Multitemporal generalization of Schwarzschild solution

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ABSTRACT

The $n$-time generalization of Schwarzschild solution is presented. The equations of geodesics for the metric are integrated and the motion of the relativistic particle is considered. The multitemporal analogue of the Newton’s gravitational law for the objects, described by the solution, is suggested. The scalar-vacuum generalization of the multitemporal solution is also presented.

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1 Introduction

In [1] the generalization of the Schwarzschild solution to the case of $n$ internal Ricci-flat spaces was obtained. (The case $n = 1$ was considered earlier in [2].) In [3] this solution was generalized on $O(d + 1)$-symmetric (Tangherlini-like) case. (In [4] the special case of the solution [3] with $n = 2$ was considered).

This paper is devoted to an interesting special case of the solution [1]. This is the $n$-time generalization of the Schwarzschild solution. We note, that the idea of considering of space-time manifolds with extra time directions was discussed earlier by different authors (see, for example, [5-12]). Some revival of the interest in this direction was inspired recently by string models [9-12].

In sec. 2 the multitemporal generalization of Schwarzschild formula is considered and corresponding geodesic equations are integrated. In sec. 3 the motion of the relativistic particle in the background of the solution is investigated and a multitemporal analogue of the Newton's formula is obtained. The sec. 4 is devoted to multitemporal generalization of Newton's mechanics and Newton's gravitational law for interacting objects described by the solution ("multitemporal hedgehogs").

2 The metric and geodesic equations

The metric generalizing the Schwarzschild solution to the multitemporal case reads

$$g = -\sum_{i=1}^{n} f^i dt^i \otimes dt^i + f^{-b} dR \otimes dR + f^{1-b} R^2 d\Omega^2,$$

(2.1)

where $f = 1 - (L/R)$, $L = \text{const}$, $d\Omega^2$ is standard metric on 2-dimensional sphere and the parameters $b, a_i$ satisfy the relations

$$b = \sum_{i=1}^{n} a_i, \quad b^2 + \sum_{i=1}^{n} a_i^2 = 2.$$

(2.2)

The metric (2.1) satisfies the Einstein equations (or equivalently $R_{MN}[g] = 0$) and may be obtained as a special case of the solution [1] or more general solution [3].

The metrics $g(a, L)$ and $g(-a, -L)$ are equivalent for any set $a = (a_1, \ldots, a_n)$, satisfying (2.2). This may be easily verified using the following transformation of the radial variable: $R = R_e + L$. So, without loss of generality we restrict our consideration by the case $L > 0$ (the case $L = 0$ is trivial).

In the case

$$a_i = \delta_{ik},$$

(2.3)

$k \in \{1, \ldots, n\}$, the metric (2.1) has the following form

$$g = g^{(k)}_{Sc,h - \sum_{i \neq k} dt^i \otimes dt^i},$$

(2.4)

i.e. it is a trivial (cylindrical) extension of the Schwarzschild solution with the time $t^k$. It describes an extended (in times) membrane-like object. Any section of this object by
hypersurface $t^i = t^i_0 = \text{const}$, $i \neq k$, is the 4-dimensional black hole, "living" in the time $t^k$. It may be proved that the solution (2.1) has a singularity at $R = L$ for all sets of parameters $(a_1, \ldots, a_n)$ except $n$ Schwarzschild-like points (2.3) (for $n = 2$ this was proved in [14]).

We consider the geodesic equations for the metric (2.1)

$$\ddot{x}^M + \Gamma^M_{NP}[g] \dot{x}^N \dot{x}^P = 0, \quad (2.5)$$

where $x^M = x^M(\tau)$, $\dot{x}^M = dx^M/d\tau$ and $\tau$ is some parameter on a curve.

These equations are nothing more than the Lagrange equations for the Lagrangian

$$L_1 = \frac{1}{2} g_{MN}(x) \dot{x}^M \dot{x}^N = \frac{1}{2} [f^{-b}(\dot{R})^2 + f^{1-b} \dot{R}^2(\dot{\theta}^2 + \sin^2 \theta \dot{\varphi}^2) - \sum_{i=1}^n f^{a_i}(\dot{i})^2]. \quad (2.6)$$

The complete set of integrals of motion for the Lagrange system (2.6) is following

$$f^{a_i} \dot{i}^i = \varepsilon^i, \quad (2.7)$$
$$f^{1-b} \dot{R}^2 = j, \quad (2.8)$$
$$f^{-b} \dot{R}^2 + f^{1-b} \dot{R}^2 \dot{\varphi}^2 - \sum_{i=1}^n f^{a_i}(\dot{i})^2 = 2E = 2L_1, \quad (2.9)$$

$i = 1, \ldots, n$. Without loss of generality we put here $\theta = \frac{\pi}{2}$.

**Multitemporal horizon.** Here we consider the null geodesics. Putting $E = 0$ in (2.9) we get for a light "moving" to the center

$$\dot{R} = -\sqrt{\sum_{i=1}^n (\varepsilon^i)^2 f^{b-a_i} - j^2 f^{1+2b} R^{-2}} \quad (2.10)$$

and consequently

$$t^i - t^i_0 = -\int_{R_0}^{R} dx \frac{\varepsilon^i[f(x)]^{-a_i}}{\sqrt{\sum_{i=1}^n (\varepsilon^i)^2 [f(x)]^{b-a_i} - j^2 [f(x)]^{1+2b} x^{-2}}}. \quad (2.11)$$

$i = 1, \ldots, n$.

Let $L > 0$, $\varepsilon = (\varepsilon^i) \neq 0$ and $a = (a_1, \ldots, a_n)$ satisfies (2.2). We say that the $\varepsilon$-horizon takes place for the metric (2.1) at $R = L$ if and only if

$$\|t - t_0\| \equiv \sum_{i=1}^n |t^i - t^i_0| \rightarrow +\infty, \quad (2.12)$$

as $R \rightarrow L$ for all $t_0$ and $j$. It may be proved [14] that for $L > 0$ and for non-Schwarzschild set $a$ the $\varepsilon$-horizon for the metric (2.1) at $R = L$ is absent for any $\varepsilon \neq 0$. For the Schwarzschild set of parameters (2.3) the $\varepsilon$-horizon takes place if $\varepsilon^k \neq 0$, i.e. light should "move" in $t^k$-direction.
3 Relativistic particle

Let us consider the motion of the relativistic particle in the gravitational field, corresponding to the metric (2.1). The Lagrangian of the particle is

\[ L_2 = -m \sqrt{-g_{MN}(x) \dot{x}^M \dot{x}^N}, \]  

(3.1)

where \( m \) is the mass of the particle. The Lagrange equations for (3.1) in the proper time gauge

\[ g_{MN}(x) \dot{x}^M \dot{x}^N = -1 \]  

(3.2)

coincide with the geodesic equations (2.5). In this case \((E^i) = (m \dot{\varepsilon}^i)\) is the energy vector and \( J = m J \) is the angular momentum. For fixed values of \( \varepsilon^i \) the 3-dimensional part of the equations of motion is generated by the Lagrangian

\[ L_* = \frac{m}{2} \left( f^{i\beta} \tilde{g}_{\text{sch}, \alpha \beta}(x) \dot{x}^\alpha \dot{x}^\beta + \sum_{i=1}^n (\varepsilon^i)^2 f^{i \alpha} \right). \]  

(3.3)

where \( \tilde{g}_{\text{sch}} \) is the space section of the Schwarzschild metric.

Now, we restrict our consideration by the non-relativistic motion at large distances: \( R \gg L \). In this approximation: \( t^i = \varepsilon^i \tau \), \( \sum_{i=1}^n (\varepsilon^i)^2 = 1 \). It follows from (3.3) that in the considered approximation we get a non-relativistic particle of mass \( m \), moving in the potential

\[ V = -\frac{m}{2} \sum_{i=1}^n (\varepsilon^i)^2 a_i L = -G \frac{m(M_{ij} \varepsilon^i)}{R}, \]  

(3.4)

where \( G \) is the gravitational constant and

\[ M_{ij} = a_i \delta_{ij} L / 2G, \]  

(3.5)

are the components of the gravitational mass matrix.

We note, that the relation (3.4) may be rewritten as following

\[ V = -G \frac{\text{tr}(MM_I)}{R} \]  

(3.6)

where \( M_I = (m \varepsilon_i \varepsilon_j) \) is the inertial mass matrix of the particle. For \( n = 1 \) the potential (3.6) coincides with the Newton’s one.

Matrix form. The solution (2.1) may be also rewritten in the matrix form

\[ g = -[(1 - L/R)^A]_{ij} d\Omega^i \otimes d\Omega^j + (1 - L/R)^{-trA} d\Omega \otimes d\Omega + (1 - L/R)^{1-trA} R^2 d\Omega^2, \]  

(3.7)

where \( A \) is a real symmetric \( n \times n \)-matrix satisfying the relation

\[ (tr A)^2 + tr(A^2) = 2. \]  

(3.8)
Here $x^A \equiv \exp(A \ln x)$ for $x > 0$. The metric (3.7) can be reduced to the metric (2.1) by the diagonalization of the $A$-matrix: $A = S^T (a_i \delta_{ij}) S$, $S^T S = 1_n$ and the reparametrization of the time variables: $S^T f^i = t^i$. In this case the gravitational mass matrix is
\[
(M_{ij}) = (A_{ij} L/2G).
\]
We may also define the gravitational mass tensor as
\[
\mathcal{M} = M_{ij} dt^i \otimes dt^j.
\]
We call the extended (in time) object, corresponding to the solution (3.7)-(3.8) as multitemporal Schwarzschild hedgehog. At large distances $R \gg L$ this object is described by the matrix analogue of the Newton’s potential
\[
\Phi_{ij} = -\frac{1}{2} L A_{ij}/R = -GM_{ij}/R.
\]
Clearly, that this potential for the diagonal case (2.1) $A = a_i \delta_{ij}$ is a superposition of the potentials, corresponding to "pure states": Schwarzschild-like membranes (2.4). So, in the post-Newtonian approximation the Schwarzschild hedgehog is equivalent to the superposition of black hole membranes (2.4), corresponding to different times.

4 Multitemporal Newton laws

The solution (3.7), (3.8) may be also rewritten as following
\[
g = -[(1 - ||L||/R)^L/||L||]_{ij} dt^i \otimes dt^j + (1 - ||L||/R)^{-tr L/||L||} dR \otimes dR + (1 - ||L||/R)^{1-(tr L/||L||)} R^2 d\Omega^2,
\]
where here $L = (L_{ij}) \neq 0$ is real symmetric $n \times n$-matrix with the norm
\[
||L|| \equiv \sqrt{\frac{1}{2} (tr L)^2 + \frac{1}{2} tr(L^2)}.
\]
We call matrix $L$ as gravitational length matrix.

Now we consider the interaction between two multitemporal hedgehogs with gravitational length matrices $L_1 = (L_{1,ij})$ and $L_2 = (L_{2,ij})$ located at large distances from each other
\[
||\vec{x}| | \gg ||L_1||_1, ||L_2||_2, \quad \vec{x} \equiv \vec{x}_1 - \vec{x}_2.
\]
We begin with the simplest case $n = 1$. In Newton’s mechanics the equations of motion for two point-like masses $M_1 = L_1/2G$ and $M_2 = L_2/2G$ with world lines $\vec{x}_1 = \vec{x}_1(t)$ and $\vec{x}_2 = \vec{x}_2(t)$ respectively are well-known:
\[
\frac{d^2 \vec{x}_1}{dt^2} = -\frac{M_2}{2||\vec{x}||^3} \vec{x}, \quad (4.4)
\]
\[
\frac{d^2 \vec{x}_2}{dt^2} = \frac{M_1}{2||\vec{x}||^3} \vec{x}, \quad (4.5)
\]
where \( \tilde{x} \) is defined in (4.3). The equations (4.4), (4.5) may be obtained from the Einstein equations, when the solutions describing the post-Newtonian (4.3), non-relativistic motion

\[
\left| \frac{d\tilde{x}_a}{dt} \right| \ll 1,
\]

(4.6)
a = 1, 2, of two black holes are considered.

Our hypothesis is that the generalization of this scheme to the multitemporal case should lead to the following equations of motion for two non-relativistic hedgehogs with gravitational length matrices \( L_1 \) and \( L_2 \) in the post-Newtonian approximation (4.3)

\[
\frac{d^2\tilde{x}_1}{dt^2} = -L_{2,ij} \frac{\tilde{x}}{2|\tilde{x}|^3},
\]

(4.7)
\[
\frac{d^2\tilde{x}_2}{dt^2} = L_{1,ij} \frac{\tilde{x}}{2|\tilde{x}|^3}.
\]

(4.8)
The functions \( \tilde{x}_a = \tilde{x}_a(t_1, \ldots , t_a), a = 1, 2, \) describe the world surfaces of two multitemporal objects in the considered approximation. The multitemporal analogue of the non-relativistic condition (4.6) reads

\[
\left| \frac{d\tilde{x}_a}{dt^i} \right| \ll 1,
\]

(4.9)
a = 1, 2, \( i = 1, \ldots , n \). Defining gravitational mass matrices

\[
(M_{a,ij}) = (L_{a,ij}/2G),
\]

(4.10)
and forces

\[
\tilde{F}_{a,ij} = M_{a,ij} \frac{d^2\tilde{x}_a}{dt^i dt^j},
\]

(4.11)
a = 1, 2, we get

\[
\tilde{F}_{1,ij} = -GM_{1,ij}M_{2,ij} \frac{\tilde{x}}{|\tilde{x}|^3},
\]

(4.12)
\[
\tilde{F}_{1,ij} = -\tilde{F}_{2,ij},
\]

(4.13)
i, j = 1, \ldots , n. Relations (4.11), (4.12) and (4.13) are multitemporal analogues of the Newton’s laws, describing the multitemporal “motion” of two interacting non-relativistic hedgehogs in the post-Newtonian approximation. (The generalization to multi-hedgehog case is quite transparent.) We note, that for \( \tilde{F}_1 = tr(\tilde{F}_{1,ij}) \) we get the formula suggested previously in [15]

\[
\tilde{F}_1 = -Gtr(M_1M_2) \frac{\tilde{x}}{|\tilde{x}|^3}.
\]

(4.14)

**Scalar-vacuum generalization.** The solution (2.1) can be easily generalized on scalar-vacuum case. In this case the field equations corresponding to the action

\[
S = \frac{1}{2} \int d^Dx \sqrt{|g|} [\tilde{R}[g] - \partial_M \varphi \partial^N \varphi g^{MN}],
\]

(4.15)
are satisfied for the metric (2.1) and the scalar field
\[ \varphi = \frac{1}{2}\gamma \ln(1 - \frac{L}{R}) + \text{const}, \]  
(4.16)
with the parameters related as following
\[ b = \sum_{i=1}^{n} a_i, \quad b^2 + \sum_{i=1}^{n} a_i^2 + q^2 = 2. \]  
(4.17)
This solution is a special case of the solution [16] or more general dilatonic-electro-vacuum solution [14,17].

Conclusion

In this paper we considered the multitemporal generalization of the Schwarzschild solution. We integrated the equations of geodesics for the metric and considered the motion of relativistic particle in the background, corresponding to the metric. We obtained the modification of Newton’s law for interaction of massive non-relativistic particle with multitemporal hedgehog (i.e., extended in time object, described by the solution). We also suggested multitemporal analogues of Newton’s formulas for non-relativistic motion of interacting hedgehogs. We note, that the main difference of the multitemporal (n-time) case from the ordinary n = 1 case is following: in the space-time with n time coordinates the gravitational and inertial masses are n × n matrices, and the energy of a relativistic particle is the n-component vector.

References


