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Quantum Superspin Chains

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Abstract

Supersymmetric formulations of spin- $\frac{1}{2}$ quantum chains, associated with integrable models, are considered. One class of super-extensions, based on low-dimensional classical and exceptional superalgebras containing $sl(2)$, is illustrated with the case of $osp(1/2)$. A more radical generalization, in which the algebra of Pauli matrices is identified with the algebra of supersymmetric quantum mechanics, is also presented, and some of its algebraic properties discussed.

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Introduction

The study of quantum spin chains has historical antecedents in the Heisenberg spin- $\frac{1}{2}$ model of (anti)-ferromagnetism,

$$\mathcal{H} = J \sum_n \sigma_n \cdot \sigma_{n+1}, \quad (1)$$

and aside from physical applications, provides an arena in which many aspects of statistical mechanics and related questions of two dimensional conformal field theory can be examined, often with the benefit of exactly solvable models as paradigms[1]. Amongst the latter are a multitude of 'higher spin' variants[2], with which are associated in special cases dynamical symmetry algebras responsible for integrability[3], and which continue to be investigated[4, 5]. A further algebraic step involves the replacement of the spins σ_n in (1) by generators of an arbitrary Lie algebra[6]. More generally, supersymmetric formulations can be considered, and in this contribution we discuss various possibilities which generalize different features of (1). In the sequel, two main types will be presented, along with some of their algebraic properties, and a comparison of some of their salient features with other studies of supersymmetric quantum chains[7, 8, 9].

I Low-Dimensional Superalgebras

An obvious super-algebraic extension involves the replacement of the spins at each site, by the generators of an arbitrary simple Lie superalgebra. However, in order to retain the physical feature of (1), that it incorporates a spin-spin interaction, it is natural to look for Lie superalgebras in which the rotation group, or at least $sl(2)$, occurs in a prominent or natural way. In practice, this suggests low-dimensional classical or exceptional superalgebras, such as $osp(1/2)$, $sl(2/1)$, $osp(3/2)$, $D(2,1;\alpha)$, $G(3)$ or $F(4)$ [10].

Rather than attempt a systematic investigation, we take up the simplest case of the five-dimensional di-spin algebra $osp(1/2)$ [11, 12] as an illustrative example. This is generated by the $su(2) \cong sp(2)$ generators $S_i \equiv J_i$, $i = 1, 2, 3$, accompanied by odd generators S_α , $\alpha = 4, 5$ such that ($\epsilon_{45} \equiv 1 \equiv \epsilon^{54}$)

$$\begin{aligned} [J_i, J_j] &= i\epsilon_{ijk} J_k \\ [J_i, S_\alpha] &= -\frac{1}{2}(\sigma_i)_\alpha^\beta S_\beta \\ \{S_\alpha, S_\beta\} &= (\sigma_i \epsilon)_{\alpha\beta} J_i \end{aligned}$$

One important feature of (1), which is sufficient for integrability in the spin- $\frac{1}{2}$ case[2], is the fact that H can be regarded essentially as the sum of a set of two-particle projection operators which satisfy the Temperley-Lieb algebra (see below). For the present case of $osp(1/2)$, we demonstrate the corresponding statement (which has also been established for simple Lie

algebras[6]), and display the associated Hamiltonian, for the fundamental representation (of dimension three). (Physically, the two spin- $\frac{1}{2}$ states at each site also become coupled to an additional scalar (spin-0) state of opposite statistics).

Irreducible representations of $osp(1/2)$ are characterized by a ‘superspin’ $\mathcal{J}$, corresponding to $sl(2)$ content $\mathcal{J}$, $\mathcal{J} - \frac{1}{2}$ (unless $\mathcal{J} = 0$) with dimension $4\mathcal{J} + 1$ and Casimir $C = \mathcal{J}(\mathcal{J} + \frac{1}{2})$, where

$$C = \mathbf{S} \cdot \mathbf{S} \equiv J_i J_i - \varepsilon^{\alpha\beta} S_\alpha S_\beta$$

In the case of the tensor product of superspin $\mathcal{J}^1$ with superspin $\mathcal{J}^2$, for total superspin $\mathcal{J}$ we have

$$X = 2\mathbf{S}^1 \cdot \mathbf{S}^2 = \mathcal{J}(\mathcal{J} + \frac{1}{2}) - \mathcal{J}^1(\mathcal{J}^1 + \frac{1}{2}) - \mathcal{J}^2(\mathcal{J}^2 + \frac{1}{2})$$

As an example, let us consider the 3-dimensional fundamental representation with superspin $\mathcal{J} = \frac{1}{2}$. The product of two superspin- $\frac{1}{2}$ irreps yields $\mathcal{J} = 0, \frac{1}{2}$, or 1, and so X has corresponding eigenvalues $-1, -\frac{1}{2}, \frac{1}{2}$ respectively, from which the superspin-0 projector is

$$P = \frac{4}{3}X^2 - \frac{1}{3} \quad (2)$$

Acting on an open chain of L sites, we may construct projectors

$$P_{n,n+1} = 1 \otimes \cdots \otimes 1 \otimes P \otimes 1 \otimes \cdots \otimes 1$$

and consider the corresponding Hamiltonian

$$\mathcal{H} = - \sum_{n=1}^{L-1} P_{n,n+1}$$

It can be verified from (2) that*

$$\begin{aligned} P_{n,n+1}^2 &= P_{n,n+1} \\ P_{n,n+1} P_{n+1,n+2} P_{n,n+1} &= P_{n,n+1} \\ [P_{m,m+1}, P_{n,n+1}] &= 0 \text{ if } |m - n| > 1 \end{aligned} \quad (3)$$

That is, the $P_{n,n+1}$ satisfy the Temperley-Lieb algebra with parameter $\sqrt{\beta} = 1$, and hence by general arguments are associated with a solution of the Yang-Baxter equation[2]. In fact, in the present case

$$\begin{aligned} J_i &\rightarrow \frac{1}{2}\tilde{\sigma}_i \equiv \frac{1}{2} \begin{pmatrix} 0 & \\ & \sigma_i \end{pmatrix}, \\ S_4 &\rightarrow \begin{pmatrix} 0 & 0 & \frac{1}{2} \\ \frac{1}{2} & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad S_5 \rightarrow \begin{pmatrix} 0 & -\frac{1}{2} & 0 \\ 0 & 0 & 0 \\ \frac{1}{2} & 0 & 0 \end{pmatrix}, \end{aligned}$$

*The graded nature of the tensor product space must be used in evaluating matrix elements of X

and introducing a convenient basis for 3×3 matrices via

$$\begin{aligned} Y &= \begin{pmatrix} 1 & 0 & 0 \\ 0 & -\frac{1}{2} & 0 \\ 0 & 0 & -\frac{1}{2} \end{pmatrix}, \\ Q_4 &= \begin{pmatrix} 0 & 0 & 0 \\ \frac{1}{2} & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad Q_5 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ \frac{1}{2} & 0 & 0 \end{pmatrix}, \\ \bar{Q}_4 &= \begin{pmatrix} 0 & \frac{1}{2} & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad \bar{Q}_5 = \begin{pmatrix} 0 & 0 & \frac{1}{2} \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \end{aligned}$$

we find, up to an additive constant,

$$P_{n,n+1} = -\frac{1}{2}\tilde{\sigma}_n \cdot \tilde{\sigma}_{n+1} - \varepsilon^{\alpha\beta}(Q_\alpha^n Q_\beta^{n+1} + \bar{Q}_\alpha^n \bar{Q}_\beta^{n+1}) + \frac{2}{3}Y_n Y_{n+1} \quad (4)$$

Thus in this example, the familiar spin-spin interaction is accompanied by contributions which weight the occupancy of the even and odd subspaces, and also by supersymmetric terms which measure the projections of the three-dimensional ‘superspin’ onto these subspaces. Aside from its $osp(1/2)$ symmetry, the Hamiltonian (4) is manifestly a function of $sl(2/1)$ generators at each site, and is therefore similar to the $t - J$ models (related to the Hubbard model for strong electron correlations) considered for example, in [13, 14, 15, 16, 17, 18, 19]. In the one-dimensional case, the continuum limit has been given via non-linear realizations of $osp(1/2, \mathbb{C})$, $sl(2/1) \supset gl(1) \times sl(2)$ or $sl(2/1) \supset s(gl(1/1) \times gl(1))$, which are integrable via the Landau equations and super-generalizations of the non-linear Schrödinger hierarchy. In the two-dimensional case, the ground states have been discussed in terms of topological solutions of an effective Chern-Simons gauge theory.

As presented here, the integrability of (4) derives from the existence of the Temperley-Lieb algebra. However, whereas *isotropic* Hamiltonians[20, 21] arising from zero spin projections of spin s have Jones parameter $\sqrt{\beta} = 2s + 1$, the chain (4) has $\sqrt{\beta} = 1$; that is, the supersymmetry is responsible for cancellation between the even and odd states at each site yielding an ‘effective’ spin $s = 0$. In fact, there exist Temperley-Lieb formulations [22, 23] of *anisotropic* (XXZ) models with generic $\sqrt{\beta}$, in particular with $\sqrt{\beta} = 1$. Thus it seems likely, that our *isotropic* $osp(1/2)$ superspin chain is in the same universality class as the corresponding XXZ chain. In particular, one might expect the Hamiltonian provided by (4) to be massless from the Temperley-Lieb equivalence.

A difference of our model (4) from the $t - J$ models, is that the supersymmetric couplings are not number-preserving in the usual sense. In conventional notation, the QQ terms are of the form

$$\sum_{n,\sigma} \sigma c_{n,\sigma}^\dagger c_{n+1,\sigma'}^\dagger + \sigma c_{n,\sigma} c_{n+1,\sigma'}$$

(where $\sigma = \pm$ and $\sigma' = -\sigma$), as against the normal “hopping” terms of the form

$$\sum_{n,\sigma} c_{n,\sigma}^\dagger c_{n+1,\sigma} + c_{n,\sigma} c_{n+1,\sigma}^\dagger$$

in which a spin σ destroyed at one site is created at the neighbouring site. However, to the extent that novel mechanisms for high temperature superconductivity already invoke modified statistics and separation of spin and charge degrees of freedom, the chain (4) provides a new integrable and supersymmetric model system.

II Supersymmetric Quantum Mechanics

For the second class of supersymmetric extensions of (1), we point out an alternative, very natural way in which the algebraic structure can be generalized, and for which, many realizations are possible. Starting with the basic spin- $\frac{1}{2}$ models, consider the following properties of the site operators:

$$\begin{aligned} \{\sigma^x, \sigma^x\} &= 2 = \{\sigma^y, \sigma^y\}, \\ \{\sigma^x, \sigma^y\} &= 0, \\ [\sigma^x, \sigma^y] &= 2i\sigma^z. \end{aligned} \quad (5)$$

Thus the 2×2 Pauli matrices may themselves be regarded, not as generators of a Lie algebra, nor as a Clifford algebra, but rather as a prototype of ($N = 2$) *supersymmetric quantum mechanics* in which the role of the two real supercharges is played by σ^x and σ^y (and the Hamiltonian is the unit matrix). Thus we consider models in which there are ‘spins’ Q^x, Q^y , commuting on different sites, and providing a realization of the algebra

$$\begin{aligned} \{Q^x, Q^x\} &= 2H = \{Q^y, Q^y\}, \\ \{Q^x, Q^y\} &= 0, \\ [H, Q^x] &= 0 = [H, Q^y]. \end{aligned} \quad (6)$$

Writing

$$\begin{aligned} Q &\equiv \frac{1}{2}(Q^x - iQ^y), \\ Q^\dagger &\equiv \frac{1}{2}(Q^x + iQ^y), \end{aligned}$$

we have the usual algebra of supersymmetric quantum mechanics[24],

$$\begin{aligned} \{Q, Q^\dagger\} &= H, & (Q)^2 &= 0 = (Q^\dagger)^2, \\ [H, Q] &= 0 = [H, Q^\dagger]. \end{aligned} \quad (7)$$

For the algebra (6) we can define an operator which plays the role of σ^z simply as the commutator

$$Q^z = \frac{1}{2i}[Q^x, Q^y].$$

Hence the complete structure of the spin- $\frac{1}{2}$ models can be lifted to this more general case, giving for example XXZ Heisenberg quantum chains with “site” supersymmetry.

To illustrate the discussion with a concrete example, we take the simplest case of the analogue to the Ising quantum spin chain[25] with L sites and zero external field:

$$H = i \sum_{j=1}^L Q_j^x Q_j^y - \lambda \sum_{j=1}^L Q_j^x Q_{j+1}^x, \quad |\lambda| \leq 1 \quad (8)$$

which leads to the following hermitean Hamiltonian (compare [25]),

$$H = - \sum_{j=1}^L [Q_j^\dagger, Q_j] - \lambda \sum_{j=1}^L (Q_j^\dagger + Q_j) (Q_{j+1}^\dagger + Q_{j+1}) \quad (9)$$

For the algebra (6) we can define a *number* operator N which has eigenvalues 0, 1 by means of†

$$N_j = \frac{1}{2} H_j^{-1} ([Q_j^\dagger, Q_j] + 1) = H_j^{-1} Q_j^\dagger Q_j, \quad (10)$$

This operator satisfies the usual relations

$$[N_j, Q_k] = -\delta_{jk} Q_k \quad [N_j, Q_k^\dagger] = \delta_{jk} Q_k^\dagger \quad (11)$$

as well as

$$[H_j, N_k] = 0 \quad (12)$$

(so that $N_j, Q_j, Q_j^\dagger$ and H_j generate a $gl(1/1)$ algebra at each site). This allows us to perform a Jordan-Wigner transformation upon the operators appearing in Hamiltonian (9) by defining operators

$$\begin{aligned} s_j &= \exp[i\pi \sum_{k<j} N_k] Q_j \\ s_j^\dagger &= \exp[-i\pi \sum_{k<j} N_k] Q_j^\dagger \end{aligned}$$

which satisfy the anti-commutation relations

$$\begin{aligned} \{s_j, s_k\} &= 0 = \{s_j^\dagger, s_k^\dagger\} \\ \{s_j^\dagger, s_k\} &= \delta_{jk} H_k \end{aligned} \quad (13)$$

In terms of these fermionic operators, assuming for simplicity periodic boundary conditions, the Hamiltonian becomes

$$\begin{aligned} H &= - \sum_{j=1}^L (s_j^\dagger s_j - s_j s_j^\dagger) - \lambda \sum_{j=1}^{L-1} (s_j^\dagger - s_j)(s_{j+1}^\dagger + s_{j+1}) \\ &\quad - (-1)^M \lambda (s_L s_1 - s_L^\dagger s_1^\dagger + s_L s_1^\dagger - s_L^\dagger s_1) \end{aligned} \quad (14)$$

†Formally, N_j is defined by (10) only on the positive part of the spectrum of H_j (the case of broken supersymmetry [26]). In the unbroken case, an extension of N_j whose domain includes the zero-energy ground state $|0\rangle_j$ can be made consistently with (11), (12) since $Q_j|0\rangle_j = 0 = Q_j^\dagger|0\rangle_j$.

where $\mathcal{M} = \sum_{i=1}^L N_i$. The Fock space decomposes into a direct sum of eigenspaces of the operator $(-1)^{\mathcal{M}}$. These are the space of even (resp. odd) number of excitations with eigenvalues 1 (-1). The Hamiltonian (14) takes a different form on each of these subspaces which we shall denote by $\mathbf{H}_e$ ($\mathbf{H}_o$). This is equivalent to re-writing (14) as a mixed-sector Hamiltonian[27]

$$\mathbf{H} = \mathbf{H}_e P_e + \mathbf{H}_o P_o$$

where P_e (P_o) is the projector onto the space of even (odd) occupation number. Certainly (14) reduces down to the Ising Hamiltonian for the choice $Q^x = \sigma^x$, $Q^y = \sigma^y$, (whence $\mathbf{H}$ is the 2×2 identity matrix).

Thus far, the discussion of the present class of models with “site” supersymmetry has been introduced at an abstract level, and it is useful to discuss concrete examples. Many models of supersymmetric quantum mechanics are known[24, 26, 28, 29, 30], but in general, in the finite-dimensional case, there are d pairs of degenerate levels at energies $\varepsilon_d \geq \varepsilon_{d-1} \geq \dots \geq \varepsilon_1 > 0$, and (in the unbroken case) a possible additional nondegenerate zero-energy state. In view of (7), a convenient normalization leads to the representation (see Figure 1. below)

$$Q^\dagger = \begin{pmatrix} 0 & 0 & \dots & 0 & 0 & \dots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & \dots & 0 & 0 & \dots & 0 \\ \sqrt{\varepsilon_1} & 0 & 0 & \dots & 0 & \dots & 0 \\ 0 & \sqrt{\varepsilon_1} & 0 & \dots & 0 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \sqrt{\varepsilon_d} & 0 & \dots & 0 \end{pmatrix} \quad (15)$$

(with an additional zero $(d+1)$ st row and column inserted in the unbroken case). Thus the spectrum of (14) may be discussed in terms of a reducible $2d$ dimensional $gl(1/1)$ representation realized by the supercharges (15).

Returning to the general analysis of models of type (8), (9) it might seem that one could now proceed to carry out a Bogoliubov transformation, in order to diagonalise such Hamiltonians in the usual way[31]. However, one soon runs into trouble. For example, suppose there existed operators t_i , $t_i^\dagger$ and G_i given by

$$\begin{aligned} t_k &= \sum_i \frac{\phi_{ki} + \psi_{ki}}{2} s_i + \frac{\phi_{ki} - \psi_{ki}}{2} s_i^\dagger \\ t_k^\dagger &= \sum_i \frac{\phi_{ki} + \psi_{ki}}{2} s_i^\dagger + \frac{\phi_{ki} - \psi_{ki}}{2} s_i \\ G_k &= \sum_i \gamma_{ki} H_i \end{aligned} \quad (16)$$

which satisfied the anti-commutation relations (13). Trying to satisfy the first of these relations, we are left with the constraint

$$\sum_i (\phi_{ki} \phi_{li} - \psi_{ki} \psi_{li}) H_i = 0 \quad \text{for any } 1 \leq k, l \leq L$$

which (along with the other constraints on ϕ_{ki} , ψ_{ki} and γ_{ki}) does not allow the usual conclusion that the (numerical) vectors $\underline{\phi}_k$ (as well as $\underline{\psi}_k$) are orthogonal, which plays a big part in diagonalization process in the normal Ising case.

Figure 1. The $2d$ -dimensional representation of $gl(1/1)$. The operators Q and $Q^\dagger$ act horizontally. The vertical labels are the eigenvalues of the operator $Q^x = H(2N - 1)$.

Nevertheless, as we shall now see, we can argue on heuristic grounds that the spectrum will consist of a *Gemisch* (mixture) of different Ising-like spectra. Consider, as an example, the 10-dimensional representation of $gl(1/1)$ on a chain with four sites, and consider the state consisting of the spins $\{+\varepsilon_4, -\varepsilon_2, -\varepsilon_3, +\varepsilon_2\}$ (see Figure 2). Now because the supercharges Q and $Q^\dagger$ just swap between the $N = 1$ and $N = 0$ sectors of each site, the Hamiltonian (14) operates on this state, producing a linear combination of the $4^2 = 16$ states $\{\pm\varepsilon_4, \pm\varepsilon_2, \pm\varepsilon_3, \pm\varepsilon_2\}$. In fact these 16 states form an invariant subspace of the 10^4 -dimensional total state space.

Figure 2. The state $\{+\varepsilon_4, -\varepsilon_2, -\varepsilon_3, +\varepsilon_2\}$ of a 4-site chain with a 10-dimensional representation of $gl(1/1)$ at each site.

In general if we have a sequence of integers $\alpha = (\alpha_1, \alpha_2, \dots, \alpha_L)$ where

$1 \leq \alpha_i \leq d$ for all i , then each of the d^L possible subspaces of spins $\{\pm \varepsilon_{\alpha_1}, \pm \varepsilon_{\alpha_2}, \dots, \pm \varepsilon_{\alpha_L}\}$ will be invariant under this Hamiltonian. Thus we can block diagonalise it into d^L Hamiltonians $H^{(\alpha)}$ which will be $2^L \times 2^L$ matrices. Moreover, we can now define

$$a_i = \frac{1}{\sqrt{\epsilon_{\alpha_i}}} s_i \quad a_i^\dagger = \frac{1}{\sqrt{\epsilon_{\alpha_i}}} s_i^\dagger \quad (17)$$

which obey ordinary fermionic anti-commutation relations. Thus following the usual trick of Lieb, Schultz and Mattis, and considering the Hamiltonian (14) acting on the states of even occupation number, we can write the Hamiltonian $H_{\epsilon}^{(a)}$ as

$$\mathcal{H}_e^{(\alpha)} = - \sum_{ij} A_{ij} a_i^\dagger a_j + \frac{1}{2} B_{ij} (a_i^\dagger a_j^\dagger - a_i a_j) + \text{const.}$$

where the $L \times L$ matrices A and B are given by

$$A = \begin{pmatrix} -\varepsilon_{\alpha_1} & -\frac{\lambda}{2}\sqrt{\varepsilon_{\alpha_1}\varepsilon_{\alpha_2}} & 0 & 0 & \dots & 0 & 0 & -\frac{\lambda}{2}\sqrt{\varepsilon_{\alpha_1}\varepsilon_{\alpha_L}} \\ & -\varepsilon_{\alpha_2} & -\frac{\lambda}{2}\sqrt{\varepsilon_{\alpha_2}\varepsilon_{\alpha_3}} & 0 & \dots & \dots & \dots & \dots \\ & & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots \\ & & & \ddots & \ddots & \ddots & \ddots & \ddots \\ & & & & & & -\varepsilon_{L-1} & -\frac{\lambda}{2}\sqrt{\varepsilon_{\alpha_{L-1}}\varepsilon_{\alpha_L}} \\ & & & & & & & -\varepsilon_L \end{pmatrix}$$

with A being symmetric, and B anti-symmetric. The eigenvalues of the Hamiltonian $H_e^{(\alpha)}$ can then be written down once the eigenvalues of the matrix $(A+B)(A-B)$ are known. Unfortunately the exact determination of these eigenvalues is not possible for a general sequence α (the exception being when $\alpha_1 = \alpha_2 = \dots = \alpha_L$, in which case this matrix becomes cyclic, as in the Ising case). Nevertheless the lowest eigenvalue of the complete Hamiltonian can be calculated (it occurs for the configuration $\alpha = (d, d, d, \dots, d)$) and, after the Hamiltonian is correctly normalized, it turns out that the central charge of the model is as it is for the Ising case, i.e., $c = \frac{1}{2}$.

It is evident from the form of (15) that there is a natural organizing superalgebra in the present models; indeed (15) can be viewed as providing an embedding in $gl(d/d)$ or $gl(d+1/d)$ of the $gl(1/1)$ algebra generated by Q , $Q^\dagger$, H and N . Since the fundamental $2d$ or $2d+1$ -dimensional representation is reducible, this does not lead directly to algebraic methods of establishing the solvability of the model (for example, the projector formalism of section

I cannot be applied easily). However, the previous discussion can be put in a more algebraic framework by considering appropriate dynamical algebras in the chains

In this language, the Bogoliubov transformation of (16) was an attempt to diagonalize $\mathbf{H}$ in the basis of L copies of $gl(1/1)$ generated by Q_j , $Q_j^\dagger$, H_j and N_j , whereas the discussion of Ising-like sectors leading to (17) can be seen as the observation that (8), (9) are diagonalized in the basis of $gl(2^{L-1}/2^{L-1})$. For each of the d^L sectors (α), and at each site, the $gl(1/1)$ algebra is equivalent via (17), to a fermion algebra for which the usual Bogoliubov transformation is guaranteed. Nevertheless, the role of $gl(d/d)$ or $gl(d+1/d)$ may be important in considering boundary conditions. Thus if $S \rightarrow \hat{S} \equiv U^\dagger S U$ is a superalgebra automorphism, the identification of the spin at site $L+1$ with that at site 1 via $v_{L+1} = U v_1$ essentially entails the boundary condition $Q_{L+1} = \hat{Q}_1$, and leads to a mixing of the separate "Ising-like" sectors discussed above. A similar involvement of such automorphisms in defects in this type of superspin chain can also be expected.

As emphasized in the introduction to this section, (8), (9) is the simplest case of a broad class derivable from spin- $\frac{1}{2}$ models wherein the Pauli algebra is supplanted by the algebra of supersymmetric quantum mechanics[†], and the general question of integrability needs to be investigated. From the analysis in terms of $gl(1/1)$ presented above, for example, there is no obvious route to an R -matrix which solves the Yang-Baxter equation. However, the emergence of free fermion algebras in the above example suggests a fundamental connection which may hold in general. On the other hand, certain realizations of supersymmetry [30] have a natural relationship with Lie algebras and hence, for $su(2)$, with spin models: for example, if $Q = a^\dagger J_-$ and $Q^\dagger = a^\dagger J_+$, then there are $2(2s+1)$ states at each site and the model (8), (9) would resemble a mixed spin system[5].

[†]Generalizations involving extended, deformed or para-supersymmetry will not be discussed here.

III Conclusions

In this note, we have considered two types of integrable generalization of the original spin- $\frac{1}{2}$ Heisenberg model of ferromagnetism to “quantum super-spin chains”. In the first class of models (I), the spin-spin nearest neighbour interaction is accompanied by additional terms measuring the relationship between the spin at each site, and additional graded components, corresponding to extending $sl(2)$ to a (simple) superalgebra. The principle was illustrated in the case of $osp(1/2)$ (with Hamiltonian (4)), which lead to a model with $sl(2/1)$ symmetry similar to that considered in reference[16].

For the second class of models (II) the structure of the Pauli matrices was viewed as a prototype of the algebra of supersymmetric quantum mechanics, leading to a very general *Ansatz* for the quantum superchain Hamiltonian with many realizations. Although the models were all shown to be Ising-like in their critical behaviour (for periodic boundary conditions), the possibility was suggested that different types of boundary conditions, or defects, could mix the different Ising-like sectors[27].

It should be pointed out that the role of the superalgebra $gl(M/N)$ is here different from that in the models with dynamical Hecke algebras [7, 8, 9], in which $M, N \rightarrow \infty$ in the thermodynamic limit. In the present case, there is an embedding $gl(1/1) \subset gl(d+1/d)$ or $gl(d/d)$ where the dimension of the spin space at each site is fixed as $2d+1$ or $2d$.

$\hat{R}$ matrices associated to irreducible representations of quantum superalgebras have been studied[32], in particular, those related to the quantum superalgebra $U_q(osp(1/2))$ [33, 34]. In the latter reference, where the spin chain Hamiltonian obtained from the $\hat{R}$ matrix for the spin- $\frac{1}{2}$ representation was written as a linear combination of the projectors $P_{1/2}$ and P_1 of the relevant tensor product decomposition, it was shown that the $\hat{R}$ matrix for the superspin- $\mathcal{J}$ representation of $U_q(osp(1/2))$ was related by a gauge transformation to spin- $2\mathcal{J}$ representation of $U_{q,1/2}(sl(2))$. In this work, we have worked to generalize the term $\sigma \cdot \sigma$ in (1) via the Temperley-Lieb algebra, as in reference [6]. As a result, we have not concerned ourselves directly with anisotropic models which would arise from considerations of quantum (super)algebras. Similarly, we have not considered models related to colour[35, 36, 37] or parastatistics.

In relation to the original work on supersymmetry and spin systems, and generally on lattice supersymmetry[24, 38], these models, of the form $H = \sum_n (a_n^* a_n + \psi_n^* \psi_n)$, are closer in the spirit to the models I, in that bosonic and fermionic operators explicitly appear at each site (nearest neighbour couplings would arise from kinetic terms etc, in the continuum), despite the fact that it is with models II, rather than I, that a form of supersymmetric quantum mechanics is realized.

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