{m-3}{t_a}) \cdots (v - u + \frac{m-1}{t_a}) & \text{for } b = a \\
1 & \text{otherwise}.
\end{cases} \quad (2.3)$$

In [28], $W_1^{(a)}(u) (1 \leq a \leq r)$ is called the fundamental representation of $Y(X_r)$. Viewed as a module over $X_r \subset Y(X_r)$, $W_1^{(a)}(u)$ is reducible in general but the contained irreducible components are independent of $u$. Thus we let simply $W_m^{(a)}$ denote the $X_r$-module so obtained. Then it is known that [16]

$$C_r;$$

$$W_m^{(a)} \simeq \begin{cases} 
\oplus V(k_1 \omega_1 + \cdots + k_a \omega_a) & \text{for } B_r, \\
V(m\omega_r) & \text{for } D_r,
\end{cases} \quad 1 \leq a \leq r - 1, \quad a = r$$

(2.4a)

$$B_r \text{ and } D_r;$$

$$W_m^{(a)} \simeq \oplus V(k_\alpha_\omega_a + k_{\alpha_{a+2}} \omega_{a+2} + \cdots + k_\omega_a) \quad 1 \leq a \leq r',$$

$$r' = \begin{cases} 
 r & \text{for } B_r, \\
r - 2 & \text{for } D_r, \quad a_0 \equiv a \mod 2, \quad a_0 = 0 \text{ or } 1,
\end{cases}$$

$$W_m^{(a)} \simeq V(m\omega_a) \quad a = r - 1, r \text{ only for } D_r.$$
Here $\omega_0 = 0$ and $V(\lambda)$ denotes the irreducible $X_r$-module with highest weight $\lambda$. The sum in (2.4a) is taken over non-negative integers $k_1, \ldots, k_a$ that satisfy $k_1 + \cdots + k_a \leq m, k_j \equiv m\delta_{ja} \mod 2$ for all $1 \leq j \leq a$. The sum in (2.4b) extends over non-negative integers $k_{a_0}, k_{a_0+2}, \ldots, k_a$ obeying the constraint $t_\alpha(k_{a_0} + k_{a_0+2} + \cdots + k_a) + k_a = m$. If one depicts the highest weights in the sum (2.4a) and (resp. (2.4b)) by Young diagrams as usual, they correspond to those obtained from the $a \times m$ rectangular one by successively removing $1 \times 2$ and (resp. $2 \times 1$) pieces.

As mentioned in section 3.2 of [8], we assume in this paper that there exists a natural $q$-analogue of these modules over the quantum affine algebra $U_q(X^{(1)}_r)$, which will also be denoted by $W^{(a)}_m$. When referring it as an $X_r$-module, it means that the corresponding $Y(X_r)$-module in the $q \to 1$ limit has been regarded so.

### 2.2 T-system

Consider the transfer matrix (1.2) acting on the quantum space $\otimes_{j=1}^N W^{(p)}_j(w_j)$. We shall reserve the letters $p$ and $s$ for this meaning throughout the paper. (See also the end of section 2.4.) In [8], a set of functional relations, the $T$-system, was conjectured for $U_q(X^{(1)}_r)$ symmetry models for any $X_r$. For $X_r = B_r$, $C_r$, and $D_r$ they read as follows.

\begin{align}
B_r : \\
T^{(a)}_m(u - 1)T^{(a)}_m(u + 1) &= T^{(a)}_{m+1}(u)T^{(a)}_{m-1}(u) + g^{(a)}_m(u)T^{(a-1)}_m(u)T^{(a+1)}_m(u) \\
& \quad 1 \leq a \leq r - 2, \\
T^{(r-1)}_m(u - 1)T^{(r-1)}_m(u + 1) &= T^{(r-1)}_{m+1}(u)T^{(r-1)}_{m-1}(u) + g^{(r-1)}_m(u)T^{(r-2)}_m(u)T^{(r)}_m(u), \\
T^{(r)}_{2m}(u - \frac{1}{2})T^{(r)}_{2m}(u + \frac{1}{2}) &= T^{(r)}_{2m+1}(u)T^{(r)}_{2m-1}(u) + g^{(r)}_{2m}(u)T^{(r-1)}_m(u - \frac{1}{2})T^{(r-1)}_m(u + \frac{1}{2}), \\
T^{(r)}_{2m+1}(u - \frac{1}{2})T^{(r)}_{2m+1}(u + \frac{1}{2}) &= T^{(r)}_{2m+2}(u)T^{(r)}_{2m}(u) + g^{(r)}_{2m+1}(u)T^{(r-1)}_m(u)T^{(r-1)}_{m+1}(u).
\end{align}

\begin{align}
C_r : \\
T^{(a)}_m(u - \frac{1}{2})T^{(a)}_m(u + \frac{1}{2}) &= T^{(a)}_{m+1}(u)T^{(a)}_{m-1}(u) + g^{(a)}_m(u)T^{(a-1)}_m(u)T^{(a+1)}_m(u) \\
& \quad 1 \leq a \leq r - 2, \\
T^{(r-1)}_{2m}(u - \frac{1}{2})T^{(r-1)}_{2m}(u + \frac{1}{2}) &= T^{(r-1)}_{2m+1}(u)T^{(r-1)}_{2m-1}(u) + g^{(r-1)}_{2m}(u)T^{(r-2)}_m(u)T^{(r)}_m(u - \frac{1}{2})T^{(r)}_m(u + \frac{1}{2}), \\
T^{(r-1)}_{2m+1}(u - \frac{1}{2})T^{(r-1)}_{2m+1}(u + \frac{1}{2}) &= T^{(r-1)}_{2m+2}(u)T^{(r-1)}_{2m}(u) + g^{(r-1)}_{2m+1}(u)T^{(r-2)}_m(u)T^{(r)}_m(u)T^{(r)}_{m+1}(u), \\
T^{(r)}_m(u - 1)T^{(r)}_m(u + 1) &= T^{(r)}_{m+1}(u)T^{(r)}_{m-1}(u) + g^{(r)}_m(u)T^{(r-1)}_2T^{(r-1)}_2.
\end{align}
\[ D_r : \]
\[
T_m^{(a)}(u + 1) = T_{m+1}^{(a)}(u)T_{m-1}^{(a)}(u) + g_m^{(a)}(u)T_m^{(a-1)}(u)T_m^{(a+1)}(u)
\]
\[ 1 \leq a \leq r - 3, \]  
\[
T_m^{(r-2)}(u + 1) = T_{m+1}^{(r-2)}(u)T_{m-1}^{(r-2)}(u)
\]
\[
+ g_m^{(r-2)}(u)T_m^{(r-3)}(u)T_m^{(r-1)}(u)T_m^{(r)}(u),
\]
\[
T_m^{(a)}(u + 1) = T_{m+1}^{(a)}(u)T_{m-1}^{(a)}(u) + g_m^{(a)}(u)T_m^{(a-2)}(u)
\]
\[ a = r - 1, r. \]  

Here the subscripts of the transfer matrices in the lhs are taken to be positive and we assume that \( T_m^{(0)}(u) = T_0^{(a)}(u) \equiv \text{Id}. \) \( g_m^{(a)}(u) \) is a scalar function that depends on \( W_s^{(p)} \) and satisfies

\[
g_m^{(a)}(u - \frac{1}{t_a})g_m^{(a)}(u + \frac{1}{t_a}) = g_m^{(a)}(u)g_m^{(a)}(u + 1). \]  

See eq. (3.18) in [8]. We have slightly changed the convention from [8] so that \( T_m^{(a)}(u + j) \)
there corresponds to \( T_m^{(a)}(u + 2j) \) here, etc. A wealth of consistency for the \( T \)-system have been observed in [8,9,10,11] for any \( X_r \) and we shall assume (2.5) henceforth. Owing to the commutativity (1.3), one can regard (2.5) as the functional relations on the eigenvalues \( \Lambda_m^{(a)}(u) \). \( \Lambda_0^{(a)}(u) = \Lambda_0^{(a)}(u + 1) = 1 \) Then it can be recursively solved for \( \Lambda_m^{(a)}(u) \) in terms of \( \Lambda_1^{(a)}(u + \text{shift}), \ldots, \Lambda_r^{(a)}(u + \text{shift}) \). In fact, \( \Lambda_m^{(a)}(u) \) will be obtainable within a polynomial
in these functions as argued in [8]. This process corresponds to the Step 2 mentioned in section 1.5.

2.3 Bethe ansatz equation. As in (1.4), the eigenvalues \( \Lambda_m^{(a)}(u) \) will be expressed by
the solutions to the BAE [26,21]:

\[
- \frac{\phi(iu_k^{(a)} + \frac{x}{t_p}\delta_{ap})}{\phi(iu_k^{(a)} - \frac{x}{t_p}\delta_{ap})} = \prod_{b=1}^{r} \frac{Q_b(iu_k^{(a)} + (\alpha_a, \alpha_b))}{Q_b(iu_k^{(a)} - (\alpha_a, \alpha_b))},
\]  

where \( s \) and \( p \) are the labels of the quantum space \( \otimes_{j=1}^{N} W_s^{(p)}(w_j) \), \( \phi(u) \) is given in (1.4b) and \( Q_{a}(u) \) is defined by

\[
Q_a(u) = \prod_{j=1}^{N_a} [u - iu_{j}^{(a)}] \quad 1 \leq a \leq r. \]  

Here \( N_a \) is a non-negative integer analogous to \( n \) in (1.4b). The system size \( N \) in \( \phi(u) \) and \( N_a \) are to be taken so that \( \omega^{(p)}_{a} \equiv Ns_{w_p} - \sum_{a=1}^{r} N_a \alpha_a \in \sum_{a=1}^{r} R_{\geq 0} \omega_a \). The BAE (2.7) is imposed on the numbers \( \{ u_k^{(a)} \mid 1 \leq a \leq r, 1 \leq k \leq N_a \} \). In section 5, we will consider a slightly modified version of (2.7) that suits the diagram automorphism symmetry in \( X_r = D_r \).

2.4 Empirical rules in Analytic Bethe ansatz. As in (1.4a), the functions \( Q_{a}(u) \) and \( \phi(u) \) are the constituents of the dress and the vacuum parts in the analytic Bethe ansatz, respectively. In handling the formulas like (1.10) or (1.11), we find it convenient to specify
these parts as $dr(\text{tableau})$ and $vac(\text{tableau})$. For example, from the first equation in (1.9) one has

$$
1 = dr[1] vac[1], \quad dr[1] = \frac{Q(u - 1)}{Q(u + 1)}, \quad vac[1] = \phi(u + 2). \tag{2.9a}
$$

In general the DVF reads

$$
\Lambda^{(a)}_m(u) = \sum Q_{a_1}(u + x_1) \cdots Q_{a_n}(u + x_n) \phi(u + z_1) \cdots \phi(u + z_k), \tag{2.9b}
$$

in which ratios of $Q_a$'s are the dress parts and products of $\phi$'s are the vacuum parts. Using these notations we now introduce three hypotheses, “dress universality”, “top term” and “coupling rule” in the analytic Bethe ansatz. They are the properties of mathematical interest rendering valuable insights into the auxiliary space $W^{(a)}_m$ as the $U_q(X_r^{(1)})$ or the Yangian modules. Roughly speaking, the latter two are the information on the “highest weight vector” and the “action” of the Chevalley-like generators. The hypotheses have been confirmed in several examples and we believe they should rightly be added to the postulates (i)-(iii) explained in section 1.3.

**Dress universality.** Let $T^{(a)}_m(u)$ and $T^{(a)}_m(u)$ be the transfer matrices with the same auxiliary space $W^{(a)}_m(u)$ but acting on the different quantum spaces

$$
\otimes_{k=1}^N W^{(p)}_s(w_k) \quad \text{and} \quad \otimes_{k=1}^{N'} W^{(p')}_{s'}(w'_k),
$$

respectively. Denote by $Q_a(u)$ and $Q'_a(u)$ the functions (2.8) specified from the solutions to the BAE (2.7) for these quantum space choices. Suppose one got their eigenvalues in the DVF's

$$
\Lambda^{(a)}_m(u) = \sum_{j=1}^{\dim W^{(a)}_m} \text{tab}_j, \quad \Lambda^{(a)}_m(u) = \sum_{j=1}^{\dim W^{(a)}_m} \text{tab}'_j, \tag{2.10}
$$

where tab$_j$ and tab$_j'$ denote the terms whose vacuum parts correspond to the same (i.e., “$j$-th”) vector from $W^{(a)}_m$ in the trace (1.2). Then the dress universality is stated as

$$
dr(\text{tab}_j) = dr(\text{tab}'_j)|_{Q'_a(u) \rightarrow Q_a(u)} \quad \text{for all } j. \tag{2.11}
$$

Namely, the dress part is independent of the quantum space choice if it is expressed in terms of $Q_a(u)$. On the contrary, one has $vac(\text{tab}_j) \neq vac(\text{tab}'_j)|_{N' \rightarrow N, w'_k \rightarrow w_k}$ in general if $(p', s') \neq (p, s)$.

**Top term.** Among the dim $W^{(a)}_m$ terms in (2.10), let tab$_1$ denote the one corresponding to the “highest weight vector” in $W^{(a)}_m$. By this we mean more precisely the unique vector of weight $m w_a$ when $W^{(a)}_m$ is regarded as an $X_r$-module in the sense of section 2.1. Plainly, tab$_1$ is the analogue of the first term on the rhs of (1.4a). Then the top term hypothesis reads

$$
dr(\text{tab}_1) = \frac{Q_a(u - \frac{m}{l_a})}{Q_a(u + \frac{m}{l_a})} \quad \text{for all } j. \tag{2.12}
$$
in (2.10), which is certainly consistent to the dress universality. It follows from (2.12) that

\[
\Lambda_m^{(a)}(u)(\Lambda_m^{(a)}(-u))_{w_j \cdots w_j, u_h^{(a)} \cdots u_h^{(a)}} = \Phi(u)(\Phi(-u))_{w_j \cdots w_j} + \cdots,
\]

where \( \Phi(u) = vac(\text{tab}_1) \) is a product of \( \phi \)'s. This is essentially eq.(5) in [21], which is a consequence of the first inversion relation of the relevant \( R \)-matrix.

**Coupling rule.** Regard the auxiliary space \( W_m^{(a)} \) as an \( X_r \)-module in the sense of section 2.1 and let \( \lambda \) be a weight without multiplicity

\[
\text{mult}_\lambda W_m^{(a)} = 1. \tag{2.13}
\]

Then it makes sense to denote by \( \boxed{\lambda} \) the term in (2.10) corresponding to the \( \lambda \)-weight vector from \( W_m^{(a)} \). Thus \( \Lambda_m^{(a)}(u) = \cdots + \boxed{\lambda} + \cdots \). Now the coupling rule is stated as follows.

If \( \lambda \) and \( \mu \) are multiplicity-free weights such that \( \lambda - \mu = \alpha_a \), then

(a) \( \boxed{\lambda} \) and \( \boxed{\mu} \) share common poles of the form \( 1/Q_a(u + \xi) \)

for a certain \( \xi \) depending on \( \lambda \) and \( a \). \tag{2.14a}

(b) The BAE (2.7) guarantees \( Res_{u = -\xi + \iota u_h^{(a)}}(\boxed{\lambda} + \boxed{\mu}) = 0 \) in such a way that

\[
\frac{d_r \boxed{\mu}}{d_r \boxed{\lambda}} = \prod_{b=1}^r \frac{Q_b(u + \xi + (\alpha_a | \alpha_b))}{Q_b(u + \xi - (\alpha_a | \alpha_b))}. \tag{2.14b}
\]

The hypothesis tells that for \( \lambda - \mu = \alpha_a \), spurious “poles of color a” in \( \boxed{\lambda} \) and \( \boxed{\mu} \) couple into a pair yielding zero residue in total. This is a more detailed information than just saying that the BAE assures pole-freeness as in (ii) in section 1.3. To determine \( \xi \) is a non-trivial task in general. From (2.14b), (2.7) and \( \boxed{\lambda} = d_r \boxed{\lambda} vac(\lambda) \) etc, one deduces

\[
\frac{vac(\lambda)}{vac(\mu)} = \frac{\phi(u + \xi + \frac{\alpha}{r_p} \delta_{ap})}{\phi(u + \xi - \frac{\alpha}{r_p} \delta_{ap})} \tag{2.15}
\]

for the vacuum parts. The last equation in (2.14b) excludes the possibility to exchange \( \lambda \) and \( \mu \) in (2.14b) and (2.15) simultaneously, in which case the BAE could also have ensured the pole-freeness. The coupling rule is certainly valid in (1.8) and (1.9) since \( \boxed{1} \) corresponds to the weight \( \omega_1 \) and \( \boxed{2} \) to \( -\omega_1 = \omega_1 - \alpha_1 \) in \( sl(2) \). We will visualize (2.14) and (2.15) as

\[
\boxed{\lambda} \quad \boxed{\mu} \quad \boxed{1} \quad \boxed{2}
\]

where the number over the arrow signifies the color of the pole shared by the two boxes.

There are two more postulates that embody the asymptotics and the second inversion properties mentioned in (iii) in section 1.3. The first one is stated as

**Character limit.** As said in the end of section 1.4, the eigenvalue \( \Lambda_m^{(a)}(u) \) is a Yang-Baxterization \( (u\text{-dependent}) \) version of the character of the auxiliary space \( W_m^{(a)}(u) \) viewed
as an $X_r$-module in the sense of section 2.1. Indeed, the latter can be recovered from the former as

$$
\lim_{u \to \sigma_1 \infty (|u|^2 \to 1)} q^{r(\sigma_1, \sigma_2)} \Lambda_m^{(a)}(u) = \sum_{\lambda} (\text{mult}_\lambda W_m^{(a)}) q^{2\sigma_1 \sigma_2 (\omega_s^{(p)}) |\lambda)} \sigma_1 \sigma_2 = \pm 1, \quad (2.16)
$$

where the sum extends over all the weights in $W_m^{(a)}$, $q^{r(\sigma_1, \sigma_2)}$ is some convergence factor and $\omega_s^{(p)}$ has been specified after (2.8). One readily sees that (2.16) is consistent with (2.14b) and (2.15) by computing the asymptotics of $\Lambda \Lambda$. Eq. (2.16) is also asserting that DVF’s always contain $Q_s \lambda$ via the combination $Q_s(u + \cdots)/Q_s(u + \cdots)$ as in (2.9b) and that they are homogeneous polynomials w.r.t $\phi(u + \cdot \cdot \cdot)$. Thus $k$ is common in all the terms in (2.9b). In [8, 9, 29], the rhs of (2.16) was denoted by $Q_m^{(a)}(\omega_s^{(p)})$. It obeys the $Q$-system, the character identity in [16], which was extensively used to formulate the conjectures on dilogarithm identity [29, 30, 8, 9], $q$-series formula for an $X_r^{(1)}$ string function [31] and to find the $T$-system [8]. The limit (2.16) is essentially eq.(12) in [21]. Now we state the second postulate.

Crossing symmetry. Most $R$-matrices enjoy the so-called crossing symmetry, eq.(4) in [21], from which the second inversion relation follows. The eigenvalue $\Lambda_m^{(a)}(u)$ inherits the following property form it.

$$
\Lambda_m^{(a)}(u) = (-)^{kN} \Lambda_m^{(a)}(-g - u)|_{w_j \to -w_j, u_i^{(a)} \to -u_i^{(a)}}. \quad (2.17)
$$

Here $g$ is defined in (2.1), $k$ is the order of the DVF w.r.t $\phi$ as in (2.9b) and $N$ is the number of lattice sites entering $\phi$ via (1.4b). This is essentially eq.(6) in [21], which we call the crossing symmetry as well. Note that the BAE (2.7) remains unchanged under the simultaneous replacement $w_j \to -w_j$ and $u_i^{(a)} \to -u_i^{(a)}$. In particular, if $\pm \lambda$ are multiplicity-free weights of $W_m^{(a)}$, the combination $\Lambda + \Lambda$ in $\Lambda_m^{(a)}(u)$ becomes same on both sides of (2.17) as

$$
\Lambda = (-)^{kN} \Lambda|_{u \to -g - u, w_j \to -w_j, u_i^{(a)} \to -u_i^{(a)}}. \quad (2.18)
$$

From the definitions of $\phi(u)$ (1.4b) and $Q_s(u)$ (2.8), the rhs of (2.17) is then obtained from (2.9b) by the simultaneous replacements

$$
x_i \to g - x_i, \; y_i \to g - y_i, \; z_i \to g - z_i. \quad (2.19)
$$

The dress universality, top term, coupling rule, character limit and crossing symmetry severely limit the possible form of the DVF in the analytic Bethe ansatz. In particular if all the weights in $W_m^{(a)}$ are multiplicity-free, (2.12), (2.14) and (2.15) determine the DVF for $\Lambda_m^{(a)}(u)$ completely up to an overall scalar multiple. In such cases, one even does not need the vacuum parts a priori hence can avoid a tedious computation of the $R$-matrices. The DVF’s given in the subsequent sections have actually been derived in that manner for such cases. Except a few cases, it is yet to be verified if those DVF’s with $\forall Q_s(u) = 1$ yield the actual vacuum eigenvalues obtainable from the relevant $R$-matrix as in (1.6). In
a sense we have partially absorbed the postulate (i) of section 1.3 into (2.11)-(2.15) here, which may be viewed as a modification of the analytic Bethe ansatz itself.

Let us include a remark before closing this section. Suppose one has found the DVF when the the quantum space is $\otimes_{j=1}^{N} W_4^{(p)}(w_j)$. Then, the one for $\otimes_{j=1}^{N} W_{s}^{(p)}(w_j)$ can be deduced from it by the replacement

$$\phi(u) \rightarrow \phi_s(u) \overset{\text{def}}{=} \prod_{k=1}^{s} \phi(u + \frac{s + 1 - 2k}{t_p}).$$

(2.20)

To see this one just notes that the lhs of (2.7) is equal to $-\frac{\phi_s(iu^{(a)} + \delta_{ap}/t_p)}{\phi_s(iu^{(a)} - \delta_{ap}/t_p)}$. See also (2.15).

In view of this we shall hereafter consider the $s = 1$ case only with no loss of generality.

3. Eigenvalues for $C_r$

3.1 Eigenvalue $\Lambda_1^{(1)}(u)$. The family of $U_q(C_r^{(1)})$-modules \( \{ W_m^{(a)} \mid 1 \leq a \leq r, m \in \mathbb{Z}_{\geq 1} \} \) is generated by decomposing tensor products of $W_4^{(1)}$ as suggested in [8]. In terms of the eigenvalues, it implies that all the $\Lambda_m^{(a)}(u)$ are contained in a suitable product $\prod_k \Lambda_1^{(1)}(u + c_k)$. Thus we first do the analytic Bethe ansatz for the fundamental eigenvalue $\Lambda_1^{(1)}(u)$. The relevant auxiliary space is $W_1^{(1)} \simeq V(\omega_1)$ as an $C_r$-module from (2.4a), which is the vector representation. Then all the weights are multiplicity-free and one can apply the coupling rule (2.14). To be concrete, we introduce the orthogonal vectors $\epsilon_a, 1 \leq a \leq r$ normalized as $\langle \epsilon_a | \epsilon_b \rangle = \delta_{ab}/2$ and realize the root system as follows.

$$\alpha_a = \begin{cases} \epsilon_a - \epsilon_{a+1} & \text{for } 1 \leq a \leq r - 1, \\ 2\epsilon_r & \text{for } a = r. \end{cases}$$

(3.1)

$$\omega_a = \epsilon_1 + \cdots + \epsilon_a.$$ 

Then the weights in $V(\omega_1)$ are $\epsilon_a$ and $-\epsilon_a (1 \leq a \leq r)$, which we will abbreviate to $a$ and $\bar{a}$, respectively. In this notation the set of weights reads

$$J = \{ 1, 2, \ldots, r, \bar{r}, \ldots, 2, 1 \}.$$ 

(3.2)

Starting from the top term (2.12), one successively applies the coupling rule (2.14) to find the DVF

$$\Lambda_1^{(1)}(u) = \sum_{a \in J} \bar{a},$$

(3.3)

with the elementary boxes defined by

$$\begin{align*}
\square &= \psi_a(u) \frac{Q_{a-1}(u + \frac{a+1}{2})Q_{a}(u + \frac{a-1}{2})}{Q_{a-1}(u + \frac{a-1}{2})Q_{a}(u + \frac{a+1}{2})} \quad 1 \leq a \leq r - 1, \\
\square' &= \psi_r(u) \frac{Q_{r-1}(u + \frac{r+1}{2})Q_{r}(u + \frac{r-3}{2})}{Q_{r-1}(u + \frac{r-3}{2})Q_{r}(u + \frac{r+1}{2})}, \\
\square'' &= \psi_r(u) \frac{Q_{r-1}(u + \frac{r+1}{2})Q_{r}(u + \frac{r+5}{2})}{Q_{r-1}(u + \frac{r+3}{2})Q_{r}(u + \frac{r+1}{2})}, \\
\square &= \psi_a(u) \frac{Q_{a-1}(u + \frac{2r-a+1}{2})Q_{a}(u + \frac{2r-a+4}{2})}{Q_{a-1}(u + \frac{2r-a+4}{2})Q_{a}(u + \frac{2r-a+2}{2})} \quad 1 \leq a \leq r - 1,
\end{align*}$$

(3.4a)
where we have set $Q_0(u) = 1$. In the above, the vacuum part $\psi_{\text{vac}}(u) = \frac{\text{vac}}{a}$ is given by

$$
\psi_{\text{vac}}(u) = \begin{cases} 
\phi(u + \frac{p+1}{2})\phi(u + \frac{2r-p-3}{2}) & 1 \leq a \leq p \\
\phi(u + \frac{p}{2})\phi(u + \frac{2r-p-3}{2}) & p + 1 \leq a \leq p + 1 \\
\phi(u + \frac{p}{2})\phi(u + \frac{2r-b-1}{2}) & \bar{p} \leq a \leq \bar{r}
\end{cases}
$$

(3.4b)

depending on the quantum space $\otimes_{j=1}^N W_{1}^{(p)}(w_j)$. The symbol $\prec$ here stands for a total order in the set $I$ specified as

$$
1 \prec 2 \prec \cdots \prec r \prec r \prec \cdots \prec 2 \prec 1.
$$

(3.5)

When $p = r$, the second possibility in (3.4b) is absent. The case $p = 1$ was obtained in [21]. Note that $\bar{r}$ is the totipotent term (2.12). By the construction, $p$ enters only the vacuum part (3.4b) hence the dres universality (2.11) is valid. The crossing symmetry (2.18) holds between $[\bar{r}]$ and $[a]$. Under the BAE (2.7), (3.3) is pole-free because the coupling rule (2.14) and (2.15) as follows.

$$
Res_{u = \frac{b}{2} + i u_{b}}(\frac{b}{2} + \frac{b+1}{2}) = 0 \quad 1 \leq b \leq r - 1,
$$

(3.6a)

$$
Res_{u = \frac{r}{2} + i u_{r}}(\frac{r}{2} + \frac{r}{2}) = 0,
$$

(3.6b)

$$
Res_{u = \frac{2r - b + 2}{2} + i u_{b}}(\frac{b+1}{2} + \frac{b}{2}) = 0 \quad 1 \leq b \leq r - 1.
$$

(3.6c)

Following section 2.4, this can be summarized in the diagram

$$
1 \rightarrow 2 \rightarrow \cdots \rightarrow r-1 \rightarrow r \rightarrow r-1 \rightarrow \cdots \rightarrow 2 \rightarrow 1 \rightarrow 1
$$

This turns out to be identical with the crystal graph [24,25].

3.2 Eigenvalue $\Lambda_{1}^{(a)}(u)$. Let us proceed to $\Lambda_{1}^{(a)}(u)$, which can be constructed from the elementary boxes (3.4). For $1 \leq a \leq r$, let $T_{1}^{(a)}$ be the set of the tableaux of the form

$$
\begin{array}{cccccccc}
& & & & & & i_1 & \\
& & & & & & \vdots & \\
& & & & & & i_a & \\
\end{array}
$$

(3.7a)

with entries $i_k \in J$ obeying the following conditions

$$
1 \leq i_1 \prec i_2 \prec \cdots \prec i_a \leq \bar{r},
$$

(3.7b)

If $i_k = c$ and $i_l = c$, then $r + k - l \geq c$.

(3.7c)

We remark that these constraints are very similar but different from the crystal base [24,25], where (3.7c) is replaced by $a + 1 + k - l \leq c$. We identify each element (3.7a) of $T_{1}^{(a)}$ with
the product of (3.4) with the spectral parameters \( u + \frac{a_k}{a}, u + \frac{a_k}{a^2}, \ldots, u - \frac{a_k}{a^a} \) from the top to the bottom, namely,

\[
\prod_{k=1}^{a} \left[ u - \frac{a_k}{a} \right]. 
\tag{3.8}
\]

Then the analytic Bethe ansatz yields the following DVF.

\[
\Lambda_1^{(a)}(u) = \sum_{T \in \mathcal{T}_1^{(a)}} T \quad 1 \leq a \leq r, 
\tag{3.9}
\]

which reduces to (3.3) when \( a = 1 \). Let us observe consistency of this result before proving that it is pole-free in section 3.3. Firstly, the dress part of

\[
\begin{array}{c}
1 \\
\vdots \\
a
\end{array}
\]

is \( Q_a(u - \frac{1}{a})/Q_a(u + \frac{1}{a}) \), telling that the above tableau indeed gives the top term (2.12).

Secondly, the set \( \mathcal{T}_1^{(a)} \) is invariant under the interchange of the two tableaux

\[
\begin{array}{c}
i_1 \\
\vdots \\
i_a
\end{array}
\quad
\begin{array}{c}
i_g \\
\vdots \\
i_1
\end{array}
\]

and the crossing symmetry (2.18) is valid among them. Thirdly, the character limit (2.16) can be proved. This is essentially done by showing

\[
\#\mathcal{T}_1^{(a)} = \dim V(\omega_a) = \left( \frac{2r}{a} \right) - \left( \frac{2r}{a-2} \right). 
\tag{3.10}
\]

which corresponds to the \( q \to 1 \) limit of (2.16) since \( W_1^{(a)} \simeq V(\omega_a) \) as a \( C_r \)-module by (2.4a). We have verified (3.10) by building injections in both directions between the sets of depth \( a \) tableaux (3.7a) breaking (3.7c) and the depth \( a - 2 \) ones only obeying the constraint as (3.7b). Once (3.10) is established, the weight counting in (2.16) for \( q \neq 1 \) is shown consistent with eq.(2.2.2) of [25] by noting that the injections are weight preserving and \( \lim_{u \to -\infty} \langle \gamma \rangle = q^{2(\omega(\ell) \mid \kappa_a)} \) for some \( \ast \).

3.3 Pole-freeness of \( \Lambda_1^{(a)}(u) \). The quantity (3.9) passes the crucial condition in the analytic Bethe ansatz, namely,

**Theorem 3.3.1.** \( \Lambda_1^{(a)}(u)(1 \leq a \leq r) \) (3.9) is free of poles provided that the BAE (2.7) (for \( s = 1 \)) is valid.

For the proof we prepare a few Lemmas.
Lemma 3.3.2. For $1 \leq b \leq r - 1$, the products

$$\frac{b}{b+1}, \quad \frac{b+1}{b}$$

(3.11)

with the spectral parameter $v$ ($v - 1$) for the upper (lower) box do not involve $Q_b$ function.

The proof is straightforward by using the explicit form (3.4). It is also elementary to check.

Lemma 3.3.3. For $1 \leq b \leq r - 1$, put

$$\frac{b}{b+1}, \quad \frac{b+1}{b}$$

(3.12a)

$$\frac{b}{b+1}, \quad \frac{b+1}{b}$$

(3.12b)

$$\frac{b}{b+1}, \quad \frac{b+1}{b}$$

(3.12c)

where the indices specify the spectral parameters attached to the boxes (3.4). Then $X_i$‘s do not involve $Q_b$ function.

The point is that (3.12a) and (3.12c) have only one $Q_b$ function in their denominators after some cancellations owing to the spectral parameter choice $v, v - r + b$.

Lemma 3.3.4. For $1 \leq b \leq r - 1$, let the tableaux

$$\begin{array}{c|c|c}
\xi & \xi & \\
\hline
b & b+1 & \\
\eta & \eta & \\
\hline
b+1 & b & \\
\zeta & \zeta & \\
\end{array}$$

or

(3.13)

be the elements in $T_{r-b}^{(s)}$ such that the columns $[\xi, \eta, \zeta]$ do not contain the boxes with entries $b, b + 1, b + 1$ and $b$. Then the length of $[\eta]$ is less than $r - b$.

Proof. We shall show this with respect to the first tableau in (3.13). The proof for the second one is similar. Suppose on the contrary that the length $L$ of $[\eta]$ satisfies

$$L \geq r - b.$$  

(3.14)

Due to (3.7b) and (3.5), $[\eta]$ then consists of the elementary boxes with entries from $\{b + 2, b + 3, \ldots, r, r - 1, \ldots, b + 2\}$. By (3.14), there must be at least one letter $c \in \{b +
$2, b + 3, \ldots, r$ such that both $\boxed{c}$ and $\boxed{c}$ occur in $\eta$. Let $c_0$ be the smallest among such $c$’s. Then $\eta$ has the structure

$$
\begin{array}{c}
\eta_1 \\
c_0 \\
\eta_2 \\
c_0' \\
\eta_3
\end{array}
$$

(3.15)

By the definition, the columns $\eta_1$ and $\eta_3$ contain only boxes $\boxed{q}$ and $\boxed{$ for $b+2 \leq q \leq c_0-1$, respectively. Moreover, $\boxed{q}$ and $\boxed{$} must not be present simultaneously. Thus the total length $L_{13}$ of $\eta_1$ and $\eta_3$ should satisfy

$$
L_{13} \leq (c_0 - 1) - (b + 2) + 1 = c_0 - b - 2.
$$

(3.16)

Since (3.15) is a part of the first tableau in (3.13) belonging to $T_1^{(a)}$, the condition (3.7c) must be valid for the $c_0, c_0'$ pair. In terms of the length $L_2$ of the column $\eta_2$, (3.7c) reads

$$
r - L_2 - 1 \geq c_0.
$$

(3.17)

Combining (3.14), (3.16) and (3.17), we have the contradiction

$$
r - b \leq L = L_{13} + L_2 + 2 \leq (c_0 - b - 2) + (r - c_0 - 1) + 2 = r - b - 1,
$$

owing to the onset assumption (3.14), and thus finish the proof.

With these Lemmas, we proceed to

Proof of Theorem 3.3.1. We shall show that color $b$ singularity is spurious, i.e.,

$Res_{u = i_{b+1}}^{(b)} \cdots \Lambda_1^{(a)}(u) = 0$ for each $2 \leq b \leq r - 1$. The remaining cases $b = 1$ and $r$ can be verified similarly and more easily. Among the elementary boxes (3.4a), the factor $1/Q_b(u + \cdots)$ enters only $\boxed{b}, \boxed{b+1}, \boxed{b+1}$ and $\boxed{b}$. Thus one has to keep track of only these four boxes appearing in (3.7a). Accordingly, let us write (3.9) as $\Lambda_1^{(a)}(u) = S_0 + S_1 + \cdots + S_4$, where $S_k$ denotes the partial sum over the tableaux (3.7a) containing precisely $k$ boxes among the above four. Obviously $S_0$ is free of $1/Q_b(u + \cdots)$. So is $S_4$ because the relevant tableaux involve both of the $2 \times 1$ patterns in (3.11) and therefore do not contain $Q_b$ by Lemma 3.3.2. Next consider $S_1$ which is the sum over the tableaux of the form

$$
\begin{array}{c}
\xi \\
b \\
\eta
\end{array}
\quad \begin{array}{c}
\xi \\
b + 1 \\
\eta
\end{array}
\quad \begin{array}{c}
\xi \\
b + 1 \\
\eta
\end{array}
\quad \begin{array}{c}
\xi \\
b \\
\eta
\end{array}
$$

Here $\boxed{\xi}$ and $\boxed{\eta}$ stand for columns with total length $a - 1$ and they do not contain $\boxed{b}, \boxed{b+1}, \boxed{b+1}$ and $\boxed{b}$. From (3.6), color $b$ residues in the first and second (third and fourth) tableaux sum up to zero. By the same reason $S_3$ is free of color $b$ singularities
since the relevant tableaux must contain one of (3.11). Thus we are left with $S_2$, whose
summands are classified into the following four types:

\[
\begin{array}{cccc}
\xi & \xi & \xi & \xi \\
\eta & \eta & \eta & \eta \\
\xi & \xi & \xi & \xi \\
\eta & \eta & \eta & \eta \\
\end{array}
\]

(3.18)

Here, $[\xi]$, $[\eta]$ and $[\zeta]$ are columns without $[b]$, $[b+1]$ and $[b]$. We are going to show
that the sum of the four tableaux (3.18) is free of color $b$ singularity for any fixed $[\xi]$, $[\eta]$ and $[\zeta]$. Denoting their lengths by $k-1$, $l-1$ and $a-l$, respectively, we consider the
cases $r+k-l \geq b+1$, $r+k-l = b$ and $r+k-l \leq b-1$ separately. If $r+k-l \geq b+1$, all
the four tableaux (3.18) actually belong to $T_1^{(a)}$ and the pole-freeness of their sum follows
straightforwardly from (3.6). If $r+k-l = b$, the third tableau in (3.18) is absent since it breaks (3.7c). Up to an overall factor not containing $Q_b$, the remaining three terms are proportional to those in (3.12) for some $r$. From Lemma 3.3.3, their sum has zero residue
both at $v = -\frac{b}{2} + i u_k^{(b)}$ by (3.6a) and at $v = -\frac{b}{2} - 1 + i u_k^{(b)}$ by (3.6c). Finally, we consider the case $r+k-l \leq b-1$, when the second and third tableaux in (3.18) do not exist because they both break (3.7c). In fact, the first and the fourth ones are also absent. This is because $r+k-l \leq b-1$ is equivalent to saying that the length of $[\eta]$ is not less than $r-b$ against Lemma 3.3.4. Thus $S_2$ is free of color $b$ poles, which completes the proof of the Theorem.

3.4 Eigenvalue $\Lambda^{(1)}_{m}(u)$. The result (3.9) accomplishes Step 1 in section 1.5 already with the tableau language sought in Step 3. The remaining task is Step 2, i.e., to find the eigenvalues $\Lambda^{(a)}_{m}(u)$ for higher $m$ by solving the $T$-system (2.5b) with

\[
g^{(a)}_{m}(u) = 1 \quad \text{for } 1 \leq a \leq r-1, \\
g^{(r)}_{m}(u) = \prod_{k=1}^{m} g^{(r)}_{1}(u + m + 1 - 2k), \\
g^{(r)}_{1}(u) = \psi(u + \frac{r+1}{2})\psi\left(u - \frac{r+1}{2}\right),
\]

under the initial conditions $\Lambda^{(a)}_{0}(u) = 1$ and (3.9). So far we have done this only partially
to get a conjecture on $\Lambda^{(1)}_{m}(u)$. To present it we introduce a set $T_{m}^{(1)}$ ($m \in \mathbb{Z}_{\geq 1}$) of the tableaux having the form

\[
\begin{array}{cccccccc}
*, & \cdots, & i_1, & \overline{r}, & \cdots, & \overline{r}, & \overline{j_1}, & \cdots, & \overline{j_1} \\
\end{array}
\]

(3.19a)

with the conditions

\[
k, n, l \geq 0, \quad k + 2n + l = m, \quad 1 \leq i_1 \leq i_2 \leq \cdots \leq i_k \leq r, \quad 1 \leq j_1 \leq j_2 \leq \cdots \leq j_l \leq r.
\]

(3.19b) (3.19c)
Writing (3.19a) simply as \( \prod_{k=1}^{m} \prod_{u=-u}^{u-u-m+\frac{1}{2}+2k} \), with \( i_k \in J \) (3.2), we identify it with the product of (3.4) with the spectral parameters \( u - \frac{m-1}{2}, u - \frac{m-2}{2}, \ldots, u + \frac{m-1}{2} \) from the left to the right, namely,

\[
\prod_{k=1}^{m} \prod_{u=-u}^{u-u-m+\frac{1}{2}+2k}. \tag{3.20}
\]

Then we conjecture that the \( T \)-system (2.5b) with the initial condition (3.9) leads to

\[
\Lambda_m^{(1)}(u) = \sum_{T \in T_m^{(1)}} T \quad m \in \mathbb{Z}_{\geq 1}. \tag{3.21}
\]

This is just (3.3) when \( m = 1 \). We remark that (3.21) consists of the correct number of terms,

\[
\#T_m^{(1)} = \dim W_m^{(1)} \tag{3.22}
\]

To see this, note from (2.4a) that the rhs is equal to

\[
dim V(m\omega_1) + \dim V((m-2)\omega_1) + \cdots + \begin{cases} \dim V(0) & m \text{ even} \\ \dim V(\omega_1) & m \text{ odd} \end{cases}. \tag{3.23}
\]

On the other hand, the set \( T_m^{(1)} \) is the disjoint union of those tableaux (3.19a) with \( n = 0, 1, 2, \ldots \). Thus it suffices to check

\[
dim V(m\omega_1) = \# \{(3.19a) \in T_m^{(1)} \mid n = 0 \}. \tag{3.24}
\]

Obviously the rhs is \( \binom{m+2r-1}{m} \), which agrees with the lhs calculated from Weyl’s dimension formula.

### 3.5 \( C_2 \) case.

For \( C_2 \) it is possible to provide the full solution \( \Lambda_m^{(1)}(u), \Lambda_m^{(2)}(u) \) to the \( T \)-system [10]. In terms of the tableaux, \( \Lambda_m^{(1)}(u) \) in [10] is certainly given by (3.21) up to an inessential overall scalar reflecting a different convention on \( \Lambda_0^{(1)}(u) \). To present the other eigenvalue \( \Lambda_m^{(2)}(u) \) there, we introduce a set \( T_m^{(2)} \) of \( 2 \times m \) tableaux

\[
\begin{array}{ccc}
i_1 & \cdots & i_m \\
j_1 & \cdots & j_m
\end{array} \tag{3.25a}
\]

obeying the conditions

\[
\begin{align*}
\text{Every column belongs to } T_1^{(2)} \text{ (3.7) for } C_2, \tag{3.25b} \\
i_1 \preceq \cdots \preceq i_m, \quad \text{and} \quad j_1 \preceq \cdots \preceq j_m, \tag{3.25c} \\
\text{The column } \begin{array}{c} 1 \\
T \end{array} \text{ is contained at most once.} \tag{3.25d}
\end{align*}
\]

We identify each element (3.25a) in \( T_m^{(2)} \) with the product of (3.4) with the spectral parameters as follows,

\[
\prod_{k=1}^{m} \prod_{u=-u}^{u-u-m+\frac{1}{2}+2k} \prod_{k=1}^{m} \prod_{u=-u}^{u-u-m+\frac{3}{2}+2k}. \tag{3.26}
\]
Namely, the shifts increase by 2 from the left to the right, decrease by 1 from the top to the bottom and their average is 0. Then the result in [10] reads

\[
\Lambda_m^{(2)}(u) = \sum_{T \in T_m^{(2)}} T. \tag{3.27}
\]

4. Eigenvalues for \( B_r \)

As in the \( C_r \) case we first introduce elementary boxes attached to the vector representation. Using them as the building blocks, we will construct the DVF for \( \Lambda_1^{(a)}(u)(1 \leq a \leq r-1) \) and prove its pole-freeness under the BAE. We also conjecture an explicit form of \( \Lambda_m^{(a)}(u)(1 \leq a \leq r - 1) \) in terms of tableaux made of these boxes.

Compared with the \( C_r \) case, a distinct feature in \( B_r \) (and \( D_r \)) is the existence of the spin representation. Any finite dimensional irreducible \( B_r \)-module is generated by decomposing a tensor product of the spin representation. Thus we introduce another kind of elementary boxes attached to the spin representation. It enables a unified description of the DVFs for all the fundamental eigenvalues \( \Lambda_1^{(a)}(u)(1 \leq a \leq r) \). An explicit relation will be given between the two kinds of the elementary boxes.

4.1 Eigenvalue \( \Lambda_1^{(1)}(u) \).

Let \( \epsilon_a, 1 \leq a \leq r \) be the orthonormal vectors \( (\epsilon_a|\epsilon_b) = \delta_{ab} \) realizing the root system as follows.

\[
\alpha_a = \begin{cases} 
\epsilon_a - \epsilon_{a+1} & \text{for } 1 \leq a \leq r - 1 \\
\epsilon_a & \text{for } a = r 
\end{cases}, \\
\omega_a = \begin{cases} 
\epsilon_1 + \cdots + \epsilon_a & \text{for } 1 \leq a \leq r - 1 \\
\frac{1}{2}(\epsilon_1 + \cdots + \epsilon_r) & \text{for } a = r 
\end{cases}. \tag{4.1}
\]

The auxiliary space relevant to \( \Lambda_1^{(1)}(u) \) is \( W_1^{(1)} \cong V(\omega_1) \) as an \( B_r \)-module. This is the vector representation, whose weights are \( \epsilon_a, -\epsilon_a (1 \leq a \leq r) \) and 0. By abbreviating them to \( a, \bar{a} \) and 0, the set of weights is given by

\[
J = \{1, 2, \ldots, r, 0, \bar{a}, \ldots, \bar{1}\}. \tag{4.2}
\]

All the weights are multiplicity-free, therefore one can determine the DVF from (2.12) and (2.14). The result reads

\[
\Lambda_1^{(1)}(u) = \sum_{\mathbf{a} \in J} [\mathbf{a}], \tag{4.3}
\]

which is formally the same with (3.3). The elementary boxes here are defined by

\[
[\mathbf{a}] = \psi_a(u) \frac{Q_{a-1}(u + a + 1)Q_a(u + a - 2)}{Q_{a-1}(u + a - 1)Q_a(u + a)} 1 \leq a \leq r, \\
[0] = \psi_0(u) \frac{Q_r(u + r - 2)Q_r(u + r + 1)}{Q_r(u + r)^2}, \tag{4.4a} \\
[\bar{a}] = \psi_{\bar{a}}(u) \frac{Q_{a-1}(u + 2r - a - 2)Q_a(u + 2r - a + 1)}{Q_{a-1}(u + 2r - a)Q_a(u + 2r - a - 1)} 1 \leq a \leq r,
\]

where \( Q_n(t) = \prod_{j=1}^{n} (t - \lambda_j) \).
where we have set $Q_0(u) = 1$. The vacuum parts $\psi_a(u)$ depend on the quantum space 
$\otimes_{j=1}^{N} W_{1}^{(p)}(w_j)$ and are given by

$$\psi_a(u) = \begin{cases} 
\phi(u + p + \frac{1}{t_p})\phi(u + 2r - p - 1 + \frac{1}{t_p})\Phi_p^r(u) & \text{for } 1 \leq a \leq p \\
\phi(u + p - \frac{1}{t_p})\phi(u + 2r - p - 1 - \frac{1}{t_p})\Phi_p^r(u) & \text{for } p + 1 \leq a \leq \overline{p + 1} 
\end{cases} \quad (4.4b)$$

where

$$\Phi_p^r(u) = \prod_{j=1}^{p-1} \phi(u + p - 2j - \frac{1}{t_p})\phi(u + 2r - p + 2j - 1 + \frac{1}{t_p}) \quad (4.4c)$$

$$= \Phi_p^r(-2r + 1 - u) |_{w_k \rightarrow w_k}.$$ 

The common factor $\Phi_p^r(u)$ here will play a role in section 4.7, where the boxes here are related to those in section 4.5. The order $\prec$ in the set $J$ (4.2) is defined by

$$1 \prec 2 \prec \cdots \prec r \prec 0 \prec \overline{r} \prec \cdots \prec \overline{\overline{p}}.$$ 

(4.5)

Note the top term $[1]$ (2.12), the dress universality (2.11) and the crossing symmetry (2.18) for the pairs $[a] \leftrightarrow [\overline{a}]$ and $[0] \leftrightarrow [0]$. Under the BAE (2.7), (4.3) is pole-free because the coupling rule (2.14) and (2.15) have been embodied as

$$Res_{u=-b+iu_k^\ell}([\overline{b} + \overline{b + 1}]) = 0 \quad 1 \leq b \leq r - 1, \quad (4.6a)$$

$$Res_{u=-r+iu_k^\ell}([0 + 0]) = 0, \quad (4.6b)$$

$$Res_{u=-r+1+iu_k^\ell}([0 + \overline{r}]) = 0, \quad (4.6c)$$

$$Res_{u=-2r+b+1+iu_k^\ell}([\overline{b} + \overline{b + 1} + \overline{b}]) = 0 \quad 1 \leq b \leq r - 1. \quad (4.6d)$$

Thus we have a diagram

$$\begin{array}{cccccccccccccccccc}
1 & 2 & 2 & \cdots & r-1 & r & r & r & r & r & 2 & 2 & 1 & 1 \\
\end{array}$$

This is again identical with the crystal graph [24,25]. For $p = 1$, (4.3) has been known earlier in [21].

4.2 Eigenvalue $\Lambda_1^{(a)}(u)$ for $1 \leq a \leq r - 1$. For $1 \leq a \leq r - 1$, let $T_1^{(a)}$ be the set of the tableaux of the form (3.7a) with $i_k \in J$ (4.2) obeying the condition

$$i_k \prec i_{k+1} \text{ or } i_k = i_{k+1} = 0 \quad \text{for any } 1 \leq k \leq a - 1.$$ 

(4.7)

Namely, the entries must increase strictly from the top to the bottom in the sense of (4.5) except a possible segment of consecutive 0’s. We identify each element (3.7a) of $T_1^{(a)}$ with the product of (4.4a) with the spectral parameters $u + a - 1, u + a - 3, \ldots, u - a + 1$ from the top to the bottom

$$\prod_{k=1}^{a} \left[ l_k \right]_{u=a+1-2k}.$$ 

(4.8)
Then the analytic Bethe ansatz yields the following DVF.

\[ \Lambda_1^{(a)}(u) = \frac{1}{F_a^{(p,r)}(u)} \sum_{T \in \mathcal{T}_1^{(a)}} T \quad 1 \leq a \leq r - 1, \]  

(4.9a)

where the scalar \( F_a^{(p,r)}(u) \) is defined by

\[
F_a^{(p,r)}(u) = \prod_{j=1}^{a-1} \prod_{k=0}^{p-1} \phi(u + p + a - 1 - \frac{1}{t_p} - 2j - 2k) \phi(u + 2r - p - a + \frac{1}{t_p} + 2j + 2k) \quad (4.9b)
\]

\[
= F_a^{(p,r)}(-2r + 1 - u)|_{w_k \rightarrow -w_k}.
\]

Notice that \( F_a^{(p,r)}(u) = 1 \) hence (4.9a) reduces to (4.3) when \( a = 1 \). From (4.4c) and (4.27c) in section 4.5, (4.9b) can also be written as

\[
F_a^{(p,r)}(u) = \prod_{j=1}^{a-1} \phi(u + p + a - 1 - \frac{1}{t_p} - 2j) \phi(u + 2r + p + a - 2 + \frac{1}{t_p} - 2j)
\]

\[
\times \prod_{j=1}^{a-1} \Phi_p(u + a - 1 - 2j)
\]

\[
= \prod_{j=1}^{a-1} \psi_0^{(p,r)}(u + r - a - \frac{1}{2} + 2j) \psi_p^{(p,r)}(u - r + a + \frac{1}{2} - 2j).
\]

(4.10a)

(4.10b)

By using (4.10a), it can be checked that each summand \( T \) in (4.9a) contains the factor \( F_a^{(p,r)}(u) \) and \( \Lambda_1^{(a)}(u) \) is homogeneous of order \( 2p \) w.r.t \( \phi(u + \cdots) \). This will be seen more manifestly in Theorem 4.7.1. One can observe the top term and the crossing symmetry in the DVF (4.9a) as done after (3.9). The character limit (2.16) is also valid. To see this, we introduce a map \( \chi \) from \( \mathcal{T}_1^{(a)} \) to the Laurent polynomials \( \mathbb{C}[z_1, z_1^{-1}, \ldots, z_r, z_r^{-1}] \) by

\[
\chi \left( \begin{array}{c} 1 \\
\vdots \\
\ell_a 
\end{array} \right) = y_1 \cdots y_a,
\]

(4.11a)

where

\[
y_0 = 1, \ y_a = z_a, \ y_a = z_a^{-1}, \ 1 \leq a \leq r.
\]

(4.11b)

In view of \( \lim_{u \rightarrow -\infty, |q| > 1} q^u [a] = q^{2(\omega_p)|a|} \) for some \( \ast \), this corresponds to taking the limit (2.16) of the element (3.7a). Since \( W_1^{(a)} \simeq V(\omega_a) \oplus V(\omega_a^{-2}) \oplus \cdots \) from (2.4b), we are to show

\[
\sum_{T \in \mathcal{T}_1^{(a)}} \chi(T) = chV(\omega_a) + chV(\omega_a^{-2}) + \cdots,
\]

(4.11c)
for $1 \leq a \leq r - 1$. Here $chV$ denotes the classical character of the $B_r$-module $V$ on letters $z_1, \ldots, z_r$. This can be easily proved from (4.7) and the known formula

$$
chV(\omega_a) = \sum_{i_1, \ldots, i_a \in J} y_{i_1} \cdots y_{i_a}, 
$$

(4.11d)

for $1 \leq a \leq r - 1$. Eq.(4.11d) originates in $so(2r + 1) \leftrightarrow gl(2r + 1)$.

4.3 Pole-freeness of $\Lambda_1^{(a)}(u)$ for $1 \leq a \leq r - 1$. The purpose of this section is to show

**Theorem 4.3.1.** $\Lambda_1^{(a)}(u)(1 \leq a \leq r - 1)$ (4.9) is free of poles provided that the BAE (2.7) (for $s = 1$) is valid.

For the proof we need

**Lemma 4.3.2.** For $n \in \mathbb{Z}_{\geq 0}$, put

$$
\prod_{j=0}^{n} [\square]_{-2j} = \frac{Q_r(v + r + 1)Q_r(v + r - 2n - 2)}{Q_r(v + r)Q_r(v + r - 2n - 1)}X_1, \tag{4.12a}
$$

$$
\prod_{j=1}^{n} [\square]_{-2j} = \frac{Q_r(v + r - 1)Q_r(v + r - 2n - 2)}{Q_r(v + r)Q_r(v + r - 2n - 1)}X_2, \tag{4.12b}
$$

$$
\prod_{j=0}^{n-1} [\square]_{-2j} = \frac{Q_r(v + r + 1)Q_r(v + r - 2n)}{Q_r(v + r)Q_r(v + r - 2n - 1)}X_3, \tag{4.12c}
$$

$$
\prod_{j=1}^{n-1} [\square]_{-2j} = \frac{Q_r(v + r - 1)Q_r(v + r - 2n)}{Q_r(v + r)Q_r(v + r - 2n - 1)}X_4, \tag{4.12d}
$$

where the indices specify the spectral parameters attached to the boxes (4.4). Then

$$
X_i \text{'s do not involve } Q_r \text{ function}, \tag{4.13a}
$$

$$
\frac{Q_r(v + r \pm 1)}{Q_r(v + r)} \text{ comes from the box } [\square], \tag{4.13b}
$$

$$
\frac{Q_r(v + r - 2n - 1 \pm 1)}{Q_r(v + r - 2n - 1)} \text{ comes from the box } [\square]_{-2n}, \tag{4.13c}
$$

where $* = r, \tau$ or 0.

This can be verified by a direct calculation.

**Lemma 4.3.3.** If the BAE (2.7) ($s = 1$) is valid, then

$$
Res_{v = -r + i u_k^{(r)}} ((4.12a) + (4.12b)) = Res_{v = -r + i u_k^{(r)}} ((4.12c) + (4.12d)) = 0, \tag{4.14}
$$

$$
Res_{v = -r + 2n + 1 + i u_k^{(r)}} ((4.12a) + (4.12c)) = Res_{v = -r + 2n + 1 + i u_k^{(r)}} ((4.12b) + (4.12d)) = 0.
$$
This follows from (4.13b,c) and (4.6b,c). Now we proceed to
Proof of Theorem 4.3.1. As remarked after (4.10), there is no pole originated from the
overall scalar $1/F_a^{(p,r)}(u)$ in (4.9a). Thus one has only to show that the apparent color
b poles $1/Q_b(u + \cdots)$ in $\sum_{T \in T_1^{[a]}} T$ are spurious for all $1 \leq b \leq r$ under the BAE. For
$1 \leq b \leq r - 1$, this can be done similarly to the proof of Theorem 3.3.1. In fact the present
case is much easier since (4.7) is so compared with (3.7b,c). Henceforth we focus on the
$b = r$ case which needs a separate consideration. From (4.4a), we have only to keep track
of the boxes $[4] [0]$ and $[6]$ containing $Q_r$. Let us classify the tableaux (3.7a) in $T_1^{(a)}$ (4.7)
into the sectors labeled by the number $n$ of $[1]$'s contained in them. In each sector, we
further divide the tableaux into four types according to the entries $(u, d)$ in the boxes just
above and below the consecutive $[1]$'s.

- type $1_n: u \neq r$ and $d \neq r$,
- type $2_n: u = r$ and $d \neq r$,
- type $3_n: u \neq r$ and $d = r$,
- type $4_n: u = r$ and $d = r$.

Thus we have

\[
\sum_{T \in T_1^{[a]}} T = \sum_{n=0}^{a} \sum_{i=1}^{4} S_{n,i},
\]

(4.15a)

\[
S_{n,i} = \sum_{T \in \text{type } i_n} T,
\]

(4.15b)

\[
S_{a,2} = S_{a,3} = S_{a,4} = S_{a-1,4} = 0.
\]

(4.15c)

Consider the following quartet of the tableaux of types $1_{n+1}$, $2_n$, $3_n$ and $4_{n-1}$, respectively.

\[
\begin{array}{cccc}
\xi & \xi & \xi & \xi \\
0 & r & 0 & r \\
0 & 0 & 0 & 0 \\
\vdots & \vdots & \vdots & \vdots \\
0 & 0 & \bar{r} & \bar{r} \\
\eta & \eta & \eta & \eta \\
\end{array}
\]

(4.16)

Here, $[\xi]$ and $[\eta]$ are the columns with total length $a - n - 1$ and they do not contain $[r] [0]$ and $[\bar{r}]$. In view of (4.8) and (4.13a), the tableaux (4.16) are proportional to the
four terms (4.12) with some $v$ up to an overall factor not containing $Q_r$. Thus from Lemma
4.3.3, their sum is free of color $r$ singularity. This is true for any fixed $[\xi]$ and $[\eta]$ such
that the tableaux (4.16) belong to $T_1^{(a)}$. Therefore $S_{n+1,1} + S_{n,2} + S_{n,3} + S_{n-1,4}$ is free of color $r$ singularity for each $1 \leq n \leq a - 1$. Due to (4.15c), the remaining terms in (4.15a) are $S_{1,1}, S_{0,1}, S_{0,2}$ and $S_{0,3}$. By the definition $S_{0,1}$ is independent of $Q_r$ and it is straightforward to check that $S_{1,1} + S_{0,2} + S_{0,3}$ is free of color $r$ singularity by using (4.6b,c). This establishes the Theorem.

4.4 Eigenvalue $\Lambda_m^{(a)}(u)$ for $1 \leq a \leq r - 1$. Starting from (4.9) and $\Lambda_1^{(r)}(u)$ that will be described in section 4.5, we are to solve the $T$-system (2.5a) with

$$g_m^{(a)}(u) = 1 \quad \text{for } 2 \leq a \leq r,$$

$$g_m^{(1)}(u) = \prod_{k=1}^{m} g_1^{(1)}(u + m + 1 - 2k),$$

$$g_1^{(1)}(u) = F_2^{(p,r)}(u),$$

where the last quantity has been given in (4.9b) and (4.10). The solution will yield a DVF for the general eigenvalue $\Lambda_m^{(a)}(u)$. Here we shall present the so derived conjecture for $1 \leq a \leq r - 1$.

Let $T_m^{(a)}(1 \leq a \leq r - 1)$ be the set of the $a \times m$ rectangular tableaux containing $j,k$ $i_{jk} \in J$ at the $(j,k)$ position.

\[
\begin{array}{cccc}
  i_{11} & \cdots & i_{1m} \\
  i_{21} & \cdots & i_{2m} \\
  \vdots \\
  i_{a1} & \cdots & i_{am}
\end{array}
\]

The entries are to obey the conditions

$$i_{jk} < i_{j+1k} \text{ or } i_{jk} = i_{j+1k} = 0 \text{ for any } 1 \leq j \leq a - 1, 1 \leq k \leq m,$$  

$$i_{jk} < i_{j,k+1} \text{ or } i_{jk} = i_{j,k+1} \in J \setminus \{0\} \text{ for any } 1 \leq j \leq a, 1 \leq k \leq m - 1.$$  

(4.18a)

(4.18b)

Notice that (4.18a) is equivalent to saying that each column belongs to $T_1^{(a)}$ defined in (4.7). We identify each element of $T_m^{(a)}$ as above with the following product of (4.4a):

$$\prod_{j=1}^{a} \prod_{k=1}^{m} [j,k]_{u - u + a - m - 2j + 2k}.$$  

(4.19)

Then we conjecture that the $T$-system (2.5a) with (4.17) and the initial condition (4.9) leads to

$$\Lambda_m^{(a)}(u) = \frac{1}{\prod_{k=1}^{m} F_2^{(p,r)}(u - m - 1 + 2k)} \sum_{T \in T_m^{(a)}} T \quad 1 \leq a \leq r - 1, m \in \mathbb{Z}_{\geq 1}.$$  

(4.20)
From (4.18a) and the remark after (4.10), the rhs is homogeneous of degree \(2pm\) w.r.t \(\phi\).
The conjecture (4.20) reduces to (4.9a) when \(m = 1\). For \(B_2\), (4.20) is certainly true because 
\(\Lambda_m^{(1)}(u)\) of \(B_2\) equals \(\Lambda_m^{(2)}(u)\) of \(C_2\) given in (3.27) under the exchange \(Q_1(u) \leftrightarrow Q_2(u)\).
The cases \(m = 2, a = 1, 2\) have also been checked directly for \(B_3\) and \(B_4\). As a further support, we have verified 
\(\# T_m^{(a)} = \dim W_m^{(a)}\) by computer for several values of \(a\) and \(m\). For example, both sides yields 247500 for \(B_3, a = m = 3\). We emphasize that the set \(T_m^{(a)}\) is specified by a remarkably simple rule (4.18). It would deserve to be a proper base of the \(U_q(B_1^{(1)})\)
or \(Y(B_r)\) module \(W_m^{(a)}\) having the classical content (2.4b).

4.5 Eigenvalue \(\Lambda_1^{(r)}(u)\). From (2.4b) the relevant auxiliary space is \(W_1^{(r)} \simeq V(\omega_r)\) as 
a \(B_r\)-module. This is the spin representation, whose weights are all multiplicity-free and given by
\[
\frac{1}{2}(\mu_1 \epsilon_1 + \cdots + \mu_r \epsilon_r), \quad \mu_1, \ldots, \mu_r = \pm. \tag{4.21}
\]
Thus we shall introduce another kind of elementary boxes \([\mu_1, \mu_2, \ldots, \mu_r]\) by which the
DVF can be written as
\[
\Lambda_1^{(r)}(u) = \sum_{\{\mu_j = \pm\}}^{r} \left[ \begin{array}{c} \mu_1, \mu_2, \ldots, \mu_r \end{array} \right]. \tag{4.22}
\]

We let \(T_1^{(r)}\) denote the set of \(\dim W_1^{(r)} = 2^r\) boxes as above. The indices \(r\) and \(p\) here 
ecognize the rank of \(B_r\) and the quantum space \(\otimes_{j=1}^{N} W_1^{(p)}(w_j)\), respectively. Each box is 
identified with a product of dress and vacuum parts that are defined via certain recursion 
relations w.r.t these indices. To describe them we introduce the operators \(\tau_\gamma^u, \tau_\gamma^Q\) and \(\tau_\gamma^C\)
acting on the DVF (2.9b) as follows.
\[
\begin{align*}
\tau_\gamma^u & : u \to u + \gamma, \tag{4.23a} \\
\tau_\gamma^Q & : Q_a(u) \to Q_{a+1}(u), \tag{4.23b} \\
\tau_\gamma^C & : Q_a(u+x) \to Q_a(u+\gamma-x), \phi(u+x) \to \phi(u+\gamma-x) \text{ for any } x. \tag{4.23c}
\end{align*}
\]
By the definition they obey the relations
\[
\begin{align*}
\tau_\gamma^Q \tau_\gamma^u & = \tau_\gamma^u \tau_\gamma^Q, \quad \tau_\gamma^Q \tau_\gamma^C = \tau_\gamma^C \tau_\gamma^Q, \tag{4.24a} \\
\tau_\gamma^C \tau_\gamma^C & = \tau_{\gamma-\gamma}^u, \quad \tau_\gamma^u \tau_\gamma^\gamma = \tau_{\gamma+\gamma}^u. \tag{4.24b}
\end{align*}
\]
In view of (2.8), \(\tau_\gamma^Q\) is equivalent to \(N_a \to N_{a+1}\) and \(u_j^{(a)} \to u_j^{(a+1)}\). It is to be understood 
as replacing \(Q_a(u)\) with \(1 \leq a \leq r \leq -1\) for \(B_{r-1}\) by \(Q_{a+1}(u)\) for \(B_r\). The operator \(\tau_\gamma^C\) will 
be used to describe the transformation (2.19) concerning the crossing symmetry. Now the 
recursion relations read,
\[
\left[ +, +, x \right]_p^r = \phi(u + r + p - \frac{3}{2} + \frac{1}{t_p} \tau_\gamma^Q \left[ +, x \right]_{p-1}^{r-1}. \tag{4.25a}
\]
\[
\begin{align*}
\phi(u + r + p - \frac{3}{2}) = & \frac{1}{t_p} \frac{Q_1(u + r - \frac{5}{2})}{Q_1(u + r - \frac{1}{2})}, \\
& \text{vac} \mu_{\pm} = \begin{cases} 
\phi(u + \frac{5}{2}) & p = 1 \\
\phi(u + 1)\phi(u + 3) & p = 2
\end{cases}, \\
& \text{vac} \mu_{\mp} = \begin{cases} 
\phi(u + \frac{5}{2}) & p = 1 \\
\phi(u)\phi(u + 3) & p = 2
\end{cases}
\end{align*}
\]

(4.25b)

\[
\begin{align*}
\phi(u + r - p + \frac{1}{2}) = & \frac{1}{t_p} \frac{Q_1(u + r + \frac{3}{2})}{Q_1(u + r + \frac{1}{2})}, \\
& \text{vac} \mu_{+} = \begin{cases} 
\phi(u + \frac{1}{2}) & p = 1 \\
\phi(u)\phi(u + 3) & p = 2
\end{cases}, \\
& \text{vac} \mu_{-} = \begin{cases} 
\phi(u + \frac{1}{2}) & p = 1 \\
\phi(u)\phi(u + 2) & p = 2
\end{cases}
\end{align*}
\]

(4.26a)

where \(\xi\) denotes arbitrary sequence of \(\pm\) symbols with length \(r - 2\). The recursions (4.25) are valid for \(1 \leq p \leq r\) and \(r \geq 3\). The initial condition is given by

\[
\text{vac} \mu_{\pm} = \begin{cases} 
\phi(u + \frac{5}{2}) & p = 1 \\
\phi(u + 1)\phi(u + 3) & p = 2
\end{cases},
\]

(4.26b)

As for the dress parts we simply let \(\text{dr} \mu_{\pm} \) be the same for any \(0 \leq p \leq r\).

This is consistent with (4.26a) and the dress universality (2.11). Under these setting the recursions (4.25) and the initial condition (4.26) provide a complete characterization of our \(\mu_{\pm}\) for any \(0 \leq p \leq r, r \geq 2\) and \(\{\mu_j\}\). Thus we have presented the DVF (4.22) for the eigenvalue \(\Lambda_{1}^{(r)}(u)\). In the rational case \((q \to 1)\) with \(p = 1\), a similar recursive description is available in [5].

Let us observe various features of our DVF (4.22) before proving that it is pole-free in section 4.6. Firstly, it is easy to calculate the vacuum parts explicitly.

\[
\text{vac} \mu_{\pm} = \psi_{\pm}^{(p,r)}(u),
\]

(4.27a)
\[ n = \# \{ j \mid \mu_j = -1, 1 \leq j \leq p \}, \quad (4.27b) \]
\[
\psi_n^{(p,r)}(u) = \prod_{j=0}^{n-1} \phi(u + r - p + 2j + \frac{1}{2} - \frac{1}{t_p})
\prod_{j=n}^{p-1} \phi(u + r - p + 2j + \frac{1}{2} + \frac{1}{t_p}) \quad 0 \leq n \leq p. \quad (4.27c)
\]

This is order \( p \) w.r.t \( \phi \). Secondly, the top term is given by
\[
\underbrace{+_{p}^{r}}_{p} = \psi_n^{(p,r)}(u) \frac{Q_r(u - \frac{1}{2})}{Q_r(u + \frac{1}{2})}. \quad (4.28)
\]

This is consistent with (2.12) since the above box is associated with the highest weight \( (\epsilon_1 + \cdots + \epsilon_r)/2 = \omega_r \) from (4.1) and (4.21). Thirdly, the crossing symmetry \( \Lambda_1^{(r)}(u) = (-)^{pN} \Lambda_1^{(r)}(-2r + 1 - u) \mid_{u_j \rightarrow -u_j, \omega_j \rightarrow -\omega_j} \) is valid, which is precisely (2.17) with the order \( k = p \) as remarked above. At the level of the boxes, this is due to
\[
\tau_{2r-1}^{C} \underbrace{\mu_1, \ldots, \mu_r}_{p} = \underbrace{-\mu_1, \ldots, -\mu_r}_{p}, \quad (4.29)
\]
where the effect of \( (-)^{pN} \) has been absorbed into \( \tau_{2r-1}^{C} \) as explained in (2.19). In the sequel, we will write such \( \pm \) sequences as above simply as \( \mu \) and \( \overline{\mu} \), etc. As an warming-up exercise let us show (4.29) by induction on \( r \). In view of \( (n^C)^2 = 1 \), it suffices to check the two cases \( (\mu_1, \mu_2) = (+, +) \) and \( (+, -) \). We shall do the former case and leave the latter to the readers. Put \( \mu = (+, +, \nu) \) with \( \nu \) being a length \( r - 2 \) sequence of \( \pm \). Then the l.h.s of (4.29) becomes
\[
\tau_{2r-1}^{C} \underbrace{+_{p}^{r}, +, \nu}_{p} = \tau_{2r-1}^{C} \left( \phi(u + r + p - 3 \frac{1}{2} + \frac{1}{t_p}) \tau_{p}^{Q} \underbrace{+_{p}^{r-1}, \nu}_{p} \right)
\]
\[
= \phi(u + r - p + \frac{1}{2} - \frac{1}{t_p}) \tau_{2r-1}^{C} \tau_{p}^{Q} \tau_{2r-3}^{C} \underbrace{-_{p}^{r-1}}_{p}
\]
where we have used (4.25a) in the first line and the induction assumption in the second line. By means of (4.24) one may substitute \( \tau_{2r-1}^{C} \tau_{p}^{Q} \tau_{2r-3}^{C} = \tau_{p}^{u \cdot Q} \) into the latter. From (4.25d), the result is equal to \( \underbrace{-_{p}^{r}}_{p} \), which is the r.h.s of (4.29).

4.6 **Pole-freeness of** \( \Lambda_1^{(r)}(u) \). In section 4.5, we have formally allowed \( p = 0 \) in the boxes that consist of the DVF (4.22). Correspondingly, we find it convenient to consider the BAE with \( p = 0 \) as the one obtained from (2.7) by setting its l.h.s always \(-1 \). We shall quote (2.7) as BAE\(_p^r\). Our aim here is to establish
\textbf{Theorem 4.6.1.} For \( r \geq 2 \) and \( 0 \leq p \leq r \), \( \Lambda_1^{(r)}(u) \) (4.22) is free of poles provided that the BAE \( \kappa_p \) (2.7) (for \( s = 1 \)) is valid.

We are to show that color \( a \) poles \( 1/Q_a \) are spurious for each \( 1 \leq a \leq r \). The poles are located by

\textbf{Lemma 4.6.2.} For \( 1 \leq a \leq r - 1 \) the factor \( 1/Q_a \) is contained in the box \( \mu_1, \ldots, \mu_r \)

if and only if \( (\mu_a, \mu_{a+1}) = (+, -) \) or \( (-, +) \). Any two such boxes \( \eta, +, -, \xi \) and \( \eta, -, +, \xi \) share a common color \( a \) pole \( 1/Q_a(u + y) \) for some \( y \). The factor \( 1/Q_r \)

is contained in all the boxes. For \( \epsilon = \pm \), any two boxes \( \zeta, \epsilon, \epsilon \) and \( \zeta, \epsilon, -\epsilon \) share a common color \( r \) pole \( 1/Q_r(u + z) \) for some \( z \).

The assertions are immediate consequences of (4.25) and (4.26). If one puts

\( \lambda = (\eta, +, -, \xi) \), \( \mu = (\eta, -, +, \xi) \) and identifies them with the weights via (4.21), one has \( \lambda - \mu = \epsilon_a - \epsilon_{a+1} = \alpha_a \) for \( 1 \leq a \leq r - 1 \) by (4.21). A similar relation holds for \( a = r \) as well. Thus the above Lemma is another example of the coupling rule (2.14a). In this view Theorem 4.6.1 is a corollary of

\textbf{Theorem 4.6.3.} For \( 1 \leq a \leq r - 1 \), let \( \eta, \xi \) and \( \zeta \) be any \( \pm \) sequences with lengths \( a - 1, r - a - 1 \) and \( r - 2 \), respectively. If the BAE \( \kappa_p \) (2.7) (for \( s = 1 \)) is valid, then

\[
\text{Re} s_{u = -y + iu_1^{(a)}} \begin{pmatrix} \eta, +, -, \xi \end{pmatrix} + \begin{pmatrix} \eta, -, +, \xi \end{pmatrix} = 0, \tag{4.30a}
\]

\[
\text{Re} s_{u = -z + iu_1^{(r)}} \begin{pmatrix} \zeta, \pm, \pm \end{pmatrix} + \begin{pmatrix} \zeta, \pm, \mp \end{pmatrix} = 0, \tag{4.30b}
\]

where \( y \) and \( z \) are those in Lemma 4.6.2.

The rest of the present subsection will be devoted to a proof of this Theorem assuring that all the color \( a \) poles are spurious. In fact the proof will be done essentially by establishing (2.14b) and (2.15). It follows that the character limit (2.16) is also valid for \( \Lambda_1^{(r)}(u) \) (4.22). We prepare

\textbf{Lemma 4.6.4.} Let \( \xi \) be any sequence of \( \pm \) with length \( r - 1 \). Then

\[
\tau_{2r+1}^{C} \begin{pmatrix} \mu, -, \xi \end{pmatrix} = \left( \frac{\phi(u + r + p + \frac{1}{2} + \frac{1}{p})}{\phi(u + r - p + \frac{1}{2} - \frac{1}{p})} \right)^{1-\delta_{p0}} Q_1(u + r - \frac{1}{2}) \begin{pmatrix} \mu, -, \xi \end{pmatrix} \tag{4.31}
\]
for $0 \leq p \leq r$.

**Proof.** We show this for $\xi = (+, \nu)$. The case $\xi = (-, \nu)$ is similar. Suppose $p \geq 1$. Then the lhs of (4.31) is rewritten as

$$
\tau_{2r+1}^C \begin{array}{c}
\tau_{2r+1}^{C} \\
p
\end{array} \tau_{2r}^{C} \begin{array}{c}
+,-,\nu \\
p
\end{array} = \phi(u+r+p+\frac{1}{2} + \frac{1}{t_p} \tau^Q_{2r} \begin{array}{c}
+,-,\nu \\
p
\end{array}

(4.32)
$$

by means of (4.24b), (4.25a) and (4.29). In the rhs of (4.31), the box part is $\begin{array}{c}
+,+,\nu \\
p
\end{array}$.

Replacing this by the rhs of (4.25c) with $\xi = \nu$, one finds that the resulting expression coincides with the last line in (4.32). The case $p = 0$ follows from this and (4.26b).

Finally we give

**Proof of Theorem 4.6.3.** It is straightforward to check (4.30) for $r = 2$ by (4.26a). We assume that Theorem is true for $B_{r-1}$ and use induction on $r$. We shall verify (4.30a) only. Eq. (4.30b) can be shown more easily by a similar method. In the sequel the cases $a \geq 3$, $a = 2$ and $a = 1$ are considered separately.

**The case** $a \geq 3$. Put $\eta = (\eta_1, \eta')$. Then (4.25) transforms the sum of the two boxes in (4.30a) into

$$
X_1 \tau_{1-\eta_1}^u \tau^Q \begin{array}{c}
r-1 \\
p-1
\end{array} + \begin{array}{c}
r-1 \\
p-1
\end{array},
$$

where $X_1$ involves only $\phi$ and possibly $Q_1/Q_1$. Then Lemma 4.6.2 implies that the poles $1/Q_a(u+y)$ in (4.30a) must originate in the factor $1/Q_{a-1}(u+y-1+\eta_1)$ shared by the above two boxes. Thus the lhs of (4.30a) is proportional to $Res_{u=-y+1-\eta_1+i\nu^Q_{a-1}}$ of the sum of the above boxes. But this is a case of (4.30a) for $B_{r-1}$ hence 0 by the induction assumption.

**The case** $a = 2$. In (4.30a) the length of $\eta$ is 1. We consider the case $\eta = +$. The proof for $\eta = -$ is almost identical. First we rewrite the two boxes in (4.30a) by (4.25) as follows.

$$
\begin{array}{c}
+,+,\nu \\
p
\end{array} = X_2 \tau^Q \begin{array}{c}
r \\
p
\end{array} \begin{array}{c}
r-1 \\
p-1
\end{array} \begin{array}{c}
+,+,\xi \\
p
\end{array} = X_2 \frac{Q_1(u+r-\frac{5}{2})}{Q_1(u+r-\frac{1}{2})} \tau^Q \begin{array}{c}
r-1 \\
p-1
\end{array} \begin{array}{c}
+,\xi \\
p
\end{array}

(4.33)
$$

where $X_2 = \phi(u+r+p-\frac{3}{2} + \frac{1}{t_p})$ is independent of $Q_b$’s. By further using (4.25), one finds that

$$
\begin{array}{c}
r-1 \\
p-1
\end{array} \begin{array}{c}
+,\xi \\
p
\end{array} = X_3 \frac{Q_1(u+r-\frac{7}{2})}{Q_1(u+r-\frac{3}{2})}, \quad \begin{array}{c}
r-1 \\
p-1
\end{array} \begin{array}{c}
-,+,\xi \\
p
\end{array} = X_4 \frac{Q_1(u+r+\frac{1}{2})}{Q_1(u+r-\frac{3}{2})},

(4.34)
$$

29
where \( X_3 \) and \( X_4 \) do not involve \( Q_1 \). From the induction assumption, the sum of these boxes must be ensured to be regular at \( u = -r + \frac{3}{2} + i u_k^{(1)} \) via the \( \text{BAE}^r_{p-1} \). Therefore

\[
\begin{align*}
&\left[ \begin{array}{l}
+,-,+,\xi \\
p
\end{array} \right] / \left[ \begin{array}{l}
+,+,+,\xi \\
p-1
\end{array} \right] \\
&= Q_1(u + r + \frac{1}{2})Q_2(u + r - \frac{5}{2})\phi(u + r - \frac{3}{2} - \frac{1}{r_p} \delta_{1p-1}) \\
&\quad \cdot Q_1(u + r - \frac{7}{2})Q_2(u + r - \frac{1}{2})\phi(u + r - \frac{3}{2} + \frac{1}{r_p} \delta_{1p-1})
\end{align*}
\]

should hold in order that the lhs be evaluated as \(-1\) at \( u = -r + \frac{3}{2} + i u_k^{(1)} \) from the \( \text{BAE}^r_{p-1} \). In deriving (4.35) we have used the fact that \( t_{p-1} \) of \( B_{r-1} \) is equal to \( t_p \) of \( B_r \). Combining (4.33) and (4.35) one deduces

\[
\begin{align*}
&\left[ \begin{array}{l}
+,+,+,\xi \\
p
\end{array} \right] / \left[ \begin{array}{l}
+,-,+,\xi \\
p
\end{array} \right] \\
&= \frac{Q_1(u + r - \frac{5}{2})Q_2(u + r + \frac{1}{2})Q_3(u + r - \frac{5}{2})\phi(u + r - \frac{3}{2} - \frac{1}{r_p} \delta_{2p})}{Q_1(u + r - \frac{7}{2})Q_2(u + r - \frac{1}{2})Q_3(u + r - \frac{1}{2})\phi(u + r - \frac{3}{2} + \frac{1}{r_p} \delta_{2p})}
\end{align*}
\]

Thanks to \( \text{BAE}^r_p \) (2.7), this is indeed \(-1\) at \( u = -r + \frac{3}{2} + i u_k^{(2)} \), proving that color \( a = 2 \) poles are spurious.

The case \( a = 1 \). The two boxes in (4.30a) are \( \left[ \begin{array}{l}
+,-,\xi \\
p
\end{array} \right] \) and \( \left[ \begin{array}{l}
-,+\xi \\
p
\end{array} \right] \). From (4.25b,c) they share a color \( a = 1 \) pole at \( u = -r + \frac{1}{2} + i u_k^{(1)} \). Let us rewrite the latter as follows.

\[
\begin{align*}
&\left[ \begin{array}{l}
+,-,\xi \\
p
\end{array} \right] = \tau^C_{2r-1} \left[ \begin{array}{l}
-,+,\xi \\
p
\end{array} \right] \\
&\quad \cdot \phi(u + r - p + \frac{1}{2} - \frac{1}{r_p}) \frac{Q_1(u + r + \frac{3}{2})Q_2(u + r - \frac{1}{2})}{Q_1(u + r - \frac{3}{2})Q_2(u + r + \frac{1}{2})} \tau^C_{2r-1} \left[ \begin{array}{l}
-,+,\xi \\
p
\end{array} \right]
\end{align*}
\]

where we have used (4.29) and (4.25b). In the last line, \( \tau^C_{2r-1} \left[ \begin{array}{l}
-,+,\xi \\
p
\end{array} \right] \) can be further rewritten by applying Lemma 4.6.4 with \( r \to r - 1, p \to p - 1 \). Dividing the resulting expression by the rhs of (4.25b) we obtain

\[
\begin{align*}
&\left[ \begin{array}{l}
-,+,\xi \\
p
\end{array} \right] / \left[ \begin{array}{l}
+,-,\xi \\
p
\end{array} \right] = \frac{Q_1(u + r + \frac{3}{2})Q_2(u + r - \frac{3}{2})}{Q_1(u + r - \frac{3}{2})Q_2(u + r + \frac{1}{2})} \left( \frac{\phi(u + r - p + \frac{1}{2} - \frac{1}{r_p})}{\phi(u + r + p - \frac{3}{2} + \frac{1}{r_p})} \right) \delta_{1p}
\end{align*}
\]
At the pole location \( u = -r + \frac{1}{2} + i \alpha_k^{(1)} \), this is just \(-1\) owing to \( \text{BAE}_p \) (2.7) with \( a = 1 \). Therefore (4.30a) is free of color 1 poles. This completes the proof of Theorem 4.6.3 hence Theorem 4.6.1.

4.7 Relations between two kinds of boxes. Here we clarify the relation between the two kinds of the boxes \( [\alpha] \) and \( [\mu_1, \ldots, \mu_r] \) introduced in section 4.1 and 4.5, respectively. In terms of the relevant auxiliary spaces, they are associated with the vector and the spin representations. To infer their relation, recall the classical tensor product decomposition

\[
V(\omega_r) \otimes V(\omega_r) = V(2\omega_r) \oplus V(\omega_{r-1}) \oplus \cdots \oplus V(\omega_1) \oplus V(0). \tag{4.37}
\]

Correspondingly, there exists an \( U_q(B^{(1)}_q) \) quantum \( R \)-matrix \( R_{W_1^{(r)}, W_1^{(r)}}(u) \) [32] acting on the \( q \)-analogue of the above. On each component \( V(\omega) \) of the rhs, it acts as a constant \( \rho_\omega(u) \) that depends on the spectral parameter \( u \). A little investigation of the spectrum \( \rho_\omega(u) \) in [32] tells that only \( \rho_{\omega_2}(u), \rho_{\omega_3}(u), \ldots \) are non zero at \( u = -2(r-a)+1 \) for \( 1 \leq a \leq r-1 \). From this and (2.4b) we see that the specialized \( R \)-matrix \( R_{W_1^{(r)}, W_1^{(r)}}(-2(r-a)+1) \) yields the embedding

\[
W_1^{(a)}(u) \hookrightarrow W_1^{(r)}(u + r - a - \frac{1}{2}) \otimes W_1^{(r)}(u - r + a + \frac{1}{2}) \tag{4.38}
\]

in the notation of [8]. According to the arguments there, (4.38) imposes the following functional relation among the transfer matrices having the relevant auxiliary spaces:

\[
T_1^{(r)}(u + r - a - \frac{1}{2})T_1^{(r)}(u - r + a + \frac{1}{2}) = T_1^{(a)}(u) + T'(u) \quad \text{for } 1 \leq a \leq r - 1. \tag{4.39}
\]

Here \( T'(u) \) denotes some matrix commuting with all \( T_m^{(b)}(v) \)'s. When \( a = r - 1 \), (4.39) is just the last equation in (2.5a) with \( m = 0 \), hence \( T'(u) = T_2^{(a)}(u) \). Viewed as a relation among the eigenvalues, (4.39) implies that each term in the \( \text{DVF} \) (4.9a) can be expressed as a product of certain two boxes in section 4.5 with the spectral parameters \( u + r - a - \frac{1}{2} \) and \( u - r + a + \frac{1}{2} \). Actually we have

**Theorem 4.7.1.** For \( 1 \leq a \leq r - 1, k, n, l \in \mathbb{Z}_{\geq 0} \) such that \( k + n + l = a \), take any integers \( 1 \leq i_1 < \cdots < i_k \leq r \) and \( 1 \leq j_1 < \cdots < j_l \leq r \). Then the following equality holds between the elements of \( T_1^{(a)} \) and \( \mathcal{T}_1^{(r)} \) defined in (4.7,8) and (4.25,26), respectively.

\[
\left( \tau_{-r+a+\frac{1}{2}}^{u} \prod_{p=1}^{r} \frac{\mu_p}{\mu_{p+1}} \right) \left( \tau_{r-a-\frac{1}{2}}^{u} \prod_{p=1}^{r} \frac{\nu_p}{\nu_{p+1}} \right),
\]

\[
(4.40a)
\]
where there are $n$ $\boxed{0}$s in the lhs and $F_a^{(p,r)}(u)$ is defined in (4.9b) and (4.10). The $\pm$ sequences in the rhs are specified by

$$
\mu_b = \begin{cases} 
+ & \text{if } b \in \{i_1, \ldots, i_k\}, \\
- & \text{otherwise}
\end{cases},
$$

$$
\nu_b = \begin{cases} 
- & \text{if } b \in \{j_1, \ldots, j_l\}, \\
+ & \text{otherwise}
\end{cases}.
$$

(4.40b)

Note that both sides of (4.40a) are of order $2p$ w.r.t $\phi$ and carry the same weight $\epsilon_{i_1} + \cdots + \epsilon_{i_k} - \epsilon_{j_1} - \cdots - \epsilon_{j_l}$. The Theorem is again proved by induction on the rank $r$.

To do so we write the boxes (4.4a) as $\boxed{b}_p$ to mark the $r, p$ dependence explicitly. Then they enjoy the recursive property as follows.

**Lemma 4.7.2.** For $1 \leq p \leq r$ and $2 \leq b \leq r$, the boxes (4.4) fulfill

$$
\boxed{b}_p = X_1 \left( \frac{Q_1(u + 3)}{Q_1(u + 1)} \right)^{\delta_{2b}} \tau_{1_r}^u R^Q \left( \frac{r-1}{p-1} \right),
$$

$$
X_1 = \phi(u - p + 2 - \frac{1}{l_p}) \phi(u + 2r + p - 3 + \frac{1}{l_p}).
$$

(4.41)

This can be checked directly by using the explicit form (4.4). In particular, one uses $X_1 = \Phi(u)/\Phi^{-1}(u + 1)$.

**Proof of Theorem 4.7.1 for $a = k = 1$ and $n = l = 0$.** We illustrate an inductive proof w.r.t $r$ in this case. General cases can be verified based on it by a similar idea through tedious calculations. Eq.(4.40) can be directly checked for $r = 2$. By letting $i_1 = b$ and noting that $F_1^{(p,r)}(u) = 1$, (4.40) reads

$$
\boxed{b}_p = \left( \tau_{-r+\frac{b}{p}}^{u} \tau_{-\frac{1}{p-1}}^{u} \tau_{\frac{r}{p}}^{u} \right) \left( \tau_{r-\frac{b}{p}}^{u} \tau_{\frac{r-1}{p-1}}^{u} \tau_{\frac{r}{p-1}}^{u} \right),
$$

(4.42)

where the $+$ symbol in the first box on the rhs is located at $b$-th position from the left. Below we focus on the case $b \geq 2$ and leave $b = 1$ case to the readers. Then, by applying the recursions (4.25) and (4.24) the rhs is rewritten as

$$
X_1 \left( \frac{Q_1(u + 3)}{Q_1(u + 1)} \right)^{\delta_{2b}} \tau_{1_r}^u R^Q \left( \tau_{-\frac{r}{p-1}}^{u} \tau_{\frac{r-1}{p-1}}^{u} \tau_{\frac{r}{p-1}}^{u} \right),
$$

(4.43)

where $X_1$ is the one in (4.41). The $+$ symbol in the first box is now at the $b - 1$ th position. The quantity in the largest parenthesis of (4.43) is precisely the rhs of (4.42) with $r \rightarrow r - 1$ and $p \rightarrow p - 1$. By induction one may replace it with $\boxed{b}_p$. The resulting expression is just the rhs of (4.41) hence equal to $\boxed{b}_p$ by Lemma 4.7.2. This completes the induction step hence the proof.
5. Eigenvalues for \( D_r \)

Our results for \( D_r = so(2r) \) are quite parallel with those for \( B_r = so(2r + 1) \) in many respects. In fact many formulas here becomes those in section 4 through a formal replacement \( r \rightarrow r + \frac{1}{2} \). Thus we shall state them without a proof, which can be done in a similar manner to the \( B_r \) case. We will introduce two kinds of boxes associated with the vector and the spin representations and clarify their relation. A distinct feature in \( D_r \) is that there are two representations of the latter kind, \( V(\omega_{r-1}) \) and \( V(\omega_r) \), each having the quantum affine analogue \( W_1^{(r-1)} \) and \( W_1^{(r)} \), respectively. They are interchanged under the Dynkin diagram automorphism. In order to respect the symmetry under it, we modify the quantum spaces \( \otimes_{j=1}^N W_1^{(p_j)}(w_j) \) for \( p = r - 1 \) and \( r \) into \( \otimes_{j=1}^N W_1^{(\pm)}(w_j) \) where \( W_1^{(\pm)}(w) = W_1^{(r)}(w + 2) \otimes W_1^{(r-1)}(w + 2) \). Pictorially, one may view this as arranging the vertical lines on the square lattice endowed with the modules \( V(\omega_r), V(\omega_{r-1}) \) alternately and with the inhomogeneity as \( w_1 = 2, w_1 = 2 + 2, w_2 = 2, w_2 = 2, \ldots \). This pattern has been introduced to utilize the degeneracy of the spin-conjugate spin \( R \)-matrix [32] \( \text{Im} R_{W_1^{(r)}, W_1^{(r-1)}}(u = 4) \simeq V(\omega_r + \omega_{r-1}) \), where the image becomes manifestly symmetric under the automorphism. The BAE (2.7) (with \( s = 1 \) is thereby unchanged as long as \( p = 1, 2, \ldots, r - 2 \). Instead of \( p = r - 1 \) and \( r \), we now take \( p = \pm \), for which the BAE reads

\[
- \frac{\phi_+^p(iu_k^{(a)} + \delta_{ar})\phi_-^p(iu_k^{(a)} + \delta_{ar-1})}{\phi_+^p(iu_k^{(a)} - \delta_{ar})\phi_-^p(iu_k^{(a)} - \delta_{ar-1})} = \prod_{b=1}^r \frac{Q_b(iu_k^{(a)} + (\alpha_a | \alpha_b))}{Q_b(iu_k^{(a)} - (\alpha_a | \alpha_b))}.
\]

(5.1)

Here the functions in the lhs are defined via \( \phi(u) \) (1.4b) by

\[
\phi_\pm^p(u) = \phi(u + 2), \quad \phi_\pm^p(u) = \phi(u - 2).
\]

(5.2)

5.1 Eigenvalue \( \Lambda_1^{(1)}(u) \). Let \( \epsilon_a, 1 \leq a \leq r \) be the orthonormal vectors \( (\epsilon_a | \epsilon_b) = \delta_{ab} \) realizing the root system as follows.

\[
\alpha_a = \begin{cases} 
\epsilon_a - \epsilon_{a+1} & \text{for } 1 \leq a \leq r - 1, \\
\epsilon_r + \epsilon_r & \text{for } a = r
\end{cases}
\]

\[
\omega_a = \begin{cases} 
\frac{1}{2}(\epsilon_1 + \cdots + \epsilon_{r-1} + \epsilon_r) & \text{for } a = 1 \leq a \leq r - 2, \\
\frac{1}{2}(\epsilon_1 + \cdots + \epsilon_{r-1} + \epsilon_r) & \text{for } a = r
\end{cases}
\]

(5.3)

The auxiliary space relevant to \( \Lambda_1^{(1)}(u) \) is \( W_1^{(1)} \simeq V(\omega_1) \) as an \( D_r \)-module by (2.4b). This is the vector representation, whose weights are all multiplicity-free and given by \( \epsilon_a \) and \( -\epsilon_a (1 \leq a \leq r) \). By abbreviating them to \( a \) and \( \bar{a} \), the set of weights and the DVF are given as follows.

\[
J = \{1, 2, \ldots, r, \bar{r}, \ldots, \bar{1}\},
\]

\[
\Lambda_1^{(1)}(u) = \sum_{a \in J} \bar{a}
\]

(5.4)

(5.5)
This is formally the same with (3.2-3). The elementary boxes are defined by

\[ a = \psi_a(u) \frac{Q_{a-1}(u + a + 1)Q_a(u + a - 2)}{Q_{a-1}(u + a - 1)Q_a(u + a)} \quad 1 \leq a \leq r - 2, \]

\[ r - 1 = \psi_{r-1}(u) \frac{Q_{r-2}(u + r)Q_{r-1}(u + r - 3)Q_r(u + r - 3)}{Q_{r-2}(u + r - 2)Q_{r-1}(u + r - 1)Q_r(u + r - 1)}, \]

\[ r = \psi_r(u) \frac{Q_{r-1}(u + r + 1)Q_{r-1}(u + r - 1)Q_r(u + r - 1)}{Q_{r-1}(u + r + 1)Q_{r-1}(u + r - 1)Q_r(u + r - 1)}, \]

\[ \bar{r} = \psi_{\bar{r}}(u) \frac{Q_{r-1}(u + r - 3)Q_{r-1}(u + r - 1)Q_r(u + r - 1)}{Q_{r-1}(u + r - 3)Q_{r-1}(u + r - 1)Q_r(u + r - 1)}, \]

\[ \bar{r} - 1 = \psi_{\bar{r}-1}(u) \frac{Q_{r-2}(u + r + 1)Q_{r-1}(u + r + 1)}{Q_{r-2}(u + r)Q_{r-1}(u + r - 1)Q_r(u + r - 1)}, \]

\[ \overline{a} = \psi_{\overline{a}}(u) \frac{Q_{a-1}(u + 2r - a - 3)Q_a(u + 2r - a)}{Q_{a-1}(u + 2r - a - 1)Q_a(u + 2r - a - 2)} \quad 1 \leq a \leq r - 2, \]

where we have set \( Q_0(u) = 1 \). The vacuum part \( \psi_a(u) \) depends on the quantum space \( \otimes_{j=1}^{N} W_{1}^{(p)}(w_j) \) and is given by

if \( 1 \leq p \leq r - 2 \)

\[ \psi_a(u) = \begin{cases} 
\phi(u + p + 1)\phi(u + 2r - p - 1)\Phi_p(u) & \text{for } 1 \leq a \leq p \\
\phi(u + p - 1)\phi(u + 2r - p - 1)\Phi_p(u) & \text{for } p + 1 \leq a \leq p + 1 \\
\phi(u + p - 1)\phi(u + 2r - p - 3)\Phi_p(u) & \text{for } p \leq a \leq 1 
\end{cases} \]

if \( p = \pm \)

\[ \psi_a(u) = \begin{cases} 
\phi_p^+(u + r)\phi_p^{-}(u + r)\Phi_{r+1}^+(u) & \text{for } 1 \leq a \leq r - 1 \\
\phi_p^+(u + r)\phi_p^{-}(u + r - 2)\Phi_{r+1}^+(u) & \text{for } a = r \\
\phi_p^+(u + r - 2)\phi_p^{-}(u + r - 2)\Phi_{r+1}^+(u) & \text{for } a = \overline{r} \\
\phi_p^+(u + r - 2)\phi_p^{-}(u + r - 2)\Phi_{r+1}^+(u) & \text{for } \overline{r} - 1 \leq a \leq 1 
\end{cases} \]

where

\[ \Phi_p^+(u) = \prod_{j=1}^{p-1} \phi(u + p - 2j - 1)\phi(u + 2r - p + 2j - 1) \]

\[ = \Phi_p^+(2r - 2 - u)|_{w_z = -w_z}. \]

The order \( \prec \) in the set \( J \) (5.4) is specified by

\[ 1 \prec 2 \prec \cdots \prec r - 1 \prec \overline{r} \prec \overline{r - 1} \prec \cdots \prec \overline{2} \prec \overline{1}. \]

We impose no order between \( r \) and \( \overline{r} \). The DVF (5.5) possesses all the features explained in section 2.4. In particular it is pole-free under the BAE (2.7) and (5.1) thanks to the coupling rule (2.14). It can be summarized in the diagram
in the same sense with those in sections 3.1 and 4.1. This is again identical with the crystal graph [24,25]. For \( p = 1 \), the DVF (5.5-6) has been known earlier in [21].

5.2 Eigenvalue \( \Lambda_1^{(a)}(u) \) for \( 1 \leq a \leq r - 2 \). For \( 1 \leq a \leq r - 2 \), let \( T_1^{(a)} \) be the set of the tableaux of the form (3.7a) with \( i_k \in J \) (5.4) obeying the condition

\[
i_k < i_{k+1} \text{ or } (i_k, i_{k+1}) = (r, r) \text{ or } (i_k, i_{k+1}) = (r, r) \text{ for any } 1 \leq k \leq a - 1.
\] (5.8)

We identify each element (3.7a) of \( T_1^{(a)} \) with the product of (5.6a) with the spectral parameters \( u + a - 1, u + a - 3, \ldots, u - a + 1 \) from the top to the bottom as in (4.8). Then the analytic Bethe ansatz yields the following DVF.

\[
\Lambda_1^{(a)}(u) = \frac{1}{F_a^{(p,r)}(u)} \sum_{T \in T_1^{(a)}} T \quad 1 \leq a \leq r - 2.
\] (5.9a)

Here the function \( F_a^{(p,r)}(u) \) is defined by

\[
F_a^{(p,r)}(u) = \left\{ \begin{array}{ll}
\prod_{j=1}^{a-1} \psi_j^{(p,r)}(u + r - a - 1 + 2j) \psi_p^{(p,r)}(u - r + a + 1 - 2j) & \text{for } 1 \leq p \leq r - 2 \\
\prod_{j=1}^{a-1} \psi_j^{(p,r)}(u + r - a - 1 + 2j) \psi_{p-1,-1}^{(p,r)}(u - r + a + 1 - 2j) & \text{for } p = \pm
\end{array} \right.
\] (5.9b)

where \( \psi_n^{(p,r)}(u) \) and \( \psi_{n,\pm}^{(p,r)}(u) \) are specified in (5.15) in section 5.4. Notice that \( F_1^{(p,r)}(u) = 1 \) hence (5.9a) reduces to (5.5) when \( a = 1 \). It can be shown that each summand \( T \) in (5.9a) contains the factor \( F_a^{(p,r)} \). This will be seen manifestly in Theorem 5.5.1. The DVF (5.9) for \( \Lambda_1^{(a)}(u) \) is homogeneous w.r.t \( \phi \) of order \( 2p \) if \( 1 \leq p \leq r - 2 \) and order \( 2r + 2 \) if \( p = \pm \).

One can observe the top term and the crossing symmetry in the DVF (5.9) as done after (3.9). To check the character limit (2.16) is also similar to (4.11). From (2.4b) we must show (4.11c) again for \( 1 \leq a \leq r - 2 \) under the absence of \( y_0 \) in (4.11b). But this is straightforward from (5.8) and by noting that the character formula (4.11d) is still valid for \( D_r \) if \( J \) is taken as (5.4).

By a similar method to Theorem 4.3.1 one can prove

**Theorem 5.2.1.** \( \Lambda_1^{(a)}(u)(1 \leq a \leq r - 2) \) (5.9) is free of poles provided that the BAE (2.7) (with \( s = 1 \)) for \( 1 \leq p \leq r - 2 \) and (5.1) for \( p = \pm \) are valid.

5.3 Eigenvalue \( \Lambda_m^{(1)}(u) \). Starting from (5.9) and the DVFs of \( \Lambda_1^{(r)}(u), \Lambda_1^{(r-1)}(u) \) that will be given in section 5.4, we are to solve the \( T \)-system (2.5c). The scalar \( g_m^{(a)}(u) \) there
is to be taken as (4.17) with $F^{(p, r)}_2(u)$ determined from (5.9b). The solution will yield a DVF for the general eigenvalue $\Lambda_m^{(a)}(u)$. This program is yet to be executed completely. Here we shall only present a conjecture on $\Lambda_m^{(1)}(u)$.

For $m \in \mathbb{Z}_{\geq 1}$, let $T_m^{(1)}$ denote the set of tableaux of the form

\[
\begin{array}{c}
\vdots \\
1 & \cdots & m
\end{array}
\]

with $i_k \in J$ (5.4) obeying the condition

\[i_k \leq i_{k+1} \quad \text{for any } 1 \leq k \leq m - 1, \quad r \text{ and } \Phi \text{ do not appear simultaneously.} \tag{5.10}\]

We identify each element of $T_m^{(1)}$ as above with the product of (5.6a) as follows.

\[
\prod_{k=1}^{m} [i_k - a - m - 1 + 2k].
\]

Then we have the conjecture

\[
\Lambda_m^{(1)}(u) = \sum_{T \in T_m^{(1)}} T, \tag{5.11}
\]

which reduces to (5.5) when $m = 1$. It is easy to prove $\# T_m^{(1)} = \dim W_m^{(1)}$. We have checked (5.11) up to $m = 4$ for $D_4$ and $m = 3$ for $D_5$.

5.4 Eigenvalues $\Lambda_1^{(r-1)}(u)$ and $\Lambda_1^{(r)}(u)$. Now the relevant auxiliary spaces are $W_1^{(r-1)} \simeq V(\omega_{r-1})$ and $W_1^{(r)} \simeq V(\omega_r)$ as $D_r$-modules. They are the two spin representations, whose weights are all multiplicity-free and given by (4.21) for $V(\omega_{r-1})$ if $\mu_1 \mu_2 \cdots \mu_r = -$ and for $V(\omega_r)$ if $\mu_1 \mu_2 \cdots \mu_r = +$. As in the $B_r$ case we shall build the boxes $\begin{array}{c}
\mu_1, \mu_2, \ldots, \mu_r
\end{array}$ by which the DVF can be written as

\[
\Lambda_1^{(r-1)}(u) = \sum_{\{\mu_j = \pm 1\}} \left\{ \mu_1, \mu_2, \ldots, \mu_r \right\}, \tag{5.12a}
\]

\[
\Lambda_1^{(r)}(u) = \sum_{\{\mu_j = \pm 1\}} \left\{ \mu_1, \mu_2, \ldots, \mu_r \right\}. \tag{5.12b}
\]

We let $T_1^{(r-1)}$ and $T_1^{(r)}$ denote the sets of dim $W_1^{(r-1)} = \dim W_1^{(r)} = 2r^{-1}$ boxes in (5.12a) and (5.12b), respectively. The indices $r$ and $p \in \{1, 2, \ldots, r - 2, +, -\}$ signify the rank of $D_r$ and the quantum space $\otimes_{j=1}^{N} W_1^{(p)}(w_j)$, respectively. The boxes are again defined by the recursion relations w.r.t these indices. By using the operators (4.24), they read, for $1 \leq p \leq r - 2$,
\[ +,+,\xi_p = \phi(u + r + p - 1)\tau Q +,\xi_p \]  

\[ +,-,\xi_p = \phi(u + r + p - 1)\frac{Q_1(u + r - 3)}{Q_1(u + r - 1)}\tau Q -,\xi_p \]  

\[ -,+,\xi_p = \phi(u + r - p - 1)\frac{Q_1(u + r + 1)}{Q_1(u + r - 1)}\tau_2^u\tau Q +,\xi_p \]  

\[ -, -,\xi_p = \phi(u + r - p - 1)\tau_2^u\tau Q -,\xi_p \]  

for \( p = \pm \),

\[ +,+,\xi_p = \phi(u + 2r)\tau Q +,\xi_p \]  

\[ +,-,\xi_p = \phi(u + 2r)\frac{Q_1(u + r - 3)}{Q_1(u + r - 1)}\tau Q -,\xi_p \]  

\[ -,+,\xi_p = \phi(u - 2)\frac{Q_1(u + r + 1)}{Q_1(u + r - 1)}\tau_2^u\tau Q +,\xi_p \]  

\[ -, -,\xi_p = \phi(u - 2)\tau_2^u\tau Q -,\xi_p \].  

Here \( \xi \) denotes arbitrary sequence of \( \pm \) symbols with length \( r - 2 \). The recursions (5.13) involve both boxes in \( T_1^{(r-1)} \) and \( T_1^{(r)} \) and hold for \( r \geq 5 \). As in \( B_r \) case, we formally consider boxes with \( p = 0 \) and fix them by (4.26b) and the convention explained after it. We are yet to specify the initial condition, i.e., data for \( D_4 \) case. As for the dress parts,
they are given by

\[
\begin{align*}
&dr_{+++-} = \frac{Q_4(u - 1)}{Q_4(u + 1)}, \\
&dr_{++-+} = \frac{Q_2(u)Q_4(u + 3)}{Q_2(u + 2)Q_4(u + 1)}, \\
&dr_{+-++} = \frac{Q_1(u + 1)Q_2(u + 4)Q_3(u + 1)}{Q_1(u + 3)Q_2(u + 2)Q_3(u + 3)}, \\
&dr_{+-+-} = \frac{Q_1(u + 1)Q_3(u + 5)}{Q_1(u + 3)Q_3(u + 3)}, \\
&dr_{-+-+} = \frac{Q_1(u + 5)Q_3(u + 1)}{Q_1(u + 3)Q_3(u + 3)}, \\
&dr_{-++-} = \frac{Q_1(u + 5)Q_2(u + 2)Q_3(u + 5)}{Q_1(u + 3)Q_2(u + 4)Q_3(u + 3)}, \\
&dr_{--++} = \frac{Q_2(u + 6)Q_4(u + 3)}{Q_2(u + 4)Q_4(u + 5)}, \\
&dr_{--+-} = \frac{Q_4(u + 7)}{Q_4(u + 5)}.
\end{align*}
\]  

(5.14a)

The other 8 are deduced from the above by

\[
\begin{align*}
&dr_{\mu_1, \mu_2, \mu_3, \mu_4} = dr_{\mu_1, \mu_2, \mu_3, -\mu_4} Q_3(u) = Q_4(u),
\end{align*}
\]  

(5.14b)

which is consistent with the diagram automorphism symmetry. In fact, through the recursions (5.13), the property (5.14b) leads to

\[
\begin{align*}
&dr_{\mu_1, \ldots, \mu_{r-1}, \mu_r} = dr_{\mu_1, \ldots, -\mu_{r-1}, \mu_r} Q_{r-1}(u) = Q_r(u).
\end{align*}
\]

As for the vacuum parts, we shall give their general form that includes the initial condition \((r = 4)\) and fulfills the recursions (5.13).

\[
\begin{align*}
&vac_{\mu_1, \ldots, \mu_r} = \left\{ \begin{array}{ll}
\psi_{n,p}^{(p,r)}(u) & \text{for } 1 \leq p \leq r - 2, \\
\psi_{n,p}^{(p,r)}(u) & \text{for } p = \pm
\end{array} \right., \\
&n = \left\{ \begin{array}{ll}
\#\{ j \mid \mu_j = -, 1 \leq j \leq p \} & \text{for } 1 \leq p \leq r - 2, \\
\#\{ j \mid \mu_j = -, 1 \leq j \leq r - 1 \} & \text{for } p = \pm
\end{array} \right., \\
&\psi_n^{(p,r)}(u) = \prod_{j=0, j \neq n}^{p} \phi(u + r - p + 2j - 1), \\
&\psi_{n,+}^{(p,r)}(u) = \psi_{n,-}^{(p,r)}(u) = \prod_{j=0}^{r+1} \phi(u + 2j - 2),
\end{align*}
\]  

(5.15a, 5.15b, 5.15c, 5.15d)
\[ \psi_{n,-}^{(+,r)}(u) = \psi_{n,+}^{(-,r)}(u) = \phi(u + 2n) \prod_{j=0}^{r+1} \phi(u + 2j - 2). \]  

(5.15e)

By the definition, \( n \) ranges over \( 0 \leq n \leq p \) in (5.15c) and \( 0 \leq n \leq r - 1 \) in (5.15d,e). This completes the characterization of all the \( 2^r \) boxes hence the DVF (5.12) for any \( r \geq 4, p \in \{0, 1, \ldots, r - 2, +, -\} \). In the rational case \((q \to 1)\) with \( p = 1 \), a similar recursive description is available in [5].

Let us list a few features explained in section 2.4. Firstly, the top term (2.12) corresponds to

\[
\begin{align*}
\text{dr} \begin{array}{c} +, \ldots, +, - \end{array} & = \frac{Q_{r-1}(u - 1)}{Q_{r-1}(u + 1)}, \\
\text{dr} \begin{array}{c} +, \ldots, +, + \end{array} & = \frac{Q_r(u - 1)}{Q_r(u + 1)},
\end{align*}
\]

(5.16)

where the lhs’ are indeed associated with the highest weights \( \omega_r \) and \( \omega_{r-1} \) in view of (5.3) and (4.21). Secondly, the crossing symmetry (2.18,19) holds.

\[
\begin{align*}
\tau^C_{r-2} \begin{array}{c} \mu_1, \ldots, \mu_r \end{array} & = \begin{array}{c} -\mu_1, \ldots, -\mu_r \end{array} \\
\tau^C_{r-2} \begin{array}{c} \mu_1, \ldots, \mu_r \end{array} & = \begin{array}{c} -\mu_1, \ldots, -\mu_r \end{array} \\
\end{align*}
\]

(5.17)

for \( 1 \leq p \leq r - 2 \), for \( p = \pm \).

Thirdly, the coupling rule (2.14a) is valid due to

**Lemma 5.4.1.** For \( 1 \leq a \leq r - 1 \) the factor \( 1/Q_a \) is contained in the box \( \begin{array}{c} \mu_1, \ldots, \mu_r \end{array} \)

if and only if \((\mu_a, \mu_{a+1}) = (+, -) \) or \((-+, +)\). Any two such boxes \( \begin{array}{c} \eta, +, - , \xi \end{array} \)

and \( \begin{array}{c} \eta, +, + , \xi \end{array} \)

share a common color a pole \( 1/Q_a(u + y) \) for some \( y \). The factor \( 1/Q_r \)

is contained in the box \( \begin{array}{c} \mu_1, \ldots, \mu_r \end{array} \) if and only if \( \mu_{r-1} = \mu_r \). Any two such boxes

\( \begin{array}{c} \zeta, +, + \end{array} \) and \( \begin{array}{c} \zeta, -, - \end{array} \)

share a common color r pole \( 1/Q_r(u + z) \) for some \( z \).

As introduced in the beginning of section 4.6, let \( \text{BAE}^r_{p=0} \) be (2.7) with the lhs being always \(-1\). Under the BAE, the pair of the coupled boxes yield zero residue in total. We claim this in
Theorem 5.4.2. For $1 \leq a \leq r - 1$, let $\eta, \xi$ and $\zeta$ be any $\pm$ sequences with lengths $a - 1, r - a - 1$ and $r - 2$, respectively. If the $\text{BAE}_p^s$ (2.7) (with $s = 1$) for $0 \leq p \leq r - 2$ and (5.1) for $p = \pm$ are valid, then

$$\text{Res}_{u = -y + i u_k^{(a)}} \left( \sum_{p = 0}^{r} \eta, +, -, \xi \right) + \left( \sum_{p = 0}^{r} \eta, -, +, \xi \right) = 0, \quad (5.18a)$$

$$\text{Res}_{u = -z + i u_k^{(r)}} \left( \sum_{p = 0}^{r} \zeta, +, + \right) + \left( \sum_{p = 0}^{r} \zeta, -, - \right) = 0, \quad (5.18b)$$

where $y$ and $z$ are those in Lemma 5.4.1.

The proof is similar to that for Theorem 4.6.3. In particular (2.14b) and (2.15) can be shown, therefore the character limit (2.16) is valid for the DVFЗs (5.12). (When $p = \pm$, one modifies the $\omega_{s=1}^{(p)}$ in (2.16) suitably.) Notice that both of the coupled boxes in (5.18) belong to the same set $\mathcal{T}_1^{(r-1)}$ or $\mathcal{T}_1^{(r)}$. Thus Lemma 5.4.1 and Theorem 5.4.2 lead to

Theorem 5.4.3. For $r \geq 4$ and $p \in \{0, 1, \ldots, r - 2, +, -\}$, $\Lambda_1^{(r-1)}(u)$ and $\Lambda_1^{(r)}(u)$ in (5.12) are free of poles provided that the $\text{BAE}_p^s$ (2.7) (with $s = 1$) for $0 \leq p \leq r - 2$ and (5.1) for $p = \pm$ are valid.

5.5 Relations between two kinds of boxes. The elementary boxes $\mathcal{A}$ and $[\mu_1, \ldots, \mu_r]$ introduced in section 5.1 and 5.4 are related by

Theorem 5.5.1. For $1 \leq a \leq r - 2, k, n, l \in \mathbb{Z}_{\geq 0}$ such that $k + 2n + l = a$, take any integers $1 \leq i_1 < \cdots < i_k \leq r$ and $1 \leq j_1 < \cdots < j_l \leq r$. Then the following equality holds between the elements of $\mathcal{T}_1^{(a)}$ and $\mathcal{T}_1^{(r-1)} \cup \mathcal{T}_1^{(r)}$ defined in section 5.2 and (5.13-15), respectively.

$$\begin{pmatrix} i_1 \\ \vdots \\ i_k \\ r \end{pmatrix}^{(a,p,r)}(u) = \left( \tau_{u-r+a+1}^{r} \left[ \mu_1, \ldots, \mu_r \right] \right) \left( \tau_{u-a-1}^{r} \left[ v_1, \ldots, v_r \right] \right),$$

where there are $n$ $\tau$'s in the lhs and $F_{i}^{(p,r)}$ is defined in (5.9b). The $\pm$ sequences $\mu$ and $\nu$ in the rhs are determined by (4.40b).
Put $a \equiv r + \sigma \mod 2$ where $\sigma = 0$ or $1$. Then the tableaux in the rhs of (5.19) belong to the following sets.

$$
\begin{align*}
\mu \in \left\{ \begin{array}{ll}
T_1^{(r-\sigma)} & \text{if } l \text{ even} \\
T_1^{(r-\sigma)-1} & \text{if } l \text{ odd}
\end{array} \right., \\
\nu \in \left\{ \begin{array}{ll}
T_1^{(r)} & \text{if } l \text{ even} \\
T_1^{(r)-1} & \text{if } l \text{ odd}
\end{array} \right.
\end{align*}
(5.20)
$$

One can rewrite the rhs of (5.19) so as to interchange the parity of $l$ in (5.20). Given any $\pm$ sequences $\mu = (\mu_1, \ldots, \mu_r)$ and $\nu = (\nu_1, \ldots, \nu_r)$, we set

$$
\begin{align*}
\epsilon_k(\mu, \nu) &= \#\{ j \mid 1 \leq j \leq k, \mu_j = - \} - \#\{ j \mid 1 \leq j \leq k, \nu_j = - \}, \\
d_y(\mu, \nu) &= \min \left( \{ \infty \} \cup \{ k \mid 1 \leq k \leq r - 1, \epsilon_k(\mu, \nu) = y \} \right).
\end{align*}
(5.21a, b)
$$

Then we have

**Lemma 5.5.2.** For any $1 \leq a \leq r - 2$ and any $\pm$ sequences $\mu = (\mu_1, \ldots, \mu_r), \nu = (\nu_1, \ldots, \nu_r)$, one has

$$
\begin{align*}
\left( \tau_{-r+a+1}^u \begin{array}{c}
\mu_1, \ldots, \mu_r \end{array} \right) \left( \tau_{r-a-1}^u \begin{array}{c}
\nu_1, \ldots, \nu_r \end{array} \right) \\
= \left( \tau_{-r+a+1}^u \begin{array}{c}
\mu'_1, \ldots, \mu'_r \end{array} \right) \left( \tau_{r-a-1}^u \begin{array}{c}
\nu'_1, \ldots, \nu'_r \end{array} \right),
\end{align*}
(5.22a)
$$

where $\mu'_j$ and $\nu'_j$ are determined by

$$
(\mu'_j, \nu'_j) = \left\{ \begin{array}{ll}
(\mu_j, \nu_j) & \text{if } 1 \leq j \leq d_{r-a-1}(\mu, \nu) \\
(\nu_j, \mu_j) & \text{otherwise}
\end{array} \right.
(5.22b)
$$

The Lemma enables the interchange of those $\mu_j$ and $\nu_j$ with $j > d_{r-a-1}(\mu, \nu)$ in the products (5.22a). In case $d_{r-a-1}(\mu, \nu) = \infty$, the assertion is trivial. One may apply Lemma 5.5.2 to rewrite the rhs of (5.19). A little inspection tells that $1 \leq d_{r-a-1}(\mu, \nu) \leq r - 1$ for any those $\mu$ and $\nu$ appearing there. Moreover, for such $d = d_{r-a-1}(\mu, \nu)$ one can evaluate the difference

$$
\#\{ j \mid d < j \leq r, \mu_j = - \} - \#\{ j \mid d < j \leq r, \nu_j = - \} = 2n + 1 \in 2\mathbb{Z} + 1,
$$

in terms of the $n$ in Theorem 5.5.1. Thus Lemma 5.5.2 expresses the rhs of (5.19) by the tableaux such that

$$
\begin{align*}
\mu' \in \left\{ \begin{array}{ll}
T_1^{(r-1+\sigma)} & \text{if } l \text{ even} \\
T_1^{(r-\sigma)} & \text{if } l \text{ odd}
\end{array} \right., \\
\nu' \in \left\{ \begin{array}{ll}
T_1^{(r)} & \text{if } l \text{ even} \\
T_1^{(r)-1} & \text{if } l \text{ odd}
\end{array} \right.
\end{align*}
(5.23)
$$
which is opposite to (5.20). Based on these observations, we can give a similar argument to section 4.7 that backgrounds Theorem 5.5.1. There is a degeneracy point \( u = -2(r - a - 1) \) of the \( U_q(D_\infty^{(1)}) \) quantum \( R \)-matrix [32] where it yields embedding

\[
W_1^{(a)}(u) \leftrightarrow W_1^{(r-1)}(u + r - a - 1) \otimes W_1^{(r-1+\sigma)}(u - r + a + 1), \\
W_1^{(a)}(u) \leftrightarrow W_1^{(r)}(u + r - a - 1) \otimes W_1^{(r-\sigma)}(u - r + a + 1),
\]

(5.24)

According to [8], (5.24) implies the functional relations

\[
T_1^{(r-1)}(u + r - a - 1)T_1^{(r-1+\sigma)}(u - r + a + 1) = T_1^{(a)}(u) + T'(u), \\
T_1^{(r)}(u + r - a - 1)T_1^{(r-\sigma)}(u - r + a + 1) = T_1^{(a)}(u) + T''(u),
\]

(5.25a, b)

where \( T'(u) \) and \( T''(u) \) are some matrices commuting with all \( T_1^{(b)}(v) \)'s. In particular if \( a = r - 2 \) (\( \sigma = 0 \)), (5.25) is the last equation in (2.5c) with \( m = 1 \), hence \( T'(u) = T_1^{(r-1)}(u) \) and \( T''(u) = T_2^{(r)}(u) \). One may regard (5.25a, b) as equations on the eigenvalues and substitute (5.9a) and (5.12). Then Theorem 5.5.1 tells how one can pick up the DVF for \( \Lambda_1^{(a)}(u) \) from the lhs. For example in (5.25a), one depicts the terms in \( \Lambda_1^{(a)}(u) \) as the lhs of (5.19). Then the \( l \) odd terms are indeed contained in \( \Lambda_1^{(r-1)}(u + r - a - 1)\Lambda_1^{(r-1+\sigma)}(u - r + a + 1) \) due to (5.19) and (5.20). The \( l \) even terms can also be found by expressing the above product in terms of the tableaux in (5.23).

6. Discussions

6.1 Summary. In this paper we have constructed the dressed vacuum forms (DVF)s (2.9b) for several eigenvalues \( \Lambda_m^{(a)}(u) \) of the row-to-row transfer matrices \( T_m^{(a)}(u) \) (1.2) via the analytic Bethe ansatz. Relevant vertex models are those associated with the fusion quantum \( R \)-matrices \( R_{W_1^{(a)}, W_1^{(b)}}(u) \) for \( Y(X_r) \) or \( U_q(X_r^{(1)}) \) with \( X_r = B_r, C_r, \) and \( D_r \). We have determined the DVFs for all the transfer matrices \( T_1^{(a)}(u) \) \((1 \leq a \leq r)\) associated with the fundamental representations \( W_1^{(a)} \) in the sense of [28]. In particular, they have been proved pole-free under the Bethe ansatz equation, a crucial property in the analytic Bethe ansatz. Once the DVFs of \( \Lambda_1^{(a)}(u) \) are fixed, those for the other eigenvalues are uniquely determined from the \( T \)-system [8], a set of functional relations among the transfer matrices. Based on this we have conjectured the DVFs for several \( \Lambda_m^{(a)}(u) \) with higher \( m \). These results extend earlier ones in [5, 10, 11, 19, 20, 21].

The DVFs for \( \Lambda_m^{(a)}(u) \) are Yang-Baxterizations of the characters of the auxiliary spaces \( W_m^{(a)} \). We have found that they are described by remarkably simple rules using analogues of the semi-standard Young tableaux. We believe that the sets of tableaux \( T_m^{(a)} \) introduced in this paper are natural objects that label the base of the irreducible finite dimensional modules \( W_m^{(a)} \) over the Yangians or the quantum affine algebras.

6.2 Further extensions. Let us indicate further applications of our approach. As can be observed through sections 2 to 5, the hypotheses called the top term (2.12) and the coupling rule (2.14), (2.15) severely restrict possible DVFs. This is especially significant when as
many weight spaces as possible are multiplicity-free (2.13) in the auxiliary space. An interesting example of such a situation is Yangian analogue of the adjoint representation. Below we exclude the case $X_r = A_r$, where the DVF for general eigenvalues is already available [19]. Then it is known [12,28] that the Yangian $Y(X_r)$ admits the irreducible representation $W_{adj}$ isomorphic to $V(\theta) \oplus V(0)$ as an $X_r$-module. Here $\theta$ denotes the highest root hence $V(\theta)$ means the adjoint representation of $X_r$. One can identify $W_{adj}$ in the family $\{W_m^{(s)}\}$ by $\theta$ and the data in appendix A of [33].

$$(\theta, W_{adj}) = \begin{cases} (\omega_1, W_1^{(1)}) & E_7, E_8, F_4, G_2 \\ (2\omega_1, W_1^{(2)}) & C_r \\ (\omega_2, W_1^{(2)}) & B_r, D_r \\ (\omega_6, W_1^{(6)}) & E_6 \\ \end{cases}$$

Thus the cases $X_r = B_r, C_r$ and $D_r$ are already covered in this paper. For $G_2$, the DVF of $\Lambda^{(1)}_1(u)$ has been obtained recently [11]. Let us turn to the remaining cases, $\Lambda^{(1)}_1(u)$ of $E_{7,8}, F_4$ and $\Lambda^{(6)}_1(u)$ of $E_6$. By the definition, $\dim W_{adj} = \dim X_r + 1$. All the weights in $W_{adj}$ are multiplicity-free except the null one, $\text{mult}_0 W_{adj} = r + 1$. Thus one may try to apply the top term (2.12), the coupling rule (2.14,15) and the crossing symmetry (2.18) to possibly determine the $\dim X_r - r$ terms in the DVF corresponding to the root vectors. We have checked that this certainly works consistently and fix those terms uniquely. Moreover, we have found that pole-freeness under the BAE requires exactly $r + 1$ more terms that make the null weight contribution $(r + 1)q^0$ in the character limit (2.16). These features are equally valid in the trigonometric case as well. Thus the resulting DVFs are candidates of the transfer matrix eigenvalues for the trigonometric vertex models associated with $U_q(E_8^{(1)})$, etc. The details will appear elsewhere. It still remains to understand the hypotheses (2.12), (2.14) and (2.18) intrinsically and thus to unveil the full aspects of the analytic Bethe ansatz.

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References


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