The Standard Model with Gravity Couplings

Abstract

It has been shown by 't Hooft, and many others after him, that classical gravity
presence of instantons and the Adler-Weinberg anomaly.
and T; and discuss possible violations of these discrete symmetries, including CP, in the
In addition, we investigate the behavior of the theory under discrete transformations P, C
requires that the number of fundamental fermions in the theory must be multiples of 16.
to explore a global anomaly associated with the theory, and argue that if removed
also explore a global anomaly associated with the theory, and argue that if removed.
the Adler-Weinberg connection allows for couplings only to left-handed Weyl fermions. We
guarantees the cancellation of perturbative chiral gauge anomalies, despite the fact that the
Standard Model of particle physics can be introduced into the theory in a manner which
in four dimensions can be described equally well by (anti-)self-dual variables instead of
in this paper, we examine the coupling of matter fields to
Abstract

La-y Nam Chang
Institute for High Energy Physics
Virginia Polytechnic Institute and State University
Blacksburg, VA 24061-0435

and

Chopin Soo

Center for Gravitational Physics and Geometry
Pennsylvania State University
University Park, PA 16802-6300

I-lay Nam Chang

Abstract

The Standard Model with Gravity Couplings

la-y@vtvm1.cc.vt.edu

Hep-TH-9406188, CGP-94/6-2, VPI-HEP-94-2
Introduction

Ashtekar[1] has introduced a set of variables to describe gravity, which makes essential use of the chiral decomposition of the connection one-forms of the local Lorentz group. What has been shown is that, at least for Einstein manifolds without matter, the full set of field equations of general relativity can be recovered by use of only one of the two chiral projections of these connection forms. We may either use the self-dual or the anti-self-dual one-forms. The constraints of the theory then define what the other set has to be.

Chiral projections are used routinely in particle physics; indeed, the fermion fields that define the Standard Model are all chiral Weyl spinor fields. Within the Ashtekar context, left-handed spinor fields are coupled to one of these connection forms, say $A^{-}$, while right-handed ones are coupled to the other forms. Since only one of these is all one needs to define general relativity, the actual Lagrangian must be expressed entirely in terms of either left-handed or right-handed Weyl spinor fields.

That the Standard Model can be so described is of course already known. For example, in SO(10) grand unification schemes[2], one employs a single 16-dimensional left-handed Weyl field to describe one generation of fermions. What is less clear is how the coupling of Ashtekar fields affects the resultant physics.

In what follows, we describe some facets of these consequences. Since Ashtekar gravity makes use of only one of the chirally projected Lorentz connections, there arises the question of whether anomalies which are normally present in such theories are under control. We show that the usual conditions for anomaly cancellations for the Standard Model gauge groups remain true in the presence of Ashtekar gravity, and that the new fields do not introduce any further perturbative anomalies. However, they do introduce global anomalies. Cancellation of these obstructions in the most general context results in the rather strong constraint that the total number of fundamental fermions in the theory must be a multiple of 16. Grand unification schemes based upon groups such as SU(5) are therefore inconsistent when coupled to gravity. As a consequence, there is every likelihood that neutrinos must be massive when consistency with gravity is taken into account.

We also discuss how the usual discrete symmetries are implemented in the presence Ashtekar gravity. We shall show that it is possible to posit discrete transformation laws for the fermion and Ashtekar fields which are consistent both with our general notions of what parity, time-reversal, and charge conjugation transformations are, and with the fundamental canonical commutation relations of all of the fields. Discrete symmetries can
be implemented in the classical limit. However, what we shall show is that, inevitably, the underlying quantum action is not invariant under parity. This situation is to be contrasted with what happens in SO(10) grand unification schemes, where despite the use of a single Weyl spinor field, the resultant action is invariant under a discrete transformation, which one could identify as parity in that it maps left-handed fields to right-handed ones. The question of CPT invariance will also be briefly discussed.

The Samuel-Jacobson-Smolin action and the Ashtekar variables

We first consider spacetimes of Lorentzian signature (−, +, +, +) and start with the action proposed by Samuel, and Jacobson and Smolin\[3\]

$$S_{SJS} = \int_M \mathcal{L}_G$$

\begin{align}
S_{SJS} &= \frac{1}{8\pi G} \int_M \Sigma^\pm a \lor F_a^\pm \pm i \frac{\lambda}{3(16\pi G)} \int_M \Sigma^\pm a \lor \Sigma_a^\pm \\
\end{align}

The (anti)self-dual two-forms $\Sigma^\pm$ which obey $\ast \Sigma^\pm a = \pm i \Sigma^\pm a$, are defined as

$$\Sigma^\pm a \equiv (-e^0 \lor e^a \pm \frac{i}{2} e^a_{bc} e^b \lor e^c)$$

$F^\pm$ are the curvature two-forms of the $SO(3, C)$ Ashtekar connections i.e.

$$F_a^\pm = dA_a^\pm + \frac{1}{2} \epsilon_a^{bc} A_b^\pm \lor A_c^\pm$$

and $e_A, A = 0, \ldots, 3$ denote the vierbein one-forms in four dimensions while $\lambda$ is the cosmological constant. Lower case Latin indices $a, b, c$, which run from 1 to 3 label internal $SO(3, C)$ indices, while upper case Latin indices label flat tangent space indices and range from 0 to 3. Spacetime indices will be denoted by Greek indices.

The Ashtekar variables\[1\] are sometimes referred to as (anti)self-dual variables because the equations of motion of the Samuel-Jacobson-Smolin action with respect to $A^\pm$,

$$D \Sigma^\pm a = 0$$

implies that $A^-$ and $A^+$ are the anti-self-dual and self-dual part of the spin connection, $\omega$, respectively i.e.

$$A_a^\pm = \pm i \omega_a^\pm - \frac{1}{2} \epsilon_a^{bc} \omega_{bc}$$
It is easy to see that the action reproduces the Ashtekar variables and constraints. For convenience, we work in the spatial gauge in which the components of the vierbein and its inverse can be written in the form

\[ e_A = \begin{bmatrix} N & 0 \\ N_i e_{a_j} & e_{a_i} \end{bmatrix}, \quad E^\mu_A = \begin{bmatrix} N^{-1} & 0 \\ -N_i/N & \sigma^i_a \end{bmatrix} \]  \tag{6}

The form assumed in (6) is compatible with the ADM[4] decomposition of the metric

\[ ds^2 = \epsilon_A^\mu e^\lambda_\nu dx^\mu dx^\nu \]

\[ = -N^2(dx^0)^2 + g_{ij}(dx^i + N^i dx^0)(dx^j + N^j dx^0) \]  \tag{7}

with the spatial metric \( g_{ij} = \epsilon^a_i \epsilon^a_j \). Thus we see that the choice (6) in no way compromises the values of the lapse and shift functions, \( N \) and \( N^i \), which have geometrical interpretations in hypersurface deformations. With this decomposition, it is straightforward to re-write

\[ S_{SJS} = \frac{1}{16\pi G} \int d^4x \left\{ \pm 2i\tilde{\sigma}^i a A_{i a} + 2i A_{a i}^T D_i \tilde{\sigma}^i a + 2i N^j \tilde{\sigma}^i a F_{ij a}^T \right\} \]

\[ - \frac{1}{16\pi G} \int d^4x \left\{ \tilde{N} \left( \epsilon^{abc} \tilde{\sigma}^i a \tilde{\sigma}^j b F_{ij a}^c + \frac{\lambda}{3} \epsilon^{abc} \epsilon_{ijk} \tilde{\sigma}^i a \tilde{\sigma}^j b \tilde{\sigma}^k c \right) \right\} \]

+ boundary terms

with \( \tilde{\sigma} \) and \( \tilde{N} \) defined as

\[ \tilde{\sigma}^i a = \frac{1}{2} \epsilon^{ijk} \epsilon^{abc} \epsilon_{jb} \epsilon_{kc} \]  \tag{9a}

\[ \tilde{N} = \det(e_{ai})^{-1} N \]  \tag{9b}

The tildes above and below the variables indicate that they are tensor densities of weight 1 and -1 respectively. Therefore, \( \pm(2i\tilde{\sigma}^i a / 16\pi G) \) are readily identified as the conjugate variables to \( A_{i a}^T \), and we have the commutation relation

\[ \left[ \tilde{\sigma}^i a (\bar{x}, t), A_{j b}^T (\bar{y}, t) \right] = \pm(8\pi G)\hbar \delta_j^i \delta^b_3 \delta^3 (\bar{x} - \bar{y}) \]  \tag{10}

The variables \( A_{a i}^T, N^i \) and \( \tilde{N} \) are clearly Lagrange multipliers for the Ashtekar constraints which can be identified as Gauss’ law generating \( SO(3) \) gauge invariance

\[ G^a \equiv 2i D_i \tilde{\sigma}^i a \approx 0 \]  \tag{11}
and the “supermomentum” and “superhamiltonian” constraints

\[
\begin{align*}
H_i & \equiv 2i\tilde{\sigma}^j a_i \approx 0 \\
H & \equiv \epsilon_{abc}\tilde{\sigma}^a \tilde{\sigma}^b(F_{ij}^c + \frac{\lambda}{3}\epsilon_{ijk}\tilde{\sigma}^k) \approx 0
\end{align*}
\] (12a)  
(12b)

Ashtekar showed that these constraints and their algebra, despite their remarkable simplicity, are equivalent in content to the constraints and constraint algebra of general relativity [1]. Note that both the self-dual and anti-self-dual versions describe pure gravity equally well.

**Coupling to matter fields**

The coupling of matter fields to gravity described by the (anti)self-dual Ashtekar variables have been considered by others before [5], [6], [7]. It should be emphasized that besides self-interactions in pure gravity, only fermions couple directly to the Ashtekar-Sen connections. They are hence direct sources for the Ashtekar-Sen connection. Conventional scalars and Yang-Mills fields have direct couplings only to the metric (hence to \(\tilde{\sigma}\)) rather than to the Ashtekar-Sen connection. Thus when one substitutes the conventional gravitational action with the Samuel-Jacobson-Smolin action, complications can come from the fermionic sector, primarily because one has to make a choice between the actions described by \(A^+\) and \(A^-\) and once the choice is made, it is permissible to couple only right or left-handed fermions (but not both) to the theory [8].

We first look at the conventional Dirac action, with couplings to ordinary spin connections. We then describe couplings of Ashtekar fields to fermions, and explore the physical implications in the following sections.

Consider the conventional Dirac action for an electron or a single quark of a particular color in the presence of only the conventional gravitational field. The action can be written as

\[
S_D = -\frac{i}{2}\int_M (\ast 1)\overline{\Psi}\gamma^A E_A \left[D_{\omega}\Psi + h.c.\right] - i\int_M (\ast 1)\overline{\Psi}m(\phi)\Psi
\] (13)

where the covariant derivative with respect to the spin connection, \(\omega\), is defined by

\[
D_{\omega}\Psi = dx^\mu(\partial_\mu + \frac{1}{4}\omega_{\mu BC}\mathcal{S}^{BC})\Psi
\] (14)

with the generator \(\mathcal{S}^{AB} = \frac{1}{4}[\gamma^A, \gamma^B]\). We adopt the convention

\[
\{\gamma^A, \gamma^B\} = 2\eta^{AB}
\] (15)
with $\eta^{AB} = \text{diag}(-1, +1, +1, +1)$.

We use the chiral representation henceforth for convenience and clarity. In the chiral representation

$$\gamma^A = \begin{pmatrix} 0 & i\tau^A \\ i\tau^A & 0 \end{pmatrix}$$

(16)

where $\tau^a = -\tau^a$ ($a = 1, 2, 3$) are Pauli matrices, and $\tau^0 = \tau^0 = -I_2$. We also have

$$\gamma^5 = i\gamma^0 \gamma^1 \gamma^2 \gamma^3$$

(17a)

$$= \begin{pmatrix} I_2 & 0 \\ 0 & -I_2 \end{pmatrix}$$

(17b)

and the Dirac bispinor is expressed in terms of two-component left and right-handed Weyl spinors, $\phi_L, R$, as

$$\Psi = \begin{pmatrix} \phi_R \\ \phi_L \end{pmatrix}$$

(18)

The contractions in (13) are defined by

$$E_A \nabla \Psi \equiv E^\mu A_\mu \Psi$$

(19)

Note that the vierbein vector fields and one-forms $E_A = E^\mu A_\mu, \epsilon^A = \epsilon^A \mu dx^\mu$ satisfy $E_A |\epsilon^B = \delta_A^B$.

In the chiral representation, the covariant derivative can be written as

$$\nabla \Psi \equiv dx^\mu \left\{ \partial_\mu I_A - i \begin{pmatrix} A^+_{\mu a} \tau^a / 2 \\ 0 \end{pmatrix} \right\} \begin{pmatrix} \phi_R \\ \phi_L \end{pmatrix}$$

(20)

and in conventional coupling of fermions to spin connections $A^+$ are precisely

$$A^+_{\mu} = \pm i\omega_{\mu a} - \frac{1}{2} \epsilon_{abc} \nabla^a \omega_{bc}$$

(21)

Written in terms of the Weyl spinors, the Dirac action is

$$S_D = \int_M (\ast 1)(-\frac{i}{2}) \phi_L^j \gamma^A E_{A_1} D^- \phi_L + \int_M (\ast 1)(\frac{i}{2}) E_{A_1} (D^- \phi_L)^j \gamma^A \phi_L$$

(22)

$$+ \int_M (\ast 1)(\frac{i}{2}) \phi_{\mu} \gamma^A E_{A_1} D^\mu \phi_R + \int_M (\ast 1)(\frac{i}{2}) E_{A_1} (D^\mu \phi_R)^j \gamma^A \phi_R$$

+ mass term
where
\begin{align}
D^- \phi_L &\equiv (d - i A^-_a \tau^a/2) \phi_L, \quad D^+ \phi_R \equiv (d - i A^+_a \tau^a/2) \phi_R \tag{23a} \\
(D^- \phi_L)^\dagger &\equiv d \phi_L^\dagger + i \phi_L^\dagger A^+_a \tau^a/2, \quad (D^+ \phi_R)^\dagger \equiv d \phi_R^\dagger + i \phi_R^\dagger A^-_a \tau^a/2 \tag{23b}
\end{align}

Notice that in (22,) terms (1) and (4) contain $A^-$ but not $A^+$ while (2) and (3) involve $A^+$ but not $A^-$. Furthermore, it is well-known that right-handed spinors can be written in terms of left-handed ones through the relation
\[
\phi_R = -i \tau^2 \chi_L^a
\]
and vice-versa, so the term (4) which involves $A^-$ can be written in terms of totally left-handed (anti-self-dual) spin connections and Weyl spinors as
\[
\int_M (\ast 1)(-i/2) \chi_L^a \tau^A E_A \bar{\phi} E \bar{\phi} D^- \chi_L
\]
Similar remarks apply to the term (2) and the field $A^+$. In order to couple spinors to Ashtekar connections of only one chirality, we can now define modified “chiral” Dirac actions [5], [7]
\begin{align}
S^-_D &\equiv 2\{(1) + (4)\} + \text{mass term} \tag{26a} \\
&= \int_M (\ast 1)(-i)(\phi_L^\dagger \tau^A E_A \bar{\phi} + \chi_L^a \tau^A E_A \bar{\phi}) \\
&\quad + \text{mass term} \tag{26b}
\end{align}
\begin{align}
S^+_D &\equiv 2\{(2) + (3)\} + \text{mass term} \tag{27a} \\
&= \int_M (\ast 1)(-i)(\phi_R^\dagger \tau^A E_A \bar{\phi} + E_A((D^- \phi_L)^\dagger \tau^A \phi_L)) \\
&\quad + \text{mass term} \tag{27b}
\end{align}

Next, we make the identification that the quantities $A^\pm$, as anticipated by the notation, are precisely the connections introduced by Ashtekar in his simplification of the constraints of General Relativity, and the very same variables in the Samuel-Jacobson-Smolin action of Eqn. (1). Instead of the sum of the Einstein-Hilbert-Palatini action and the full Dirac action, $S_D$, the total action is now taken to be $(S^-_D + S^+_D S_J S)$. Remarkably, it is possible to
show that this total action reproduces the correct classical equations of motion for General Relativity with spinors [5], [9].

**Discrete transformations and spinors**

The usual discrete transformations for the second quantized spinor fields are

\[
P : \phi_L(x) \leftrightarrow -i\tau^2 \chi_L^*(-P^{-1}(x)) \quad \text{i.e.} \quad (\phi_L(x) \leftrightarrow \phi_R(P^{-1}(x)) \quad (28a)
\]

\[
C : \phi_L(x) \leftrightarrow \chi_L(x) \quad \text{i.e.} \quad (\Psi^c = C\Psi^T) \quad (28b)
\]

\[
T : \phi_L(x) \mapsto -\tau^2 \phi_L(T^{-1}(x)) \quad (28c)
\]

A set of discrete transformations for the gravity variables consistent with the ones described above for fermion fields can then be defined as follows:

\[
P : (\sigma^{ia}(\vec{x}, t), A^\mp_{ia}(\vec{x}, t)) \mapsto (\sigma^{ia}(P^{-1}(\vec{x}, t)), A^\pm_{ia}(P^{-1}(\vec{x}, t))) \quad (29a)
\]

\[
C : (\sigma^{ia}(\vec{x}, t), A^\mp_{ia}(\vec{x}, t)) \mapsto (\sigma^{ia}(\vec{x}, t), A^\mp_{ia}(\vec{x}, t)) \quad (29b)
\]

\[
T : (\sigma^{ia}(\vec{x}, t), A^\mp_{ia}(\vec{x}, t)) \mapsto (\sigma^{ia}(T^{-1}(\vec{x}, t)), A^\mp_{ia}(T^{-1}(\vec{x}, t))) \quad (29c)
\]

These transformations are consistent with the commutation relations (10). In dealing with curved spacetime, it is more convenient to rewrite these transformations in terms of their action on the one-forms \((e^A, A^\mp_a)\). These are:

\[
P : (e^0, e^a; A^\mp_a) \mapsto (e^0, -e^a; A^\pm_a) \quad (30a)
\]

\[
C : (e^A; A^\mp_a) \mapsto (e^A; A^\pm_a) \quad (30b)
\]

\[
T : (e^0, e^a; A^\mp_a) \mapsto (-e^0, e^a; A^\pm_a) \quad (30c)
\]

P and T transformations for the Ashtekar variables have been discussed previously [10]. However, that discussion did not cover CPT nor take into account the effect of coupling to fermions.

To discuss the effect of the discrete transformations in a concise manner, we first define \(A_{AB} = -A_{BA}\) such that

\[
A_{0a} \equiv \frac{1}{2i}(A_a^- - A_a^+) \quad (31a)
\]

\[
A_{bc} \equiv -\frac{1}{2}e^a_{bc}(A_a^- + A_a^+) \quad (31b)
\]
and keep in mind the effect of the discrete transformations displayed in (30).

The curvature

\[ F_{AB} \equiv dA_{AB} + A_A^C \wedge A_{CB} \]  

then has components

\[ F_{0a} = \frac{1}{2i}(F_a^- - F_a^+) \]  \hspace{1cm} (33a)

\[ F_{bc} = -\frac{1}{2}e^a_{\ bc}(F_a^- + F_a^+) \]  \hspace{1cm} (33b)

and the “torsion” \( T_A(A^\mp) \) which depends on \( A^\mp \) is defined as

\[ T^A \equiv de^A + A_A^B \wedge e^B \]  \hspace{1cm} (34)

With all this, it can be shown that the “chiral” gravitational and Dirac actions are related to the conventional ones by

\[ S^\mp_{JS} = \frac{1}{(16\pi G)} \int_M e^A \wedge e^B \wedge *F_{AB} + \frac{i}{(16\pi G)} \int_M [-d(e^A \wedge T^A) + T^A \wedge T_A] \]

\[ + \frac{\lambda}{16\pi G} \int_M e^0 \wedge e^1 \wedge e^2 \wedge e^3 \]  \hspace{1cm} (35)

and

\[ S^\mp_D = S_D + \int_M \left[ \pm \frac{i}{4} \psi^A \epsilon_{ABCD} e^C \wedge e^D \wedge T^B + \frac{i}{2(3!)} d(\psi^A \epsilon_{ABCD} e^B \wedge e^C \wedge e^D) \right] \]  \hspace{1cm} (36)

where \( \psi^A \equiv \phi^A_L \tau^A \phi_L + \chi^A_L \tau^A \chi_L \).

The combined fermionic and gravitational total action is therefore

\[ S_{total} = S^\mp_D + S^\mp_{JS} \]

\[ = S_D + \frac{1}{(16\pi G)} \int_M e^A \wedge e^B \wedge *F_{AB} \]

\[ = \mp \frac{i}{2} \int_M d\left[ -\frac{1}{(8\pi G)} e^A \wedge T_A + \frac{1}{3!} (\epsilon_{ABCD} \psi^A e^B \wedge e^C \wedge e^D) \right] \]

\[ \mp \frac{i}{(16\pi G)} \int_M \Theta_A \wedge \Theta^A + \frac{\lambda}{16\pi G} \int_M e^0 \wedge e^1 \wedge e^2 \wedge e^3 \]  \hspace{1cm} (37)

where

\[ \Theta_A \equiv T_A + (2\pi G) \epsilon_{ABCD} \psi^B e^C \wedge e^D \]  \hspace{1cm} (38)
We then find that under P (and also CP and CPT), the change is

$$\Delta S_D^- = P S_D^- D^{-1} - S_D^-$$

(39a)

$$= i \int_M \frac{1}{3!} d(\epsilon_{A B C D}\psi^A \epsilon^B \wedge \epsilon^C \wedge \epsilon^D) - \frac{1}{2} \epsilon_{A B C D} \psi^A \epsilon^C \wedge \epsilon^D \wedge T^B$$

(39b)

and the change of the total action is then

$$\Delta S_{\text{total}}^- = \Delta S_{\bar{S},JS}^- + \Delta S_D^-$$

(40a)

$$= i \int_M d\left\{ - \frac{1}{(8\pi G)} \epsilon^A \wedge T_A + \frac{1}{3!} (\epsilon_{A B C D} \psi^A \epsilon^B \wedge \epsilon^C \wedge \epsilon^D) \right\}$$

$$+ \frac{i}{(8\pi G)} \int_M \Theta_A \wedge \Theta^A$$

(40b)

It should be noted that for spacetimes with Lorentzian signatures, the Ashtekar variables are not real but rather must satisfy reality conditions which are supposed to be enforced by a suitable inner product for quantum gravity[1]. Furthermore, the actions are not explicitly real and their hermiticity could be tied to the inner product for the yet unavailable quantum theory of gravity. However, if we are interested in examining second quantized matter in a gravitational background (for that matter, in flat spacetime), we may enforce the reality conditions on $A^-$ by hand. We assume for our present purposes, that even though we do not at the moment have an inner product for quantum gravity that will enforce these reality conditions, we can pass over to the second order formulation whereby we eliminate the Ashtekar connections in terms of the vierbein and spinors through the equations of motion for $A$ and enforce the reality conditions by inspection.

So, varying the total first order action, $S_D^- + S_{\bar{S},JS}^-$, with respect to $A$ yields

$$D \Sigma^{-a} = (4\pi G) \left[ (1/3) \epsilon_{b c d} \psi^a \epsilon^b \wedge \epsilon^c \wedge \epsilon^d + (1/2) \epsilon^a_{b c d} \psi_0 \epsilon^b \wedge \epsilon^c \wedge \epsilon_0 + i \psi b \epsilon^a \wedge \epsilon_0 \wedge \epsilon_0 \right]$$

which implies that the Ashtekar connection is

$$A_a^- = i \omega_0 a - \frac{1}{2} \epsilon_a^{b c} \omega_{b c} - (2\pi G) \epsilon_{a b c} \left[ (1/2) \epsilon^c_{A B} \psi^A \epsilon^B - i \psi b \epsilon^a \right]$$

(41)

Upon imposing the usual hermiticity requirements on the spinors (which implies that $\psi^A$ is hermitian) and the requirement that $\epsilon^A$ (hence the spin connection $\omega$) is real, one finds that $\Theta = 0$, which for the case of pure gravity, reduces to the torsionless condition. Thus passing over to the second order formulation where the Ashtekar connection has been
eliminated and reality conditions have been imposed, we find that the change in the action under $P$ (and also $CP$ and $CPT$) is

\[
\Delta S_{\text{total}} = -\frac{i}{12} \int_M d[\epsilon_{ABCD} \psi^A e^B \wedge e^C \wedge e^D]
\]

\[
= -\frac{i}{2} \int_M \partial_\mu [\text{det}(\epsilon) j^\mu_5] d^4 x
\]

\[
= -\frac{i}{2} \int_M (\nabla_\mu j^\mu_5) \text{det}(\epsilon) d^4 x
\]

\[
= -\frac{i}{2} \Delta Q_5
\]

where

\[
j^\mu_5 = \bar{\Psi} \gamma^\mu A^5 \Psi
\]

\[
= -E^\mu_A (\phi^A_L \gamma^A \phi^A_L - \phi^A_R \gamma^A \phi^A_R)
\]

\[
= -E^\mu_A (\phi^A_L \gamma^A \phi^A_L + \chi^A_L \gamma^A \chi^A_L)
\]

is precisely the chiral current, and $\Delta Q_5$ is the change in the chiral charge. Therefore, it would appear that if the chiral current is not conserved, the action fails to be hermitian and invariant under $P$, $CP$ and $CPT$. In this respect, the Adler-Bell-Jackiw anomaly or the axial anomaly [11] induced by global instanton effects may lead to violations of CPT through chirality changing transitions. Chirality changing transitions due to QCD instantons have been investigated before [12], including 't Hooft [13] in his resolution of the $U(1)_A$ problem. We shall postpone the discussion on the physical implications and bounds on the apparent non-hermiticity of the action and the implied violations of discrete symmetries to a separate report [14].

Although the expressions and discussions above are for a single quark or electron, it is easy to incorporate more matter fields into the theory simply by replacing $\psi^A$ with

\[
\psi^A \equiv \sum \Phi^A_L \gamma^A \Phi_L
\]

where the sum is over all Weyl fermions, including right-handed fermions written as left-handed ones.

**The Standard Model with Ashtekar Fields**

We next examine how the quark and lepton fields of the Standard Model are to be coupled to the Ashtekar fields. Since we are allowing couplings to $A^-$ rather than the
whole spin connection, only left-handed fermions can be coupled to the theory. By writing all right-handed fields in terms of left-handed ones, it is possible to couple all the Standard Model quark and lepton fields to Ashtekar gravity in a consistent manner. We will show in the next section that this coupling will not disturb the cancellations of all perturbative anomalies, given the multiplets of the Standard Model. The question of how to deal with the global anomaly that is present in the theory will be examined in the following section.

We will label collectively by $W_i T^i$ the connection one-forms of the Standard Model. The $T^i$’s denote generators of the Standard Model gauge group, and there should not be any confusion between this index $i$ and spatial indices. The covariant derivative in $\mathcal{S}_D^\mu$ in the action is then replaced by the total covariant derivative

$$D\phi_{LJ} = \left[ (d - iA_\mu^a(\tau^a/2))\delta_{IJ} - iW_i(T^i)_{IJ} \right] \phi_{LJ} \quad (46)$$

where the index $I$ associated with $\Psi_{L_I}$ denotes internal “flavor” and/or “color”. In the modified model, the right-handed Weyl fields of the conventional Standard Model are to be written as left-handed Weyl fermions via the relation

$$\phi_R = -i\tau^2 \chi_L^*$$

So an electron or a single quark of a particular color is represented as a pair of left-handed Weyl fermions ($\phi_L$ and $\chi_L$), and a left-handed neutrino is represented by a single left-handed Weyl fermion.

The total action is still to be as in Eqn.(26) with the total covariant derivative of (46) but summed over all left-handed fermions, including “right-handed” fermions which are written as left-handed ones. Using Eqn.(47) right-handed currents can be written in terms of left-handed currents through

$$J_R^\mu = \phi_{R_i}^\dagger \tau^\mu(T^i_R)_{IJ} \phi_{R_J}$$

$$\quad = \chi_{L_i}^\dagger \tau^\mu(T^i_L)_{IJ} \chi_{L_J} \quad (48b)$$

with

$$T^\mu_L = -(T^\mu_R)^T \quad (49)$$

Notice however that terms containing the ordinary gauge connections $W$ are hermitian, so the factor 2 multiplying terms denoted by (1) and (4) in (26) takes care of the contributions from terms with $W$ in (2) and (3) which would have been present had we use the
conventional Dirac action. Similarly, this holds also for the left-handed neutrino although it is now described by only the term (1) (summed over the species) instead of (1) and (2). The net result of all this is that the difference between the “Modified Standard Model” described by $S_\vec{\rho}$ with couplings only to left-handed fermions instead of the conventional $S_\rho$ is still as in (36).

**Cancellation of Gauge Anomalies**

We shall first demonstrate that the Standard Model defined entirely in terms of left-handed fields, and the anti-self-dual Ashtekar fields $A^-$, is free of perturbative chiral gauge anomalies.

Recall that in the flat spacetime limit, the anomalies can be determined via Fujikawa’s method. In the chiral representation, we write

$$\Psi_L = \begin{pmatrix} 0 \\ \bar{\phi}_L \end{pmatrix}, \quad \Xi_L = \begin{pmatrix} 0 \\ \lambda_L \end{pmatrix}$$

(50)

The Euclidean path integral measure for the fermions is

$$\prod_{I,J} D\bar{\Psi}_L, D\Psi_L, D\Xi_{L,I}, D\Xi_{L,J}$$

(51)

The left-handed fermions can be expanded in terms of the chiral orthonormal set $\{X_n^L, X_n^R\}$ (see, for instance, Ref.[15]) with

$$X_n^L(x) = \left(\frac{1 - \gamma^5}{\sqrt{2}}\right)X_n(x), \quad \lambda_n > 0$$

$$= \left(\frac{1 - \gamma^5}{2}\right)X_0(x), \quad \lambda_n = 0$$

(52a)

$$X_n^R(x) = \left(\frac{1 + \gamma^5}{\sqrt{2}}\right)X_n(x), \quad \lambda_n > 0$$

$$= \left(\frac{1 + \gamma^5}{2}\right)X_0(x), \quad \lambda_n = 0$$

(52b)

In flat Euclidean space, the $X_n$'s are eigenvectors of the hermitian Dirac operator in presence of the Standard Model gauge fields. In our present instance, the Dirac operator contains the full set of spin connections, and remains hermitian provided $\Theta = 0$, and $X_n$'s are the eigenvectors of this modified Dirac operator. The chiral projections in (52) serve the purpose of providing a regularization scheme to select the combinations of the spin connections appropriate to the Ashtekar fields, as defined in Eqn.(21).
We now use the expansion:

\[ \Psi_L(x) = \sum_{\lambda_n \geq 0} a_n X^L_n(x) \]  
\[ \overline{\Psi}_L(x) = \sum_{\lambda_n \geq 0} \overline{T}_n X^R_{\lambda} \]  

Under a gauge transformation with generator \( T^i \),

\[ \Psi_L(x) \rightarrow e^{-i \alpha(x) T^i} \Psi_L(x) \]

the measure for each left-handed field

\[ d\mu = D\overline{\Psi}_L D\Psi_L \equiv \prod_n da_n \prod_n db_n \]

transforms as (see Ref. [15])

\[ d\mu \rightarrow d\mu \exp\{-i \int \alpha(x) A^i(x) dx\} \]

The anomaly can be computed as in Ref.[15]:

\[ A^i(x) = \sum_{all n} X^\dagger_n(x) \gamma^5 T^i X_n(x) dx \]  
\[ = -\frac{1}{16\pi} Tr\{ T^i \frac{1}{2} e^{\mu \nu \alpha \beta} G_{\alpha \beta} G_{\mu \nu} \} \]

and is proportional to \( Tr(T^i\{ T^j, T^k \}) \). Here, \( G_{\mu \nu} \) is the field strength associated with \( W^i_\mu \).

So, the condition for cancellation of perturbative gauge anomalies is that \( Tr(T^i\{ T^j, T^k \}) \) when summed over all fields coupled to \( W_i T^i \) cancel. But as we have seen, if “right-handed” spinors in the Standard Model coupled to \( W_i T^i \) are written as left-handed spinors coupled to \( W_i T^i_L \) such that the representations \( T^i_L = -(T^i_R)^T \), then

\[ Tr(T^i_L\{ T^j_L, T^k_L \}) = -Tr(T^i_R\{ T^j_R, T^k_R \}) \]

and, the condition for the perturbative chiral gauge anomalies of left and “right”-handed fermions to cancel is

\[ Tr(T^i_L\{ T^j_L, T^k_L \}) + Tr(T^i_L\{ T^j_L, T^k_L \}) = 0 \]
which is precisely equivalent to the well-known condition [16]

$$Tr(T^i_L \{ T^j_L, T^k_L \}) = Tr(T^i_R \{ T^j_R, T^k_R \})$$

(60)

The Ashtekar fields do not appear in the above because the generators of Ashtekar gauge group belong to $su(2)$, and $SU(2)$ is a “safe group” where $Tr(T^i \{ T^j, T^k \}) = 0$ for any representation. The introduction of the Ashtekar fields therefore does not disturb the usual anomaly cancellation conditions. But the theory can still be afflicted with global anomalies, which is the issue we will address next.

**Global anomaly**

Witten [17] showed that in four dimensions, theories with an odd number of Weyl fermion doublets coupled to gauge fields of the $SU(2)$ group are inconsistent. The Standard Model has four such doublets in each generation, and so it is not troubled by such a global anomaly. But what about the Ashtekar gauge fields? Strictly speaking for manifolds of Lorentzian rather than Euclidean signatures, the group is complexified $SU(2)$ and isomorphic to $SL(2, C)$. But both these groups have the homotopy group $\Pi_4(G) = Z_2$, and the non-trivial transformations of the complexified $SU(2)$ group in $\Pi_4(G)$ are associated with the rotation group. As Witten [17] has argued, the presence of non-trivial elements of this homotopy group could produce global anomalies. So it would appear that in four dimensions there could be further constraints on the particle content in order to ensure that the theory be free of the global anomaly associated with the Ashtekar gauge group.

We review the essential points of Witten’s argument. To start off, consider a suitable Wick rotation of the background spacetime into a Riemannian manifold. The Dirac operator (with ordinary spin connection) is then hermitian with respect to the inner product $<X|Y> = \int d^4x \det(e)X^\dagger(x)Y(x)$ [15]. The fermions can then be expanded in terms of the complete set of the eigenfunctions of the Dirac operator. The expansions will be the same as in Eqns. (52) and (53) but now the eigenfunctions are normalized so that

$$\int_M d^4x \det(e)X_m^\dagger(x)X_n(x) = \delta_{mn}$$

(61)

Consider next a chiral transformation by $\pi$ which maps each left-handed Weyl fermion $\Psi_L(x) \mapsto \exp(\pi \gamma_5)\Psi_L(x) = \exp(\pi)\Psi_L(x) = -\Psi_L(x)$. Obviously such a map is a symmetry of the action. However, the measure is not necessarily invariant under such a chiral
transformation because of the ABJ anomaly. Instead, for each left-handed fermion the measure transforms as

\[ d\mu \mapsto d\mu \exp(-i\pi \int_M d^4x \det(\epsilon) \sum_n X_n^\dagger(x)\gamma_5X_n(x)) \]  

(62)

The expression \( \int_M d^4x \sum_n \det(\epsilon)X_n^\dagger(x)\gamma_5X_n(x) \) is formally equal to \((n_+ - n_-)\), where \(n_+\) are the number of positive and negative chirality zero modes of the Dirac operator. Upon regularization, the expression works out to be

\[ \sum_n \det(\epsilon)X_n^\dagger(x)\gamma_5X_n(x) = \lim_{M \to \infty} \sum_n \det(\epsilon)X_n^\dagger(x)\gamma_5e^{-(\lambda_n/M)^2}X_n(x) \]

\[ = \lim_{M \to \infty} \lim_{x' \to x} Tr\gamma_5 \det(\epsilon) e^{-i(\gamma^\mu D_n/M)^2} \sum_n X_n(x)X_n(x') \]

\[ = -\frac{1}{384\pi^2} R_{\mu\nu\sigma\tau} \ast R^{\mu\nu\sigma\tau} \]  

(63)

As a result, \((n_+ - n_-) = \sum_n \int_M d^4x \det(\epsilon)X_n^\dagger(x)\gamma_5X_n(x) = -\tau/8\), where \(\tau = (1/48\pi^2) \int_M R_{\mu\nu\sigma\tau} \ast R^{\mu\nu\sigma\tau}\), is the signature of the compact four-manifold \(M\). Consequently, the change in the measure for each left-handed Weyl fermion under a chiral rotation of \(\pi\) is precisely \(\exp(i\pi\tau/8)\). If there are altogether \(N\) left-handed Weyl fermions in the theory, the total measure changes by \(\exp(iN\pi\tau/8)\). But, as emphasized in Ref. [18], the transformation of \(-1\) on the fermions can also be considered to be an \(SU(2)\) transformation. Since \(SU(2)\) is a safe group with no perturbative chiral anomalies, the measure must be invariant under all \(SU(2)\) transformations. Thus there is an inconsistency unless this phase factor, \(\exp(iN\pi\tau/8)\), is always unity. Note that in the above evaluation of the Jacobian under a chiral transformation of \(\pi\), we took into account only the gravitational coupling of the Weyl fermions because as we have mentioned earlier, the number of Weyl multiplets coupled to \(SU(2)_W\) in the Standard Model is even, and so their contributions to the Jacobian under a chiral rotation of \(\pi\) is always a factor of unity.

It is known that for consistency of parallel transport of spinors for topological four-manifolds, \(\tau\) must be a multiple of 8 for spin structures to exist[19]. A theorem due to Rohlin [20] states the signature of a smooth spin four-manifold is divisible by 16. If one restricts to only smooth four-manifolds with spin structures in quantum gravity, or in semiclassical quantum field theory with left-handed spinors, one allows only compact smooth spin four-manifold backgrounds, then under a chiral rotation of \(\pi\) the measure is
invariant regardless of N. Otherwise, N must be even if \( \tau \) is only required to be a multiple of 8.

It is not presently clear what the integration range for \( A^\tau \) should be. But, it is quite likely that it should extend over a wider class of manifolds than those which admit classical spin structures[21]. If we do admit such manifolds into the integration range for the Ashtekar fields, the above consistency condition then implies that there can only be \( 16k \) fermions in total in the theory, where \( k \) is an integer.

Now, if one counts the number of left-handed Weyl fermions that are coupled to the Ashtekar-Sen connection for the Standard Model, one finds that the number is 15 per generation, giving a total of 45 for 3 generations. This comes about because each bispinor is coupled twice to the Ashtekar connection while each Weyl spinor is coupled once (e.g. for the first generation, the number is 2 for each electron and each up or down quark of a particular color, and 1 for each left-handed neutrino.) Thus if one requires N to be even because of the above considerations, the global anomaly with respect to the Ashtekar gauge group seems to imply that there should be additional particle(s). There could be a partner for each neutrino making N to be 16 per generation, or a partner for just the \( \tau \)-neutrino or even four generations, depending upon whether we allow for the presence of generalized spin structures. In particular, in the general case with \( 16k \) fermions, grand unification schemes based upon groups such as SU(5) would be inconsistent when coupled to gravity. However notice that for the absence of this global anomaly, the extra particle(s) need only couple to the Ashtekar connection i.e. the gravitational field and nothing else. One can speculate that the additional particle(s) would nevertheless imply non-zero masses for neutrinos, thereby giving rise to neutrino oscillations, and play a significant role in the cosmological issue of “dark matter”.

**Acknowledgments**

The research for this work is supported by the Department of Energy under Grant No. DE-FG05-92ER40709, the NSF under Grant No. PHY-9396246, and research funds provided by the Pennsylvania State University. One of us (C.S.) would like to thank Abhay Ashtekar, Lee Smolin, Jose Mourao and other members of the Center for Gravitational Physics and Geometry for encouragement and helpful discussions.
References


[8] One has to make a choice between the + or − actions. We shall choose the − action. There is an ambiguity in the conventions of the Dirac matrices, γ, which allows one to couple A+ rather than A− to left-handed Weyl spinors. We have adopted a choice which couples anti-self-dual spin connections and A− to left-handed spinors. To the extent that there are no right-handed neutrinos in nature, we should describe nature with only left-handed Weyl spinors. So once the initial choice of coupling A− to the neutrino is made, the theory cannot be described by A+ and right-handed Weyl spinors.

[9] From Eqn. (42), it can be seen that in the second order formalism, the Ashtekar-Sen connection contributes a cubic fermionic term to the Dirac equation when we vary with respect to the spinors. This cubic term can be cancelled by adding a quartic term proportional to $\int_M \langle \Psi^- \Psi \rangle_{A}$ to the action. However, this extra term in the action is separately invariant under C, P and T, and hence do not affect our discussions on the discrete symmetries later on.


