Effective potential for nonrelativistic non-Abelian Chern-Simons matter system in constant background fields*

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H3C-3J7

ABSTRACT

* This work is supported in part by funds provided by the Natural Sciences and Engineering Research Council of Canada and the Fonds pour la Formation de Chercheurs et l'Aide à la Recherche.
We present the effective potential for nonrelativistic matter coupled to non-Abelian Chern-Simons gauge fields. We perform the calculation using a functional method in constant background fields to satisfy Gauss’s law and to simplify the computation. Both the quantum gauge and matter fields are integrated over. The gauge fixing is achieved with an $R_{\xi}$-gauge in the $\xi \to 0$ limit. Divergences appearing in the matter sector are regulated via operator regularization. We find no correction to the Chern-Simons coupling constant and the system experiences conformal symmetry breaking to one-loop order except at the known value of self-duality. These results agree with previous analysis of the non-Abelian Aharonov-Bohm scattering.
1. Introduction

Chern-Simons theories have been studied in many context in the last decade from the study of general relativity to condensed matter systems. An important line of developments occurred when it was shown that classical relativistic charged scalars minimally coupled to an Abelian Chern-Simons gauge field in (2+1) space-time dimensions have vortex (soliton) solutions for self-dual equations when the coupling constant takes special values in a $\phi^6$-theory [1,2]. The presence of vortex solutions permits the emergence of new mechanisms for anyons superconductivity [3]. Evidence has been found showing that the existence of such systems possessing vortex solutions is due to the presence of an $N = 2$ supersymmetry obtained by adding fermion fields in an appropriate way [4,5].

It is more reasonable to think that the physics of superconductors should be at lower energies and described by a nonrelativistic system. It turns out that the same statements as above can be made for the corresponding nonrelativistic field theory. Specifically, by taking the limit $c \to \infty$ ($c$ being the speed of light), one obtains a field theory of interacting nonrelativistic scalar fields minimally coupled to an Abelian Chern-Simons gauge field [6,7]. This theory also contains self-dual vortex (soliton) solutions when the coupling constant takes a special value [7]. Perhaps more surprisingly in the nonrelativistic case, the self-duality originates also from $N = 2$ supersymmetry [8].

Much work has already been done in generalizing these ideas to non-Abelian theories. Relativistic and nonrelativistic models of matter fields coupled to non-Abelian Chern-Simons field [9-11] have been studied at the classical level, however, the relation between them as the limit $c \to \infty$ has never been analysed as above. Nevertheless, non-Abelian self-dual solitons exist in the corresponding nonrelativistic Chern-Simons field theory [11]. Supersymmetric extensions for the relativistic system have proven to show the same relation between supersymmetry and self-duality as is the case for the Abelian theories [12,13] and it could be interesting to see if a supersymmetric extension of the nonrelativistic non-Abelian Chern-Simons
matter system is possible.

The quantization of the above models has been discussed in various context. In the case of the pure non-Abelian Chern-Simons theory, Pisarski et al [14] have shown using a perturbative analysis with dimensional regularization that a one-loop radiative correction to the Chern-Simons coupling constant $\kappa \rightarrow \kappa + \frac{c_2(G)}{2}$ (shifted by the Casimir of the group) occurs. The same result was then obtained by Witten [15] with a saddle point quantization around pure gauge vector potentials. Their calculations were confirmed by using a modified Pauli-Villars method [16], an $F^2$-type regulator [10] and a modified operator regularization method for determining phases of determinants [17]. However, it is possible that this shift of the Chern-Simons coupling constant be absent if a variant of dimensional regularization [10] or a BRST-invariant regulator is used [18].

When relativistic matter fields are included, Chen et al. showed that infinite renormalization for the matter fields as well as for the Chern-Simons gauge field is necessary at two loops and therefore that the fields obtain nontrivial anomalous dimensions. Also, the $\beta$-function for the gauge coupling constant is zero to two-loop order [10].

In the case when nonrelativistic matter fields are coupled to the Abelian Chern-Simons field, we know that the theory experiences conformal symmetry breaking at the quantum level unless the coupling constant takes the self-dual value and that this result holds up to three loop order [19-22]. Only recently was a perturbative analysis performed for the scattering of scalars in the nonrelativistic non-Abelian theory using Feynman’s diagrammatic [23]. Again, the conformal symmetry is restored upon choosing the self-dual point. All these computations were performed either in the conventional Feynman diagrammatic or within a functional method.

Our goal in this paper is to complement the above discussion and to compute the scalar field effective potential of the nonrelativistic non-Abelian Chern-Simons system with the help of a functional method.

We start with an $SU(2)$ non-Abelian Chern-Simons system action $[\text{diag } \eta = ...$
\[(+, -,-)\]

\[S = \int dt d^2x \left\{-\kappa e^{\alpha \beta \gamma} \text{Tr}(A_\alpha \partial_\beta A_\gamma + \frac{2}{3} A_\alpha A_\beta A_\gamma) + i \phi^\dagger D_t \phi - \frac{1}{2} |D \phi|^2 - \frac{\lambda_{pqrs}}{4} \phi_p^\dagger \phi_q^\dagger \phi_r \phi_s \right\}
\]

(1.1)

where the gauge fields belong to the su(2) Lie algebra \(A_p = i \frac{A^a_p \tau^a}{2}\), and \(D_t = \partial_t + i A^a_0 \frac{\tau^a}{2}\) and \(D = e^{-iA^a} \frac{\tau^a}{2}\) are the time and space covariant derivatives respectively. \(\phi_p\) is the two component nonrelativistic scalar field, \(p = 1, 2\). The self-interaction coupling constants satisfy \(\lambda_{pqrs} = \lambda_{qprs}\) since the fields are bosonic and \(\lambda_{pqrs} = \lambda_{rsqp}\) for the Lagrangian to be real. \(\tau^a\) are the Pauli matrices which satisfy the usual commutation relations \([\tau^a, \tau^b] = e^{abc} \tau^c\) and trace relation \(\text{Tr}(\tau^a \tau^b) = \frac{1}{2} \delta^{ab}\). We have omitted the mass parameter since in nonrelativistic systems, it is always possible to set it equal to unity. [We will use a vector notation: for instance, in the plane the cross product is \(V \times W = \epsilon^{ij} V^i W^j\), the curl of a vector is \(\nabla \times V = \epsilon^{ij} \partial_j V^i\), the curl of a scalar is \((\nabla \times S)^i = \epsilon^{ij} \partial_j S\) and we shall introduce the notation \((A \times \hat{z})^i = \epsilon^{ij} A^j\). The notation \(x = (t, x)\) will also be used unless stated otherwise.]

The action (1.1) enjoys several invariances at the classical level. It is obvious that the matter part of this action is Galilean invariant and conformally invariant [6,7]. The presence of the Chern-Simons term as the only kinematical term for gauge fields does not spoil these two sets of invariances as this term is topologically invariant i.e. it is invariant upon any space and time transformations [9,24]. Nevertheless, the interesting symmetry is gauge invariance. Let us for a moment forget the self-interacting part of the matter sector. The matter fields are minimally coupled, hence this part is gauge invariant. The self-interacting part however is gauge invariant only for

\[\lambda_{1111} = 2 \lambda_{1212} = \lambda_{2222} \equiv \lambda\]

(1.2)

with the other constants vanishing. The Chern-Simons term is not invariant against
gauge transformation; rather it changes by total derivatives. Under special circumstances the total derivatives can be set to zero. However, if we need to consider large gauge transformations, only the exponential of $i \times$ action need be gauge invariant. In this case, we speak of “gauge invariant action” if the quantization condition $4\pi\kappa = \text{integer}$ applies [9]. We will use these assessments in the course of the calculation.

To compute the scalar field effective potential of the action (1.1), we proceed with the functional method of Jackiw [25], which is a useful way to evaluate the effective potential without having to use a classical background field, in conjunction with the operator regularization method [26,27]. The difference here with other evaluations of effective potentials is that the electromagnetic field is linearly related to the matter field through Gauss’s law. The procedure involves shifting the fields present in the action by constants. Shifting the matter field by a constant implies that the magnetic field must be constant and consequently, quantal effects could emerge from the background gauge field towards the scalar field effective potential. In the Abelian version of this model, the same procedure was used to compute the effective potential for scalars [22]. However in that case, the magnetic field $B = \nabla \times \mathbf{A}(x) = \text{constant}$ was satisfied by a vector potential, which depended linearly on $x$. In the non-Abelian case, it is possible to satisfy Gauss’s law with a constant vector potential [see below].

The paper is organised as follows: In the next section, we set up the problem and provide the classical equation of motion to show how we satisfy the ones for the electromagnetic fields with constant background fields. We show that although local gauge invariance is lost through such a choice of background fields, global gauge invariance is retained; we gauge fix in a Galilean fashion in a gauge reminiscent of the $R_3$-gauge, which is globally gauge invariant. In the third section, we present the results of the calculation and in the last section, we summarize the work and
2. Constant background fields, equation of motion and gauge invariance.

The functional evaluation of the effective potential starts with the definition of a new shifted action:

\[ S_{\text{new}} = S\{ \phi_{p}(x) = \varphi_{p} + \pi_{p}(x); A_{\mu}^{a}(x) = a_{\mu}^{a} + Q_{\mu}^{a}(x) \} \]

\[ - S\{ \varphi_{p}, a_{\mu}^{a} \} - \text{terms linear in quantum fields} \quad (2.1) \]

where we shift the scalar field and the vector potential by constant fields in such a way that the classical equations of motion for the electromagnetic fields are satisfied. It is the action (2.1) that enters the path integral for the evaluation of the effective potential.

Let us for a moment look at the classical equation of motion of the action (1.1) to see how the equation of motion respond to the constant shift. The gauge covariant classical equation of motion for arbitrary fields are

\[ B^{a} \equiv \nabla \times a^{a} + \frac{1}{2} c^{abc} a^{b} \times a^{c} = - \frac{1}{2\kappa} \phi \frac{\partial}{\partial a^{a}} \phi \quad (2.2a) \]

\[ E^{a} \equiv - \nabla a^{a} - \partial_{t} a^{a} + c^{abc} a^{b} a^{c} = \frac{1}{4\kappa} J^{a} \times \dot{z} \quad (2.2b) \]

\[ i(D_{t}\phi)_{p} + \frac{1}{2}(D \cdot D\phi)_{p} - \frac{1}{2} \lambda_{pqrs} \phi_{q}^{*} \phi_{r} \phi_{s} = 0 \quad (2.2c) \]

where the current is given by \( J^{a} = \frac{1}{2\kappa} \left[ \phi \frac{\partial}{\partial a^{a}}(\frac{\nabla a^{a}}{2})(D\phi) - (D\phi) \frac{\partial}{\partial a^{a}}(\frac{\nabla a^{a}}{2})\phi \right] \) and \( D \) is the covariant derivative with respect to the background gauge field.

The equation for the magnetic field (2.2a) is recognized as Gauss’s law. Since scalar fields are shifted by constants, Eqs.(2.2) have to be read with \( \phi_{p} = \varphi_{p} = \text{constant} \). To maintain consistency with Gauss’s law, we need to choose a background vector potential \( a^{a} \) such that the magnetic field is constant throughout the plane. The simplest choice is to take a constant background vector potential \([28]\). We can also choose \( a_{\mu}^{a} \) constant with the help of Eq.(2.2b). Since Eqs.(2.2)
with constant background fields are now globally gauge covariant, without fear of
loosing generality, we can find an explicit solution to Eqs.(2.2a,b). If we choose
for instance, \( \varphi_p = (v,0) \) and if we label the SU(2) group structure with colors
\( a = (1,2,3) \equiv (Y,B,R) \) then we find that \( a_Y = \left( \sqrt{\frac{e^2}{2k}},0 \right) \), \( a_B = (0,-\sqrt{\frac{e^2}{2k}}) \),
\( a_R^0 = \frac{e^2}{16k} \) and the other components vanishing, is a solution to Eqs.(2.2a,b). Of
course, this particular solution does not satisfy the equation of motion for the
scalar field, Eq.(2.2c), unless \( \varphi_p = 0 \) or if the coupling constants satisfy \( \lambda = -\frac{5}{16k} \).
We will use the above solution for \( a^a_\mu \) and \( \varphi_p \) in the definition of \( S_{\text{new}} \) in Eq.(2.1)
and extrapolate at the end of the calculation the form of the effective potential
since it must be globally gauge invariant [see below].

We now turn to a proof that global gauge invariance is retained upon quantizing
this theory around constant background fields. We follow the discussion of Abbott
[29]. Under local gauge transformations

\[
A'_\mu = U^{-1} \partial_\mu U + U^{-1} A_\mu U \tag{2.3a}
\]

\[
\phi' = U^{-1} \phi \tag{2.3b}
\]

or infinitesimally with \( U = \exp i \omega^a \tau^a / 2 \)

\[
\delta A^a_\mu = \partial_\mu \omega^a - \epsilon^{abc} A^b_\mu \omega^c \tag{2.4a}
\]

\[
\delta \phi = -i \omega^a (\tau^a / 2)_{pq} \phi_q \tag{2.4b}
\]

the action of Eq.(1.1) transforms according to

\[
\delta S = (4 \pi \kappa)(2 \pi) w(U) \tag{2.5}
\]

where \( w(U) \) is the winding number and the usual quantization condition over
\( 4 \pi \kappa = \) integer follows if we want \( \exp \{ i \times \text{action} \} \) to be gauge invariant under
(large) gauge transformation.
In Jackiw’s approach the generating functional is defined as

\[
Z(\varphi_p; a^a_\mu) = \int \delta \pi \delta Q \ det \left[ \frac{\delta G^a}{\delta \omega^b} \right] \exp i \int dt \ d^2x \left[ \mathcal{L}(\varphi_p + \pi_p; a^a_\mu + Q^a_\mu) + \frac{1}{2\xi} \partial_\mu G^a \right. \\
\left. - \mathcal{L}(\varphi_p, a^a_\mu) - \frac{\delta \mathcal{L}}{\delta A_{\mu \nu}} n_{\mu} \cdot Q - \frac{\delta \mathcal{L}}{\delta \phi} n_{\mu \nu} \cdot \pi \right]
\]  

(2.6)

where \( \varphi_p \) and \( a^a_\mu \) are constants, and \( \frac{\delta G^a}{\delta \omega^b} \) is the ghost contribution and is given by the functional derivative of the gauge-fixing term under the infinitesimal quantum gauge transformation \( \delta Q^a_\mu = \partial_\mu \omega^a - \epsilon^{abc}(a^b_\mu + Q^b_\mu) \omega^c \). Then, just as in the conventional approach, the effective potential at vanishing external current and vanishing quantum field argument is

\[
V_{\text{eff}}[\varphi_p, a^a_\mu] = \frac{i}{\int d^3x} \ln Z[\varphi_p, a^a_\mu]
\]  

(2.7)

It remains to choose the background field gauge condition which reveals to be rather difficult for the problem at hand. The reason is as follows: when matter is not present, the Chern-Simons theory is defined without the introduction of a metric. Upon choosing the gauge-fixing condition the theory could loose its topological character [15]. Indeed, Witten chose a Lorentz-type family gauges in his derivation of the one-loop quantum correction to the pure non-Abelian Chern-Simons theory. He found, however, that the topological property of the action remained unaffected. In the case where matter is coupled to the Chern-Simons theory, we already have chosen a metric and we must preserve as many as the symmetry present there. In our case, we have to preserve the Galilean symmetry. We therefore choose

\[
G^a = \nabla \cdot \mathbf{Q}^a + \frac{i}{2} \xi \pi^\dagger \tau^a \varphi
\]  

(2.8)

and note that the gauge-fixing resembles the \( R_\xi \)-type gauge-fixing conditions.
Now, by making the following change of variables for the quantum fields

\[ Q_\mu \rightarrow Q'_\mu = U^{-1}Q_\mu U \]
\[ \pi \rightarrow \pi' = U^{-1}\pi \]  \hspace{1cm} (2.9)

where \( U \) is a gauge transformation with constant \( \omega^a \), it is easy to show that \( Z[\varphi_p, a^a_\mu] \) and hence the effective potential are invariant under the constant background gauge transformation

\[ a_\mu \rightarrow a'_\mu = U^{-1}a_\mu U \]
\[ \varphi \rightarrow \varphi' = U^{-1}\varphi \]  \hspace{1cm} (2.10)

since each term is invariant. It is interesting to note that in retaining only global gauge invariance, the gauge-fixing condition becomes simpler since it is not written in an explicit background gauge covariant form [29]. This will of course be advantageous in the course of the explicit calculation since it will enables us to integrate out the gauge and matter fields by performing determinants as they are now diagonal in Fourier space [see below]. We now turn to the calculation of the effective potential in the \( R_\xi \)-gauge.

3. The effective potential.

We perform the calculation of the scalar field effective potential following the procedure set up in the previous section. The quadratic part in quantum fields of the action appearing in Eq. (2.6) upon using the gauge-fixing condition of Eq. (2.8) and \( \varphi_p = (v, 0), a_Y = (\sqrt{\frac{\kappa v}{2\xi}}, 0), a_B = (0, -\sqrt{\frac{\kappa v}{2\xi}}), a_R^0 = \frac{v^0}{16\xi} \) and the other components vanishing is

\[ S = \int dt \; d^3x \left\{ \frac{\kappa}{2}(\dot{Q}_a) \times Q_a - \kappa Q^0_a \nabla \times Q_a + \frac{1}{2\kappa}(\nabla \cdot Q_a)^2 - \frac{\rho}{8}Q_a \cdot Q_a 
+ i\pi^\dagger(D_t)\pi - \frac{1}{2}|D\pi|^2 + \mathcal{L}_{S,1} + \frac{\xi}{8}(\pi^\dagger \tau^a \varphi)(\varphi^\dagger \tau^a \pi) \right\} \]
+ R^0 Q_R^0 + B^0 Q_B^0 + Y^0 Q_Y^0 + \mathbf{R} \cdot \mathbf{Q}_R + \mathbf{B} \cdot \mathbf{Q}_B + \mathbf{Y} \cdot \mathbf{Q}_Y \right\} \quad (3.1)

where \( \rho = v^*v \), \( \mathcal{L}_{S.I} \) stands for the quadratic self-interacting part in \( \pi \)-fields, which will be treated later, and the currents are given by

\[
\begin{align*}
R^0 &= [j_R + \kappa (a_B \times Q_Y - a_Y \times Q_B)] \\
B^0 &= j_B \\
Y^0 &= j_Y \\
\mathbf{R} &= \kappa [a_B Q_Y^0 - a_Y Q_B^0] \times \hat{z} \\
\mathbf{B} &= \frac{1}{2} a_B j_R + \kappa a_Y^0 Q_Y \times \hat{z} \\
\mathbf{Y} &= \frac{1}{2} a_Y j_R
\end{align*}
\]

with the useful definition for matter-currents

\[
\begin{align*}
j_R &= -\frac{1}{2}(\pi_1^* v + v^* \pi_1) \\
j_B &= -\frac{i}{2}(\pi_2^* v - v^* \pi_2) \\
j_Y &= -\frac{1}{2}(\pi_2^* v + v^* \pi_2)
\end{align*}
\]  

We are now ready to proceed with the functional integration in the \( \xi \to 0 \) limit and up to include \( \mathcal{O}(v^4) \) contributions [we refer to \( \mathcal{O}(v^4) \) whenever we have \( \mathcal{O}(\lambda^2), \mathcal{O}(\frac{1}{\kappa}) \) or \( \mathcal{O}(\frac{1}{\kappa^2}) \)]. The contribution coming from the ghosts to one-loop order is given by the determinant of the functional derivative of the gauge-fixing term Eq. (2.8) with respect to an infinitesimal quantum gauge transformation as above Eq. (2.7) without terms having quantum fields

\[
\det \frac{\delta G^a}{\delta \omega^b} = \det \left[ -\nabla^2 \delta^{ab} - \epsilon^{abc} a_c \cdot \nabla - \frac{\xi}{4}(v^*, 0) \tau^b \tau^a \begin{pmatrix} v \\ 0 \end{pmatrix} \right] \quad . \quad (3.4)
\]

This contribution is easily calculated since it factorizes from the path integral and the determinant is performed on a \( 3 \times 3 \) matrix. The result to \( \mathcal{O}(\xi) \) is

\[
V_{\text{ghosts}} = -\text{tr} \ln \left( 1 - \frac{(a_B \cdot \mathbf{p})^2}{p^4} - \frac{(a_Y \cdot \mathbf{p})^2}{p^4} \right) \quad . \quad (3.5)
\]
where the trace is now taken only on energy/momentum space.

Next, we integrate out the quantum gauge fields by integrating first over the R-color. The first line in Eq. (3.1) is diagonal in the (R,B,Y) colors, however, the last line in Eq. (3.1) mixes the Q’s with different colors. For instance in the R-sector, the structure of the exponent in the functional integration is $-\frac{1}{2} Q_R^{\mu} \Delta^{-1} Q_R^\nu + Q_R^{\mu} R_\mu$

where in Fourier space $(i\partial_\mu = p_\mu)$

$$\Delta^{-1}(v; \omega, p) = \begin{pmatrix} 0 & -m & n \\ m & \frac{\rho}{4} - \frac{1}{\xi} p^1 p^1 & -i\kappa \omega - \frac{1}{\xi} p^1 p^2 \\ -n & i\kappa \omega - \frac{1}{\xi} p^1 p^2 & \frac{\rho}{4} - \frac{1}{\xi} p^2 p^2 \end{pmatrix}, \quad (3.6)$$

with $m = i\epsilon p^2$ and $n = i\epsilon p^1$. Upon the usual change of variable, one obtains a contribution to the effective potential of the type $\text{det}^{-1/2} \Delta^{-1}_{\mu\nu}$ and a modification to the original action by the amount $\frac{1}{2} R^\mu \Delta_{\mu\nu} R^\nu$, which does not contain any $Q_R^\mu$ dependence with

$$\Delta(v; \omega, p) = -\frac{1}{e^2 \kappa^2 p^2} \begin{pmatrix} \frac{\rho}{4} & -m & n \\ m & 0 & 0 \\ -n & 0 & 0 \end{pmatrix} + O(\xi). \quad (3.7)$$

The contribution to the effective potential vanishes in the limit $\xi \to 0$, however the amount $\frac{1}{2} R^\mu \Delta_{\mu\nu} R^\nu$ modifies the B-sector and the Y-sector and provides also contributions exclusive to the matter sector. For instance in the B-sector, the structure of the exponent in the functional integration is now $-\frac{1}{2} Q_B^{\mu} \Delta^{-1} Q_B^\nu - \frac{1}{2} Q_B^{\mu} \Theta_{\mu\nu} Q_B^\nu + Q_B^{\mu} B'_\mu$ with currents given by

$$B'_0 = B_0 + J_R \left( \frac{ip \cdot ay}{p^2} + \frac{(ip \cdot ay)}{p^2} (\kappa a_B \times Q_Y) \right) \quad (3.8)$$

$$B' = B + (\kappa ay \times \hat{z}) \left( \frac{ip \cdot a_B}{p^2} Q_Y^0 - \frac{\rho}{4\kappa p^2} J_R ay \times \hat{z} + \frac{\rho}{4p^2} (a_B \cdot ay) Q_Y - \frac{\rho}{4p^2} a_B (ay \cdot Q_Y) \right)$$

and a matrix $\Theta$ which depends only on background fields. Upon integrating the B-sector, one gets two contributions to the effective potential, one that modifies
the structure of the action in the Y-sector, and one which contribute only to the
matter sector. The contributions to the effective potential are

\[ \ln \text{det}^{-1/2} \Delta_{\mu\nu}^{-1} \]

which vanishes in the \( \xi \to 0 \) limit and the second is

\[ \ln \text{det}^{-1/2} (1 + \Delta \times \Theta) = -\frac{1}{2} \ln \left(1 - \frac{(p \cdot a_Y)^2}{p^4}\right)^2. \]

The modification to the action in the Y-sector is

\[ \frac{1}{2} B_{\mu \nu} \left[ 1 + \Delta \times \Theta \right]^{-1} B^{\mu \nu}. \]

Upon collecting all terms that depends on the \( Q^a_Y \) variable, we can
integrate the Y-sector in the same way. Finally, the result of the Q-integration
is divided in a contribution to the effective potential and a part that modifies the
matter sector. We get

\[
V_{\text{eff}}(v, a^a_B) = \frac{1}{2} \text{tr} \ln \left(1 - \frac{(p \cdot a_Y)^2}{p^4}\right)^2 + \frac{1}{2} \text{tr} \ln \left(1 - \frac{(p \cdot a_B)^2}{p^4}\right)^2 - \text{tr} \ln \left(1 - \frac{(a_B \cdot p)^2}{p^4} - \frac{(a_Y \cdot p)^2}{p^4}\right) + \frac{i}{\text{vol}} \ln \int \mathcal{D}x \exp i S_{\text{matter}} \quad (3.9)
\]

where the first contribution in Eq.(3.9) comes from integrating the B-sector while
the second term comes from the Y-sector. The third contribution originates from
the ghosts sector. Although the limit \( \xi \to 0 \) should be taken at the end of the
calculation, we have carefully dropped terms of \( \mathcal{O}(\xi) \) to clarify the expressions.
The modified action \( S_{\text{matter}} \) is given by

\[
S_{\text{matter}} = \int dt \, d^2 x \left\{ i \pi \right\} \left(D_t \pi \right) - \frac{1}{2} |D \pi|^2 + \mathcal{L}_{\text{SI}} + \xi \frac{8}{\pi} (p^a \tau^a) (\varphi^a \tau^a \pi)
\]

\[
- \frac{\rho}{8 k^2} j_R \frac{1}{p^2} j_R
\]

\[
- \frac{\rho}{8 k^2} j_Y \frac{1}{p^2} j_Y - 2 k \left( j_R i (p \cdot a_B) j_R \right) \right) \quad (3.10)
\]

where all expressions have the operators \( p \) and \( a \) acting on the right, and the
covariant derivatives read \( D_t = -i (\omega + \frac{1}{2} a_0 R^R) \) and \( D = i (p - \frac{1}{2} a^a \tau^a) \). The first
line comes from the original action while the second is from the R-integration. The
third and fourth lines are from B and Y-integration respectively.
Some comments are in order at this point. The ghosts contribution in Eq. (3.9) cancels against the gauge field contributions to $O(v^4)$ leaving only the remaining functional integration over the matter sector. Indeed, if we had not introduced any matter fields, we would have gotten a vanishing answer in contrast with ref. [14-17] but in agreement with [10,18,23].

In any case, when matter is present, the effective potential is given by the remaining functional integration over the matter fields. It is not too difficult to see that the structure of the action (3.10) is

$$\int dt d^4x \left\{ \frac{1}{2} \pi^a_1(x) \mathcal{D}^{-1}_{ab}(x - x') \pi^b_1(x') + \frac{1}{2} \pi^a_2(x) \mathcal{E}^{-1}_{ab}(x - x') \pi^b_2(x') + J \pi^*_2 + J^* \pi_2 \right\}$$

(3.11)

where the notation for the scalar fields is $\pi^a_i = (\pi, \pi^*)$ for each $i = 1, 2$, the current mixing the $\pi$'s is given by $J = \frac{i}{2}(a_\perp \cdot p) \pi_1 + \frac{v}{4} \left( \frac{p \times a_\perp}{p^2} \right) j_R$, and the matrix for the $\pi_1$ field in Fourier space to $O(\xi)$ is

$$\mathcal{D}^{-1}(\varphi_\rho, a^\mu; \omega, p) = \begin{pmatrix} \omega - \frac{1}{2}p^2 + A + \frac{B}{p^2} + \frac{C}{p^2} & \left( -\frac{1}{2} \lambda - \frac{\rho}{16 \kappa^2 p^2} + \frac{f}{4 \kappa p^2} \right) v^2 \\ \left( -\frac{1}{2} \lambda - \frac{\rho}{16 \kappa^2 p^2} + \frac{f}{4 \kappa p^2} \right) (v^*)^2 & -\omega - \frac{1}{2}p^2 + A + \frac{B}{p^2} + \frac{C}{p^2} \end{pmatrix}$$

(3.12)

with $a_\pm = a_B \pm i a_Y$, $A = \frac{1}{8}(a_+ \cdot a_-) - \frac{a_0^0}{4} - \frac{1}{2} \lambda \rho$, $B = -\frac{\rho^2}{16 \kappa^2}$, $C = \frac{\rho}{4 \kappa} f$ and $f = -[(p \cdot a_B)(p \times a_Y) + (p \cdot a_Y)(p \times a_B)]$.

Similarly, the matrix for the $\pi_2$ field is

$$\mathcal{E}^{-1}(\varphi_\rho, a^\mu; \omega, p) = \begin{pmatrix} \omega - \frac{1}{2}p^2 + E + \frac{F}{p^2} & 0 \\ 0 & -\omega - \frac{1}{2}p^2 + E + \frac{F}{p^2} \end{pmatrix}$$

(3.13)

with $E = \frac{1}{8}(a_+ \cdot a_-) + \frac{a_0^0}{4} - \frac{1}{2} \lambda \rho$, and $F = -\frac{\rho^2}{8 \kappa^2}$. To perform the functional integration over $\pi_2$ is not difficult. Upon doing it, there remains only to perform the integration over $\pi_1$ and the result, keeping in mind that we are computing up to include $O(v^4)$ in the limit $\xi \to 0$, is

$$V_{\text{eff}}(v, a^a_\mu) \int d^3x = i \ln \int \delta \pi \exp i S_{\text{matter}} = -\frac{i}{2} \ln |\text{Det} \mathcal{E}^{-1}_{ab}| - \frac{i}{2} \ln |\text{Det} \{ \mathcal{D}^{-1}_{ab} + \mathcal{M}_{ab} \}|$$

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where the easily found $\mathcal{M}_{ab}$ matrix appears as a consequence of the mixing between the $\pi$'s and the determinant are taken functionally. Since the operators $\mathcal{E}^{-1}$ and $\mathcal{D}^{-1}$ are diagonal in Fourier space, we can write the operatorial form of the final contribution to the effective potential to $\mathcal{O}(v^4)$ and in the limit $\xi \to 0$ as

$$V_{\text{eff}}(v, a^a_\mu) = -\frac{i}{2} \text{tr} \ln \frac{1}{\mu'^4} \left\{ -\omega^2 + (p^2/2 - E)^2 - F \right\}$$

$$- \frac{i}{2} \text{tr} \ln \frac{1}{\mu'^4} \left\{ -\omega^2 + (p^2/2 - A - \frac{B}{p^2} - \frac{C}{p^2})^2 - \frac{1}{4} \lambda^2 p^2 \right.$$  

$$+ \omega(X_+ - X_-) + (A - \frac{p^2}{2})(X_+ + X_-) + X_+ X_- \right\} \tag{3.15}$$

where the trace is performed in energy/momentum space, the parameter $\mu'$ of mass dimension one is introduced for dimensional reasons [26], and

$$X_\pm = \frac{1}{4} \Delta_E^{-1} \left\{ (\mathbf{p} \cdot a_+)(\pm \omega - p^2/2 + E)(\mathbf{p} \cdot a_-) \right.$$  

$$- i \frac{\rho}{4\kappa}(\mathbf{p} \times a_+)(\pm \omega - p^2/2)(\mathbf{p} \cdot a_-) + i \frac{\rho}{4\kappa}(\mathbf{p} \times a_-)(\pm \omega - p^2/2)(\mathbf{p} \cdot a_+) \right\}$$

with $\Delta_E \equiv -\omega^2 + (p^2/2 - E)^2$.

We pose for a moment to note that so far we have not used any form of regulator to extract the information we have in Eq. (3.15). This is because we have not encountered any ultraviolet divergences so far. However, Eq. (3.15) is divergent in the ultraviolet regime and therefore requires a regulator in order to evaluate its contribution to the effective potential. We will use operator regularization [22,26,27] to perform the computation since it preserve all symmetries present at the classical level modulo anomalies.

For each logarithm in Eq. (3.15), it is necessary to identify an operator $H_0$ and an operator $H_1$. Upon using operator regularization, the n-point function is easily identified as the n-th $H_1$ insertion with $H_0$ acting as the propagator for each internal lines. Following Ref.[22], we define $H_0 = \{-\omega^2 + (p^2/2 - A_1)^2\}/\mu'^4$ for the first logarithm and $H_0 = \{-\omega^2 + (p^2/2 - E_1)^2\}/\mu'^4$ for the second one where $A = A_1 + A_2$, $E = E_1 + E_2$, $A_1 = -\lambda \rho$ and $E_1 = -\frac{1}{2} \lambda \rho$. In $H_1$, we collect the rest of the expressions for each logarithm.
Both logarithm are easily regulated via

$$\text{Tr} \ln H = - \lim_{s \to 0} \frac{d}{ds} \frac{1}{\Gamma(s)} \int_0^\infty dt \, t^{s-1} \left\{ e^{-H_0 t} + e^{-H_0 t(-t) H_I} + \cdots \right\}$$

and upon using the useful integral over the energy $d\omega$

$$I \equiv \int_{-\infty}^{\infty} d\omega \, \frac{1}{2\pi \Delta_E^{1+s}} = i \frac{(2s)!}{s!} (p^2 + 2a)^{-(1+2s)} \text{ , } (3.17)$$

the contribution to the effective potential from the first log is

$$- \frac{i}{2} \int \frac{d^2 p}{(2\pi)^2} \left( \frac{d}{ds} \right) \left\{ [2E_1 E_2 + E_2^2 - F] - p^2 E_2 \right\} i \frac{(2s)!}{s!} \frac{\mu^4}{(p^2 - 2E_1)^{1+2s}}$$

$$+ \frac{i}{4} \int \frac{d^2 p}{(2\pi)^2} \left( \frac{d}{ds} \right)^2 \left( p^4 E_2^2 i \right) \frac{(2s+2)!}{(s+1)!} \frac{\mu^4}{(p^2 - 2E_1)^{3+2s}} \text{ , } (3.18)$$

where the first term is a one-pts function while the second is a two-pts function.

Upon symmetric integration over momentum integrals, (3.18) becomes

$$- \frac{1}{8\pi} F \ln \left( \frac{\mu^2}{-2E_1} \right) \text{ . } (3.19)$$

The second logarithm is more tedious to compute as it involves many one and two-pts functions. We rewrite the second logarithm of Eq.(3.15) as

$$- \frac{i}{2} \int d\omega d^2 p \ln \frac{1}{\mu^2} \left\{ - \omega^2 + \left( \frac{p^2}{2} - A_1 \right)^2 - A_2 p^2 + 2A_1 A_2 + A_2^2 - \frac{1}{4} \lambda^2 \rho^2 - B - \frac{C}{p^2} \right\}$$

$$- \frac{i}{2\kappa} \left( \frac{p^4}{4} \right) \frac{1}{\Delta E_i} (p \times a_+)(p \cdot a_-) - (p \times a_-)(p \cdot a_+) \left| \frac{1}{\Delta E_i} \right|$$

$$+ \left( \omega^2 + \frac{p^4}{4} \right) \frac{1}{2} (p \cdot a_+)(p \cdot a_-) \frac{p^2 E_2}{\Delta E_i} - \frac{p^2}{4} (A + E) (p \cdot a_+)(p \cdot a_-) \frac{1}{\Delta E_i}$$

$$+ \left( \omega^2 + \frac{p^4}{4} \right) \frac{1}{2} (p \cdot a_+)(p \cdot a_-) \frac{1}{\Delta E_i} + \frac{1}{16} \frac{(p \cdot a_+)^2 (p \cdot a_-)}{\Delta E_i} \right\} \text{ . } (3.20)$$
Upon using the regulated form of the logarithm, Eq. (3.17) and symmetric integration over $d^2 \mathbf{p}$, we obtain for the non-vanishing one-pts functions

\[
- \frac{1}{32\pi}\left\{8A_1A_2 - 4[2A_1 A_2 + A_2^2 - \frac{1}{4}\lambda^2 \rho^2 - B] - E_2(a_+ \cdot a_-) + (A + E)(a_+ \cdot a_-) - (A_1 + E_1)(a_+ \cdot a_-) - \frac{1}{16}\mathcal{P}\right\}\ln \frac{\mu^2}{-2A_1} \tag{3.21}
\]

where $\mathcal{P} = \{(a_+ \cdot a_+)(a_- \cdot a_-) + 2(a_+ \cdot a_-)^2\}$ and from the non-vanishing two-pts function

\[
- \frac{1}{32\pi}\left\{4A_2^2 - A_2(a_+ \cdot a_-) + \frac{1}{16}\mathcal{P}\right\}\ln \frac{\mu^2}{-2A_1} \tag{3.22}
\]

where each terms in Eq. (3.22) arises separately from squaring the $A_2 \mathbf{p}^2$-term of Eq. (3.20), from crossing the $A_2 \mathbf{p}^2$-term with the one before last of Eq. (3.20), and from squaring the one before last of Eq. (3.20), respectively.

Upon collecting all contributions of Eq. (3.19,21,22), we obtain for the unnormalized effective potential

\[
V_{\text{eff}}(\rho, a_+^a) = \frac{1}{4}\lambda \rho^2 + c_1 \rho^2 - \frac{1}{8\pi}\left(F + \left(\frac{1}{4}\lambda^2 \rho^2 + B\right)\right)\ln \frac{\mu^2}{-2A_1} \\
= \frac{1}{4}\lambda \rho^2 + c_2 \rho^2 + \frac{1}{8\pi}\left(4\lambda^2 - \frac{3}{\kappa^2}\right)\frac{\rho^2}{16}\ln \frac{\rho}{\mu^2} \tag{3.23}
\]

where now global gauge invariance is restored with $\rho = v_\mu^\dagger v_\mu$. In obtaining Eq. (3.19) and Eqs. (3.21-22), we drop an unimportant (const.$\rho^2$)-term arising from the first term independent of $H_1$ in the regulated form of the logarithm. We have inserted this contribution in the $c_1 \rho^2$-term in Eq. (3.23) together with a term of the same form which arises from Eq. (3.19) because $A_1 = 2E_1$. The $c_2 \rho^2$-term collects the $c_1 \rho^2$-term with the term proportional to $\rho^2 \ln 2\lambda$. In any case, the $c_2 \rho^2$-term disappear upon normalizing the effective potential. Note that no ultraviolet divergences occur in Eq. (3.23) as expected upon using operator regularization.
After imposing the normalization condition

$$\frac{d^2}{d\rho^2} V_{\text{eff}}|_{\rho = \mu^2} = \frac{1}{2} \lambda(\mu), \quad (3.21)$$

the normalized effective scalar field potential in the R_\xi-gauge in \( \xi \to 0 \) limit up to include \( \mathcal{O}(v^4) \) contributions is

$$V_{\text{eff}}(\rho, a^a_\mu) = \frac{1}{4} \rho^2 \left[ \lambda(\mu) + \frac{1}{8\pi} \left( \lambda^2(\mu) - \frac{4}{\kappa^2} \alpha^2 \right) \left( \ln \frac{\rho}{\mu^2} - \frac{3}{2} \right) \right]. \quad (3.25)$$

where the appearance of the group theoretical factor \( \alpha^2 = 3/16 \) is a consequence of the \( \text{su}(2) \) Lie algebra: 3 corresponds to the number of generators and 1/16 to a normalization of the generators. Note that the background gauge fields do not contribute to the scalar field effective potential.

4. Summary and conclusions

We computed the scalar field effective potential of a nonrelativistic non-Abelian Chern-Simons field theory possessing various classical symmetries such as Galilean, conformal and gauge symmetries. We applied the traditional functional method using constant background gauge and matter fields in order to satisfy Gauss’s law. Simplifications in the course of the calculation are manifest when constant background gauge and matter fields are used since the determinants are taken on \( 3 \times 3 \) constant matrix, which are diagonal in Fourier space, and when an \( R_\xi \) gauge-fixing condition is imposed, which respect Galilean invariance and global gauge invariance. We have regulated the divergences in the matter sector using operator regularization. We note that the scalar field effective potential does not depend, to the order considered, on the background gauge field, which satisfies Gauss’s law and that our result is in agreement with a diagrammatic analysis of the non-Abelian Aharonov-Bohm scattering. As a spin off of our calculation, we find that there are no infinite nor finite renormalization of the Chern-Simons coupling constant \( \kappa \) in our method in contrast to the results of ref.[14-17] but in agreement with [10,18,23].

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We note that the effective potential presented in Eq. (3.25) is a generalization of the effective potential found in the Abelian version of the model (1.1), which can be retrieved by setting $\alpha^2 = 1$ [22].

We did not discuss here the gauge parameter dependence of our result Eq. (3.25). However, in the Abelian version of the model (1.1), the effective potential was also computed with the $R_s$ gauge-fixing condition and with a Coulomb gauge with arbitrary $\xi$. We found in that case, that the effective potential was the same in either gauge-fixing conditions and was independent of the gauge parameter $\xi$ [22]. We therefore expect that our result for the effective potential presented in Eq. (3.25) to be gauge parameter independent.

Finally, we analyse the scale anomaly. Conformal symmetry is related to the $\beta$-function. A non-vanishing $\beta$-function indicates conformal symmetry breaking. Using the renormalization group equation

$$0 = \mu \frac{d}{d\mu} V_{\text{eff}}(\rho) = \left[ \mu \frac{\partial}{\partial \mu} + \beta(\lambda(\mu)) \frac{\partial}{\partial \lambda(\mu)} \right] V_{\text{eff}}(\rho)$$

(4.1)

the $\beta$-function reads

$$\beta(\lambda(\mu)) = \frac{1}{4\pi} \left( \lambda^2(\mu) - \frac{4}{\kappa^2} \frac{3}{16} \right) .$$

(4.2)

For unrelated coupling constants the theory loses conformal symmetry. At the self-dual point $\lambda(\mu) = -\frac{\sqrt{2}}{2\kappa}$ and at $\lambda(\mu) = \frac{\sqrt{3}}{2\kappa}$ the $\beta$-function vanishes; hence, the theory is conformally symmetric, recovering the result of Bak and Bergman [23].
Acknowledgements

We thank R.B. MacKenzie, D.G.C. McKeon and M.B. Paranjape for useful comments.

References